Topological **M**ethods in **N**onlinear **A**nalysis
Volume 41, No. 1, 2013, 85–112

THE ROLE OF EQUIVALENT METRICS IN FIXED POINT THEORY

Adrian Petruşel — Ioan A. Rus — Marcel-Adrian Şerban

Abstract. Metrical fixed point theory is accomplished by a wide class of terms:

- operators (bounded, Lipschitz, contraction, contractive, nonexpansive, noncontractive, expansive, dilatation, isometry, similarity, Picard, weakly Picard, Bessaga, Janos, Caristi, pseudocontractive, accretive, etc.),
- convexity (strict, uniform, hyper, etc.),
- deffect of some properties (measure of noncompactness, measure of nonconvexity, minimal displacement, etc.),
- data dependence (stability, Ulam stability, well-posedness, shadowing property, etc.),
- attractor,
- basin of attraction ...

The purpose of this paper is to study several properties of these concepts with respect to equivalent metrics.

1. Introduction

Metrical structures play an important role in topology ([5], [9], [16], [17], [26], [32], [35], [55], [24], [66], [67], [82], [86], [101], [102], etc.), functional analysis

2010 *Mathematics Subject Classification.* 47H10, 54H25, 47J25, 47H04, 54E35.

Key words and phrases. Fixed point theory, topologically equivalent metric, strongly equivalent metric, Picard operator, weakly Picard operator, contraction, generalized contraction, well-posedness, shadowing property, stability, Ulam stability, invariant subset.

For the first and the third author this research work was supported by a grant of the Romanian Research Council.

and operator theory ([1], [3], [7], [10], [26], [30], [31], [33], [39], [42], [47], [53], [60], [72], [99], [115], etc.), as well as some other topics of pure and applied mathematics ([2], [6], [8], [11], [18], [53], [93], [111], [116], etc.).

In fixed point theory, the role of the metric is fruitful and complex and it is strongly defined by several terms, such as:

- convexity (strict, uniform, hyper, etc.),
- deffect of some properties (measure of noncompactness, measure of nonconvexity, minimal displacement, etc.),
- data dependence (stability, Ulam stability, well-posedness, shadowing property, etc.),
- attractor,
- basin of attraction...

The purpose of this paper is to study several properties of above mentioned terms with respect to equivalent metrics.

Throughout this paper, the standard notations and terminologies in nonlinear analysis are used. For the convenience of the reader we recall some of them.

Let X be a nonempty set and d, ρ be two metrics on X. Then, by definition, d, ρ are called:

(a) topologically equivalent if d and ρ define the same topology on X, i.e.

$$U \subset X \text{ is } d\text{-open} \Leftrightarrow U \subset X \text{ is } \rho\text{-open}.$$

(b) strongly (or Lipschitz or uniformly or metric) equivalent if there exists $c_1, c_2 > 0$ such that

$$c_1\rho(x,y) \leq d(x,y) \leq c_2\rho(x,y), \quad \text{for all } x, y \in X.$$

Notice that, if X is a nonempty set and d and ρ are two metrics on X such that d and ρ are strongly equivalent, then d and ρ are topologically equivalent too.

Let X be a nonempty set and $f\colon X \to X$ an operator. Then $f^0 := 1_X$, $f^1 := f$, $f^{n+1} := f \circ f^n$, $n \in N$ denote the iterate operators of the operator f. By $\mathrm{F}_f := \{x \in X \mid f(x) = x\}$ we will denote the fixed point set of the operator f.

Definition 1.1. Let (X, d) be a metric space. An operator $f\colon X \to X$ is *Picard operator* (briefly PO) if:

(a) $\mathrm{F}_f = \{x^*\}$;
(b) $(f^n(x))_{n\in\mathbb{N}} \to x^*$ as $n \to \infty$, for all $x \in X$.

Definition 1.2. Let (X, d) be a metric space. An operator $f\colon X \to X$ is *weakly Picard operator* (briefly WPO) if the sequence $(f^n(x))_{n\in N}$ converges for all $x \in X$ and the limit (which may depend on x) is a fixed point of f.

If $f: X \to X$ is a WPO, then we may define the operator $f^\infty: X \to X$ by $f^\infty(x) := \lim_{n\to\infty} f^n(x)$. Obviously $f^\infty(X) = \mathrm{F}_f$. Moreover, if f is a PO and we denote by x^* its unique fixed point, then $f^\infty(x) = x^*$, for each $x \in X$.

Definition 1.3. Let (X, d) be a metric space and $f: X \to X$ be a WPO. Then, f is called a *ψ-weakly Picard operator* (briefly ψ-WPO) if and only if $\psi: \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing and continuous in 0 function with $\psi(0) = 0$ and

$$d(x, f^\infty(x)) \le \psi(d(x, f(x))), \quad \text{for all } x \in X.$$

In particular, if there exists $c > 0$ such that $\psi(t) = ct$, for all $t \in \mathbb{R}_+$, then f is said to be a c-WPO.

For the theory of POs and WPOs see [88]. See also [90], [94], [96], [98] and [99]. For the convergence of the iterates see [8], [10], [63], [94], [107], [4].

Let (X, d) be a metric space. We will also use the following symbols:
$P(X) = \{Y \subset X \mid Y \text{ is nonempty}\}$, $P_{\mathrm{b}}(X) := \{Y \in P(X) \mid Y \text{ is bounded}\}$,
$P_{\mathrm{cl}}(X) := \{Y \in P(X)\ Y \text{ is closed}\}$, $P_{\mathrm{b,cl}}(X) := P_{\mathrm{b}}(X) \cap P_{\mathrm{cl}}(X)$,
$P_{\mathrm{cp}}(X) := \{Y \in P(X) \mid Y \text{ is compact}\}$.
If $T: X \to P(X)$ is a multivalued operator then the graph of the multifunction T is denoted by $\mathrm{Graph}(T) := \{(x, y) \in X \times X \mid y \in T(x)\}$. Also throughout the paper $\mathrm{F}_T := \{x \in X \mid x \in T(x)\}$ (respectively, $(\mathrm{SF})_T := \{x \in X \mid \{x\} = T(x)\}$) denotes the fixed point set, (respectively, the strict fixed point set) of the multivalued operator T.

The following (generalized) functionals are used in the main section of the paper.

- The *gap functional*:

$$D_d: P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}, \quad D_d(A, B) := \inf\{d(a, b) \mid a \in A,\ b \in B\}.$$

- The *δ generalized functional*:

$$\delta_d: P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}, \quad \delta_d(A, B) := \sup\{d(a, b) \mid a \in A,\ b \in B\}.$$

- The *excess generalized functional*:

$$\rho_d: P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}, \quad \rho_d(A, B) := \sup\{D_d(a, B) \mid a \in A\}.$$

- The *Hausdorff–Pompeiu generalized functional*:

$$H_d: P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}, \quad H_d(A, B) := \max\{\rho_d(A, B), \rho_d(B, A)\}.$$

Some important concepts are recalled now.

Definition 1.4. Let (X,d) be a metric space, and $T\colon X \to P_{\rm cl}(X)$ be a multivalued operator. By definition, T is a *multivalued weakly Picard* (briefly MWP) *operator* if for each $x \in X$ and each $y \in T(x)$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that:

(a) $x_0 = x$, $x_1 = y$;
(b) $x_{n+1} \in T(x_n)$, for each $n \in \mathbb{N}$;
(c) the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent and its limit is a fixed point of T.

Remark 1.5. A sequence $(x_n)_{n\in\mathbb{N}}$ satisfying the conditions (a) and (b) in the Definition 1.4 is called a sequence of successive approximations of T starting from $(x,y) \in \mathrm{Graph}(T)$.

If $T\colon X \to P(X)$ is a MWP operator, then we define $T^\infty\colon \mathrm{Graph}(T) \to P(\mathrm{F}_T)$ by the formula $T^\infty(x,y) := \{z \in \mathrm{F}_T \mid$ there exists a sequence of successive approximations of T starting from (x,y) that converges to $z\}$.

Definition 1.6. Let (X,d) be a metric space and $T\colon X \to P(X)$ be a MWP operator. Then, T is called a *ψ-multivalued weakly Picard operator* (briefly ψ-MWP operator) if and only if $\psi\colon \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing and continuous in 0 function with $\psi(0) = 0$ and there exists a selection t^∞ of T^∞ such that

$$d(x, t^\infty(x,y)) \le \psi(d(x,y)), \quad \text{for all } (x,y) \in \mathrm{Graph}(T).$$

In particular, if there exists $c > 0$ such that $\psi(t) = ct$, for all $t \in \mathbb{R}_+$, then T is said do be a c-MWP operator.

Let (X,d), (Y,ρ) be metric spaces and $T\colon X \to P_{\rm cl}(Y)$ be a multivalued operator. Then T is said to be an a-contraction if $a \in \,]0,1[$ and $H_\rho(F(x_1), F(x_2)) \le ad(x_1,x_2)$, for all $x_1, x_2 \in X$. Notice that, if T is a self multivalued contraction on X, then T is a $\frac{1}{1-a}$-MWP operator.

For the theory of multivalued weakly Picard operators see [76]. See also [78] and [99], [21], [43], [57], [72], [81], [84], [106].

For the metrical fixed point theory see [60], [78], [1], [10], [42], [47], [56], [62], [93], [94], [19], [20], [57], [59], [61], [74], [75], [91], [105], [109].

2. Constructing metrics with a given property

We start this section by presenting the following problem.

Problem 2.1. Let (X,d) be a metric space and $\theta\colon \mathbb{R}_+ \to \mathbb{R}_+$ a function. In which conditions on θ we have:

(a) $\theta \circ d$ is a metric on X?
(b) d and $\theta \circ d$ are topologically (respectively, strongly) equivalent metrics?

Let (X, d) be a metric space and $\theta\colon \mathbb{R}_+ \to \mathbb{R}_+$ a function. It is easy to see that $\theta \circ d$ satisfies the first axiom of the metric if and only if $\theta^{-1}(0) = \{0\}$ and the second axiom of the metric is always satisfied. The major problem remains to ensure the third axiom of the metric, namely the triangle inequality. We will need some notions and results.

Definition 2.2. A function $\theta\colon \mathbb{R}_+ \to \mathbb{R}_+$ is called a metric preserving function if and only if for every nonempty set X and every metric d on X, the mapping $\theta \circ d$ is a metric on X. Moreover, if for every nonempty set X and every metric d on X, the mapping $\theta \circ d$ is topologically equivalent with d, then the mapping $\theta\colon \mathbb{R}_+ \to \mathbb{R}_+$ is called a strongly metric preserving function.

The notion of the metric preserving function seems to be introduced for the first time by W.A. Wilson in [114], while the first detailed study of such functions was made by T.K. Sreenivasan in [104]. Since then many papers have been written concerning the properties and the characterization of these functions (see [14], [15], [22], [27]–[29], [108], [110]; see also [16], [17], [32], [35], [36]).

Recall also that $\theta\colon \mathbb{R}_+ \to \mathbb{R}_+$ is called an amenable function if and only if $\theta^{-1}(0) = \{0\}$. Throughout the paper we will denote by

$$\mathcal{A} = \{\theta\colon \mathbb{R}_+ \to \mathbb{R}_+ \mid \theta^{-1}(0) = \{0\}\}$$

the set of amenable functions.

Definition 2.3 (F. Terpe [108], P. Corazza [22]). A triple (a, b, c) of nonnegative real numbers is called a triangle triplet if and only if

$$a \le b + c, \quad b \le a + c, \quad c \le a + b$$

or equivalently

$$|a - b| \le c \le a + b.$$

We have the following characterization of metric preserving functions.

Theorem 2.4 (J. Borsík, J. Doboš [14], J. Doboš [27], P. Corazza [22]). *Let $\theta \in \mathcal{A}$. Then the following statements are equivalent*:

(a) *θ is metric preserving function*;
(b) *if (a, b, c) is a triangle triplet then so is $(\theta(a), \theta(b), \theta(c))$*;
(c) *if (a, b, c) is a triangle triplet then $\theta(a) \le \theta(b) + \theta(c)$*;
(d) *for every $a, b \in \mathbb{R}_+$ we have* $\max\{\theta(c) : |a - b| \le c \le a + b\} \le \theta(a) + \theta(b)$.

We have the following characterization of strongly metric preserving functions.

Theorem 2.5 (J. Borsík, J. Doboš [14], J. Doboš [27], P. Corazza [22]). *Let θ be a metric preserving function. Then the following statements are equivalent*:

(a) *θ is strongly metric preserving function*;
(b) *θ is continuous in* 0;
(c) *θ is continuous on* $\mathbb{R}_+$;
(d) *for each $\varepsilon > 0$ there exists an $a > 0$ with $\theta(a) < \varepsilon$.*

Let us consider now the second problem of this section.

Problem 2.6. Let (X, d) be a metric space and $f\colon X \to X$ an operator. In which condition on f there exists an equivalent (topologically, strongly) metric ρ on X with respect to which f is a contraction?

For this problem we can present the following well known relevant examples.

Example 2.7 (Bielecki (1956)). Let $K \in C([a;b] \times [a;b] \times \mathbb{R})$ satisfies the Lipschitz condition ($L_K > 0$)

$$|K(t,s,u) - K(t,s,u)| \le L_k \cdot |u - v|,$$

for all $t, s \in [a;b]$ and $u, v \in \mathbb{R}$. We consider on $C[a;b]$ the following metrics

$$d(x,y) = \max_{a \le t \le b} |x(t) - y(t)|$$

and, for $\tau > 0$,

$$d_\tau(x,y) = \max_{a \le t \le b} (|x(t) - y(t)| \cdot e^{-\tau(t-a)}).$$

The metrics d and d_τ are strongly equivalent. Let us consider on $C[a;b]$ the operator $A\colon C[a;b] \to C[a;b]$ defined by

$$A(x)(t) = \int_a^t K(t,s,x(s))\, ds, \quad t \in [a;b].$$

A lipschitz constant of the operator A with respect to the metric d is $L_k(b-a)$ and with respect to the metric d_τ is L_K/τ. So, for suitable τ, the operator A is contraction with respect to the metric d_τ.

For the Bielecki method see [23], [6], [45], [64], [31] and [85].

Example 2.8 (W. Walter (1976), E. Bohl (1970), J.K. Hale and O. Lopes (1972)). Let (X, d) be a metric space and $f\colon X \to X$ an operator. We suppose that:

(a) f is L_f-Lipschitz;
(b) there exists $n_0 \in \mathbb{N}^*$ such that f^{n_0} is $\sqrt[n_0]{\alpha}$-contraction.

Then the functional $\rho: X \times X \to \mathbb{R}_+$ defined by

$$\rho(x,y) = d(x,y) + \sqrt[n_0]{\alpha^{-1}} d(f(x), f(y)) + \ldots + \sqrt[n_0]{\alpha^{1-n_0}} d(f^{n_0-1}(x), f^{n_0-1}(y))$$

is a strongly equivalent metric with d and the operator f is α-contraction with respect to the metric ρ.

EXAMPLE 2.9 (L. Janos (1967)). Let (X,d) be a compact metric space and $f: X \to X$ an operator. We suppose that:

(a) f is continuous;
(b) $\bigcap\limits_{n\in\mathbb{N}} f^n(X) = \{x^*\}$.

Then the functional $\rho: X \times X \to \mathbb{R}_+$ defined by

$$\rho(x,y) = \sup\{d(f^n(x), f^n(y)) \mid n \in \mathbb{N}\}$$

is a topologically equivalent metric with d and the operator f is nonexpansive operator with respect to the metric ρ.

An abstract example was given by Meyers as follows.

THEOREM 2.10 (P.R. Meyers [70]). *Let (X,d) be a metric space and $f: X \to X$ be a continuous operator such that:*

(a) $\mathrm{F}_f = \{x^*\}$;
(b) $f^n(x) \to x^*$ *as* $n \to +\infty$;
(c) *there exists an open neighbourhood U of x^* with the property that for any open set V containing x^* there exists $n_0 \in \mathbb{N}$ such that $f^n(U) \subset V$ for all $n \geq n_0$.*

Then for each $\lambda \in (0;1)$ there exists a topologically equivalent metric d_λ on X such that f is λ-contraction.

First of all, we remark that the condition (c) from Theorem 2.10 is equivalent with

(c') *there exists $r > 0$ with the property that, for any $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that $f^n(B(x^*, r)) \subset B(x^*, \varepsilon)$ for all $n \geq n_\varepsilon$.*

The above result of Meyers lead us to our next problem.

PROBLEM 2.11. Let (X,d) be a metric space and $f : X \to X$ be a PO operator. Under which assumptions on f the condition (c) from Theorem 2.10 is satisfied?

THEOREM 2.12. *Let (X,d) be a complete metric space and $f: X \to X$ be a φ-contraction, i.e. $\varphi: \mathbb{R}_+ \to \mathbb{R}_+$ is a comparison function and*

$$d(f(x), f(y)) \leq \varphi(d(x,y)), \quad \textit{for all } x, y \in X.$$

Then for each $\lambda \in (0;1)$ there exists a topologically equivalent metric d_λ on X such that f is λ-contraction.

Proof. If $f\colon X \to X$ is a φ-contraction then f is PO, i.e. $\mathrm{F}_f = \{x^*\}$ and $f^n(x) \to x^*$ as $n \to +\infty$, for all $x \in X$, (see [99]). Let $r > 0$. Then, for $x \in B(x^*, r)$ we have

$$d(f^n(x), x^*) \le \varphi^n(d(x, x^*)) \le \varphi^n(r) \to 0 \quad \text{as } n \to +\infty.$$

Thus, given an $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that $\varphi^n(r) < \varepsilon$, for all $n \ge n_\varepsilon$. This implies that $f^n(B(x^*, r)) \subset B(x^*, \varepsilon)$, for all $n \ge n_\varepsilon$. Hence, by Theorem 2.10, we get the conclusion. □

We can prove a similar result for the case of Ćirić–Reich–Rus operators.

Theorem 2.13. *Let (X,d) be a complete metric space and $f\colon X \to X$ be a continuous operator for which there exist $a, b \in \mathbb{R}_+$ with $a + 2b < 1$ such that*

$$d(f(x), f(y)) \le a d(x,y) + b[d(x, f(x)) + d(y, f(y))],$$

for all $x, y \in X$. Then for each $\lambda \in (0;1)$ there exists a topologically equivalent metric d_λ on X such that f is λ-contraction.

Proof. If $f\colon X \to X$ is a Ćirić–Reich–Rus operator then f is PO, (see [99]). We denote by x^* the unique fixed point of f. First of all, we remark that

$$d(f(x), x^*) \le \frac{a+b}{1-b} \cdot d(x, x^*), \quad \text{for all } x \in X.$$

We have that $\alpha = (a+b)/(1-b) < 1$ and, now, we can apply the same technique as in the previous theorem. Let $r > 0$, we have for $x \in B(x^*, r)$

$$d(f^n(x), x^*) \le \alpha^n d(x, x^*) \le \alpha^n \cdot r \to 0 \quad \text{as } n \to +\infty,$$

So, given an $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that $f^n(B(x^*, r)) \subset B(x^*, \varepsilon)$, for all $n \ge n_\varepsilon$. The conclusion follows now by Theorem 2.10. □

We can formulate a more general result for Ćirić type operators.

Theorem 2.14. *Let (X,d) be a complete metric space and $f\colon X \to X$ be a continuous operator.*

(a) *Suppose there exists $a \in [0,1[$ such that, for all $x, y \in X$, we have:*

$$d(f(x), f(y)) \le a \max\{d(x,y), d(y, f(y)), d(x, f(y)), d(y, f(x))\}.$$

Then, for each $\lambda \in (0;1)$ there exists a topologically equivalent metric d_λ on X such that f is λ-contraction.

(b) *Suppose there exists $a \in [0, 1/2[$ such that, for all $x, y \in X$, we have:*

$$d(f(x), f(y)) \le a \max\{d(x,y), d(x, f(x)), d(y, f(y)), d(x, f(y)), d(y, f(x))\}.$$

Then, for each $\lambda \in (0;1)$ there exists a topologically equivalent metric d_λ on X such that f is λ-contraction.

Based on the above ideas, we will prove now an abstract result.

Theorem 2.15. *Let (X,d) be a metric space and $f : X \to X$ be a continuous* PO *for which there exist a comparison function $\psi\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$d(f(x),x^*) \le \psi(d(x,x^*)), \quad \text{for all } x \in X.$$

Then for each $\lambda \in (0;1)$ there exists a topologically equivalent metric d_λ on X such that f is λ-contraction.

Proof. Let $r > 0$. Then, for $x \in B(x^*,r)$ we have

$$d(f^n(x),x^*) \le \psi^n(d(x,x^*)) \le \psi^n(r) \to 0 \quad \text{as } n \to +\infty.$$

Thus, for $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that $\psi^n(r) < \varepsilon$, for all $n \ge n_\varepsilon$. Thus, $f^n(B(x^*,r)) \subset B(x^*,\varepsilon)$, for all $n \ge n_\varepsilon$. Hence, by Theorem 2.10, we get the conclusion. □

For other results of this type see [47], [49]–[54], [72], [116], [65], [12], [18], [71] and [87].

3. Well-posedness for fixed point problems

The notion of well-posed fixed point problem for a singlevalued operator was defined by F.S. De Blasi, J. Myjak [25] and S. Reich, A.J. Zaslavski [83] and also studied by I.A. Rus in [89]. The case of multivalued operators is considered in A. Petruşel, I.A. Rus [77] and A. Petruşel, I.A. Rus, J.-C. Yao [79] (see also [92] and [117]).

We will define and then study the well-posedness of the fixed point problem for singlevalued and multivalued operators with respect to equivalent metrics.

We give first two definitions for a well-posed fixed point problem.

Definition 3.1. Let (X,d) be a metric space, $Y \in P(X)$ and $f\colon Y \to X$ be an operator. The fixed point problem is well-posed for f with respect to d if and only if:

(a$_1$) $\mathrm{F}_f = \{x^*\}$.
(b$_1$) If $x_n \in Y$, $n \in \mathbb{N}$ and $d(x_n, f(x_n)) \to 0$, as $n \to +\infty$, then $x_n \to x^*$ as $n \to +\infty$.

Definition 3.2. Let (X,d) be a metric space, $Y \in P(X)$ and $T\colon Y \to P_{\mathrm{cl}}(X)$ be a multivalued operator. The fixed point problem is well-posed for T with respect to D_d if and only if:

(a$_1$) $\mathrm{F}_T = \{x^*\}$.

(b_1) If $x_n \in Y$, $n \in \mathbb{N}$ and $D_d(x_n, T(x_n)) \to 0$, as $n \to +\infty$ then $x_n \to x^*$, as $n \to +\infty$.

Definition 3.3. Let (X, d) be a metric space, $Y \in P(X)$ and $T: Y \to P_{\mathrm{cl}}(X)$ be a multivalued operator. The fixed point problem is well-posed for T with respect to H_d if and only if:

(a_2) $(\mathrm{SF})_T = \{x^*\}$.

(b_2) If $x_n \in Y$, $n \in \mathbb{N}$ and $H_d(x_n, T(x_n)) \to 0$, as $n \to +\infty$ then $x_n \to x^*$, as $n \to +\infty$.

Some abstract results are given now.

Theorem 3.4. *Let X be a nonempty set and d, ρ two metrics on X. Suppose that d, ρ are strongly equivalent. Let $T: Y \subseteq X \to P(X)$ be a multivalued operator. Then*:

(a) *The fixed point problem for T is well-posed with respect to D_d if and only if it is well-posed with respect to D_ρ.*

(b) *The fixed point problem for T is well-posed with respect to H_d if and only if it is well-posed with respect to H_ρ.*

Proof. (a) Let $c_1, c_2 > 0$ such that $d \le c_1\rho$ and $\rho \le c_2 d$. Then $D_d \le c_1 D_\rho$ and $D_\rho \le c_2 D_d$.

Let $x^* \in Y$ be the unique fixed point of T. Let $x_n \in Y$, $n \in \mathbb{N}$ be such that $D_\rho(x_n, T(x_n)) \to 0$, as $n \to +\infty$. Then

$$D_d(x_n, T(x_n)) \le c_1 D_\rho(x_n, T(x_n)) \to 0, \quad \text{as } n \to +\infty.$$

Since the fixed point problem is well-posed for D_d we get that $x_n \xrightarrow{d} x^*$, as $n \to +\infty$. As consequence we have $\rho(x_n, x^*) \le c_2 d(x_n, x^*) \to 0$, as $n \to +\infty$. In a similar way, interchanging the roles of d and ρ we get the conclusion.

(b) The second conclusion can be established in a similar way, by taking into account that if $d \le c_1\rho$ and $\rho \le c_2 d$ then $\delta_d \le c_1\delta_\rho$ and $\delta_\rho \le c_2\delta_d$. $\square$

Remark 3.5. In particular, if $f: Y \subseteq X \to X$ is a singlevalued operator, we obtain similar results with respect to d and ρ.

In a similar way, we have the following result.

Theorem 3.6. *Let X be a nonempty set and d, ρ two metrics on X. Suppose that d, ρ are topologically equivalent and there exists $c > 0$ such that $d \le c\rho$. Let $T: Y \subseteq X \to P(X)$ be a multivalued operator. Then*:

(a) *If the fixed point problem for T is well-posed with respect to D_d then it is well-posed with respect to D_ρ.*

(b) *If the fixed point problem for T is well-posed with respect to H_d then it is well-posed with respect to H_ρ.*

REMARK 3.7. In particular, if $f: Y \subseteq X \to X$ is a singlevalued operator, we obtain similar results with respect to d and ρ.

REMARK 3.8. Let (X, d) be a complete metric space and $f: X \to X$ be an a-contraction. Then the fixed point problem is well posed for the operator f.

EXAMPLE 3.9. Let X be a nonempty set and $f: X \to X$ be a Bessaga operator, i.e.

$$\mathrm{F}_{f^n} = \{x^*\}, \quad \text{for each } n \in \mathbb{N}^*.$$

Then there exists a metric d on X such that the fixed point problem is well posed for the operator f with respect to the metric d.

Indeed, from Bessaga's theorem, if $a \in [0, 1[$, then there exists a complete metric d on X such that $f: (X, d) \to (X, d)$ is an a-contraction. Now, for the conclusion we apply the above remark.

4. Shadowing property

We will define first the limit shadowing property for singlevalued and multivalued operators (see [80], [38], [92]).

DEFINITION 4.1. Let (X, d) be a metric space and $f: X \to X$ be an operator. Then:

(a) f has the shadowing property with respect to d if for each $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that the following implication holds:

$$(x_k)_{k\in\mathbb{N}} \subset X, d(x_{k+1}, f(x_k)) \leq \delta(\varepsilon),\ \forall k \in \mathbb{N}$$
$$\Rightarrow \exists\, x \in X \text{ with } d(f^k(x), x_k) \leq \varepsilon,\ \forall k \in \mathbb{N}.$$

(b) f has the Lipschitz shadowing property with respect to d if there exists $L > 0$ such that for each $\varepsilon > 0$ the following implication holds:

$$(x_k)_{k\in\mathbb{N}} \subset X, d(x_{k+1}, f(x_k)) \leq \varepsilon,\ \forall k \in \mathbb{N}$$
$$\Rightarrow \exists\, x \in X \text{ with } d(f^k(x), x_k) \leq L\varepsilon,\ \forall k \in \mathbb{N}.$$

(c) f has the limit shadowing property with respect to d if for each sequence $(x_k)_{k\in\mathbb{N}} \subset X$ such that

$$d(x_{k+1}, f(x_k)) \to 0 \quad \text{as } k \to \infty,$$
$$\text{there exists } x \in X \text{ such that } d(f^k(x), x_k) \to 0 \quad \text{as } k \to \infty.$$

Definition 4.2. Let (X, d) be a metric space and $T: X \to P(X)$ be a multivalued operator. By definition

(a) T has the shadowing property with respect to d if for each $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that for each sequence $(y_n)_{n\in\mathbb{N}} \subset X$ with

$$D_d(y_{n+1}, T(y_n)) \leq \delta(\varepsilon), \quad \text{for all } n \in \mathbb{N}.$$

there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ of successive approximations for T starting from arbitrary $x_0 \in X$ (i.e. for each $n \in \mathbb{N}$ one have $x_{n+1} \in T(x_n)$), such that

$$d(x_n, y_n) \leq \varepsilon, \quad \text{for all } n \in \mathbb{N}.$$

(b) T has the Lipschitz shadowing property with respect to d if there exists $L > 0$ such that for each $\varepsilon > 0$ and for each sequence $(y_n)_{n\in\mathbb{N}} \subset X$ such that

$$D_d(y_{n+1}, T(y_n)) \leq \varepsilon, \quad \text{for all } n \in \mathbb{N}.$$

There exists a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ of successive approximations for T starting from arbitrary $x_0 \in X$, such that

$$d(x_n, y_n) \leq L\varepsilon, \quad \text{for all } n \in \mathbb{N}.$$

(b) T has the limit shadowing property with respect to d if for each sequence $(y_n)_{n\in\mathbb{N}} \subset X$ such that

$$D_d(y_{n+1}, T(y_n)) \to 0 \quad \text{as } n \to \infty.$$

There exists a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ of successive approximations for T starting from arbitrary $x_0 \in X$ (i.e. for each $n \in \mathbb{N}$ one have $x_{n+1} \in T(x_n)$), such that $d(x_n, y_n) \to 0$ as $n \to \infty$.

Some abstract results are given now.

Theorem 4.3. *Let X be a nonempty set and d, ρ two metrics on X. Suppose that d, ρ are strongly equivalent. Let $f: X \to X$ be an operator. Then*:

(a) *f has the shadowing property with respect to d if and only if f has the shadowing property with respect to ρ.*
(b) *f has the Lipschitz shadowing property with respect to d if and only if f has the Lipschitz shadowing property with respect to ρ.*
(c) *f has the limit shadowing property with respect to d if and only if f has the limit shadowing property with respect to ρ.*

Proof. Let $c_1, c_2 > 0$ such that

$$c_1\rho(x, y) \leq d(x, y) \leq c_2\rho(x, y), \quad \text{for each } x, y \in X.$$

(a) Suppose that f has the shadowing property with respect to d. We will show that f has the shadowing property with respect to ρ. Indeed, let $\varepsilon > 0$. We know that for each $\varepsilon' := c_1\varepsilon > 0$ there exists $\delta(\varepsilon') > 0$ such that:

$$(x_k)_{k\in\mathbb{N}} \subset X, d(x_{k+1}, f(x_k)) \le \delta(\varepsilon'), \forall k \in \mathbb{N} \\ \Rightarrow \exists\, x \in X \text{ with } d(f^k(x), x_k) \le \varepsilon',\ \forall k \in \mathbb{N}.$$

Define $\delta_1(\varepsilon) := \delta(\varepsilon')/c_2 > 0$. Suppose $(x_k)_{k\in\mathbb{N}} \subset X$ with $\rho(x_{k+1}, f(x_k)) \le \delta_1(\varepsilon)$, for all $k \in \mathbb{N}$. Then

$$d(x_{k+1}, f(x_k)) \le c_2\rho(x_{k+1}, f(x_k)) \le c_2\delta_1(\varepsilon) \le \delta(\varepsilon'), \quad \text{for all } k \in \mathbb{N}.$$

By hypothesis, there exists $x \in X$ such that $d(f^k(x), x_k) \le \varepsilon'$, for all $k \in \mathbb{N}$. Then

$$\rho(f^k(x), x_k) \le \frac{1}{c_1}\, d(f^k(x), x_k) \le \frac{\varepsilon'}{c_1} = \varepsilon.$$

(b) and (c) can be obtained in a similar way. □

For the case of multivalued operators we have the following result.

THEOREM 4.4. *Let X be a nonempty set and d, ρ two metrics on X. Suppose that d, ρ are strongly equivalent. Let $T\colon X \to P(X)$ be a multivalued operator. Then*:

(a) *T has the shadowing property with respect to d if and only if T has the shadowing property with respect to ρ.*
(b) *T has the Lipschitz shadowing property with respect to d if and only if T has the Lipschitz shadowing property with respect to ρ.*
(c) *T has the limit shadowing property with respect to d if and only if T has the limit shadowing property with respect to ρ.*

PROOF. Let $c_1, c_2 > 0$ such that $d \le c_1\rho$ and $\rho \le c_2 d$. Then we have $D_d \le c_1 D_\rho$ and $D_\rho \le c_2 D_d$. The conclusions follow by a similar approach to Theorem 4.3. □

We will need, for the proof of the next theorems, the following auxiliary result, known as Cauchy's Lemma (see [100]).

LEMMA 4.5. *Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be two sequences of non-negative real numbers, such that $\sum\limits_{k=0}^{+\infty} a_k < +\infty$ and $\lim\limits_{n\to+\infty} b_n = 0$. Then*

$$\lim_{n\to+\infty}\left(\sum_{k=0}^{n} a_{n-k}b_k\right) = 0.$$

Now, some concrete results concerning the shadowing property are the following

Theorem 4.6. *Let (X,d) be a complete metric space and $f\colon X \to X$ be an a-contraction. Then f has the limit shadowing property with respect to d.*

Proof. Since f is a Picard operator, we have that $\mathrm{F}_f=\{x^*\}$ and $f^k(x)\to x^*$ as $k \to +\infty$, for each $x \in X$. Then

$$\begin{aligned} d(x_k,x^*) &\le d(x_k,f(x_{k-1}))+d(f(x_{k-1}),x^*)\\ &= d(x_k,f(x_{k-1}))+d(f(x_{k-1}),f(x^*))\\ &\le d(x_k,f(x_{k-1}))+ad(x_{k-1},x^*)\le \dots\\ &\le d(x_k,f(x_{k-1}))+ad(x_{k-1},f(x_{k-2}))+\dots+a^kd(x_0,x^*). \end{aligned}$$

Then, from Cauchy's Lemma we get that $d(x_k,x^*)\to 0$ as $k\to+\infty$. Hence

$$d(x_k,f^k(x))\le d(x_k,x^*)+d(x^*,f^k(x))\to 0 \quad \text{as } k\to+\infty, \text{ for all } x\in X. \qquad \square$$

For the multivalued case we have the following theorem.

Theorem 4.7. *Let (X,d) be a complete metric space and $T\colon X \to P_{\mathrm{cl}}(X)$ be a multivalued α-contraction with $(\mathrm{SF})_T \neq \emptyset$. Then, the multivalued operator T has the limit shadowing property with respect to d.*

Proof. Notice first that $\mathrm{F}_T=(\mathrm{SF})_T=\{x^*\}$. Let $(y_n)_{n\in\mathbb{N}}$ be a sequence in X such that $D(y_{n+1},T(y_n))\to 0$ as $n\to\infty$.

We shall prove first that $d(y_n,x^*)\to 0$ as $n\to+\infty$. We successively have:

$$\begin{aligned} d(x^*,y_{n+1}) &\le H(x^*,T(y_n))+D(y_{n+1},T(y_n))\\ &\le \alpha d(x^*,y_n)+D(y_{n+1},T(y_n))\\ &\le \alpha[\alpha d(x^*,y_{n-1})+D(y_n,T(y_{n-1}))]+D(y_{n+1},T(y_n))\le\dots\\ &\le \alpha^{n+1}d(x^*,y_0)+\alpha^nD(y_1,T(y_0))+\dots+D(y_{n+1},T(y_n)). \end{aligned}$$

By Cauchy's Lemma, the right hand side tends to 0 as $n\to+\infty$. Thus, $d(x^*,y_{n+1})\to 0$ as $n\to+\infty$.

On the other hand, since T is a multivalued weakly Picard operator, we know that there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of successive approximations for T starting from arbitrary $(x_0,x_1)\in \mathrm{Graph}(T)$ which converge to a fixed point $x^*\in X$ of the operator T. Since, the fixed point is unique, we get that $d(x_n,x^*)\to 0$ as $n\to+\infty$. Hence, for such a sequence $(x_n)_{n\in\mathbb{N}}$, we have

$$d(y_n,x_n)\le d(y_n,x^*)+d(x^*,x_n)\to 0 \quad \text{as } n\to+\infty. \qquad \square$$

For other considerations on shadowing property see [33], [38], [73], [80], [92].

5. Equations with ψ-weakly Picard operators

The notion of WPO is a topological one, while the concept of ψ-WPO is a metric one. Thus, in the forthcoming part of the paper we will focus on the notion of ψ-WPO.

Let X be a nonempty set and, d and ρ two metrics on X. We have the following abstract results.

Lemma 5.1. *Let $f\colon X \to X$ be an operator. We suppose that*:

(a) *f is ψ-WPO with respect to the metric d*;

(b) *there exist α_1, $\alpha_2 > 0$ such that $d \le \alpha_1 \rho$ and $\rho \le \alpha_2 d$.*

Then the operator f is ψ_1-WPO with respect to the metric ρ, where

$$\psi_1(t) := \alpha_2 \psi(\alpha_1 t), \quad t \in \mathbb{R}_+.$$

Proof. First we remark f is WPO with respect to the metric ρ. From (a) and (b) we have:

$$\rho(x, f^\infty(x)) \le \alpha_2 d(x, f^\infty(x)) \le \alpha_2 \psi(d(x, f(x))) \le \alpha_2 \psi(\alpha_1 \rho(x, f(x))),$$

for all $x \in X$. Notice that ψ_1 is an increasing function which is continuous in 0 and $\psi_1(0) = 0$. So, f is ψ_1-WPO with respect to the metric ρ. $\square$

Lemma 5.2. *Let $f, g\colon X \to X$ be two operators. We suppose that*:

(i) *f is ψ-WPO with respect to the metric d*;

(ii) *there exists $\eta > 0$ such that $d(f(x), g(x)) \le \eta$, for all $x \in X$*;

(iii) $\mathrm{F}_g \neq \emptyset$;

(iv) *there exist α_1, $\alpha_2 > 0$ such that $d \le \alpha_1 \rho$ and $\rho \le \alpha_2 d$.*

Then:

(a) *$d(x_f^*, x_g^*) \le \psi(\eta)$, where x_f^* is the unique fixed point of f and x_g^* is a fixed point of g*;

(b) *$\rho(x_f^*, x_g^*) \le \psi_1(\alpha_2 \eta) := \alpha_2 \psi(\alpha_1 \alpha_2 \eta)$.*

Proof. For (a) see [20] or [94]. From (a) and Lemma 5.1, the statement of (b) follows. $\square$

Example 5.3. Let $X := C[a;b]$,

$$d(x,y) := \max_{a \le t \le b} |x(t) - y(t)| =: \|x - y\|_\infty$$

and for $\tau > 0$,

$$\rho_\tau(x,y) := \max_{a \le t \le b} (|x(t) - y(t)|\, e^{-\tau(t-a)}) =: \|x - y\|_\tau\,.$$

In the above setting, we have now the following result.

Theorem 5.4. *Let* $X := C[a;b]$ *and* $f\colon X \to X$ *be an operator. Suppose there exists* $l > 0$ *such that*

$$|f(x)(t) - f(y)(t)| \leq l \int_a^t |x(s) - y(s)|\, ds, \quad t \in [a;b]. \tag{5.1}$$

Then:

(a) $\mathrm{F}_f = \{x_f^*\}$;
(b) f *is* PO *on* $\Big(C[a;b], \xrightarrow{\text{unif}}\Big)$;
(c) f *is* $(1 - l/\tau)^{-1}$-PO *with respect to* $\|\cdot\|_\tau$ *for* $l < \tau$;
(d) f *is* $e^{\tau(b-a)}(1 - l/\tau)^{-1}$-PO *with respect to* $\|\cdot\|_\infty$ *for* $l < \tau$;
(e) *If* $g\colon X \to X$ *is such that*

$$|f(x)(t) - g(x)(t)| \leq \eta, \quad \textit{for all } t \in [a;b], \ \textit{for all } x \in X,$$

and $x_g^* \in \mathrm{F}_g$, *then*

$$\|x_f^* - x_g^*\|_\infty \leq e^{\tau(b-a)} \left(1 - \frac{l}{\tau}\right)^{-1} \eta.$$

Proof. First we remark that

$$\|\cdot\|_\infty \leq e^{\tau(b-a)} \|\cdot\|_\tau \quad \text{and} \quad \|\cdot\|_\tau \leq \|\cdot\|_\infty, \quad \text{for all } \tau > 0.$$

Now, the proof follows from Lemma 5.2. □

Remark 5.5. Let us consider the functional-integral equation

$$x(t) = \int_0^t K(t, s, x(s))\, ds + h(t), \quad t \in [0;1],$$

where $K \in C([0;1] \times [0;1] \times \mathbb{R})$, $h \in C[0;1]$ and

$$|K(t, s, u) - K(t, s, v)| \leq l|u - v|, \quad \text{for all } u, v \in \mathbb{R}.$$

Then the operator $f\colon C[0;1] \to C[0;1]$ defined by

$$f(x)(t) := \int_0^t K(t, s, x(s))\, ds + h(t)$$

satisfies the condition (5.1).

In a similar way, in the case of multivalued ψ-WPO we have some corresponding properties. For example, we have the following abstract result.

Lemma 5.6. *Let* $F\colon X \to P(X)$ *be a multivalued operator. We suppose that*:

(a) F *is* ψ-MWPO *with respect to the metric* d;
(b) *there exist* α_1, $\alpha_2 > 0$ *such that* $d \leq \alpha_1 \rho$ *and* $\rho \leq \alpha_2 d$.

Then the operator F *is* ψ_1-MWPO *with respect to the metric* ρ, *where*

$$\psi_1(t) := \alpha_2 \psi(\alpha_1 t), \quad t \in \mathbb{R}_+.$$

6. Ulam stability of the fixed point equations

Following [95], we present now some notions and results with respect to Ulam stability theory. The role of strongly equivalent metrics is also investigated.

Definition 6.1. Let (X, d) be a metric space and $f: X \to X$ be an operator. By definition, the fixed point equation

$$x = f(x) \tag{6.1}$$

is *Ulam–Hyers stable* if there exists a real number $c > 0$ such that: for each $\varepsilon > 0$ and each solution y^* of the inequation

$$d(y, f(y)) \leq \varepsilon \tag{6.2}$$

there exists a solution x^* of the equation (5.1) such that

$$d(y^*, x^*) \leq c\varepsilon.$$

Definition 6.2. If the equation (6.1) is Ulam–Hyers stable and c is as in Definition 6.1, then, by definition, the equation (6.1) is *c-Ulam–Hyers stable*.

Definition 6.3. The equation (6.1) is *generalized Ulam–Hyers stable* if there exists $\psi: \mathbb{R}_+ \to \mathbb{R}_+$ increasing and continuous in 0 with $\psi(0) = 0$ such that: for each $\varepsilon > 0$ each solution y^* of (6.2) there exists a solution x^* of the equation (6.1) such that

$$d(y^*, x^*) \leq \psi(\varepsilon).$$

Definition 6.4. If the equation (6.1) is generalized Ulam–Hyers stable and ψ is as in Definition 6.3, then, by definition, the equation (6.1) is *ψ-generalized Ulam–Hyers stable*.

We have the following general result.

Theorem 6.5. *Let X be a nonempty set, $f: X \to X$ an operator, d and ρ be two metrics on X. We suppose that*:

(i) *f is c-WPO with respect to the metric d;*
(ii) *there exist α_1, $\alpha_2 > 0$ such that $d \leq \alpha_1 \rho$ and $\rho \leq \alpha_2 d$.*

Then:

(a) *the equation (6.1) is c-Ulam–Hyers stable with respect to the metric d;*
(b) *the equation (6.1) is $\alpha_1\alpha_2 c$-Ulam–Hyers stable with respect to the metric ρ.*

Proof. The statement (a) is the Remark 2.1 in [94]. From Lemma 5.1 the operator f is $\alpha_1\alpha_2 c$-WPO with respect to the metric ρ. So, from (i), the equation (6.1) is $\alpha_1\alpha_2 c$-Ulam–Hyers stable with respect to the metric ρ. □

By a similar approach we get the following theorem.

Theorem 6.6. *Let X be a nonempty set, $f\colon X \to X$ be an operator, d and ρ be two metrics on X. We suppose that*:

(i) *f is ψ-*WPO *with respect to the metric d*;
(ii) *there exist α_1, $\alpha_2 > 0$ such that $d \leq \alpha_1 \rho$ and $\rho \leq \alpha_2 d$.*

Then:

(a) *the equation* (6.1) *is ψ-generalized Ulam–Hyers stable with respect to d*;
(b) *the equation* (6.1) *is $\alpha_2\psi(\alpha_1(\,\cdot\,))$-generalized Ulam–Hyers stable with respect to ρ.*

We can get some similar results in the case of multivalued operators. For example, we present the following

Definition 6.7. Let (X,d) be a metric space and $F\colon X \to P(X)$ a multivalued operator. By definition, the fixed point equation

$$x \in F(x) \tag{6.3}$$

is *generalized Ulam–Hyers stable* if there exists an increasing function $\psi\colon \mathbb{R}_+ \to \mathbb{R}_+$ continuous in 0 with $\psi(0) = 0$ such that: for each $\varepsilon > 0$ and each solution u^* of the inequation

$$D_d(u, F(u)) \leq \varepsilon \tag{6.4}$$

there exists a solution x^* of the equation (6.3) such that

$$d(y^*, x^*) \leq \psi(\varepsilon).$$

Definition 6.8. If the equation (6.4) is generalized Ulam–Hyers stable and ψ is as in Definition 6.7, then, by definition, the equation (6.3) is *ψ-generalized Ulam–Hyers stable.*

We can prove now the following general result.

Theorem 6.9. *Let X be a nonempty set, $F\colon X \to P(X)$ a multivalued operator, d and ρ be two metrics on X. We suppose that*:

(i) *there exist α_1, $\alpha_2 > 0$ such that $d \leq \alpha_1 \rho$ and $\rho \leq \alpha_2 d$*;
(ii) *$F(x) \in P_{\mathrm{cp}}(X,d)$, for all $x \in X$*;
(iii) *F is ψ-*MWPO *with respect to the metric d.*

Then:

(a) *the equation* (6.3) *is ψ-generalized Ulam–Hyers stable with respect to d*;
(b) *the equation* (6.3) *is $\alpha_2\psi(\alpha_1(\,\cdot\,))$-generalized Ulam–Hyers stable with respect to ρ.*

Remark 6.10. For more considerations on Ulam stability see [92], [95] and the references therein.

7. Invariant partition of a set with respect to an operator

Definition 7.1. Let X be a set and $f\colon X \to X$ an operator. A partition, $X = \bigcup_{i\in I} X_i$ is an invariant partition of X with respect to f if $f(X_i) \subset X_i$, for all $i \in I$.

Example 7.2 (see [8], [46], [107]). Let X be a nonempty set and $f\colon X \to X$ an operator. Let us define on X the following equivalence relation by $x \approx y$ if and only if there exist n and m in $\mathbb{N}$ such that

$$f^n(x) = f^m(y).$$

We denote the equivalence class containing an element x by $[x]_f$ and we call $[x]_f$ great orbit of x with respect to the operator f. The partition of X defined by this equivalence relation is, by definition, the orbital partition of X with respect to the operator f. Notice that the orbital partition of X is an invariant partition of X with respect to the operator f.

The aim of this section is to study a special class of invariant partition related to the fixed point of f. First, we present some considerations on the invariant subset under an operator (see [8], [46], ...).

Let X be a nonempty set and $f\colon X \to X$ an operator. By definition a subset $Y \subset X$ is:

(a) invariant under f if and only if $f(Y) \subset Y$;
(b) forward invariant under f if and only if $f(Y) = Y$;
(c) backward invariant under if and only if $f^{-1}(Y) = Y$;
(d) completely invariant under f if and only if $f(Y) = Y = f^{-1}(Y)$.

Now we introduce a new class of invariant subset.

Definition 7.3. By definition, a subset $Y \subset X$ is *orbital invariant under f* if for all $y \in Y$ the great orbit of y with respect to the operator f, $[y]_f \subset Y$.

Example 7.4. Let $X := [0; 1]$ and $f(x) := x/2$. In this case, for $Y := [0; 1]$ we have:

(a) $[0; 1]$ is an invariant subset, but it isn't a forward invariant and, thus, it is not completely invariant;
(b) $[0; 1]$ is orbital invariant;

For $Y := \{0\}$ we have:

(a) $\{0\}$ is completely invariant;
(b) $\{0\}$ is orbital invariant.

We have the following remarks.

Remark 7.5. The notion of orbital invariant set is less restrictive than that of completely invariant set.

Remark 7.6. In general, the orbit of $x \in X$ with respect to $f : X \to X$, $\mathcal{O}_f(x) := \{f^n(x) \mid n \in \mathbb{N}\}$ is not orbital invariant, but the great orbit, $[x]_f$, is orbital invariant.

Remark 7.7. If $X = \bigcup_{i \in I} X_i$ is an invariant partition of X with respect to f the each X_i is an orbital invariant subset under f.

Remark 7.8. The intersection of a family of orbital invariant subset is also orbital invariant subset.

On the other hand, the following result is given in [98] (see also [49] and [50]).

Theorem 7.9 (Theorem of equivalent statements). *Let X be a nonempty set and $f : X \to X$ an operator. The following statements are equivalent:*

(a) $\mathrm{F}_f = \mathrm{F}_{f^n} \neq \emptyset$, *for all* $n \in \mathbb{N}^*$;
(b) *there exists a complete metric d on X with respect to which the operator f is* WPO;
(c) *there exists $\alpha \in]0;1[$, a complete metric d on X and a partition X, $X = \bigcup_{i \in I} X_i$, such that:*
  (c$_1$) $f(X_i) \subset X_i$, *for all* $i \in I$;
  (c$_2$) $X_i \cap \mathrm{F}_f = \{x_i^*\}$, *for all* $i \in I$;
  (c$_3$) $f|_{X_i} : X_i \to X_i$ *is α-contraction with respect to the metric d.*

The above result suggests the following problem.

Problem 7.10. Let X be a nonempty set and $f : X \to X$ an operator such that

$$\mathrm{F}_f = \mathrm{F}_{f^n} \neq \emptyset, \quad \text{for all } n \in \mathbb{N}^*.$$

Does exist two metrics d and ρ on X with the following properties:

(a) d is not topological equivalent with ρ;
(b) the operator f is WPO with respect to d and with respect to ρ?

For this problem we have the following result.

Theorem 7.11. *Let X be a nonempty set, $f : X \to X$ an operator and $X = \bigcup_{\lambda \in \Lambda} X_\lambda$ the orbital partition of X with respect to f. We suppose that:*

(a) $\mathrm{F}_f = \mathrm{F}_{f^n} \neq \emptyset$, *for all* $n \in \mathbb{N}^*$;
(b) $\mathrm{card}\,\Lambda > \mathrm{card}\,\mathrm{F}_f \geq 2$.

Then there exist two metrics d and ρ with the properties (a) *and* (b) *from Problem* 7.10.

Proof. Let $x^* \in F_f$. For $x \in F_f$, $x \neq x^*$, let $X_x := [x]_f$ and $X_{x^*} := [x^*]_f \cup (X \setminus X_x)$. From (a) and (b) we have that:

- $X = \bigcup_{z \in F_f} X_z$ is an invariant partition of X with respect to the operator f;
- $F_f \cap X_z = F_{f^n} \cap X_z = \{z\}$, for all $z \in F_f$, for all $n \in \mathbb{N}$.

From the theorem of Bessaga there exists a metric d_z such that (X_z, d_z) is a complete metric space and the restriction of f to X_z, $f|_{X_z}: X_z \to X_z$ is a contraction.

Let $d: X \times X \to \mathbb{R}_+$ be defined by

$$d(u,v) := \begin{cases} d_z(u,v) & \text{if } u,v \in X_z, \\ d_z(u,z) + d_y(v,y) & \text{if } u \in X_z,\ v \in X_y,\ z \neq y. \end{cases}$$

Notice that d is a complete metric on X and f is WPO with respect to d. Moreover,

$$(\mathrm{AD})_f(z) = [z]_f \quad \text{for } z \in F_f, z \neq x^*$$

and

$$(\mathrm{AD})_f(x^*) = [x^*]_f \cup \Bigg(\bigcup_{\substack{z \in F_f \\ z \neq x^*}} (X \setminus X_z) \Bigg),$$

where $(\mathrm{AD})_f(x^*) := \{x \in X \mid f^n(x) \to x^* \text{ as } n \to +\infty\}$.

Now, let x_1^*, $x_2^* \in F_f$, $x_1^* \neq x_2^*$. Let d be the metric corresponding to x_1^* as above and ρ be the metric corresponding to x_2^*. Since $(\mathrm{AD})_f(x_1^*)$ with respect to the metric d is different from $(\mathrm{AD})_f(x_1^*)$ with respect to the metric ρ, hence the metric d is not topological equivalent with the metric ρ. So, the metrics d and ρ are as in the Problem 7.10. □

Remark 7.12. The multivalued version of the above problem (see Problem 7.10) seems to be a very difficult one.

For the theory of invariant subsets see [97], [3], [8], [46], [113], [30], [43], [60], [93] and [99].

8. Defect of some properties via equivalent metrics

Let X be a nonempty set, $f: X \to X$ an operator and d, ρ two metrics on X such that $d \leq \alpha_1 \rho$ and $\rho \leq \alpha_2 d$ for some α_1, $\alpha_2 > 0$. In this case $P_b(X) := P_b(X,d) = P_b(X,\rho)$.

8.1. Defect of fixed point: minimal displacement. (See K. Goebel [39] and [40]; see also [42], [7], [37].)

Following K. Goebel, the minimal displacement of f with respect to the metric d is defined by

$$(\mathrm{md})(f;d) := \inf\{d(x,f(x)) \mid x \in X\}.$$

Notice that

$$(\mathrm{md})(f;d) \le \alpha_1 (\mathrm{md})(f;\rho) \quad \text{and} \quad (\mathrm{md})(f;\rho) \le \alpha_2 (\mathrm{md})(f;d).$$

Hence, the minimal displacement of f with respect to the metric d is 0 if and only if the minimal displacement of f with respect to the metric ρ is 0.

8.2. Defect of compactness: Kuratowski's measure of noncompactness. (K. Kuratowski (1930), see [11], [97], [42]; see also [2], [30], [99], [3].)

The diameter functional on X with respect to d is

$$\delta_d : P_{\mathrm{b}}(X) \to \mathbb{R}_+, \quad \delta_d(Y) := \sup\{d(x,y) \mid x, y \in Y\}.$$

Notice that $\delta_d \le \alpha_1 \delta_\rho$ and $\delta_\rho \le \alpha_2 \delta_d$.

The Kuratowski measure of noncompactness on X with respect to the metric d is $\alpha_{K,d} : P_{\mathrm{b}}(X) \to \mathbb{R}_+$ defined by

$$\alpha_{K,d}(Y) := \inf\left\{\varepsilon > 0 \;\middle|\; Y = \bigcup_{i=1}^{n} Y_i,\ \delta_d(Y) \le \varepsilon,\ i = \overline{1,n},\ n \in \mathbb{N}\right\}.$$

From this definition we have $\alpha_{K,d} \le \alpha_1 \alpha_{K,\rho}$ and $\alpha_{K,\rho} \le \alpha_2 \alpha_{K,d}$. Indeed, let $Y = \bigcup_{i=1}^{n} Y_i$ such that $\delta_d(Y) \le \varepsilon$, $i = \overline{1,n}$. This implies that $\delta_\rho(Y) \le \alpha_2 \varepsilon$. Hence

$$\alpha_2\left\{\varepsilon > 0 \;\middle|\; Y = \bigcup_{i=1}^{n} Y_i,\ \delta_d(Y) \le \varepsilon,\ n \in \mathbb{N}\right\} \subset \left\{\varepsilon > 0 \;\middle|\; Y = \bigcup_{i=1}^{n} Y_i,\ \delta_\rho(Y) \le \varepsilon,\ n \in \mathbb{N}\right\}.$$

So, $\alpha_{K,\rho}(Y) \le \alpha_2 \alpha_{K,d}(Y)$, for all $Y \in P_{\mathrm{b}}(X)$.

8.3. The defect of convexity: Eisenfeld–Lakshmikantham measure of nonconvexity. (Eisenfeld–Lakshmikantham (1975); see [7], [97], [99].)

Let $(X, +, \mathbb{R})$ be a linear space and $\mathrm{co} : P(X) \to P(X)$ the convexity closure operator on X corresponding to the linear structure of X. Let d and ρ two metrics on X such that $d \le \alpha_1 \rho$ and $\rho \le \alpha_2 d$ with some α_1, $\alpha_2 > 0$. By definition the Eisenfeld-Lakshmikantham's measure on nonconvexity with respect to the metric d is the following functional

$$\beta_{\mathrm{EL},d} : P_{\mathrm{b}}(X,d) \to \mathbb{R}_+, \quad \beta_{\mathrm{EL},d}(Y) := H_d(Y, \mathrm{co}\, Y).$$

Notice that $P_{\mathrm{b}}(X,d) = P_{\mathrm{b}}(X,\rho)$ and $\beta_{\mathrm{EL},d} \le \alpha_1 \beta_{\mathrm{EL},\rho}$ and $\beta_{EL,\rho} \le \alpha_2 \beta_{\mathrm{EL},d}$.

Remark 8.1. If on (X,d) we consider a convexity structure given in terms of d (see [9] and [103]), then the convexity closure operator, co, depends of d and we denote it by co_d. In this case the following problems arise:

Let X be a nonempty set, d and ρ two metrics on X and, co_d and co_ρ the corresponding convexity closure operators.

Problem 8.2. In which conditions $\mathrm{co}_d = \mathrm{co}_\rho$?

Problem 8.3. If d and ρ are strongly equivalent the problem is to compare $\beta_{\mathrm{EL},d}$ with $\beta_{\mathrm{EL},\rho}$ on $P_{\mathrm{b}}(X,d) \cap P_{\mathrm{b}}(X,\rho)$.

We have considered here only a few examples of defect properties via equivalent metrics. For some other possible open problems see [7].

Acknowledgements. This work was possible with the financial support of a grant of the Romanian National Authority for Scientific Research, CNCSUE-FISCDI, project number PN-II-ID-PCE-2011-3-0094.

References

[1] R.P. Agarwal, M. Meehan and D. O'Regan, *Fixed Point Theory and Applications*, Cambridge Univ. Press, 2001.

[2] J. Andres and L. Górniewicz, *Topological Fixed Point Principles for Boundary Value Problems*, Kluwer, Dordrecht, 2003.

[3] J. Appell, E. De Pascale and A. Vignoli, *Nonlinear Spectral Theory*, W. de Gruyter, Berlin, 2004.

[4] M. Arov, S. Reich and A.J. Zaslavski, *Uniform convergence of iterates for a class asymptotic contraction*, Fixed Point Theory **8** (2007), 3–9.

[5] C.E. Aull and R. Lowen, *Handbook of the History of General Topology*, Kluwer Acad. Publ., Dordrecht, 2001.

[6] P.B. Bailey, L.F. Shampine and P.E. Waltman, *Nonlinear Two Point Boundary Values Problems*, Academic Press, New York, 1968.

[7] A.I. Ban and S.G. Gal, *Defects of Properties in Mathematics*, World Scientific, New Jersey, 2002.

[8] A.F. Beardon, *Iteration of Rational Functions*, Springer, 1991.

[9] G. Beer, *Topologies on Closed and Closed Convex Sets*, Kluwer, 1993.

[10] V. Berinde, *Iterative Approximations of Fixed Points*, Springer, Berlin, 2007.

[11] S.R. Bernfeld and V. Lakshmikantham, *An Introduction to Nonlinear Boundary Value Problems*, Academic Press, New York, 1974.

[12] C. Bessaga, *On the converse of the Banach fixed point principle*, Colloq. Math. **7** (1959), 41–43.

[13] E. Bohl, *Linear operator equations on a partially ordered vector space*, Aequat. Math. **4** (1970), 89–98.

[14] J. Borsík and J. Doboš, *Functions whose composition with every metric is a metric*, Math. Slovaca **31** (1981), 3–12. (in Russian)

[15] J. Borsík and J. Doboš, *On metric preserving functions*, Real Anal. Exchange **13** (1987/88), no. 1, 285–293.

[16] N. Bourbaki, *Topologie générale*, Herman, Paris, 1961.

[17] A. Brown and C. Pearcy, *An Introduction to Analysis*, Springer, New York, 1995.

[18] P.J. Bushell, *The Cayley–Hilbert metric and positive operators*, Linear Algebra Appl. **84** (1986), 271–280.

[19] T. Cardinali and P. Rubbioni, *A generalization of Caristi's fixed point theorem in metric spaces*, Fixed Point Theory **11** (2010), no. 1, 3–10.

[20] A. Chiş-Novac, R. Precup and I.A. Rus, *Data dependence of fixed points for non-self generalized contractions*, Fixed Point Theory **10** (2009), no. 1, 73–87.

[21] P. Collaco and J.C.E. Silva, *A complete comparison of* 25 *contraction conditions*, Nonlinear Anal. **30** (1997), 471–476.

[22] P. Corazza, *Introduction to metric preserving functions*, Amer. Math. Monthly **104** (1999), no. 4, 309–323.

[23] C. Corduneanu, *Bielecki method in the theory of integral equations*, Ann. Univ. Marie Curie Sklodowska **38** (1984), no. 2, 23–40.

[24] C. Costantini and S. Levi, *Metrics that generate the same hyperspace convergence*, Set-Valued Anal. **1** (1993), no. 2, 141–157.

[25] F.S. De Blasi and J. Myjak, *Sur la porosité des contractions sans point fixe*, C.R. Acad. Sci. Paris **308** (1989), 51–54.

[26] M.M. Deza and E. Deza, *Encyclopedia of Distances*, Springer, Berlin, 2009.

[27] J. Doboš, *A survey of metric preserving functions*, Questions Answers Gen. Topology **13** (1995), no. 2, 129–134.

[28] J. Doboš and Z. Piotrowski, *Some remarks on metric preserving functions*, Real Anal. Exchange **19** (1993-1994), 317–320.

[29] ______, *A note on metric preserving functions*, Internat. J. Math. Math. Sci. **19** (1996), no. 1, 199–201.

[30] T. Dominguez, M.A. Japón and G. Lopez, *Metric fixed point results concerning measures of noncompactness*, Handbook of Metric Fixed Point Theory, Kluwer Acad. Publ., Dordrecht, 2001, pp. 239–268.

[31] P.N. Dowling, C.J. Lennard and B. Turett, *Renorming of l^1 and c_0 and fixed point properties*, Handbook of Metric Fixed Point Theory, Kluwer Acad. Publ., Dordrecht, 2001, pp. 269–297.

[32] J. Dugundji, *Topology*, Allyn and Bacon, Boston, 1966.

[33] T. Eirolo, O. Nevalinna and S. Yu. Pilyugin, *Limit shadowing property*, Numer. Funct. Anal. Optim. **18** (1977), 75–92.

[34] J. Eisenfeld, V. Lakshmikantham and S. Bernfeld, *On the construction of a norm associated with the measure of noncompactness*, Nonlinear Anal. **1** (1976), no. 1, 49–54.

[35] R. Engelking, *General Topology*, PWN, Warszawa, 1977.

[36] M. Fréchet, *Les éspaces abstraits*, Gauthier–Villars, Paris, 1928.

[37] M. Furi and M. Martelli, *On the minimal displacement of point under α-Lipschitz maps in normed spaces*, Boll. Un. Mat. Ital. **9** (1974), 791–799.

[38] V. Glăvan and V. Guţu, *Shadowing in affine iterated function systems*, Fixed Point Theory **10** (2010), 229–243.

[39] K. Goebel, *Metric environment of the topological fixed point theorems*, Handbook of Metric Fixed Point Theory (W.A. Kirk and B. Sims, eds.), Kluwer Acad. Publ., Dordrecht, 2001, pp. 577–611.

[40] ______, *On the minimal displacement of points under Lipschitzian mappings*, Pacific J. Math. **48** (1973), 151–163.

[41] K. Goebel and W.A. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge Univ. Press, London, 1990.

[42] L. Górniewicz, *Topological Fixed Point Theory of Multivalued Mappings*, Kluwer Acad. Publ., Dordrecht, 1999.

[43] A. Granas and J. Dugundji, *Fixed Point Theory*, Springer, New York, 2003.

[44] J.K. Hale and O. Lopes, *Fixed point theorems and dissipative processes*, J. Differential Equations **13** (1973), 391–402.

[45] S. Heikkilä and G.F. Roach, *On equivalent norms and the contraction mapping principle*, Nonlinear Anal. **8** (1984), 1241–1252.

[46] M.C. Irwin, *Smooth Dynamical Systems*, Acad. Press, 1980.

[47] A.A. Ivanov, *Fixed Points of Metric Space Mapping*, LOMI, Leningrad, 1976. (in Russian)

[48] J. Jachymski, *An extension of A. Ostrowski's theorem on the round-off stability of iterations*, Aequat. Math. **53** (1997), 242–253.

[49] ______, *A short proof of the converse to the contraction principle and some related results*, Topol. Methods Nonlinear Anal. **15** (2000), 179–186.

[50] ______, *Converses to fixed point theorems of Zermelo and Caristi*, Nonlinear Anal. **52** (2003), 1455–1463.

[51] L. Janos, *A converse of Banach's contraction theorem*, Proc. Amer. Math. Soc. **18** (1967), 287–289.

[52] ______, *On mappings contractive in the sense of Kannan*, Proc. Amer. Math. Soc. **61** (1976), no. 1, 171–175.

[53] E. Jawari, D. Misane and M. Pouzet, *Retracts: graph and ordered set from the metric point of view*, Contemp. Math. **57** (1986), 175–226.

[54] S. Kasahara, *A remark on the converse of Banach contraction theorem*, Amer. Math. Monthly **75** (1968), 775–776.

[55] J.L. Kelley, *General Topology*, Van Nostrand, New York, 1955.

[56] M.A. Khamsi and W.A. Kirk, *An Introduction to Metric Spaces and Fixed Point Theory*, Wiley–Interscience, New York, 2001.

[57] W.A. Kirk, *Metric fixed point theory: old problems and new directions*, Fixed Point Theory **11** (2010), 45–58.

[58] ______, *Approximate fixed points of nonexpansive maps*, Fixed Point Theory **10** (2009), 275–288.

[59] ______, *Caristi's fixed point theorem and metric convexity*, Colloq. Math. **36** (1976), 81–86.

[60] W.A. Kirk and B. Sims (eds.), *Handbook of Metric Fixed Point Theory*, Kluwer Acad. Publ., Dordrecht, 2001.

[61] V. Klee, *Stability of the fixed point property*, Colloq. Math. **8** (1961), 41–46.

[62] M.A. Krasnosel'skiĭ and P. Zabreĭko, *Geometrical Methods in Nonlinear Analysis*, Springer, Berlin, 1984.

[63] T. Kuczumov, S. Reich and D. Shoikhet, *Fixed point of holomorphic mappings: a metric approach*, Handbook of Metric Fixed Point Theory, Kluwer Acad. Publ., Dordrecht, 2001, pp. 437–515.

[64] M. Kwapisz, *An extension of Bielecki's method of proving global existence and uniqueness results for functional equations*, preprint (1984), Univ. of Gdańsk.

[65] B.K. Lahiri, M.K. Chakrabarty and A. Sem, *Converse of Banach's contraction principle and stor operation*, Proc. Nat. Acad. Sci. India Sect. A **79** (2009), 367–374.

[66] A. Lechicki and S. Levi, Wijsman convergence in the hyperspace of a metric space, Boll. Un. Mat. Ital. B **1** (1987), 439–451.

[67] N. Levine, *Remarks on uniform continuity in metric spaces*, Amer. Math. Monthly **67** (1960), 562–563.

[68] Z.Q. Liu, *Fixed point theorems for condensing and compact maps*, Kobe J. Math. **11** (1994), 129–135.

[69] E. Llorens Fuster, *The fixed point property for renormings of l_2*, Seminar Math. Anal., Univ. Seville, vol. 2006, pp. 121–159.

[70] P.R. Meyers, *A converse to Banach's contraction theorem*, J. Res. Nat. Bur Standards Sect. B **71B** (1967), 73–76.

[71] ———, *On contractive semigroups and uniform asymptotic stability*, J. Res. Nat. Bur Standards Sect. B **74B** (1970), 115–120.

[72] V.I. Opoitsev, *A converse to the principle of contracting maps*, Russian Math. Surveys **31** (1976), 175–204.

[73] K. Palmer, *Shadowing in Dynamical Systems*, Kluwer Acad. Publ., Dordrecht, 2000.

[74] M. Păcurar, *An approximate fixed point proof of the Banach–Ghöhde–Kirk fixed point theorem*, Creative Math. Inform. **17** (2008), 43–47.

[75] M. Păcurar and R.V. Păcurar, *Approximate fixed point theorems for weak contractions on metric spaces*, Carpathian J. Math. **23** (2007), 149–155.

[76] A. Petruşel, *Multivalued weakly Picard operators and applications*, Sci. Math. Japon. **59** (2004), 169–202.

[77] A. Petruşel and I.A. Rus, *Well-posedness of the fixed point problem for multivalued operators*, Applied Analysis and Differential Equations (O. Cârjă and I.I. Vrabie, eds.), World Scientific, 2007, pp. 295–306.

[78] ———, *The theory of a metric fixed point theorem for multivalued operators*, Proc. of the Ninth International Conference on Fixed Point Theory and its Applications, Changhua, Taiwan-2009 (L.J. Lin, A. Petruşel and H.K. Xu, eds.), Yokohama Publ., 2010, pp. 161–175.

[79] A. Petruşel, I.A. Rus and J.-C. Yao, *Well-posedness in the generalized sense of the fixed point problems*, Taiwanese J. Math. **11** (2007), no. 3, 903–914.

[80] S.Yu. Pilyugin, *Shadowing in Dynamical Systems*, Springer, Berlin, 1999.

[81] V. Radu, *The fixed point alternative and the stability of functional equations*, Fixed Point Theory **4** (2003), 91–96.

[82] ———, *Some suitable metrics on fuzzy metric spaces*, Fixed Point Theory **5** (2004), 323–347.

[83] S. Reich and A.J. Zaslavski, *Well-posedness of fixed point problems*, Far East J. Math. Sci., Special Volume, Part III (2001), 393–401.

[84] B.E. Rhoades, *A comparison of various definitions of contractive mappings*, Trans. Amer. Math. Soc. **226** (1977), 257–290.

[85] ———, *On weighted norms and the contraction mapping principle*, NAM-Bericht **10** (1974), Univ. of Göttingen.

[86] W. Robert and L. Janos, *Constructing metrics with the Heine–Borel property*, Proc. Amer. Math. Soc. **100** (1987), 567–573.

[87] I. Rosenholtz, *Evidence of a conspiracy among fixed theorems*, Proc. Amer. Math. Soc. **53** (1975), no. 1, 213–218.

[88] I.A. Rus, *Picard operators and applications*, Sci. Math. Japon. **58** (2003), 191–219.

[89] ———, *Picard operators and well-posedness of fixed point problems*, Studia Univ. Babeş–Bolyai Math. **52** (2007), no. 3, 147–157.

[90] ———, *Weakly Picard operators and applications*, Sem. on Fixed Point Theory, Cluj-Napoca **2** (2001), 41–58.

[91] ———, *Picard operators and well-posedness of fixed point problems*, Studia Univ. Babeş–Bolyai, Mathematica **52** (2007), no. 3, 147–156.

[92] ———, *The theory of a metrical fixed point theorem: theoretical and applicative relevances*, Fixed Point Theory **9** (2008), 541–559.

[93] ———, *Principles and Applications of Fixed Point Theory*, Dacia, Cluj-Napoca, 1979. (in Romanian)

[94] ———, *Generalized Contractions and Applications*, Cluj Univ. Press, 2001.

[95] ———, *Remarks on Ulam stability of the operatorial equations*, Fixed Point Theory **10** (2009), 305–320.

[96] ———, *Some nonlinear functional differential and integral equations via weakly Picard operator theory: a survey*, Carpathian J. Math. **26** (2010), 230–258.

[97] ———, *Fixed Point Structure Theory*, Cluj Univ. Press, Cluj-Napoca, 2006.

[98] ———, *Weakly Picard mappings*, Comment. Math. Univ. Caroline **34** (1993), no. 4, 769–773.

[99] I.A. Rus, A. Petruşel and G. Petruşel, *Fixed Point Theory*, Cluj University Press, 2008.

[100] I.A. Rus and M.-A. Şerban, *Some generalizations of a Cauchy lemma and applications*, Topics in Mathematics, Computer Science and Philosophy, A Festschrift for Wolfgang W. Breckner, Cluj Univ. Press, 2008, pp. 173–181.

[101] M. Schechter, *Linking Methods in Critical Point Theory*, Birkhäuser, Boston, 1999.

[102] T. Sekowski, *On normal structure stability of fixed point property and the modulus of noncompact convexity*, Rend. Sem. Mat. Fis. Milano **56** (1986), 147–153.

[103] I. Singer, *Abstract Convex Analysis*, John Wiley and Sons, Toronto, 1997.

[104] T.K. Sreenivasan, *Some properties of distance functions*, J. Indian Math. Soc. **11** (1947), 38–43.

[105] M.A. Şerban, *Spaces with perturbed metrics and fixed point theorems*, Automat. Comput. Appl. Math. **17** (2008), no. 2, 323–334.

[106] ———, *Abstract fixed point principles via perturbed metrics* (to appear).

[107] G. Targonski, *Topics on Iteration Theory*, Vandenhoeck and Ruprecht, 1981.

[108] F. Terpe, *Metric preserving functions*, Proc. Conf. Topology and Measure IV, Greifswald, 1984, pp. 189–197.

[109] M. Turinici, *Volterra functional equations via projective techniques*, J. Math. Anal. Appl. **103** (1984), 211–229.

[110] R.W. Vallin, *On metric preserving functions and infinite derivatives*, Acta Math. Univ. Comen. **. 67v** (1989), no. 2, 373–376.

[111] A.I. Vasiliev, *Introducing an equivalent metric in a linear space by a family of its subnets*, Proc. Steklov Inst. Math. (2001), 235–242.

[112] W. Walter, *A note on contraction*, SIAM Review **18** (1976), 107–111.

[113] G.T. Whyburn, *Analytic Topology*, vol. 28, Amer. Math. Soc. Collog. Publ., 1942.

[114] W.A. Wilson, *On certain types of continuous transformations of metric spaces*, Amer. J. Math. **57** (1935), 62–68.

[115] Z.B. Xu and L.S. Wang, *Quantitative properties of nonlinear Lipschitz operators I. The Lip number*, Acta Math. Appl. Sinica **19** (1996), 175–184. (in Chinese)

[116] S.V. Zhestkov and P.P. Zabreĭko, *On a converse theorem to the fixed point principle in the theory of the Cauchy problem for linear normal partial differential systems*, Dokl. Akad. Nauk. Belarusi **45** (2001), 12–16. (in Russian)

[117] Y.-H. Zhou, J. Yu, H. Yang and S.-W. Xiang, *Hadamard types of well-posedness of non-self set-valued mappings for coincide points*, Nonlinear Anal. **63** (2005), e2427–e2436.

Manuscript received August 15, 2011

Adrian Petruşel, Ioan A. Rus and Marcel-Adrian Şerban
Babeş-Bolyai University Cluj-Napoca
Department of Mathematics
Kogălniceanu Street, No. 1
400084 Cluj-Napoca, ROMANIA

E-mail address: petrusel@math.ubbcluj.ro
iarus@math.ubbcluj.ro
mserban@math.ubbcluj.ro