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RANDOM TOPOLOGICAL DEGREE AND RANDOM DIFFERENTIAL INCLUSIONS

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Dedicated to the memory of Francesco S. DeBlasi

Abstract. We present a random topological degree effectively applicable mainly to periodic problems for random differential inclusions. These problems can be transformed to the existence problems of random fixed points or periodic orbits of the associated Poincaré translation operators. The solvability can be so guaranteed either directly by means of nontrivial topological invariants (random degree, index of a random direct potential) or via a randomization scheme using deterministic results which are "periodicity stable" under implemented parameter values.

1. Introduction

The main aim of this paper is to develop a random topological degree applicable in an easy but effective way to problems associated with random differential equations and inclusions. This will be done via transformation of the given random problems into the deterministic case. In this way, we will be able to define random Poincaré's translation operators along the trajectories of differential inclusions whose fixed points or periodic orbits determine random periodic solutions. Our approach also allows us, besides other things, to formulate a scheme

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for randomization of many periodicity results obtained for deterministic equations and inclusions.

Our investigation relies on the fact that random differential equations, inclusions and dynamical systems represent an important tool in the study of nondeterministic dynamics. For the related standard reference sources, see e.g. [6], [29], [31], [39], and the references therein. For article references, see e.g. [2], [4], [11], [14], [20], [26], [28], [34]–[36], [38], [41]. For some practical applications of random systems, see e.g. the monograph [43].

The solutions of initial and boundary value problems for random differential inclusions can be either represented or determined by random fixed points or periodic orbits of the associated random operators. That is also why the random fixed point theory plays an important role in this field. Its study was initiated by the Prague probabilistic school, especially by the work of Antonín Špaček [40] and Otto Hanš [22]. For some further earlier results, see e.g. the survey article [9]. For more recent related references, see e.g. [10], [15], [24]–[26], [33], [36]–[38], [41], [42], [44], [45].

Our paper is organized as follows. Preliminary results are collected in the next section. They concern especially two lemmas (2.10 and 2.11) for obtaining (in a deterministic way) random fixed points and periodic orbits in terms of measurable selections. In Section 3, a random degree is defined for a rather general class of random multivalued operators. The random analogies of fixed point theorems of Borsuk and Schauder are presented there as well. Random differential inclusions are considered in Section 4. It is shown that random Cauchy (initial value) problems are solvable under natural assumptions and that the associated solution operator is a random u-mapping. Random periodic problems are investigated in Section 5 via random Poincaré translation operators. The nontrivial index of a random guiding function is proved there to guarantee the existence of periodic solutions. In Section 6, a randomization scheme is finally formulated for periodic solutions in the sense that many deterministic results which are "periodicity stable" under implemented parameter values can be randomized in this way.

2. Some preliminaries

In the entire text, all topological spaces are at least metric. Let us recall that a space X is an *absolute neighbourhood retract* (ANR) if, for each space Y and every closed $A \subset Y$, each continuous mapping $f\colon A \to X$ is extendable over some neighbourhood of A. A space X is an *absolute retract* (AR) if, for each space Y and every closed $A \subset Y$, each continuous mapping $f\colon A \to X$ is extendable over Y. A space X is called an R_δ*-set* if there exists a decreasing sequence $\{X_n\}$

of compact absolute retracts X_n such that $X = \bigcap\{X_n \mid n = 1, 2, \dots\}$. For more details, see e.g. [3], [19].

Furthermore, all multivalued maps $\varphi\colon X \multimap Y$ have nonempty values, i.e. $\varphi\colon X \to 2^Y \setminus \{\emptyset\}$. We say that a single valued map $v\colon X \to Y$ is a *selection* of the multivalued map $\varphi\colon X \multimap Y$ (written $v \subset \varphi$) if $v(x) \in \varphi(x)$, for every $x \in X$. By a *fixed point* of φ, we mean $x \in X \cap Y \neq \emptyset$ such that $x \in \varphi(x)$. The set of fixed points of φ will be denoted by $\mathrm{Fix}(\varphi) := \{x \in X \mid x \in \varphi(x)\}$.

By a *k-periodic point* of $\varphi\colon X \to X$ we could obviously mean a fixed point of the k-th iterate, i.e. $x \in \varphi^k(x)$, where $x \notin \varphi^j(x)$, for $j < k$. Rather than by periodic points of φ, we shall however deal with periodic orbits of φ. By a *k-orbit* of $\varphi\colon X \multimap Y$, we shall understand a sequence $\{x_i\}_{i=0}^{k-1}$, where $x_i \in X$, $i = 0, \dots, k-1$, such that

- (i) $x_{i+1} \in \varphi(x_i)$, $i = 0, \dots, k-2$, and $x_0 \in \varphi(x_{k-1})$,
- (ii) the sequence $\{x_i\}_{i=0}^{k-1}$ is not formed by going p-times around a shorter subsequence of m consecutive elements, where $mp = k$.

If still

- (iii) $x_i \neq x_j$, $i \neq j$; $i, j = 0, \dots, k-1$, then we speak about a *primary k-orbit*.

By a *measurable space*, we shall mean as usually the pair (Ω, Σ), where a set Ω is equipped with a σ-algebra Σ of subsets. We shall use $\mathbb{B}(X)$ to denote the Borel σ-algebra on X. The symbol $\Sigma \otimes \mathbb{B}(X)$ denotes the smallest σ-algebra on $\Omega \times X$ which contains all the sets $A \times B$, where $A \in \Sigma$ and $B \in \mathbb{B}(X)$.

Denoting, for $\varphi\colon X \multimap Y$, by

$$\varphi^{-1}(B) := \{x \in X \mid \varphi(x) \subset B\} \text{ and } \varphi_+^{-1}(B) := \{x \in X \mid \varphi(x) \cap B \neq \emptyset\}$$

the small and large counter-images of $B \subset Y$, we can define (weakly) measurable multivalued maps as follows.

Definition 2.1. Let (Ω, Σ) be a measurable space and Y be a separable metric space. A map $\varphi\colon \Omega \multimap Y$ with closed values is called *measurable* if $\varphi^{-1}(B) \in \Sigma$, for each open $B \subset Y$, or equivalently, if $\varphi_+^{-1}(B) \in \Sigma$, for each closed $B \subset Y$. It is called *weakly measurable* if $\varphi_+^{-1}(B) \in \Sigma$, for each open $B \subset Y$, or equivalently, if $\varphi^{-1}(B) \in \Sigma$, for each closed $B \subset Y$.

It is well known that, for compact-valued maps $\varphi\colon \Omega \multimap Y$, the notions of measurability and weak measurability coincide. Moreover, if φ and ψ are measurable, then so is their Cartesian product $\varphi \times \psi$. For more properties and details, see [3, Proposition 3.45 in Chapter I] and [12].

As an important tool in our investigations, we shall employ a version of the Aumann selection theorem in [24, Theorem 2.2.14] which we state here in the form of lemma.

Lemma 2.2. *If $\varphi\colon \Omega \multimap Y$, where Ω is a complete measure space and Y is a complete separable metric space, is a multivalued map whose graph*

$$\Gamma_\varphi := \{(\omega, y) \in \Omega \times Y \mid y \in \varphi(\omega)\}$$

is measurable, i.e. $\Gamma_\varphi \in \Sigma \otimes \mathbb{B}(Y)$, then φ possesses a measurable (single-valued) selection $f \subset \varphi$.

Remark 2.3. If $\varphi\colon \Omega \multimap Y$ is measurable with closed values like in the Kuratowski–Ryll–Nardzewski theorem (see e.g. [3], [12], [19], [24]), then its graph Γ_φ is measurable (cf. e.g. [24, Proposition 1.7]), and subsequently φ possesses a measurable selection $f \subset \varphi$.

We shall also consider more regular semicontinuous maps.

Definition 2.4. A map $\varphi\colon X \multimap Y$ with closed values is said to be *upper semicontinuous* (u.s.c.) if, for every open $B \subset Y$, the set $\varphi^{-1}(B)$ is open in X, or equivalently, if $\varphi_+^{-1}(B)$ is closed in X. It is said to be *lower semicontinuous* (l.s.c.) if, for every open $B \subset Y$, the set $\varphi_+^{-1}(B)$ is open in X, or equivalently, if $\varphi^{-1}(B)$ is closed in X. If it is both u.s.c. and l.s.c., then it is called *continuous*.

Of course, if φ is u.s.c. or l.s.c., then it is measurable. If a single-valued $f\colon X \to Y$ is u.s.c. or l.s.c., then it is continuous. If a compact-valued $\varphi\colon X \multimap Y$ is u.s.c. and $A \subset X$ is a compact subset of X, then $\varphi(A)$ is compact. Moreover, for compact u.s.c. maps φ, $\mathrm{Fix}(\varphi)$ is compact. For more properties and details, see e.g. [3, Chapter I.3].

The notions of a random operator, a random fixed point and a random orbit are essential in this paper. In the sequel, Ω will be always a complete measure space and X be always a complete separable metric space.

Definition 2.5. Let $A \subset X$ be a closed subset and $\varphi\colon \Omega \times A \multimap X$ be a multivalued map with closed values. We say that φ is a *random operator* if it is product-measurable (measurable in the whole), i.e. measurable w.r.t. minimal σ-algebra $\Sigma \otimes \mathbb{B}(X)$, generated by $\Sigma \times \mathbb{B}(X)$, where $\mathbb{B}(X)$ denotes the Borel sets of X. If $\varphi(\omega, \cdot)\colon A \multimap X$ is still u.s.c. (or l.s.c.), then φ is called a *random u-operator* (or a *random l-operator*).

Remark 2.6. For the definition of a random operator, it is usually still required φ to be compact-valued (cf. [19]), and $\varphi(\omega, \cdot)\colon A \multimap X$ to be u.s.c. (cf. [19, Chapter III.31]) or h-continuous (cf. [24, Chapter 5.6]), for almost all $\omega \in \Omega$. Since these restrictions are not necessary for us, we omitted them in Definition 2.5.

Definition 2.7. Let $A \subset X$ be a closed subset and $\varphi\colon \Omega \times A \multimap X$ be a random operator. We say that φ has a *random fixed point* ξ if there exists a measurable mapping $\xi\colon \Omega \to A$ such that:

$$\xi(\omega) \in \varphi(\omega, \xi(\omega)), \quad \text{for every } \omega \in \Omega.$$

We let $\mathrm{Fix}^{\mathrm{ra}}(\varphi) = \{\xi\colon \Omega \to A \mid \xi \text{ is a random fixed point for } \varphi\}$.

Given k, $m \in \mathbb{N}$, we write $m|k$ to mean that m is a divisor of k. The set of all divisors of k will be denoted by $d(k)$.

Definition 2.8. Let $A \subset X$ be a closed subset and $\varphi\colon \Omega \times A \multimap X$ be a random operator. A sequence of measurable maps $\{\xi_i\}_{i=0}^{k-1}$, where $\xi_i\colon \Omega \to A$, $i = 0, \ldots, k-1$, is called a *random k-orbit*, associated to φ, if there exists a partition of Ω such that (μ denotes a measure):

$$\tag{2.1} \begin{cases} \Omega = \Omega_0 \cup \bigcup\limits_{m \in d(k)} \Omega_m, \text{ where } \mu(\Omega_0) = 0, \text{ all } \Omega_m\text{'s are measurable,} \\ \text{there are } i_0, i_1, \ldots, i_l \text{ such that } \mu(\Omega_{i_j}) > 0, \text{ for all } i_j\text{'s,} \\ \text{and the least common multiple of } i_j\text{'s is } k, \end{cases}$$

and $\{\xi_i(\omega)\}_{i=0}^{k-1}$ is, for each fixed $\omega \in \Omega_{i_j}$, a (deterministic) i_j-orbit in the sense of Definition 2.7.

Remark 2.9. Observe that for random k-orbits in Definition 2.8, the following two conditions are obviously satisfied:

(a) $\xi_{i+1}(\omega) \in \varphi(\omega, \xi_i(\omega))$, $i = 0, \ldots, k-2$, and $\xi_0(\omega) \in \varphi(\omega, \xi_{k-1}(\omega))$, for almost all $\omega \in \Omega$,

(b) the sequence $\{\xi_i\}_{i=0}^{k-1}$ is not formed by going p-times around a shorter subsequence of m consecutive elements, where $mp = k$.

One can readily check that the notion of a random 1-orbit coincides with the one of a random fixed point.

The following lemma is crucial in our considerations.

Lemma 2.10. *Let X be a separable space, A a closed subset of X and $\varphi\colon \Omega \times X \multimap X$ a measurable map with nonempty closed values. We let $\varphi_\omega\colon A \multimap X$, $\varphi_\omega(x) := \varphi(\omega, x)$. Assume further that, for every $\omega \in \Omega$, the set $\mathrm{Fix}\,\varphi_\omega := \{x \in X \mid x \in \varphi_\omega(x)\}$ of fixed points of φ_ω is nonempty and closed. Then the map $\Gamma\colon \Omega \multimap X$, given by $\Gamma(\omega) = \mathrm{Fix}\,\varphi_\omega$, has a measurable selection.*

Proof. Firstly, we define the function $f\colon \Omega \times A \to [0, \infty)$ by putting

$$f(\omega, x) := \mathrm{dist}(x, \varphi(\omega, x)) = \inf\{d(x, y) \mid y \in \varphi(\omega, x)\}.$$

Since φ is measurable, so is f (cf. e.g. [19, Proposition (19.16)], [24, Proposition 1.4]).

Now, it is obvious that the graph

$$\Gamma_F = \{(\omega, x) \in \Omega \times X \mid x \in F(\omega)\}$$

of F is equal to

$$f^{-1}(0) = \{(\omega, x) \in \Omega \times A \mid f(\omega, x) = 0\}.$$

Since f is measurable, so is the set $\Gamma_F = f^{-1}(0)$, and consequently $F\colon \Omega \multimap X$ is measurable on the graph. By virtue of Aumann's selection theorem (see Lemma 2.2), there exists a measurable selection $v\colon \Omega \to X$ of F which completes the proof. $\square$

Note that if φ is a random l-operator, then it is sufficient to assume in Lemma 2.10 only that $\varphi(\,\cdot\,, x)$ is measurable, for every $x \in X$.

In order to formulate the generalization of Lemma 2.10 for random orbits, it will be useful to make a partition of Ω as in Definition 2.8. The way of partition depends on a concrete situation considered in Lemma 2.11 below.

Defining the multivalued maps $\mathbb{O}_k\colon \Omega \multimap A^k$, $k \in \mathbb{N}$, and $\mathbb{O}_m^{k/m} \restriction_{\Omega_m}\colon \Omega_m \multimap A^k$ by

$$\mathbb{O}_k(\omega) := \{\{x_i\}_{i=0}^{k-1} \in A^k \mid \{x_i\}_{i=0}^{k-1} \text{ is a } k\text{-orbit of } \varphi(\omega, \,\cdot\,)\},$$

and

$$\mathbb{O}_m^{k/m} \restriction_{\Omega_m} (\omega) := \left\{ \{x_i\}_{i=0}^{k-1} \in A^k \;\middle|\; \begin{array}{l} \{x_i\}_{i=0}^{m-1} \text{ is an } m\text{-orbit of } \varphi(\omega, \,\cdot\,) \\ \text{and } x_{i+tm} = x_i, \text{ for } t = 1, \dots, \dfrac{k}{m} \end{array} \right\}$$

i.e. $\mathbb{O}_m^{k/m}(\omega) \restriction_{\Omega_m}$, $\omega \in \Omega_m$, is a set of m-orbits repeated (k/m)-times, we are ready to give the following crucial statement, whose "if-part" was proved in [1] (the "only if-part" follows directly from the definition of a random k-orbit).

Lemma 2.11. *Assume that $\varphi\colon \Omega \times A \multimap X$ is a random operator. Then φ admits a random k-orbit, $k \in \mathbb{N}$, if and only if $\mathbb{O}_m(\omega)$ is, under* (2.1)*, nonempty, for all $\omega \in \Omega_m$, where $m|k$.*

In particular, we can still give the following corollary (cf. [1, Corollary 1]).

Corollary 2.12. *If the set $\mathbb{O}_k(\omega)$ of orbits of $\varphi(\omega, \,\cdot\,)$ is nonempty, for almost every $\omega \in \Omega$, then φ admits a random k-orbit.*

Remark 2.13. Observe that, for $k = 1$, Corollary 2.12 generalizes Lemma 2.10 in the sense that the set $\operatorname{Fix} \varphi_\omega$ need not be closed.

3. Random degree

In this section, a random topological degree will be defined for a suitable class of random operators.

Let $\mathbb{R}^n$, $n \geq 1$, be as usually an n-dimensional real Euclidean space, with the inner product $\langle \cdot\,, \cdot \rangle$ and the induced norm $\|\cdot\|$. A closed (resp. open) ball in $\mathbb{R}^n$ with center x and radius $r > 0$ is denoted by $B^n(x,r)$ (resp. $B_0^n(x,r)$). Furthermore, put

$$\begin{aligned} B^n(r) &= B^n(0,r), & B_0^n(r) &= B_0^n(0,r), \\ S^{n-1}(r) &= B^n(r) \setminus B_0^n(r), & \mathbb{P}^n &= \mathbb{R}^n \setminus \{0\}; \end{aligned}$$

$\mathbb{Z}$ stands for the set of all integers.

For any $X \in \mathrm{ANR}$, we let

$$J^{\mathrm{ra}}(\Omega \times B^n(r), X) := \{F\colon \Omega \times B^n(r) \multimap X \mid F \text{ is a random } u\text{-operator with } R_\delta\text{-values}\}.$$

For any $X \in \mathrm{ANR}$ and any continuous function $f\colon X \to \mathbb{R}^n$, we put

$$J_f^{\mathrm{ra}}(\Omega \times B^n(r), \mathbb{R}^n) := \{\varphi\colon \Omega \times B^n(r) \multimap \mathbb{R}^n \mid \varphi = f \circ F, \text{ for some } F \in J^{\mathrm{ra}}(\Omega \times B^n(r), X), \text{ and } \varphi(\Omega \times S^{n-1}(r)) \subset \mathbb{P}^n\}.$$

Finally, we define

$$CJ^{\mathrm{ra}}(\Omega \times B^n(r), \mathbb{R}^n) := \bigcup\{J_f^{\mathrm{ra}}(\Omega \times B^n(r), \mathbb{R}^n \mid f\colon X \to \mathbb{R}^n \text{ is continuous and } X \in \mathrm{ANR}\}.$$

The aim of this section is to introduce the notion of a random topological degree for the class $CJ^{\mathrm{ra}}(\Omega \times B^n(r), \mathbb{R}^n)$. Before doing it, we need an appropriate notion of a homotopy in $CJ^{\mathrm{ra}}(\Omega \times B^n(r), \mathbb{R}^n)$.

Definition 3.1. Let $\varphi_1, \varphi_2 \in CJ^{\mathrm{ra}}(\Omega \times B^n(r), \mathbb{R}^n)$ be two maps of the form:

$$\begin{aligned} \varphi_1 &= f_1 \circ F_1, \qquad \Omega \times B^n(r) \xrightarrow{F_1} X \xrightarrow{f_1} \mathbb{R}^n, \\ \varphi_2 &= f_2 \circ F_2, \qquad \Omega \times B^n(r) \xrightarrow{F_2} X \xrightarrow{f_2} \mathbb{R}^n. \end{aligned}$$

We say that φ_1 and φ_2 are *homotopic* in $CJ^{\mathrm{ra}}(\Omega \times B^n(r), \mathbb{R}^n)$ if there exists a random u-operator with R_δ-values $\chi\colon \Omega \times B^n(r) \times [0,1] \multimap X$ and a continuous homotopy $h\colon X \times [0,1] \to \mathbb{R}^n$ such that:

(a) $\chi(\omega, x, 0) = F_1(\omega, x)$, for every $\omega \in \Omega$ and $x \in B^n(r)$,
(b) $\chi(\omega, x, 1) = F_2(\omega, x)$, for every $\omega \in \Omega$ and $x \in B^n(r)$,
(c) $h(x,0) = f_1(x)$, $h(x,1) = f_2(x)$, for every $x \in B^n(r)$,
(d) for every $(\omega, u, t) \in \Omega \times S^{n-1}(r) \times [0,1]$ and $x \in \chi(\omega, u, t)$, we have $h(x,t) \neq 0$.

The mapping $H\colon \Omega \times B^n(r) \times [0,1] \multimap \mathbb{R}^n)$ given by $H(\omega, x, t) = h(\chi(\omega, x, t), t)$ is called a *homotopy* in $CJ^{\mathrm{ra}}(\Omega \times B^n(r), \mathbb{R}^n)$ between φ_1 and φ_2.

Now, observe that if $\varphi \in CJ^{\mathrm{ra}}(\Omega \times B^n(r), \mathbb{R}^n)$, then $\varphi_\omega = \varphi(\omega, \cdot) \in CJ^{\mathrm{ra}}(\{\omega\} \times B^n(r), \mathbb{R}^n)$, for every $\omega \in \Omega$, and so the topological degree $\mathrm{Deg}(\varphi_\omega)$

of φ_ω is well defined (see e.g. [3], [8], [13], [18]). Therefore, we are allowed to define:

Definition 3.2. We define a multivalued map $D: CJ^{\mathrm{ra}}(\Omega\times B^n(r),\mathbb{R}^n) \multimap \mathbb{Z}$ by putting

$$D(\varphi) := \{\mathrm{Deg}(\varphi_\omega) \mid \omega \in \Omega\}.$$

The map D is called the *random topological degree* of φ on $CJ^{\mathrm{ra}}(\Omega\times B^n(r),\mathbb{R}^n)$.

In what follows, we say that the random topological degree $D(\varphi)$ of φ is *different from zero* (written $D(\varphi)\neq 0$) if $\mathrm{Deg}(\varphi_\omega)\neq 0$, for every $\omega\in\Omega$.

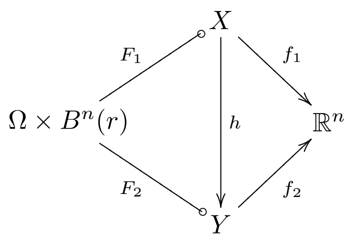
Below, we collect the most important properties of the random topological degree.

Theorem 3.3. *The multivalued map $D: CJ^{\mathrm{ra}}(\Omega \times B^n(r),\mathbb{R}^n) \multimap \mathbb{Z}$ defined in* (3.2) *has the following properties:*

(a) (Existence) *If $D(\varphi)\neq 0$, then there exists a measurable function $\xi:\Omega\to B^n(r)$ such that $0\in\varphi(\omega,\xi(\omega))$, for every $\omega\in\Omega$.*
(b) (Excision) *If $\varphi\in CJ^{\mathrm{ra}}(\Omega\times B^n(r),\mathbb{R}^n)$ and $\{(\omega,x)\in\Omega\times B^n(r)\mid 0\in\varphi(\omega,x)\}\subset\Omega\times B^n_0(\widetilde r)$, for some $0<\widetilde r<r$, then the restriction $\widetilde\varphi$ of φ to $\Omega\times B^n(\widetilde r)$ is in $CJ^{\mathrm{ra}}(\Omega\times B^n(\widetilde r),\mathbb{R}^n)$ and $D(\varphi)=D(\widetilde\varphi)$.*
(c) (Factorization) *Let $\varphi_1,\varphi_2\in CJ^{\mathrm{ra}}(\Omega\times B^n(r),\mathbb{R}^n)$ be two maps of the form:*

$$\begin{aligned}\varphi_1 &= f_1\circ F_1, &\qquad \Omega\times B^n(r) \overset{F_1}{\multimap} X \xrightarrow{f_1} \mathbb{R}^n,\\ \varphi_2 &= f_2\circ F_2, &\qquad \Omega\times B^n(r) \overset{F_2}{\multimap} Y \xrightarrow{f_2} \mathbb{R}^n,\end{aligned}$$

where $X,Y\in\mathrm{ANR}$. If there exists a continuous map $h: X\to Y$ such that the diagram



is commutative, i.e. $F_2 = h\circ F_1$ and $f_1=f_2\circ h$, then $D(\varphi_1)=D(\varphi_2)$.
(d) (Homotopy) *If φ_1 and φ_2 are homotopic in $CJ^{\mathrm{ra}}(\Omega\times B^n(r),\mathbb{R}^n)$, then $D(\varphi_1)=D(\varphi_2)$.*

Proof. Note that the properties (b)–(d) immediately follow from the respective properties of the function Deg on $CJ^{\mathrm{ra}}(\{\omega\}\times B^n(r),\mathbb{R}^n)$, i.e., for $\varphi_\omega\in CJ^{\mathrm{ra}}(\{\omega\}\times B^n(r),\mathbb{R}^n)$ and each $\omega\in\Omega$.

For the proof of (a), observe that for every $\varphi \in CJ^{\rm ra}(\Omega \times B^n(r), \mathbb{R}^n)$, we can associate the random vector field $\widehat{\varphi}\colon \Omega \times B^n(r) \to \mathbb{R}^n$ defined as follows:

$$\widehat{\varphi}(\omega, x) := x - \varphi(\omega, x), \quad \text{for every } (\omega, x) \in \Omega \times B^n(r).$$

If we assume that $D(\varphi) \neq 0$ then, for every $\omega \in \Omega$, in view of the existence property for the deterministic topological degree (see e.g. [3, Proposition (8.9.1)]), we get that $\mathrm{Fix}(\widehat{\varphi}_\omega)$ is a nonempty and compact subset of $B^n(r)$.

By applying Lemma 2.10, we get that $\xi \in \mathrm{Fix}^{\rm ra}(\widehat{\varphi})$ which satisfies the following condition:

$$0 \in \varphi(\omega, \xi(\omega)), \quad \text{for every } \omega \in \Omega,$$

and the proof is completed. □

It is well known that, from the topological degree theory, one can deduce many topological results like fixed point theorems, theorem on antipodes, theorem on invariance domains, etc.

The same is possible to deduce, under suitable assumptions, for the random topological degree. Nevertheless, we restrict our considerations to the formulation of the random version of the theorem on antipodes.

Theorem 3.4 (Random Theorem on Antipodes). *Let $\varphi \in CJ^{\rm ra}(\Omega \times B^n(r), \mathbb{R}^n)$ be a random u-operator such that:*

(3.4.1) *for every $x \in S^{n-1}(r)$ and for every $\omega \in \Omega$, we have:*

$$L_x \cap \varphi(\omega, x) = \emptyset \quad \textit{or} \quad L_x \cap \varphi(\omega, -x) = \emptyset,$$

where $L_x := \{tx \mid t \geq 0\}$.

Then $D(\varphi) \neq 0$.

Proof. For every $\omega \in \Omega$, the map $\varphi_\omega\colon B^n(r) \to \mathbb{R}^n$ satisfies the assumptions of the deterministic Borsuk Antipodal Theorem (see e.g. [19]). Thus, for every $\omega \in \Omega$, $\mathrm{Deg}(\varphi_\omega) \neq 0$ and our theorem is proved. □

Finally, we shall sketch the random topological degree theory in Banach spaces. Let E be a separable Banach space. We let:

$$\begin{aligned} B(r) &= \{x \in E \mid \|x\| \leq r\}, \\ B_0(r) &= \{x \in E \mid \|x\| < r\}, \\ S(r) &= \{x \in E \mid \|x\| = r\}. \end{aligned}$$

We define $CJ^{\rm ra}(\Omega \times B(r), E)$ in the same way as $CJ^{\rm ra}(\Omega \times B(r), \mathbb{R}^n)$ but we have assumed that, for every $\omega \in \Omega$, the map $\varphi_\omega = f \circ F_\omega : B(r) \to E$ is compact, i.e. $\overline{\varphi_\omega(B(r))}$ is a compact subset of E and $\mathrm{Fix}\,\varphi_\omega \subset B_0(r)$, for every $\omega \in \Omega$.

As before, with every $\varphi \in CJ^{\mathrm{ra}}(\Omega \times B(r), E)$, we associate the random vector field $\widehat{\varphi}\colon \Omega \times B(r) \multimap E$ by putting:

$$\widehat{\varphi}(\omega, x) = x - \varphi(\omega, x).$$

We let

$$V^{\mathrm{ra}}(\Omega \times B(r), E) := \{\psi : \Omega \times B(r) \multimap E \mid \psi = \widehat{\varphi} \text{ and } \varphi \in CJ^{\mathrm{ra}}(\Omega \times B(r), E)\}.$$

Then we define the map $D\colon V^{\mathrm{ra}}(\Omega \times B(r), E) \multimap Z$ by putting:

$$D(\psi) = \{\mathrm{Deg}(\psi_\omega) \mid \omega \in \Omega\}, \tag{3.1}$$

where $\mathrm{Deg}(\psi_\omega)$ is the deterministic topological degree of ψ_ω (see e.g. [3, p. 104]).

The random topological degree defined in (3.1) has all the properties formulated in Theorem 3.3. As a standard consequence of the above random degree theory (cf. Lemma 2.10), we can formulate:

Theorem 3.5 (Random Schauder Fixed Point Theorem). *Let $X \in \mathrm{AR}$ be a closed subset of a separable Banach space E and let $\varphi\colon \Omega \times X \multimap X$ be a random u-operator with R_δ-values such that $\varphi_\omega\colon X \multimap X$ is compact, for every $\omega \in \Omega$. Then* $\mathrm{Fix}^{\mathrm{ra}}(\varphi) \neq \emptyset$.

Note that Theorem 3.5 immediately follows from the deterministic Schauder Fixed Point Theorem (see e.g. [18] and Lemma 2.10).

For other topological consequences of Theorem 3.5, see e.g. [3], [8], [18], [19].

Remark 3.6. Let us observe that if $\varphi(\Omega \times S^{n-1}(r)) \subset S^{n-1}(\overline{r})$, for some $\overline{r} > 0$, then condition (3.4.1) can be replaced by the following one:

$$\varphi(\omega, x) \cap \varphi(\omega, -x) = \emptyset, \quad \text{for every } (\omega, x) \in \Omega \times S^{n-1}(r).$$

We recommend [3], [19], for other formulations of the Borsuk Antipodal Theorem for multivalued maps in the deterministic case. All the mentioned results have adequate random formulations.

Let us note that, for mappings with convex values, a random coincidence topological degree is considered in [41]. See also [10], [15], [25], [33], [36], [37], [45], for random fixed point theorems of multivalued operators, and [9], [22], [26], [44], for those of single-valued operators.

Remark 3.7. Randomizing the deterministic topological invariants for periodic orbits like the generalized Euler characteristic (cf. [3, Chapter II.6]), sometimes also called the Fuller index, or various indices of periodicity (cf. [27]) seems to be a delicate open problem. The most promissible appropriate tool might be with this respect the Lefschetz number of period k (see e.g. [32] and the references therein).

4. Random differential inclusions

Let $\varphi\colon \Omega \times [0,a] \times \mathbb{R}^n \multimap \mathbb{R}^n$ be a random u-operator, defined in an analogous way as above on $\Omega \times [0,a] \times \mathbb{R}^n$.

Definition 4.1. A random u-operator $\varphi\colon \Omega \times [0,a] \times \mathbb{R}^n \multimap \mathbb{R}^n$ with convex, compact values is called a *random u-Carathéodory map* if:

(4.1.1) there exists a map $\mu\colon \Omega \times [0,a] \to [0,\infty)$ such that $\mu(\omega,\,\cdot\,)$ is Lebesque integrable, $\mu(\,\cdot\,,t)$ is measurable and

$$\|\varphi(\omega,t,x)\| \leq \mu(\omega,t)(1+\|x\|),$$

for every $\omega \in \Omega$, $t \in [0,a]$ and $x \in \mathbb{R}^n$.

Now, for a given random u-Carathéodory map $\varphi\colon \Omega \times [0,a] \times \mathbb{R}^n \multimap \mathbb{R}^n$ and a measurable map $\xi_0\colon \Omega \to \mathbb{R}^n$, we shall consider the following Cauchy problem:

$$(C_{\varphi,\xi_0}) \qquad \begin{cases} x'(\omega,t) \in \varphi(\omega,t,x(\omega,t)), \\ x(\omega,0) = \xi_0(\omega), \end{cases}$$

where the *solution* $x\colon \Omega \times [0,a] \to \mathbb{R}^n$ is a map such that $x(\,\cdot\,,t)$ is measurable, $x(\omega,\,\cdot\,)$ is absolutely continuous and the derivative $x'(\omega,t)$ is considered w.r.t. t. In what follows, we shall denote by $S(\varphi,\xi_0)$ the set of all solutions of (C_{φ,ξ_0}).

Theorem 4.2. *If $\varphi\colon \Omega \times [0,a] \times \mathbb{R}^n \multimap \mathbb{R}^n$ is a random u-Carathéodory map, then $S(\varphi,\xi_0) \neq \emptyset$, for any measurable $\xi_0\colon \Omega \to \mathbb{R}^n$.*

Proof. For the proof, consider the map $F\colon \Omega \times C([0,a],\mathbb{R}^n) \multimap C([0,a],\mathbb{R}^n)$ defined as follows:

$$F(\omega,x) := \bigg\{\xi_0(\omega) + v \;\bigg|\; v(t) = \int_0^t u(\tau)d\tau \text{ and } u(\tau) \in \varphi(\omega,\tau,x(\tau)) \text{ is a Lebesque integrable selection of } \varphi\bigg\}.$$

It follows from the Kuratowski–Ryll–Nardzewski selection theorem (see Lemma 2.2 and Remark 2.3) and (4.1.1) that the map F is well defined. It can be easily seen that if $\xi\colon \Omega \to C([0,a],\mathbb{R}^n)$ is a random fixed point of F, then the map $x\colon \Omega \times [0,a] \to \mathbb{R}^n$, where $x(\omega,t) = \xi(\omega)(t)$, is a solution of (C_{φ,ξ_0}).

Therefore, it is sufficient to show that F satisfies all assumptions of Theorem 3.5. It is a standard fact (see e.g. [7], [3]) that $F(\omega,\,\cdot\,)$ is a u.s.c. compact map with convex values. So, it suffices to show that F is measurable. To do it, we define the mapping $\overline{F}\colon \Omega \times C([0,a],\mathbb{R}^n) \multimap C([0,a],\mathbb{R}^n)$ in the following way:

$$\overline{F}(\omega,x) := F(\omega,x) - \xi_0(\omega).$$

Obviously, F is measurable if and only if so is $\overline{F}$. Thus, to show that $\overline{F}$ is measurable, we consider the diagram

$$\Omega \times C([0,a],\mathbb{R}^n) \xrightarrow{G} L_1([a,b],\mathbb{R}^n) \xrightarrow{L} C([0,a],\mathbb{R}^n)$$

in which:

$$G(\omega,x) := \{\mu \in L_1([a,b],\mathbb{R}^n) \mid \mu(\tau) \in \varphi(\omega,\tau,x(\tau)),\ \text{for every } \tau\},$$
$$L(u)(t) := \int_0^t u(\tau)\,d\tau,$$

where $L_1([a,b],\mathbb{R}^n)$ denotes the space of Lebesgue integrable functions with the integral norm.

Since $\overline{F} = L \circ G$ and L is a continuous single-valued map, we have only to prove that G is measurable. To do it, we define the function:

$$f\colon \Omega \times C([0,a],\mathbb{R}^n) \times L_1([0,a],\mathbb{R}^n \to [0,\infty), \quad f(\omega,x,u) = \operatorname{dist}_{L_1}(u,G(\omega,x)).$$

Thus, we have

$$f(\omega,x,u) = \int_0^a \operatorname{dist}_{\mathbb{R}^n}(u(t),\varphi(\omega,t,x(t))\,dt.$$

Since φ is measurable, so is the map $(\omega,t,x,u) \to \operatorname{dist}_{\mathbb{R}^n}(u(t),\varphi(\omega,,t,x(t))$ (see [23, Theorem 3.3] and cf. also [38]). Thus, by the Fubini theorem, the map f is measurable and again, in view of Theorem 3.3 in [23], we infer that G is measurable which completes the proof. □

Having a random u-Carathéodory map $\varphi\colon \Omega \times [0,a] \times \mathbb{R}^n \multimap \mathbb{R}^n$, for every $\omega \in \Omega$ and $y \in \mathbb{R}^n$, we can consider the following deterministic Cauchy problem:

$$(C_{\varphi_\omega,y}) \qquad \begin{cases} x'(t) \in \varphi_\omega(t,x(t)) := \varphi(\omega,t,x(t)), \\ x(0) = y. \end{cases}$$

It is well known (see [7], [3], [13], that the set $S(\varphi_\omega,y)$ of all solutions of $(C_{\varphi_\omega,y})$ is an R_σ-set.

We define the map $P\colon \Omega \times \mathbb{R}^n \multimap C([0,a],\mathbb{R}^n)$, by putting:

$$(4.1) \quad P(\omega,y) := S(\varphi_\omega,y) = \{x \in C([0,a],\mathbb{R}^n) \mid x'(t) \in \varphi_\omega(t,x(t)) \text{ and } x(0) = y\}.$$

We can state the following important proposition:

Proposition 4.3. *Under the above assumptions, the mapping*

$$P\colon \Omega \times \mathbb{R}^n \multimap C([0,a],\mathbb{R}^n)$$

defined in (4.1) *is a random u-operator.*

Proof. It is well known (see e.g. [3], [13], [19]) that $P(\omega,\,\cdot\,)$ is u.s.c. with R_σ-values. So, it is sufficient to show that P is measurable. We shall proceed similarly as in the proof of Theorem 4.2.

Consider the diagram:

$$\begin{array}{ccc} & L_1([0,a],\mathbb{R}^n) & \\ {\scriptstyle T}\nearrow & & \searrow{\scriptstyle L} \\ \Omega\times\mathbb{R}^n & \xrightarrow[P]{\qquad\qquad} & C([0,a],\mathbb{R}^n) \end{array}$$

in which

$$T(\omega,y) := \left\{ u\in L_1([0,a],\mathbb{R}^n) \;\middle|\; u(t)\in\varphi\Big(\omega,t,y+\int_0^t u(\tau)\,d\tau\Big)\right\}$$

but, this time, $L(u)(t) = y+\int_0^t u(\tau)\,d\tau$. Then $P = L\circ T$ and again it is sufficient to show that T is measurable.

For this, we can proceed quite analogously as in the proof of the measurability of the mapping G in Theorem 4.2; cf. also Lemma 1 in [38]. □

Observe that the measurability of the operator $P\colon\Omega\times\multimap C([0,a],\mathbb{R}^n)$ says that for any measurable $\xi\colon\Omega\to\mathbb{R}^n$, in view of the Kuratowskii–Ryll–Nardzewski selection theorem, there exists a measurable selection $\eta\colon\Omega\times\mathbb{R}^n\to C([0,a],\mathbb{R}^n)$ such that $\eta(\omega,x)\subset P(\omega,\xi(\omega))$. Thus, the map $x\colon\Omega\times[0,a]\to\mathbb{R}^n$ defined as follows:

$$x(\omega,t) := \eta(\omega,x)(t) \tag{4.2}$$

is a solution of $(C_{\varphi,\xi})$.

Note that (4.2) can be reinterpreted in the sense that deterministic solutions define random solutions.

Remark 4.4. Above, we used two times the following fact from the measure theory. If $\xi\colon\Omega\to X$ and $\varphi\colon\Omega\times X\multimap Y$ are two measurable maps, then the map $\widehat{\varphi}\colon\Omega\times X\multimap Y$, $\widehat{\varphi}(\omega,x)=\varphi(\omega,\xi(\omega))$ is mesaruable, too.

In fact, we have the diagram:

$$\Omega \xrightarrow{\widehat{\xi}} \Omega\times X \overset{\varphi}{\multimap} Y,$$

where $\widehat{\xi}(\omega)=(\omega,\xi(\omega))$. Then $\hat{\varphi}=\varphi\circ\widehat{\xi}$. Observe that, for any measurable $D\subset\Omega\times X$, the set $\widehat{\xi}^{-1}(D)$ is measurable. Indeed, we can assume without any loss of generality that $D=C\times B$, where $C\subset\Omega$ and $B\subset X$ are measurable. Thus, $\widehat{\xi}^{-1}(D)=C\cap\xi^{-1}(B)$ and $\widehat{\xi}$ has the needed property, because ξ is measurable.

Now, for every measurable $U\subset Y$, we have $\widehat{\varphi}^{-1}(U)=\widehat{\xi}^{-1}(\varphi^{-1}(U))$. Since $\varphi^{-1}(U)$ is measurable, our claim holds true.

Note that Remark 4.4 will be also used in the following Section 5.

5. Periodic problem for random differential inclusions

For a random u-Carathéodory map $\varphi\colon \Omega\times[0,a]\times\mathbb{R}^n \multimap \mathbb{R}^n$, we shall consider the following periodic problem:

$$(Q_\varphi)\qquad \begin{cases} x'(\omega,t)\in\varphi(\omega,t,x(\omega,t)),\\ x(\omega,0)=x(\omega,a).\end{cases}$$

To study the periodic problem $(Q\varphi)$ for such a map φ, we shall follow an approach based on the random topological degree theory (for the deterministic case see e.g. [3], [13], [19], [30]). To do it, consider the random operator $P\colon\Omega\times\mathbb{R}^n\multimap C([0,a],\mathbb{R}^n)$ defined in (4.1) (cf. Proposition 4.3). Moreover, let us consider the evaluation map $e_a\colon C([0,a],\mathbb{R}^n)\to\mathbb{R}^n$, $e_a(x)=x(a)$. Then the composition

$$P_a := e_a\circ P\colon \Omega\times\mathbb{R}^n\multimap\mathbb{R}^n \tag{5.1}$$

is called the *random Poincaré operator* along the trajectories of (Q_φ).

Assume $\mathrm{Fix}^{\mathrm{ra}}(P_a)\neq\emptyset$. This implies that the map: $\widehat{P}\colon\Omega\times\mathbb{R}^n\multimap C([0,a],\mathbb{R}^n)$ given by

$$\widehat{P}(\omega,y):=\{x\in P(\omega,y)\mid x(0)=x(a)=y\}$$

is well defined, i.e. $\widehat{P}(\omega,y)$ is compact and nonempty.

We claim that $\widehat{P}\colon\Omega\times\mathbb{R}^n\multimap C([0,a],\mathbb{R}^n)$ is measurable. Hence, let A be a closed subset of $C([0,a],\mathbb{R}^n)$. Then we get:

$$\widehat{P}^{-1}(A)=P^{-1}(A\cap\widetilde{e}^{-1}(0)),$$

where $\widetilde{e}\colon C([0,a],\mathbb{R}^n)\multimap\mathbb{R}^n$, defined by $\widetilde{e}(x)=x(0)-x(a)$, is a continuous map. So $\widehat{P}$ is measurable and, in view of the Kuratowski–Ryll–Nardzewski selection theorem, there exists a measurable selection $\eta\colon\Omega\times\mathbb{R}^n\to C([0,a],\mathbb{R}^n)$ of $\widehat{P}$ which defines a solution of (Q_φ) by putting:

$$x\colon\Omega\times[0,a]\to\mathbb{R}^n,\qquad x(\omega,t):=\eta(\omega,\xi(\omega))(t),$$

where $\xi\in\mathrm{Fix}^{\mathrm{ra}}(P_a)$ (cf. Remark 4.4).

Conversely, if we have a solution x of (Q_φ), then the mapping $\xi\colon\Omega\to\mathbb{R}^n$, where $\xi(\omega)=x(\omega,0)$, is a fixed point of P_a. Hence, we have proved:

Proposition 5.1. *Problem (Q_φ) has a solution if and only the random Poincaré operator $P_a\colon\Omega\times\mathbb{R}^n\multimap\mathbb{R}^n$ has a random fixed point.*

To find a fixed point of P_a, we associate with P_a the random vector field $\widetilde{P}_a\colon\Omega\times\mathbb{R}^n\multimap\mathbb{R}^n$ defined as follows:

$$\widetilde{P}_a(\omega,x)=x-P_a(\omega,x).$$

Now, we can assume without any loss of generality that $\widetilde{P}_a \in CJ^{\mathrm{ra}}(\Omega \times B^n, \mathbb{R}^n)$; if not, then $O \in \widetilde{P}_a(\omega, x)$, for some $\|x\| = r$ and every $\omega \in \Omega$, and so P_a has a fixed point or, equivalently, our problem (Q_φ) has a solution.

Proposition 5.1 can be still improved in the following way.

Proposition 5.2. *Assume that $\widetilde{P}_a \in CJ^{\mathrm{ra}}(\Omega \times B^n(r), \mathbb{R}^n)$, for some $r > 0$. If $D(\widetilde{P}_a) \neq 0$, then problem (Q_φ) has a solution.*

Proposition 5.2 follows immediately from Proposition 5.1 by means of Theorem 3.3(a).

In order to show that $D(\widetilde{P}_a) \neq 0$, we shall adopt to the random case the guiding potential method introduced by Liapunov and subsequently developed by Krasnosel'skiĭ ([30]) and others (see e.g. [3], [13], [19], and the references therein).

Definition 5.3. A map $V\colon \Omega \times \mathbb{R}^n \to \mathbb{R}$ is called a *random potential* if the following two conditions are satisfied:

(a) $V(\,\cdot\,, x)$ is measurable, for every $x \in \mathbb{R}$,
(b) $V(\omega, \,\cdot\,)$ is a C^1-map, for every $\omega \in \Omega$.

If $V\colon \Omega \times \mathbb{R}^n \to \mathbb{R}$ is a random potential, then we define a *random vector field* $\partial V\colon \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ as follows:

$$\partial V(\omega, x) := \left(\frac{\partial V}{\partial x_1}(\omega, x), \ldots, \frac{\partial V}{\partial x_n}(\omega, x)\right),$$

for every $(\omega, x) \in \Omega \times \mathbb{R}^n$.

Definition 5.4. Let $V\colon \Omega \times \mathbb{R}^n \to \mathbb{R}$ be a random potential. If, for some $r_0 > r$, V satisfies the following condition:

$$0 \notin \partial V(\Omega \times S^{n-1}(r)), \quad \text{for every } r \geq r_0,$$

then V is called a *random direct potential.*

Let $V\colon \Omega \times \mathbb{R}^n \to \mathbb{R}$ be a random direct potential. Observe that $\partial V \in CJ^{\mathrm{ra}}((\Omega \times B^n(r), \mathbb{R}^n)$, for every $r \geq r_0$.

So, by Theorem 3.3, $D(\partial V)$ is well defined and, in view of the homotopy property Theorem 3.3(d), it is independent of r. Hence, it makes sense to define the *index* $I(V)$ of the random direct potential V, by putting:

$$I(V) = D(\partial V),$$

where $\mathrm{Deg}(\partial V)$ in Definition 3.2 is considered for $\partial V \in CJ^{\mathrm{ra}}((\{\omega\} \times B^n(r), \mathbb{R}^n)$ with $r \geq r_0$ and fixed $\omega \in \Omega$.

Some cases of random direct potentials with nonzero index can be found similarly as in [30], for deterministic potentials. We restrict our considerations to the following proposition (cf. [3], [13], [19], [30]).

Proposition 5.5. *If* $V:\Omega\times\mathbb{R}^n\to\mathbb{R}$ *is a random direct potential satisfying:*

$$\lim_{\|x\|\to\infty} V(\omega,x)=\infty, \quad \text{for every } \omega\in\Omega,$$

then $I(V)=\{1\}$.

Proposition 5.5 follows immediately from the deterministic case.

Definition 5.7. Let $\varphi:\Omega\times[0,a]\times\mathbb{R}^n\multimap\mathbb{R}^n$ be a random u-Carathéodory operator and let $V:\Omega\times\mathbb{R}^n\times\mathbb{R}$ be a random direct potential. We say that V is a *random guiding function* for φ if the following condition is satisfied:

$$\exists r_0>0\ \forall(\omega,t,x)\in\Omega\times[0,a]\times\mathbb{R}^n \text{ with } \|x\|\ge r_0\ \exists y\in\varphi(\omega,t,x):\\ \langle y,\partial V(\omega,x)\rangle\le 0 \text{ or } \langle y,\partial V(\omega,x)\rangle\ge 0. \tag{5.2}$$

Now, we are ready to state the main result of this section.

Theorem 5.7. *If* $\varphi:\Omega\times[0,a]\times\mathbb{R}^n\multimap\mathbb{R}^n$ *is a random u-Carathéodory operator which possesses a random guiding function* $V:\Omega\times\mathbb{R}^n\to\mathbb{R}$ *such that* $I(V)\neq 0$ *(cf. e.g. Proposition* 5.5*), then problem* (Q_φ) *has a solution.*

Sketch of the proof. To prove Theorem 5.7, we need a random version of Lemma 4.5 in [21]. This can be done by making only technical changes in the mentioned lemma. Then the proof of Theorm 5.7 is quite analogous to the proof of Theorem 4.4 in [21]. Instead of the deterministic topological degree, we use here random topological degree presented in Section 3. □

Remark 5.8. For a non-smooth (e.g. locally Lipschitz) guiding function V, the analogy of Theorem 5.7 can be given by means of the deterministic Theorem 3.2 in [13] (cf. also [3, Chapter III.8.c], [19]), provided (5.2) is replaced by a stronger condition, namely

$$\exists r_0>0\ \forall(\omega,t,x)\in\Omega\times[0,a]\times\mathbb{R}^n \text{ with } \|x\|\ge r_0:\\ \langle\varphi(\omega,t,x),\partial V(\omega,x)\rangle\le 0 \text{ or } \langle\varphi(\omega,t,x),\partial V(\omega,x)\rangle\ge 0.$$

Example 5.9. For $V(\omega,x)\equiv V(x):=\frac{1}{2}\|x\|^2$, we have $V:\mathbb{R}^n\to\mathbb{R}$, $\partial V(x)=x$, $\|\partial V(S^{n-1}(r))\|=r\ge r_0>0$ and $\lim_{\|x\|\to\infty}=\infty\Rightarrow I(V)=\{1\}$. Thus, problem (Q_φ) possesses, according to Theorem 5.7, a random solution, provided $\langle\varphi(\omega,t,x),x\rangle\le 0$ or $\langle\varphi(\omega,t,x),x\rangle\ge 0$, for all $\omega\in\Omega$, $t\in[0,a]$ and $\|x\|\ge r_0>0$, where r_0 is a suitable constant.

6. Scheme for randomization of periodicity results

Summing up the above investigations, we can formulate the following proposition.

Proposition 6.1. *Let one-parameter family of deterministic problems*

$$(Q_{\varphi_\omega}) \qquad \begin{cases} x'(t) \in \varphi_\omega(t, x(t)) := \varphi(\omega, t, x(t)), \\ x(0) = x(a), \end{cases}$$

where $\varphi\colon \Omega \times [0,a] \times \mathbb{R}^n \multimap \mathbb{R}^n$ *is a random u-Carathéodory map and* Ω *is a complete measure space, be solvable, for every fixed* $\omega \in \Omega$*. Then random problem* (Q_φ) *admits a solution.*

Proof. If (Q_{φ_ω}) has a solution, for each $\omega \in \Omega$, then there obviously exists a fixed point $x_{0,\omega} \in \mathbb{R}^n$ of the deterministic Poincaré translation operator $P_{a,\omega}$ along the trajectories of $(C_{\varphi_{\omega,y}})$, i.e. $x_{0,\omega} \in P_{a,\omega}(x_{0,\omega})$, where $P_{a,\omega}\colon \mathbb{R}^n \multimap \mathbb{R}^n$ and

$$P_{a,\omega}(y) := \{x_\omega(a) \mid x_\omega(\,\cdot\,) \text{ is a solution of } (C_{\varphi_{\omega,y}})\}. \tag{6.1}$$

Since $P_a\colon \Omega \times \mathbb{R}^n \multimap \mathbb{R}^n$ defined in (5.1) was verified in the foregoing section to be a random Poincaré operator with compact values along the trajectories of (C_{φ,ξ_0}) and since $P_a(\omega, y) = P_{a,\omega}(y)$, for every $\omega \in \Omega$ and $y \in \mathbb{R}^n$, it follows from Lemma 2.10 that P_a possesses a random fixed point. This in turn implies that, according to Proposition 5.1, problem (Q_φ) admits a solution, as claimed. □

To generalize Proposition 6.1 for random subharmonic solutions, i.e. random ka-periodic solutions with $k \in \mathbb{N}$, i.e. $x(\omega, t) \equiv x(\omega, t+ka)$, but at the same time it is not true that for any $m|k$ $x(\omega, t) \equiv x(\omega, t + ma)$, by means of Lemma 2.11 is, however, a delicate problem. The main obstruction consists rather curiously in the fact that deterministic ka-periodic solutions imply the existence of deterministic m-orbits of the associated deterministic Poincaré operators, where only $m|k$, but not necessarily $m = k$.

On the other hand, the following lemma enables us to guarantee the existence of random subharmonic solutions by means of random periodic orbits of the associated random Poincaré operators.

Lemma 6.2. *Random inclusion*

$$x'(\omega, t) \in \varphi(\omega, t, x(\omega, t)) \ \ [\equiv \varphi(\omega, t + a, x(\omega, t))]\,, \ \ a > 0, \tag{6.2}$$

where $\varphi\colon \Omega \times [0,a] \times \mathbb{R}^n \multimap \mathbb{R}^n$ *is a random u-Carathéodory map and* Ω *is a complete measure space, possesses a random ma-periodic solution,* $m \in \mathbb{N}$*, provided the random Poincaré operator* P_a *along the trajectories of* (C_{φ,ξ_0}) *defined in* (5.1) *has a random m-orbit in the sense of Definition* 2.8*.*

Proof. Consider the inclusion

$$x'(t) \in \varphi_\omega(t, x(t)) := \varphi(\omega, t, x(t)) \equiv \varphi(\omega, t + a, x(t)), \quad a > 0, \tag{6.3}$$

where φ is as above and define the solution operators $S_r\colon \Omega\times\mathbb{R}^n \multimap C([0,ma],\mathbb{R}^n)$ by the formula

$$S_r(\omega,x_r) := \{x\in C([0,ma],\mathbb{R}^n) \mid x \text{ is a solution of (6.3) with } x(ra)=x_r\},$$

where $r=0,\dots,m$.

Since

$$S_r(\omega,x_r)\big|_{[0,ra]} = \widetilde{S}_r(\omega,x_r)\big|_{[-ra,0]},$$

where

$$\widetilde{S}_r(\omega,x_r) := \{x\in C([-ma,0],\mathbb{R}^n) \mid \ x \text{ is a solution of } x'(-t)\in -\varphi_\omega(-t,x(-t)) \text{with } x(-ra)=x_r\},$$

S_r must be a product-measurable operator, for every $r=0,\dots,m$. This follows from the product-measurability of $S_r\big|_{[ra,ma]}$ and $\widetilde{S}_r\big|_{[-ra,0]}$, $r=0,\dots,m$, proved in (a slightly modified, after time rescaling) Proposition 4.3, and the fact that the only common point of the graphs of the maps $S_r\big|_{[0,ra]}$ and $S_r\big|_{[ra,ma]}$ consists of a singleton $\{(ra,x_r)\}$.

Now, for a given random m-orbit $\{\xi\}_{i=0}^{m-1}$, define the intersection

$$S\colon \Omega \multimap C([0,ma],\mathbb{R}^n)$$

of composition $S_r(\omega,\xi_r(\omega))$, $r=0,\dots,m$, i.e.

$$S(\omega) := \bigcap_{r=0}^{m} S_r(\omega,\xi_r(\omega)), \quad \text{where } \xi_0(\omega)\equiv\xi_m(\omega).$$

The definition of a random m-orbit (see Definition 2.8) guarantees that S has nonempty values. Moreover, since $S_r\big|_{[0,ra]} = \widetilde{S}_r\big|_{[-ra,0]}$ and $S_r\big|_{[ra,ma]}$ are random u-operators (see Proposition 4.3)) with compact values (see e.g. [3]), the set of values must be compact.

Since the product-measurability implies a superpositional measurability and the intersection of product-measurable operators is also product-measurable (cf. [3, Chapter I.3]), S must be a measurable operator.

Thus, applying the Kuratowski–Ryll–Nardzewski selection theorem (cf. Remark 2.3), there exists a single-valued measurable selection $x\subset S$,

$$x\colon \Omega\to C([0,ma],\mathbb{R}^n),$$

which represents the desired random ma-periodic solution x of (6.2), where $x(\omega,i)=\xi_i(\omega)$, $i=0,\dots,m-1$. □

Remark 6.3. Despite Lemma 6.2, in view of the above obstructions, to guarantee a random ka-periodic solution of (6.2), just by means of deterministic ka-periodic solutions of (6.3), where $k>1$, can fail in general. Nevertheless, we have proved in [2] that if (6.2) admits via Lemma 6.2, for $n=1$, an ma-periodic solution with $m > 1$, then the coexistence of random ka-periodic solutions of (6.2) occurs, for each $k \in \mathbb{N}$.

In view of Proposition 6.1, the randomization scheme can be formulated as follows.

Randomization scheme 6.4. *If the given deterministic inclusions are "periodicity stable", under the implementation of parameter values $\omega \in \Omega$, i.e. if they preserve a-periodic solutions of* (6.3)*, then the related random inclusions* (6.2) *admit random a-periodic solutions.*

Many differential equations and inclusions in [3], [16], [17], [30], etc. can be found to be "periodicity stable", i.e. many periodicity results for them can be randomized by means of the scheme 6.4. For the simplest illustrative examples, we can put $\Omega \subset \mathbb{R}^l$.

We can conclude by the following simple illustrative example.

Example 6.5. Consider the random inclusion

$$x'(\omega, t) + A(t)x(\omega, t) \in F(\omega, t, x(\omega, t)). \tag{6.4}$$

Assume that $A\colon [0, a] \to \mathcal{L}(\mathbb{R}^n)$ is a Lebesgue integrable $(n \times n)$-matrix function, whose Floquet multipliers are different from 1, such that $A(t) \equiv A(t + a)$. Let $F\colon \Omega \times [0, a] \times \mathbb{R}^n \multimap \mathbb{R}^n$ be a random u-Carathéodory map satisfying

$$\|F(\omega, t, x)\| \le m(t) + K\|x\|, \quad \text{for all } \omega \in \Omega,\ t \in [0, a],\ x \in \mathbb{R}^n,$$

where $K \ge 0$ is a constant such that

$$K < \left[\max_{t \in [0,a]} \int_0^a \|G(t, s)\|\, ds \right]^{-1},$$

G is the Green function associated with the homogenous problem

$$\begin{cases} x'(t) + A(t)x(t) = 0, \\ x(0) = x(a), \end{cases}$$

and $m \in L^1([0, a], \mathbb{R})$. Assume still that $F(\omega, t, x) \equiv F(\omega, t + a, x)$.

Under the above assumptions, we can prove that (6.4) admits, by means of Proposition 6.1, a random periodic solution. Indeed. It follows from the Floquet theory that the above homogeneous problem has only the trivial solution. Thus, in view of the Fredholm alternative, the deterministic inclusion

$$x'(t) + A(t)x(t) \in F_\omega(t, x(t)) := F(\omega, t, x(t))$$

possesses an a-periodic solution $x_\omega(\,\cdot\,)$ of the form

$$x_\omega(t) = \int_0^a G(t, s) f_\omega(s, x_\omega(s))\, ds, \quad \omega \in \Omega,$$

where $f_\omega \subset F_\omega$ is a measurable selection of F_ω, for each $\omega \in \Omega$. This can be checked in a standard way (see e.g. [3, Chapter III.5]) by means of a multivalued version of the Schauder fixed point theorem. Hence, applying Proposition 6.1, inclusion (6.4) admits a random a-periodic solution, as claimed.

We can also consider this problem in a reverse way, namely starting from the deterministic inclusion

$$x'(t) + A(t)x(t) \in F(t, x(t))$$

and implementing into it the parameters $\omega \in \Omega$, we can easily check that, under the above assumptions, it is "periodicity stable" w.r.t. such an implementation. Therefore, scheme 6.4 leads to the same conclusion.

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