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EXISTENCE AND MULTIPLICITY RESULTS FOR A NON-HOMOGENEOUS FOURTH ORDER EQUATION

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Abstract. In this paper we investigate the problem of existence and multiplicity of solutions for a non-homogeneous fourth order Yamabe type equation. We exhibit a family of solutions concentrating at two points, provided the domain contains one hole and we give a multiplicity result if the domain has multiple holes. Also we prove a multiplicity result for vanishing positive solutions in a general domain.

1. Introduction and statements of the main results

In this paper we will study the existence and the multiplicity of positive solutions for a non-homogeneous problem of the form:

$$\text{(P)} \qquad \begin{cases} \Delta^2 u = |u|^{p-1}u + f & \text{on}\Omega. \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded set of $\mathbb{R}^n$ and $p = (n+4)/(n-4)$ is the so-called critical exponent. These kind of problems were deeply studied in the case of the Laplacian (see for instance [1], [11], [19]). Let us recall that problem (P) was studied by Selmi [26] and Ben Ayed–Selmi [9] where the authors prove the existence of a one-bubble solution to the problem under assumptions on f. Here we will show that we can get two-bubble solutions if the domain contains small

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holes, and vanishing type solutions for a small generic perturbation f in the C^0 sense.

We recall that for $f = 0$, this problem has a deep geometrical meaning, in fact if (M, g) is an n-dimensional compact closed riemannian manifold with $n \geq 5$, we can define the Q-curvature

$$Q := \frac{n^3 - 4n^2 + 16n - 16}{8(n-2)^2(n-1)^2} R^2 - \frac{2}{(n-2)^2}|\mathrm{Ric}|^2 + \frac{1}{2(n-1)} \Delta R,$$

where R is the scalar curvature and Ric is the Ricci curvature. After a conformal change of the metric one gets for $\widetilde{g} = u^{4/(n-4)} g$,

$$Q_{\widetilde{g}} u^{(n+4)/(n-4)} = P_g u,$$

where P_g is the Paneitz operator, defined by

$$P_g u := \Delta_g^2 u - \mathrm{div}\bigg(\bigg(\frac{(n-2)^2 + 4}{2(n-2)(n-1)} Rg - \frac{4}{n-2} \mathrm{Ric}\bigg) du\bigg) + \frac{n-4}{2} Qu.$$

This gives rise to the problem of prescribing the Q-curvature, as the analogous problem on the scalar curvature (see [12], [13] and [23]). We remark that in the flat case, for instance if we consider an open set of $\mathbb{R}^n$, the problem of prescribing constant Q-curvature coincides with (P) with $f = 0$, namely

$$\Delta^2 u = |u|^{p-1} u. \tag{1.1}$$

The variational formulation of (1.1) under Navier boundary conditions in a bounded set was deeply studied, especially with the methods of critical points at infinity theory, introduced by Bahri [3] (see [13], [18] and [17]). We also remark the fact that this problem is not compact, namely, for the case $f = 0$ it corresponds exactly to the limiting case of the Sobolev embedding $H^2(\Omega) \cap H_0^1(\Omega) \hookrightarrow L^{2n/(n-4)}$, (see [27]), and thus we loose the compact embedding, so the variational setting in the classical spaces fails to show existence of solutions: in fact as in the case of the Laplacian, if the domain is star shaped we know that it has no positive solutions ([27], [28]). Finally we recall that in the recent paper [22], we studied the same Yamabe type problem, with a slightly super-critical exponent.

This work contains two main parts. In the first one we deal with a perturbation of the form εf, that is

$$\begin{cases} \Delta^2 u = |u|^{p-1} u + \varepsilon f & \text{on } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P$_\varepsilon$}$$

where f is a positive function in $C^\alpha(\Omega)$, $0 < \alpha < 1$, and $\Omega = \mathcal{D} - \overline{B(P, \mu)}$, for a given domain $\mathcal{D}$ and $P \in \mathcal{D}$. In this setting we have the following result:

THEOREM 1.1. *There exists a constant $\mu_0 = \mu_0(\mathcal{D}, f) > 0$ such that for each $0 < \mu < \mu_0$ fixed, there exist $\varepsilon_0 > 0$ and a family of solutions u_ε of* (1.3) *for $0 < \varepsilon < \varepsilon_0$, having exactly two concentration points, namely:*

$$u_\varepsilon(x) = c_n\bigg(\frac{\varepsilon^{2/(n-4)}\lambda_{1,\varepsilon}}{\varepsilon^{4/(n-4)}\lambda_{1,\varepsilon}^2 + |x-\xi_1^\varepsilon|^2}\bigg)^{(n-4)/2} + c_n\bigg(\frac{\varepsilon^{2/(n-4)}\lambda_{2,\varepsilon}}{\varepsilon^{4/(n-4)}\lambda_{2,\varepsilon}^2 + |x-\xi_2^\varepsilon|^2}\bigg)^{(n-4)/2} + \theta_\varepsilon(x)$$

and $\theta_\varepsilon(x) \to 0$ as $\varepsilon \to 0$ uniformly.

Indeed one gets more information about the solutions along the proof, for instance we will see that $\theta_\varepsilon(x) = \varepsilon w + o(\varepsilon)$, where w is the solution of:

$$\begin{cases} \Delta^2 w = f & \text{on } \Omega, \\ w = \Delta w = 0 & \text{on } \partial\Omega. \end{cases}$$

And within the proof we have that the point $((\xi_1^\varepsilon, \xi_2^\varepsilon), (a_n(\lambda_1^\varepsilon)^{n-4}, a_n(\lambda_2^\varepsilon))^{n-4})$ is a critical point of the function Ψ defined by:

$$\Psi(\xi,\Lambda) = \frac{1}{2}\bigg(\sum_{i=1}^{2}\Lambda_i^2 H(\xi_i,\xi_i) - 2\Lambda_1\Lambda_2 G(\xi_1,\xi_2)\bigg) + \sum_{i=1}^{2}\Lambda_i w(\xi_i),$$

where G is the Green's function of the Ω and H its regular part.

Moreover, if we consider a domain with multiple holes we obtain a multiplicity result. In fact, if $\Omega = \mathcal{D} - \bigcup_{1\le i\le k} \overline{B}(P_i,\mu)$ with $P_1,\dots,P_k \in \Omega$, the previous result can be generalized as in [14] and [22] to the following:

THEOREM 1.2. *Let $1 \le m \le k$. There exists a constant $\mu_0 = \mu_0(\mathcal{D}, f) > 0$ such that for each $0 < \mu < \mu_0$ fixed, there exist $\varepsilon_0 > 0$ and a family of solutions u_ε of* (P_ε) *for $0 < \varepsilon < \varepsilon_0$, of the following form:*

$$u_\varepsilon(x) = c_n\sum_{i=1}^{k}\sum_{j=1}^{2}\bigg(\frac{\varepsilon^{2/(n-4)}\lambda_{i,j,\varepsilon}}{\varepsilon^{4/(n-4)}\lambda_{i,j,\varepsilon}^2 + |x-\xi_{i,j}^\varepsilon|^2}\bigg)^{(n-4)/2} + \theta_\varepsilon(x)$$

and $\theta_\varepsilon(x) \to 0$ as $\varepsilon \to 0$ uniformly.

In particular for a domain with k holes we have at least $2^k - 1$ two-bubble solutions.

In the second part of the paper we deal with the problem

$$(\mathrm{P}_f) \qquad \begin{cases} \Delta^2 u = |u|^{p-1}u + f & \text{on } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases}$$

with no topological constraint on the domain Ω and $f \ge 0$ non identically zero. We prove the following:

Theorem 1.3. *There exist a residual subset $D \subset C^2(\overline{\Omega})$ and $\varepsilon > 0$, such that if $f \in D$ and $|f|_{C(\bar{\Omega})} < \varepsilon$, the problem* ($\mathrm{P}_\varepsilon$) *has at least* $\sum_{i=0}^{\infty} \dim H_i(\Omega) + 1$ *positive solutions.*

Here $H_*(\Omega)$ denotes the singular homology of Ω. We have additional information for these solutions as well. In fact we will see that they vanish when $|f|_{C(\overline{\Omega})} \to 0$, and they have energy smaller than the energy of a single bubble; in contrast with the solutions of the first theorem, where the energy of the solutions is greater than the one of the bubbles, and the solutions blow-up as $\varepsilon \to 0$.

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2. Preliminaries and first estimates

Let us start by defining the following functions:

$$\overline{U}_{(\xi,\lambda)}(x) = \left(\frac{\lambda}{\lambda^2 + |x-\xi|^2}\right)^{(n-4)/2},$$

where $\lambda > 0$ and $\xi \in \Omega$. For $u \in D^2(\Omega)$, we will write Pu for the projection of u on $H^2(\Omega) \cap H_0^1(\Omega)$, defined as the unique solution of the problem

$$\begin{cases} \Delta^2 v = u & \text{on } \Omega, \\ v = \Delta v = 0 & \text{on } \partial\Omega. \end{cases}$$

We also recall that the Green's function of Δ^2 for a set Ω, with Navier boundary conditions is defined as the solution of

$$\begin{cases} \Delta_x^2 G(x,y) = \delta_y & \text{on } \Omega, \\ G(x,y) = \Delta_x G(x,y) = 0 & \text{on } \partial\Omega. \end{cases}$$

This function can be written as

$$G(x,y) = \frac{a_n}{|x-y|^{n-4}} - H(x,y), \quad \text{for all } x, y \in \Omega \text{ and } x \neq y,$$

where a_n is a positive constant depending on n and H the positive smooth solution to

$$\begin{cases} \Delta_x^2 H(x,y) = 0 & \text{on } \Omega, \\ H(x,y) = \dfrac{1}{|x-y|^{n-4}}, \ \Delta H(x,y) = \Delta \dfrac{1}{|x-y|^{n-4}} & \text{on } \partial\Omega. \end{cases}$$

Now let ξ_1, ξ_2 be two points in Ω, and $\lambda_1, \lambda_2 > 0$, we will write $\overline{U}_i = \overline{U}_{(\xi_i,\lambda_i)}$ and $U_i = P\overline{U}_i$. Then one has $U_i = \overline{U}_i - \theta_i$ and

$$\theta_i(x) = H(x,\xi_i)\lambda_i^{(n-4)/2} \int_{\mathbb{R}^n} \overline{U}^p(y)\, dy + o(\lambda_i^{(n-4)/2}).$$

Away from $x = \xi$, we have

$$U_i(x) = G(x, \xi_i)\lambda_i^{(n-4)/2} \int_{\mathbb{R}^n} \overline{U}^p(y)\,dy + o(\lambda_i^{(n-4)/2}).$$

For more details about these estimates we refer to the Appendix.

Let us set now J to be the functional defined by

$$J(u) = \frac{1}{2}\int_\Omega |\Delta u|^2 - \frac{1}{p+1}\int_\Omega |u|^p,$$

and let us find an expansion of

$$J(U_1 + U_2) = \frac{1}{2}\int_\Omega |\Delta(U_1 + U_2)|^2 - \frac{1}{p+1}\int_\Omega (U_1 + U_2)^p.$$

For that we define the set

$$O_\delta(\Omega) = \{(\xi_1, \xi_2) \in \Omega \times \Omega;\ |\xi_1 - \xi_2| > \delta,\ d(\xi_i, \partial\Omega) > \delta\},$$

where $\delta > 0$ is a small fixed number and we put

$$C_n = \frac{1}{2}\int_\Omega |\Delta\overline{U}|^2 - \frac{1}{p+1}\int_\Omega \overline{U}^p.$$

Then we have the following:

LEMMA 2.1. *For (ξ_1, ξ_2) in $O_\delta(\Omega)$ we have*

$$\begin{aligned} J(U_1 + U_2) = {} & 2C_n + \frac{1}{2}\bigg(\int_{\mathbb{R}^n} \overline{U}^p\bigg) \\ & \cdot (H(\xi_1, \xi_1)\lambda_1^{n-4} + H(\xi_2, \xi_2)\lambda_2^{n-4} - 2\lambda_1^{(n-4)/2}\lambda_2^{(n-4)/2}G(\xi_1, \xi_2)) \\ & + o(\max(\lambda_1, \lambda_2)^{n-4}). \end{aligned}$$

PROOF. The proof follows from the following estimates (see the Appendix):

$$\int_\Omega |\Delta U_i|^2 = \int_{\mathbb{R}^n} |\Delta\overline{U}|^2 - \bigg(\int_{\mathbb{R}^n} \overline{U}^p\bigg)^2 H(\xi_i, \xi_i)\lambda_i^{n-4} + o(\lambda_i^{n-4})$$

and

$$\begin{aligned} \int_\Omega \Delta U_1 \Delta U_2 &= \bigg(\int_{\mathbb{R}^n} \overline{U}^p\bigg)^2 \lambda_1^{(n-4)/2}\lambda_2^{(n-4)/2}G(\xi_1, \xi_2) + o(\max(\lambda_1, \lambda_2)^{n-4}), \\ \frac{1}{p+1}\int_\Omega U_i^{p+1} &= \frac{1}{p+1}\int_\Omega \overline{U}^{p+1} - \bigg(\int_{\mathbb{R}^n} \overline{U}^p\bigg)^2 H(\xi_i, \xi_i)\lambda_i^{n-4} + o(\lambda_i^{n-4}), \end{aligned}$$

$$\begin{aligned} \frac{1}{p+1}\int_\Omega & (U_1 + U_2)^{p+1} - U_1^{p+1} - U_2^{p+1} \\ &= 2\bigg(\int_{\mathbb{R}^n} \overline{U}^p\bigg)^2 \lambda_1^{(n-4)/2}\lambda_2^{(n-4)/2}G(\xi_1, \xi_2) + o(\max(\lambda_1, \lambda_2)^{n-4}). \end{aligned}$$

Therefore one has

$$\begin{aligned}
J(U_1+U_2) &= \frac{1}{2}\int_\Omega |\Delta(U_1+U_2)|^2 - \frac{1}{p+1}\int_\Omega (U_1+U_2)^p \\
&= \sum\bigg(\frac{1}{2}\int_\Omega |\Delta U_i|^2 - \frac{1}{p+1}U_i^{p+1}\bigg) \\
&\quad + \int_\Omega \Delta U_1 \Delta U_2 - \frac{1}{p+1}\int_\Omega (U_1+U_2)^{p+1} - U_1^{p+1} - U_2^{p+1} \\
&= \sum \frac{1}{2}\bigg(\int_{\mathbb{R}^n} |\Delta \overline{U}|^2 - \bigg(\int_{\mathbb{R}^n} \overline{U}^p\bigg)^2 H(\xi_i,\xi_i)\lambda_i^{n-4}\bigg) - \frac{1}{p+1}\int_\Omega \overline{U}^{p+1} \\
&\quad + \sum\bigg(\int_{\mathbb{R}^n} \overline{U}^p\bigg)^2 H(\xi_i,\xi_i)\lambda_i^{n-4} \\
&\quad + \bigg(\int_{\mathbb{R}^n} \overline{U}^p\bigg)^2 \lambda_1^{(n-4)/2}\lambda_2^{(n-4)/2} G(\xi_1,\xi_2) \\
&\quad - 2\bigg(\int_{\mathbb{R}^n} \overline{U}^p\bigg)^2 \lambda_1^{(n-4)/2}\lambda_2^{(n-4)/2} G(\xi_1,\xi_2) + o(\max(\lambda_1,\lambda_2)^{n-4}) \\
&= 2C_n + \frac{1}{2}\bigg(\int_{\mathbb{R}^n} \overline{U}^p\bigg)^2 (H(\xi_1,\xi_1)\lambda_1^{n-4} + H(\xi_2,\xi_2)\lambda_2^{n-4} \\
&\quad - 2\lambda_1^{(n-4)/2}\lambda_2^{(n-4)/2} G(\xi_1,\xi_2)) + o(\max(\lambda_1,\lambda_2)^{n-4}). \qquad \square
\end{aligned}$$

Now, we set $\Omega_\varepsilon = \varepsilon^{-2/(n-4)}\Omega$, and we put:

$$v(x') = \varepsilon u(\varepsilon^{2/(n-4)}x')$$

Then every solution u of (P_ε) corresponds to a solution v, by means of the previous rescaling, of the following problem:

$$\begin{cases} \Delta^2 v = |v|^{p-1}v + \varepsilon^{p+1}\widetilde{f} & \text{on } \Omega_\varepsilon, \\ v = \Delta v = 0 & \text{on } \partial\Omega_\varepsilon \end{cases}$$

where $\widetilde{f}(x') = f(\varepsilon^{2/(n-4)}x')$. Hence we define the following perturbed energy functional:

$$J_\varepsilon(u) = \frac{1}{2}\int_{\Omega_\varepsilon} |\Delta u|^2 - \frac{1}{p+1}\int_{\Omega_\varepsilon} |u|^p - \varepsilon^{p+1}\int_{\Omega_\varepsilon} \widetilde{f}u.$$

We consider the function w defined by

$$\begin{cases} \Delta^2 w = f & \text{on } \Omega, \\ w = \Delta w = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

We obtain the following proposition. Set $\Lambda = (\Lambda_1,\Lambda_2)$ and $\lambda_i^2 = (a_n^{-1}\Lambda_i)^{2/(n-4)}$.

Proposition 2.2. *Let V be the sum of U_1, U_2 rescaled on Ω_ε, then for $(\xi_1,\xi_2)\in O_\delta(\Omega)$, one has*

$$J_\varepsilon(V)=2C_n+\varepsilon^2\Psi(\xi,\Lambda)+o(\varepsilon^2),$$

where

$$\Psi(\xi,\Lambda)=\frac{1}{2}\bigg(\sum_{i=1}^{2}\Lambda_i^2H(\xi_i,\xi_i)-2\Lambda_1\Lambda_2G(\xi_1,\xi_2)\bigg)+\sum_{i=1}^{2}\Lambda_i w(\xi_i).$$

Proof. The only term we need to estimate is

$$\begin{aligned}\int_\Omega f(U_1+U_2)&=\int_\Omega(\Delta^2w)(U_1+U_2)\\&=\sum_{i=1}^{2}\int_\Omega(\Delta^2w)\bigg(G(x,\xi_i)\lambda_i^{(n-4)/2}\int_{\mathbb{R}^n}\overline{U}^p(y)\,dy\bigg)+o(\lambda_i^{(n-4)/2})\\&=\sum_{i=1}^{2}w(\xi_i)\lambda_i^{(n-4)/2}\int_{\mathbb{R}^n}\overline{U}^p(y)\,dy+o(\lambda_i^{(n-4)/2}).\end{aligned}$$

The conclusion follows. □

3. Reduction process

From now on let $\Omega_\varepsilon=\varepsilon^{-2/(n-4)}\Omega$. We will consider points $\xi_i'\in\Omega_\varepsilon$ and numbers $\Lambda_i>0$, for $i=1,2$, such that $|\xi_1'-\xi_2'|>\delta\varepsilon^{-2/(n-4)}$, $d(\xi_i',\partial\Omega_\varepsilon)>\delta\varepsilon^{-2/(n-4)}$ and $\delta<\Lambda_i<\delta^{-1}$. Here we will adopt the same notations as in [14], that is $\overline{V}_i(x)=\overline{U}_{\xi_i',\Lambda_i^*}$ for $\Lambda_i^*=(c_n\Lambda_i^2)^{1/(n-4)}$; the related projections on $H^2(\Omega_\varepsilon)\cap H_0^1(\Omega_\varepsilon)$ will be denoted by V_i. Consider the functions

$$\overline{Z}_{ij}=\frac{\partial\overline{V}_i}{\partial\xi_{ij}},\quad i=1,\dots,n\quad\text{and}\quad\overline{Z}_{in+1}=\frac{\partial\overline{V}_i}{\partial\Lambda_i^*}$$

and their projections $Z_{ij}=P\overline{Z}_{ij}$. Let $V=V_1+V_2$ and $\overline{V}=\overline{V}_1+\overline{V}_2$.

For a given smooth function h, we want to solve the following linear problem:

$$\begin{cases}\Delta^2\varphi-pV^{p-1}\varphi=h+\displaystyle\sum_{i,j}c_{ij}V_i^{p-1}Z_{ij} & \text{on }\Omega_\varepsilon,\\ \varphi=\Delta\varphi=0 & \text{on }\partial\Omega_\varepsilon,\\ \langle V_i^{p-1}Z_{ij},\varphi\rangle:=\displaystyle\int_{\Omega_\varepsilon}V_i^{p-1}Z_{ij}\varphi=0 & \text{for } i=1,2,\ j=1,\dots,n+1.\end{cases}\tag{3.1}$$

We define the following weighted L^∞ norms : for a function u defined on Ω_ε

$$\|u\|_*=\|(w_1+w_2)^{-\beta}u\|_{L^\infty}+\|(w_1+w_2)^{-\beta-1/(n-4)}\nabla u\|_{L^\infty}$$

where $w_i=(1/(1+|x-\xi_i'|^2))^{(n-4)/2}$, $\beta=4/(n-4)$, and

$$\|u\|_{**}=\|(w_1+w_2)^{-\gamma}u\|_{L^\infty}$$

where $\gamma = 8/(n-4)$. We define also the set

$$O'_\delta(\Omega_\varepsilon) = \{(\xi_1,\xi_2) \in \Omega_\varepsilon \times \Omega_\varepsilon;\ |\xi_1-\xi_2| > \delta\varepsilon^{-2/(n-4)},\ d(\xi_i,\partial\Omega) > \delta\varepsilon^{-2/(n-4)}\}.$$

We refer to [22] for the proof of the following:

Proposition 3.1. *There exist $\varepsilon_0 > 0$ and $C > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ and all $h \in C^\alpha(\Omega_\varepsilon)$, the problem* (3.1) *admits a unique solution $\varphi = L_\varepsilon(h)$. Moreover, we have*

$$\|L_\varepsilon(h)\|_* \le C\|h\|_{**}, \quad |c_{ij}| \le C\|h\|_{**},$$

and

$$\|\nabla_{(\xi',\Lambda)} L_\varepsilon(h)\|_* \le C\|h\|_{**}.$$

To split the difficulties, we start by finding a solution of

$$\begin{cases} \Delta^2(V+\eta) - (V+\eta)_+^p - \varepsilon^{p+1}\widetilde{f} = \sum_{i,j} c_{ij} V_i^{p-1} Z_{ij} & \text{on } \Omega_\varepsilon, \\ \eta = \Delta\eta = 0 & \text{on } \partial\Omega_\varepsilon, \\ \langle V_i^{p-1} Z_{ij}, \eta\rangle = -\langle V_i^{p-1} Z_{ij}, \varphi\rangle & \text{for } i = 1,2, \\ & j = 1,\ldots,n+1, \end{cases}$$

where φ is the solution of

$$\begin{cases} \Delta^2\varphi = \varepsilon^{p+1}\widetilde{f} & \text{on } \Omega_\varepsilon, \\ \varphi = \Delta\varphi = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases}$$

If we take $\eta = \overline{\eta} + \varphi$, then the equation on $\overline{\eta}$ reads as follows:

$$\Delta^2\overline{\eta} - pV^{p-1}\overline{\eta} = N_\varepsilon(\overline{\eta}) - R_\varepsilon + \sum_{i,j} c_{ij} V_i^{p-1} Z_{ij} \tag{3.2}$$

with $N_\varepsilon(\overline{\eta}) = |V + \overline{\eta} + \varphi|^{p-1}(V + \overline{\eta} + \varphi)_+ - pV^{p-1}(\overline{\eta} + \varphi) - V^p$, and $R_\varepsilon = V^p - \overline{U}_1^p - \overline{U}_2^p - p|V|^{p-2}\varphi$. Therefore, taking $\psi = -L_\varepsilon(R_\varepsilon)$ and $\overline{\eta} = \psi + v$, we get an equation on v of the following form:

$$\Delta^2 v - pV^{p-1}v = N_\varepsilon(\overline{\eta}) + \sum_{i,j} c_{ij} V_i^{p-1} Z_{ij}.$$

Lemma 3.2. *There exists $C > 0$ such that for $\varepsilon > 0$ small enough and $\|v\|_* \le 1/4$, we have*

$$\|N_\varepsilon(\psi + v)\|_{**} \le \begin{cases} C(\|v\|_*^2 + \varepsilon\|v\|_* + \varepsilon^{p+1}) & \text{if } n \le 12, \\ C(\varepsilon^{2\beta-1}\|v\|_*^2 + \varepsilon^{2\beta}\|v\|_* + \varepsilon^{3p}) & \text{if } n > 12. \end{cases}$$

Proof. First, we recall that $\|\psi\|_* \le C\varepsilon^2$ and since $|\varphi| \le C\varepsilon^{p+1}$, we have

$$|\varphi|\overline{V}^{-\beta} \le C\varepsilon^{p+1}\overline{V}^{-\beta} \le C\varepsilon^2$$

hence $\|\varphi\|_* \le C\varepsilon^2$ and we can choose ε small enough so that

$$\|\overline{\eta}\|_* \le \|\psi\|_* + \|v\|_* < 1.$$

Now, we have

$$N_\varepsilon(\overline{\eta}) = \frac{p(p-1)}{2}(V + t(\overline{\eta}+\varphi))^{p-2}(\overline{\eta}+\varphi)^2,$$

for a certain $t \in (0,1)$ and hence if $n \le 12$ we have

$$|\overline{V}^{-8/(n-4)} N_\varepsilon(\overline{\eta})| \le C\overline{V}^{2\beta-8/(n-4)}\overline{V}^{p-2}\|\overline{\eta}+\varphi\|_*^2 \le C\|\overline{\eta}+\varphi\|_*^2$$

If $n > 12$ we have to distinguish two cases. First consider $\delta > 0$ and take the region $d(y, \partial\Omega_\varepsilon) > \delta\varepsilon^{-(n+2)/(n-4)}$, then one has the existence of $C_\delta > 0$ such that $V > C_\delta \overline{V}$ and therefore we get

$$|N_\varepsilon(\overline{\eta})\overline{V}^{-8/(n-4)}| \le C\overline{V}^{2\beta-8/(n-4)+p-2}\|\overline{\eta}+\varphi\|_*^2 \le C\varepsilon^{p-2}\|\overline{\eta}+\varphi\|_*^2.$$

If $d(y, \partial\Omega_\varepsilon) \le \delta\varepsilon^{-(n+2)/(n-4)}$ we have, by using Hopf lemma, that for δ sufficiently small $V(y) \sim \frac{\partial V}{\partial \nu}\, d(y, \partial\Omega_\varepsilon)$, (recall that $|\nabla V| = |\nabla \overline{V}| + o(1)$) and $|\nabla V| \ge C\varepsilon^{(n-3)/(n-4)}$, for ε small enough. Thus $V(y) \ge C\varepsilon^{2(n-3)/(n-4)}\, d(y, \partial\Omega_\varepsilon)$, therefore

$$\begin{aligned}
|N_\varepsilon(\overline{\eta})\overline{V}^{-8/(n-4)}| &\le C\overline{V}^{-8/(n-4)}(\varepsilon^{2(n-3)/(n-4)} d(y, \partial\Omega_\varepsilon))^{p-2}(\overline{\eta}+\varphi)^2 \\
&\le C\overline{V}^{-8/(n-4)}(\varepsilon^{2(n-3)/(n-4)} d(y, \partial\Omega_\varepsilon))^{p-2}(\overline{\eta}+\varphi)^2 \\
&\le C(\varepsilon^{2(n-3)/(n-4)-(n+2)/(n-4)})^{p-2}\|\overline{\eta}+\varphi\|_*^2 \\
&\le C\varepsilon^{2\beta-1}\|\overline{\eta}+\varphi\|_*^2.
\end{aligned}$$

Finally

$$\|N_\varepsilon(\psi+v)\|_{**} \le \begin{cases} C(\|\psi+v+\varphi\|_*^2) & \text{if } n \le 12, \\ C(\varepsilon^{2\beta-1}\|\psi+v+\varphi\|_*^2) & \text{if } n > 12, \end{cases}$$

which finishes the proof. □

Now we want to find a solution to (3.2). The problem can be seen as a fixed point problem if we write it in the following way

$$v = -L_\varepsilon(N_\varepsilon(\psi+v)) = A_\varepsilon(v). \tag{3.3}$$

We have the following:

Proposition 3.3. *There exists $C > 0$ such that for $\varepsilon > 0$ small enough, the problem* (3.3) *has a unique solution v, with $\|v\|_* < C\varepsilon^2$. Moreover, the map $(\xi', \Lambda) \to v$ is C^1 with respect to the norm $\|\cdot\|_*$, and $\|\nabla_{(\xi',\Lambda)} v\|_* \le C\varepsilon^2$.*

Proof. Let $F = \{u \in H^2(\Omega) \cap H_0^1(\Omega), \|u\|_* < \varepsilon^2\}$, and then consider $A_\varepsilon\colon F \to H^2(\Omega) \cap H_0^1(\Omega)$. By using the previous lemma and Proposition 3.1 we get

$$\|A_\varepsilon(u)\|_* \le C\|N_\varepsilon(u+\psi)\|_{**} \le \begin{cases} C(\|u\|_*^2+\varepsilon\|u\|_*+\varepsilon^{p+1}) & \text{if } n\le 12,\\ C(\varepsilon^{2\beta-1}\|u\|_*^2+\varepsilon^{2\beta}\|u\|_*+\varepsilon^{3p}) & \text{if } n>12,\end{cases}$$

$$\le \begin{cases} C\varepsilon^3 & \text{if } n\le 12,\\ C\varepsilon^{2\beta+3} & \text{if } n>12,\end{cases}$$

so for $\varepsilon>0$ small enough, we have that A_ε maps F into itself. Now we estimate $\|A_\varepsilon(a)-A_\varepsilon(b)\|_*$ for $a,b\in F$. Since

$$\|A_\varepsilon(a)-A_\varepsilon(b)\|_* \le C\|N_\varepsilon(a+\psi)-N_\varepsilon(b+\psi)\|_{**},$$

it suffices to show that N_ε is a contraction to finish the proof of the proposition. Note that by construction we have

$$D_uN_\varepsilon(u+\psi)=p|V+u+\psi+\varphi|^{p-2}(V+u+\psi+\varphi)-pV^{p-1}.$$

Then arguing as in [22], we obtain that N_ε is a contraction. Hence the existence and uniqueness of v follows. Next we prove that the map is C^1. We will apply the implicit function theorem to the map K defined by

$$K(\xi',\Lambda,v)=v-A_\varepsilon(v).$$

We recall that

$$D_{\xi'}N_\varepsilon(u)=p[|V+u+\varphi|^{p-2}(V+u+\varphi)-(p-1)V^{p-2}(u+\varphi)-V^{p-1}]D_{\xi'}V$$

same goes for $D_\Lambda N_\varepsilon(u)$. Also,

$$D_uK(\xi',\Lambda,u)h=h+L_\varepsilon(D_uN_\varepsilon(u+\psi)h)=h+M(h).$$

Now

$$\|M(h)\|_*\le\|D_uN_\varepsilon(u+\psi)h\|_{**}\le C\|\overline{V}^{-8/(n-4)+\beta}D_uN_\varepsilon(u+\psi)\|_\infty\|h\|_*$$

and since

$$|\overline{V}^{-8/(n-4)+\beta}D_uN_\varepsilon(u+\psi)|\le C\overline{V}^{2\beta-1}\|u+\psi\|_*,$$

we get

$$\|\overline{V}^{-8/(n-4)+\beta}D_uN_\varepsilon(u+\psi)\|_\infty\le C\begin{cases}\varepsilon^2 & \text{if } n\le 12,\\ \varepsilon^{2\beta+1} & \text{if } n>12,\end{cases}$$

hence

$$\|M(h)\|_*\le C\varepsilon^{\min(2,2\beta+1)}\|h\|_*.$$

Therefore by using the implicit function theorem, we have that φ depends continuously on the parameter (ξ',Λ). On the other hand if we differentiate with respect to ξ' we get

$$D_{\xi'}K(\xi',\Lambda,u)=D_{\xi'}u+D_{\xi'}L_\varepsilon(N_\varepsilon(u+\psi))$$

From Proposition 3.1 we get that

$$\|D_{\xi'}L_\varepsilon(h)\|_* \le C\|h\|_{**}.$$

Thus we need to compute

$$D_{\xi'}\psi = (D_{\xi'}L_\varepsilon)(R_\varepsilon) + L_\varepsilon(D_{\xi'}R_\varepsilon),$$

but

$$D_{\xi_1'}R_\varepsilon = pV^{p-1}D_{\xi_1'}V - p\overline{U}_1^{\,p-1}D_{\xi_1'}\overline{U}_1 - p(p-2)|V|^{p-3}D_{\xi_1'}V\varphi$$

which depends continuously on the parameters, and this is enough to prove that v is C^1 with respect to the parameters (ξ', Λ). Moreover, we have

$$\begin{aligned} D_{\xi'}v = -(D_vK(\xi',\Lambda,v))^{-1}[&(D_{\xi'}L_\varepsilon)(N_\varepsilon(v+\psi)) \\ &+ L_\varepsilon(D_{\xi'}(N_\varepsilon(v+\psi))) + L_\varepsilon(D_v(N_\varepsilon)(v+\psi)D_{\xi'}\psi)], \end{aligned}$$

hence

$$\|D_{\xi'}v\|_* \le C(\|N_\varepsilon(v+\psi)\|_{**} + \|D_{\xi'}(N_\varepsilon(v+\psi))\|_{**} + \|D_v(N_\varepsilon)(v+\psi)D_{\xi'}\psi\|_{**}).$$

Now, from Lemma 3.2, we know that

$$\|N_\varepsilon(v+\psi)\|_{**} \le \begin{cases} C\varepsilon^3 & \text{if } n \le 12, \\ C\varepsilon^{2\beta+3} & \text{if } n > 12, \end{cases}$$

and also

$$\begin{aligned} |D_{\xi'}(N_\varepsilon(u))| = &\, p[|V+u+\varphi|^{p-2}(V+u+\varphi) \\ &- (p-1)V^{p-2}(u+\varphi) - V^{p-1}]D_{\xi'}V \\ \le &\, CV^{p-2}|D_{\xi'}V||u| \le C\overline{V}^{\,p-2+(n-3)/(n-4)+\beta}|u|_*. \end{aligned}$$

We get

$$\overline{V}^{\,-8/(n-4)}|D_{\xi'}(N_\varepsilon(u))| \le C\overline{V}^{\,(n-3)/(n-4)+\beta-1}|u|_*,$$

therefore

$$|D_{\xi'}(N_\varepsilon(v+\psi))|_{**} \le C\varepsilon^2.$$

A similar estimate gives

$$|D_v(N_\varepsilon)(v+\psi)D_{\xi'}\psi|_{**} \le C\varepsilon^2.$$

Since there is no difference in the case of the differentiation with respect to Λ, we omit it. □

4. Reduction of the functional

Here we want to go back to our original set Ω, therefore we will denote $\xi_i' = \varepsilon^{-2/(n-4)}\xi_i$ where $\xi_i \in \Omega$ and we remark that if we take ξ_i and Λ so that $c_{ij} = 0$, then we obtain a solution of our original problem. Let $\mathcal{I}_\varepsilon$ be the functional defined by

$$\mathcal{I}_\varepsilon(u) = \frac{1}{2}\int_\Omega |\Delta u|^2 - \frac{1}{p+1}\int_\Omega |u|^{p+1} - \varepsilon \int_\Omega fu$$

so that $u = V + v + \varphi + \psi$ is a solution for our problem if and only if it is a critical point for this functional. Let us consider the functions defined on Ω by

$$\begin{aligned} \widehat{v}(\xi,\Lambda)(x) &= \varepsilon^{-1} v(\varepsilon^{-2/(n-4)}\xi,\Lambda)(\varepsilon^{-2/(n-4)}x), \\ \widehat{\psi}(x) &= \varepsilon^{-1}\psi(\varepsilon^{-2/(n-4)}x), \\ \widehat{\varphi}(x) &= \varepsilon^{-1}\varphi(\varepsilon^{-2/(n-4)}x), \\ \widehat{U}_i(x) &= \varepsilon^{-1}V_i(\varepsilon^{-2/(n-4)}x). \end{aligned}$$

Therefore if we set $\widehat{U}(x) = \widehat{U}_2(x) + \widehat{U}_1(x)$ and $I(\xi,\Lambda) = \mathcal{I}_\varepsilon(\widehat{U} + \widehat{\psi} + \widehat{v}(\xi,\Lambda) + \widehat{\varphi})$ then

$$I(\xi,\Lambda) = J_\varepsilon(V + \psi + v + \varphi).$$

Next we state the following result and we refer to [22] for the proof.

Lemma 4.1. *$u = \widehat{U} + \widehat{\psi} + \widehat{v}(\xi,\Lambda) + \widehat{\varphi}$ is a solution of the problem* (1.1) *if and only if (ξ,Λ) is a critical point of I.*

Now we define

$$\sigma_f = \int_\Omega fw,$$

and we obtain

Proposition 4.2. *We have the following expansion:*

$$I(\xi,\Lambda) = 2C_n + \varepsilon^2(\Psi(\xi,\Lambda) + \sigma_f) + o(\varepsilon^2),$$

where $o(\varepsilon^2) \longrightarrow 0$ as $\varepsilon \to 0$ in the C^1 sense, uniformly in $O_\delta(\Omega) \times (\delta,\delta^{-1})^2$.

Proof. Let us show first that

$$I(\xi,\Lambda) - \mathcal{I}_\varepsilon(\widehat{U} + \widehat{\psi} + \widehat{\varphi}) = o(\varepsilon^2),$$

and

$$\nabla_{(\xi,\Lambda)}(I(\xi,\Lambda) - \mathcal{I}_\varepsilon(\widehat{U} + \widehat{\psi} + \widehat{\varphi})) = o(\varepsilon^2).$$

Indeed, using a Taylor expansion we have

$$J_\varepsilon(\widehat{U} + \widehat{\psi} + \widehat{v}(\xi,\Lambda) + \widehat{\varphi}) - J_\varepsilon(\widehat{U} + \widehat{\psi} + \widehat{\varphi}) = \int_0^1 tD^2 J_\varepsilon(\widehat{U} + \widehat{\psi} + \widehat{\varphi} + t\widehat{v})[\widehat{v},\widehat{v}]\,dt$$

and this holds since $DJ_\varepsilon(\widehat{U}+\widehat{\psi}+\widehat{\varphi}+\widehat{v})=0$. Therefore, we have

$$\int_0^1 tD^2J_\varepsilon(\widehat{U}+\widehat{\psi}+\widehat{\varphi}+t\widehat{v})[\widehat{\varphi},\widehat{\varphi}]\,dt=\int_0^1 t\bigg[\int_{\Omega_\varepsilon}|\nabla v|^2-p(V+\psi+\varphi+tv)^{p-1}v^2\bigg]\,dt$$
$$=\int_0^1 t\int_{\Omega_\varepsilon}p[V^{p-1}-(V+\psi+\varphi+tv)^{p-1}]v^2+N_\varepsilon(v+\psi)v\,dt.$$

We have $|v|_*+|\varphi|_*+|\psi|_*=O(\varepsilon^2)$, and by using Lemma 3.2, we get

$$\int_{\Omega_\varepsilon}N_\varepsilon(v+\psi)v\le\int_{\Omega_\varepsilon}\overline{V}^{\,p-1+\beta}|N_\varepsilon(v+\psi)|_{**}|v|_*\le C\varepsilon^3\int_{\Omega_\varepsilon}\overline{V}^{\,p-1+\beta}\le C\varepsilon^3.$$

Now, the remaining part can be estimated as follows

$$\int_{\Omega_\varepsilon}[V^{p-1}-(V+\psi+\varphi+tv)^{p-1}]v^2$$
$$\le C\varepsilon^4\int_{\Omega_\varepsilon}\overline{V}^{\,2\beta}[V^{p-1}-(V+\psi+t\varphi)^{p-1}]\le C\varepsilon^4,$$

Same estimates hold if we differentiate with respect to ξ. In fact we have

$$D_\xi(J_\varepsilon(\widehat{U}+\widehat{\psi}+\widehat{v}(\xi,\Lambda)+\widehat{\varphi})-J_\varepsilon(\widehat{U}+\widehat{\psi}+\widehat{\varphi}))$$
$$=\varepsilon^{-2/(n-4)}\int_0^1 t\int_{\Omega_\varepsilon}pD_{\xi'}([V^{p-1}-(V+\psi+\varphi+tv)^{p-1}]v^2)+D_{\xi'}(N_\varepsilon(v+\psi)v)\,dt,$$

and the conclusion follows again from Lemma 3.2. Next step is to prove that

$$\mathcal{I}_\varepsilon(\widehat{U}+\widehat{\psi}+\widehat{\varphi})-\mathcal{I}_\varepsilon(\widehat{U}+\widehat{\varphi})=o(\varepsilon^2)$$

and

$$D_\xi(\mathcal{I}_\varepsilon(\widehat{U}+\widehat{\psi}+\widehat{\varphi})-\mathcal{I}_\varepsilon(\widehat{U}+\widehat{\varphi}))=o(\varepsilon^2),$$

so we start by writing

$$\mathcal{I}_\varepsilon(\widehat{U}+\widehat{\psi}+\widehat{\varphi})-\mathcal{I}_\varepsilon(\widehat{U}+\widehat{\varphi})=I_\varepsilon(U+\psi+\varphi)-I_\varepsilon(U+\varphi)$$
$$=\int_0^1(1-t)([p\int_{\Omega_\varepsilon}(V+\varphi+t\psi)^{p-1}\psi^2-\int_{\Omega_\varepsilon}|\Delta\psi|^2]$$
$$-\int_{\Omega_\varepsilon}(|V|^p-|V+\varphi|^p+p|V|^{p-1}\varphi)\psi+\int_{\Omega_\varepsilon}R^\varepsilon\psi).$$

Also

$$D_\xi(\mathcal{I}_\varepsilon(\widehat{U}+\widehat{\psi}+\widehat{\varphi})-\mathcal{I}_\varepsilon(\widehat{U}+\widehat{\varphi}))$$
$$=\varepsilon^{-2/(n-4)}\bigg[\int_0^1(1-t)\bigg(D_{\xi'}\bigg[p\int_{\Omega_\varepsilon}(V+\varphi+t\psi)^{p-1}\psi^2-\int_{\Omega_\varepsilon}|\Delta\psi|^2\bigg]\,dt$$
$$-D_{\xi'}\int_{\Omega_\varepsilon}(|V|^p-|V+\varphi|^p+p|V|^{p-1}\varphi)\psi+D_{\xi'}\int_{\Omega_\varepsilon}R^\varepsilon\psi\bigg)\bigg].$$

Again, by using the fact that $|\psi|_* + |R^\varepsilon|_{**} + |\nabla_{(\xi,\Lambda)}\psi|_* + |\nabla_{(\xi,\Lambda)}R^\varepsilon|_{**} \le C\varepsilon^2$, with $|\varphi|_* \le C\varepsilon^p$ if $n \le 12$ and $|\varphi|_* \le C\varepsilon^2$ if $n > 12$, we get the desired result. The final steps, namely showing

$$\mathcal{I}_\varepsilon(\widehat{U} + \widehat{\varphi}) - \mathcal{I}_\varepsilon(\widehat{U}) = \varepsilon^2\sigma_f + o(\varepsilon^2),$$

and

$$D_\xi(\mathcal{I}_\varepsilon(\widehat{U} + \widehat{\varphi}) - \mathcal{I}_\varepsilon(\widehat{U})) = o(\varepsilon^2),$$

are also obtained by using the same kind of estimates. □

5. Analysis of the exterior domain

Let us consider here $\Omega = \mathcal{D} - \overline{B(0,\mu)}$ for $\mu > 0$ small enough. Also for $E = \mathbb{R}^n - \overline{B(0,1)}$ define the set

$$\mathcal{V} = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n;\, G_E(x,y) - H_E^{1/2}(x,x)H_E^{1/2}(y,y) < 0\} \cap (\mu^{-1}\Omega),$$

where G_E and H_E are the Green's function and its regular part on the set E.

Let us take $f = 1$ and $\mathcal{F}_a = \{x \in \mathbb{R}^n;\ 1 < |x| < a,\ a > 1\}$, then the solution of

$$\begin{cases} \Delta^2 w_a = f & \text{on } \mathcal{F}_a, \\ w_a = \Delta w_a = 0 & \text{on } \partial\mathcal{F}_a, \end{cases}$$

is given by

$$w_a(x) = -\frac{1}{8n(n+2)}\bigg(\frac{a^4-1}{a^{4-n}-1}|x|^{4-n} - |x|^4 + a^{4-n}\frac{(1-a^n)}{a^{4-n}-1}\bigg).$$

It is easy to see that it has a maximum for

$$|x_a| = \bigg(\frac{4(1-a^{4-n})}{(n-4)(a^4-1)}\bigg)^{-1/n},$$

and $|x_a| \to \infty$ as $a \to \infty$. Now we consider the function $\varphi_{\mathcal{F}_a}$ defined, on the set $\mathcal{F}_a$ by

$$\varphi_{\mathcal{F}_a}(x,y) = \frac{1}{2}\frac{H_{\mathcal{F}_a}(x,x)w_a(y)^2 + H_{\mathcal{F}_a}(y,y)w_a(x)^2 + 2G_{\mathcal{F}_a}(x,y)w_a(y)w_a(x)}{-H_{\mathcal{F}_a}(x,x)H_{\mathcal{F}_a}(y,y) + G_{\mathcal{F}_a}^2(x,y)},$$

we will extend it to the full exterior domain $E = \{x \in \mathbb{R}^n;\ 1 < |x|\}$, for that we just extend w_a by zero for $|x| > a$. Hence knowing that

$$H_E(x,y) = \frac{a_n}{||y|(x-\overline{y})|^{n-4}}$$

where $\overline{y} = y/|y|^2$, and since w_a is radially symmetric, we get that φ_E has a critical point (x,y) if and only if $\sin(\theta) = 0$ where θ is the angle between x and y. Now we set $x = se$ and $y = -te$, where e is a unit vector and s and t are real number greater than 1. We write

$$\widetilde{\varphi}_E(s,t) = \varphi_E(se, -te).$$

Explicitly:

$$2a_n\widetilde{\varphi}_E(s,t)=\bigg(\frac{\widetilde{w}_a(t)^2}{(s^2-1)^{n-4}}+\frac{\widetilde{w}_a(s)^2}{(t^2-1)^{n-4}}$$
$$+2\widetilde{w}_a(t)\widetilde{w}_a(s)\bigg(\frac{1}{(s+t)^{n-4}}-\frac{1}{(st+1)^{n-4}}\bigg)\bigg)$$
$$\bigg(\bigg(\frac{1}{(s+t)^{n-4}}-\frac{1}{(st+1)^{n-4}}\bigg)^2-\bigg(\frac{1}{(t^2-1)^{n-4}(t^2-1)^{n-4}}\bigg)\bigg)^{-1}.$$

We recall now (see [22]) that the function defined by

$$\widetilde{\rho}(s,t)=a_n\bigg(-\frac{1}{(t^2-1)^{(n-4)/2}(s^2-1)^{(n-4)/2}}-\frac{1}{(1+st)^{n-4}}+\frac{1}{(s+t)^{n-4}}\bigg),$$

has a unique maximum point of the form (K,K), for $s,t>1$ and a unique k satisfying $\widetilde{\rho}(k,k)=0$. we can choose $a_0>0$, big enough, such that for $a>a_0$, we have $k<K<|x_a|$. Hence we can get the following:

Lemma 5.1. *The function $\widetilde{\varphi}_E$ admits a unique minimum, of the form (τ_a,τ_a). Moreover, $\tau_a\in(k,K)$.*

Next we will work on the domain $\Omega=D-\overline{B(0,\mu)}$. We set m, (resp. M) the radius of the largest (resp. smallest) ball contained (resp. containing) D, and set $\alpha=\min_\Omega f$ and $\beta=\max_\Omega f$. Thus, by using the maximum principle, we have $z_m\le w\le z_M$ for $\mu<|x|<m$, with w as defined in (2.1),

$$z_m(x)=\alpha\mu^4w_{a_1}(\mu^{-1}x)\quad\text{and}\quad z_M(x)=\beta\mu^4w_{a_2}(\mu^{-1}x),$$

here $a_1=\mu^{-1}m$ and $a_1=\mu^{-1}M$. We obtain the following

Lemma 5.2. *For $\mu>0$ small enough the function φ_E has a relative minimum in a point $(\widetilde{x}_\mu,\widetilde{y}_\mu)$, with $|\widetilde{x}_\mu|$ and $|\widetilde{y}_\mu|$ belonging to $(k,\widetilde{k})$, and $\widetilde{k}$ independent of μ.*

The proof of this lemma follows if we show that there exist $\widetilde{k}\ge K$ satisfying

$$\frac{\widetilde{\varphi}_{\mathcal{F}_{a_1}}(\widetilde{k},\widetilde{k})}{\widetilde{\varphi}_{\mathcal{F}_{a_2}}(K,K)}\ge 1,$$

the conclusion will follow from the fact that $\varphi_{\mathcal{F}_{a_1}}\le\varphi_E\le\varphi_{\mathcal{F}_{a_2}}$ and $\varphi_{\mathcal{F}_a}$ has a unique minimum point for a big enough.

Let us Define the set

$$\mathcal{X}=\{(x,y)\in\mathcal{V},\ \text{such that } k<|x|,\ |y|<\widetilde{k}\},$$

and call $c_\mu=\varphi_E(\widetilde{x}_\mu,\widetilde{y}_\mu)$. Now we choose $\delta_\mu>c_\mu$ in such way that the set $\{(x,y)\in\mathcal{X},\varphi_E=\delta_\mu\}$ is a closed curve on which $\nabla\varphi_E\neq 0$. Observe then that if we call

$$\mathcal{J}=\{(x,y)\in\mathcal{X},\ \text{such that } \varphi_E\le\delta_\mu\},$$

two situations might happen on $\partial\mathcal{J}$: either there exists a tangential direction τ such that $\nabla\varphi_E\cdot\tau\neq 0$, or x and y point in two different directions and $\nabla\varphi_E(x,y)\neq 0$ points in the normal direction to $\partial\mathcal{J}$.

Now if we look at $E_\mu=\mathbb{R}^n-\overline{B(0,\mu)}$, then we can easily see that G_{E_μ} and H_{E_μ}, are defined by

$$G_{E_\mu}(x,y)=\mu^{4-n}G_E(\mu^{-1}x,\mu^{-1}y)\quad\text{and}\quad H_{E_\mu}(x,y)=\mu^{4-n}H_E(\mu^{-1}x,\mu^{-1}y).$$

Note that $S_\mu=\mu\mathcal{J}$, corresponds exactly to the set $\{\varphi_E(\mu^{-1}x,\mu^{-1}y)\leq\delta_\mu\}$. Also

$$G(x,y)=G_{E_\mu}(x,y)+O(1)$$

on the set $\mu\mathcal{X}$. Therefore, it follows that:

$$\varphi_\Omega(x,y)=\mu^{n+4}\varphi_E(\mu^{-1}x,\mu^{-1}y)+o(1)$$

where

$$\varphi_\Omega(x,y)=\frac{1}{2}\frac{H_\Omega(x,x)w(y)^2+H_\Omega(y,y)w(x)^2+2G_\Omega(x,y)w(y)w(x)}{G_\Omega^2(x,y)-H_\Omega(x,x)H_\Omega(y,y)}$$

and $o(1)\to 0$ as $\mu\to 0$ in the C^1 sense.

6. Proof of Theorem 1.1

Since the function Ψ defined in Section 2 is singular on the diagonal of $\Omega\times\Omega$, we replace the terms $G(\xi_1,\xi_2)$ by $G_M(\xi_1,\xi_2)=\min(G(\xi_1,\xi_2),M)$ for a constant $M>0$ to be fixed later. Hence Ψ is well defined on $S_\mu\times\mathbb{R}^2_+$.

We remark that in that set, we have

$$\rho(x,y)=H(x,x)^{1/2}H(y,y)^{1/2}-G(x,y)<0,$$

therefore the principal part of Ψ which is a quadratic form, has a negative direction. We will set $\mathbf{e}(\xi_1,\xi_2)$ the vector defining the negative direction:

We have

$$\mathbf{e}(\xi_1,\xi_2)=\bigg(\frac{H(\xi_1,\xi_1)^{1/2}}{H(\xi_2,\xi_2)^{1/2}\rho(\xi_1,\xi_2)},\frac{H(\xi_2,\xi_2)^{1/2}}{H(\xi_1,\xi_1)^{1/2}\rho(\xi_1,\xi_2)}\bigg),$$

Now we are going to consider the vector $\widetilde{\mathbf{e}}$ such that, for each (ξ_1,ξ_2), $\widetilde{\mathbf{e}}(\xi_1,\xi_2)$ is the critical point of $\Psi((\xi_1,\xi_2),\,\cdot\,)$. This vector can be written explicitly in the following form

$$\widetilde{\mathbf{e}}(\xi_1,\xi_2)=\bigg(\frac{H(\xi_2,\xi_2)w(\xi_1)+G(\xi_1,\xi_2))w(\xi_2))w(\xi_1))}{G^2(\xi_1,\xi_2)-H(\xi_2,\xi_2)H(\xi_1\xi_{2=1})},\\ \frac{H(\xi_1,\xi_1)w(\xi_2)+G(\xi_1,\xi_2))w(\xi_2))w(\xi_1))}{G^2(\xi_1,\xi_2)-H(\xi_2,\xi_2)H(\xi_1\xi_{2=1})}\bigg).$$

Therefore we can check that $\Psi((\xi_1,\xi_2),\widetilde{\mathbf{e}}(\xi_1,\xi_2))=\varphi_\Omega(\xi_1,\xi_2)$.

Now we can set the min-max scheme, in a similar way as in [1], [14] and [22]. Let us define

$$K_\mu = \{(x,y) \in \mathcal{X}, (|x|,|y|) = \mu(|\widetilde{x}_\mu|, |\widetilde{y}_\mu|)\},$$

We consider the family of curves $\mathcal{R}$, satisfying the following properties, $\gamma: K_\mu^2 \times [s, s^{-1}] \times [0,1] \to A_\mu \times \mathbb{R}_+^2$ such that:

(i) for $(\xi_1,\xi_2) \in K_\mu^2$, $t \in [0,1]$ it holds

$$\gamma(\xi_1,\xi_2,s,t) = (\xi_1,\xi_2,s\widetilde{\mathbf{e}}(\xi_1,\xi_2)),$$

and

$$\gamma(\xi_1,\xi_2,s^{-1},t) = (\xi_1,\xi_2,s^{-1}\widetilde{\mathbf{e}}(\xi_1,\xi_2)).$$

(ii) $\gamma(\xi_1,\xi_2,t,0) = (\xi_1,\xi_2,t\widetilde{\mathbf{e}}(\xi_1,\xi_2))$, for all $(\xi_1,\xi_2,t) \in K_\mu^2 \times t[s,s^{-1}]$.

Now arguing as in [22], the min-max value defined by

$$C(\Omega) = \inf_{\gamma\in\mathcal{R}} \sup_{(\xi_1,\xi_2,t)\in K_\mu^2\times[s,s^{-1}]} \Psi(\gamma(\xi_1,\xi_2,t,1)),$$

is a critical value of Ψ.

Then the proof of Theorem 1.1 follows as in [15].

7. Vanishing solutions

In this section we will prove a multiplicity result concerning problem (P_f). Let us start by introducing a slightly different notation from the previous part. We set

$$\overline{U}_{(z,a)} = c_n\left(\frac{a}{1+a^2|x-z|^2}\right)^{(n-4)/2},$$

for every $z \in \Omega$ (it corresponds to $a = 1/\lambda$ in the first part of the paper). Also, we set:

$$\overline{Z}_{(z,a),i} = \frac{\partial}{\partial z_i}\overline{U}_{(z,a)},$$

for $i = 1,\dots,n$, and

$$\overline{Z}_{(z,a),n+1} = \frac{\partial}{\partial a}\overline{U}_{(z,a)}.$$

Now we consider the functional I defined on $H^2(\Omega)\cap H_0^1(\Omega)$ by

$$I(u) = \frac{1}{2}\int_\Omega |\Delta u|^2 - \frac{1}{p+1}\int_\Omega |u^+|^{p+1}.$$

We know that critical points of this functional are positive solutions to the problem

$$\begin{cases} \Delta^2 u = u^p & \text{on } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases}$$

and, if $\Omega = \mathbb{R}^n$ then the solutions for

$$\begin{cases} \Delta^2 u = u^p \quad \text{on } \mathbb{R}^n, \\ u > 0 \text{ and } u \text{ in } D^{2,2}(\mathbb{R}^n), \end{cases}$$

are of the form $\overline{U}_{(z,a)}$. We define the set

$$S = \left\{ u \in H^2(\Omega) \cap H^1_0(\Omega) - \{0\};\ \int_\Omega |\Delta u|^2 = \int_\Omega |u^+|^{p+1} \right\}.$$

It is easy to show that for every $u \in S$, we have $I(u) > C_n/n$. Now we take $0 < d_0 < 1$ small enough so that, if $d(x, \partial\Omega) < d_0$, then there exists a unique $y \in \partial\Omega$ such that $|x-y| = d(x,\partial\Omega)$. We put $d(x) = \min(d_0, d(x,\partial\Omega))$, for every x in Ω. Next we set

$$\begin{aligned} \mathcal{O}(r) &= \{(x,a) \in \Omega \times (1,\infty);\ d(x)a = r\}, \\ \overline{\mathcal{O}}(r) &= \{(x,a) \in \Omega \times (1,\infty);\ d(x)a \geq r\}. \end{aligned}$$

If we consider the eigenvalue problem

$$\Delta^2 v = \gamma\, p\, \overline{U}^p_{(z,a)} v \quad \text{on } D^2(\mathbb{R}^n),$$

then obviously $\overline{U}_{(z,a)}$ is an eigenfunction corresponding to $\gamma_1 = 1/p$. We take

$$T_{(z,a)} = \operatorname{span}\{\overline{Z}_{(z,a),i},\ i = 1, \dots, n+1\},$$

and by using the classification in [21], we have that $T_{(z,a)}$ is exactly the eigenspace corresponding to the eigenvalue 1. We set T_0 the eigenspace corresponding to the eigenvalue γ_1 and

$$T^+_{(z,a)} = (T_0 \oplus T_{(z,a)})^\perp,$$

where orthogonality here is with respect to the scalar product $(u,v) = \int_\Omega \Delta u \Delta v$, for every $u, v \in D^2(\Omega)$. Now by means of the stereographic projection from $\mathbb{R}^n$ to the sphere, we obtain a linear eigenvalue problem on a compact manifold, with operator (Paneitz) having compact resolvent. Therefore we have the following:

Lemma 7.1. *There exists $\gamma > 0$ such that for every $(z,a) \in \Omega \times (1,\infty)$, $v \in T^+_{(z,a)}$, we have*

$$\langle v, \Delta^2 v - p\,\overline{U}^p_{(z,a)}\, v\rangle \geq \gamma \int_\Omega p\,\overline{U}^p_{(z,a)}\, v^2.$$

We are going to find a particular solution to the problem (P_f):

Lemma 7.2. *There exist $\varepsilon_0 > 0$ and $C_0 > 0$ such that if $|f|_{C(\overline{\Omega})} < \varepsilon_0$, the problem (P_f) has a unique solution $u_0 \in H^2(\Omega) \cap H^1_0(\Omega)$, satisfying*

$$|u_0|_{C^1} \leq C_0 |f|_{C(\overline{\Omega})}.$$

Moreover:

$$\frac{1}{2}\int_\Omega (\Delta u_0)^2 - \frac{1}{p+1}\int_\Omega u_0^{p+1} - \int_\Omega u_0 f < \frac{C_n}{2n}.$$

Proof. Let λ_1 be the first eigenvalue of the operator Δ^2. For a fixed $0 < \lambda < \lambda_1$, consider the function

$$h(t) = \begin{cases} |t^+|^p & \text{if } t < t_0, \\ \lambda|t| & \text{if } t \geq t_0, \end{cases}$$

where t_0 is chosen such that h is continuous. Hence, since h has a linear growth at infinity and it is non-resonant, we can always find a solution to the problem

$$\begin{cases} \Delta^2 u = h(u) + f & \text{on } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega. \end{cases}$$

Moreover, using Schauder estimates we get that $|u_0|_{C^1} \leq C_0 |f|_{C(\overline{\Omega})}$. Thus by taking $\varepsilon_0 > 0$ small enough, we have the desired result. □

Let us consider $f \geq 0$ in $C(\overline{\Omega})$ with $f \not\equiv 0$. We get, by using Hopf's lemma, that there exists $c_1 > 0$ such that

$$\frac{c_1}{2} < -\frac{\partial u_0}{\partial \nu} < c_1, \quad \text{for all } x \in \partial\Omega.$$

Therefore, there exists $c_2 > 0$ such that

$$u_0(x) \geq c_2 d(x), \quad \text{for all } x \in \partial\Omega.$$

Next we want to find solutions of the form $u_0 + v$. We define on $H^2(\Omega) \cap H_0^1(\Omega)$ the functional

$$J(u) = \frac{1}{2}\int_\Omega (\Delta u)^2 - \frac{1}{p+1}\int_\Omega ((u_0 + u)^+)^{p+1} - (p+1)u_0^p v - u_0^{p+1}.$$

We note that v is a critical point of J if and only if $u_0 + v$ is a positive solution to (P_f).

Lemma 7.3. *There exists $\varepsilon_1 > 0$ such that for $|f|_{C(\overline{\Omega})} < \varepsilon_1$, and $v \in H^2(\Omega) \cap H_0^1(\Omega)$, $v^+ \not\equiv 0$, there exists a unique $t_v > t_1 > 0$ such that $J(tv)$ is increasing on $(t_1, t_v]$, decreasing on (t_v, ∞), and $J(t_v v) = \max_{t>0} J(tv)$.*

Proof. We give a sketch of the proof: since we can pick ε_1 small enough, it suffices to prove the result for $u_0 = 0$ and then argue by continuity. The functional J is now equal to I. Let us consider then

$$I(tv) = t^2 a_1 - t^{p+1} a_2$$

where $a_1 = \frac{1}{2}\int_\Omega (\Delta v)^2$ and $a_2 = (1/(p+1))\int_\Omega (v^+)^{p+1}$. This is just a polynomial equation to study. The result follows. □

Now we define the Nehari manifold

$$\mathcal{S} = \{t_v v;\ v \in H^2(\Omega) \cap H_0^1(\Omega) - \{0\}\}.$$

We have that for v in S, $J(v) > 0$, and $\langle \nabla J(v), v\rangle = 0$ if and only if $v \in \mathcal{S} \cup \{0\}$. Therefore the critical points of J are in $\mathcal{S}$.

Lemma 7.4. *The functional J satisfies the Palais–Smale condition on $\big(0, \frac{C_n}{n}\big)$.*

Proof. Let $\{u_j\}$ be a (PS) sequence at the level $0 < d < C_n/n$. Then we know by using the concentration compactness lemma, that there exists $\overline{u}$, $z_1, \ldots, z_k \in \Omega$, $a_1, \ldots, a_k \in \mathbb{R}_+^*$ such that

$$u_j = \overline{u} + \sum_{i=1}^{k} \overline{U}_{(z_i,a_i)} + o(1)$$

in the weak sense. After localization of the blow-up points, namely by testing against a function with support around the z_i, we get that the energy $J(u_j) \geq kC_n/n$. This happens if and only if $k = 0$ since $d < C_n/n$, therefore the convergence holds. □

We will need the following estimates.

Lemma 7.5. *There exists $r_0 > 2$ such that, for every $(z,a) \in \overline{\mathcal{O}}(r_0)$,*

$$\begin{aligned}\int_\Omega u_0 U_{(z,a)}^p &\geq O(d(z)a^{-(n-4)/2}),\\ |U_{(z,a)}|_{L^{n/(n-4)}} &\leq O(a^{-n/2}|\ln(a)|),\\ \int_\Omega u_0^{n/(n-4)} U_{(z,a)}^{n/(n-4)} &\leq O(d(z)^{n/(n-4)} a^{-n/2}|\ln(a)|).\end{aligned}$$

Proof. We have (see Appendix):

$$\int_\Omega u_0 U_{(z,a)}^p \geq c\int_\Omega d(x)(\overline{U}_{(z,a)}^p - p\,\theta_{(z,a)}\overline{U}_{(z,a)}^{p-1}),$$

and

$$\begin{aligned}\int_\Omega d(x)\overline{U}_{(z,a)}^p &\geq \frac{d(z)}{2}\int_{2d(z)>d(x)>d(z)/2} \overline{U}_{(z,a)}^p\\ &\geq \frac{d(z)}{2}\int_0^{d(z)} r^{n-1}\bigg(\frac{a}{1+a^2r^2}\bigg)^{(n+4)/2} dr \geq C\frac{d(z)}{2}\,a^{(n-4)/2}.\end{aligned}$$

Moreover:

$$\int_\Omega \theta_{(z,a)}\overline{U}_{(z,a)}^{p-1} = o(a^{-(n-4)/2}).$$

Then the first inequality is proved. For the second one, we get:

$$|U_{(z,a)}|_{L^{n/(n-4)}}^{n/(n-4)} \leq |\overline{U}_{(z,a)}|_{L^{n/(n-4)}}^{n/(n-4)} \leq |\overline{U}_{(0,a)}|_{L^{n/(n-4)}(B(0,C)}^{n/(n-4)} \leq Ca^{-n/2}|\ln(a)|,$$

Finally, for the last inequality we have:

$$\int_\Omega u_0^{n/(n-4)} U_{(z,a)}^{n/(n-4)} \leq \int_\Omega u_0^{n/(n-4)}\overline{U}_{(z,a)}^{n/(n-4)},$$

and by using the fact that there exists $c > 0$ such that $u_0(x) \le cd(z)$ whenever $|x - z| \le d(z)$, we get the desired result. □

Now we define the following sets :

$$\mathcal{M} = \{U_{(z,a)}; (z,a) \in \Omega \times (1,\infty)\},$$
$$\mathcal{N} = \{\lambda U_{(z,a)}; (z,a) \in \Omega \times (1,\infty),\ \lambda \in (1/2, 2)\}$$

and we call $\overline{T}_{(z,a)}$ the tangent space to $\mathcal{N}$ at $U_{(z,a)}$. We also set $F^-_{(z,a)} = \{\lambda U_{(z,a)};\ \lambda \in \mathbb{R}\}$ and $F^+_{(z,a)} = \overline{T}^\perp_{(z,a)}$. Finally, let $F_{(z,a)} = F^+_{(z,a)} \oplus F^-_{(z,a)}$ and K be the linear operator defined by

$$Ku = u_1 - u_2,$$

for any $u = u_1 + u_2$, with $u_1 \in F^+_{(z,a)}$ and $u_2 \in F^-_{(z,a)}$. We have the following

Lemma 7.6. *There exist positive constants ε_2, r_1, δ and C_1 such that for $f \in C(\overline{\Omega})$ with $|f|_{C(\overline{\Omega})} < \varepsilon_2$, $(z,a) \in \overline{\mathcal{O}}(r_1)$ and $w \in B_\delta(U_{(z,a)})$, it holds:*

$$\langle \Delta^2 v - p(w + u_0)^p_+ v, Kv\rangle \ge C_1 \int_\Omega (\Delta v)^2, \tag{7.1}$$

for every $v \in F_{(z,a)}$.

Proof. Again it is enough to show this inequality for $u_0 = 0$ and then argue by continuity. So let us take $u_0 = 0$ and by contradiction, let us assume that the inequality does not hold. Then there exists a sequence $(z_k, a_k) \in \overline{\mathcal{O}}(r_0)$, $v_k \in F_{(z_k,a_k)}$ with $|v_k| = 1$, $d(z_k)a_k = r_k \to \infty$, and $w_k \in H^2(\Omega) \cap H^1_0(\Omega)$ such that $|w_k - U_{(z_k,a_k)}| \to 0$ as $k \to \infty$, verifying

$$\limsup \langle \Delta^2 v_k - p(w_k)^p_+ v_k, Kv_k\rangle \le 0.$$

We can always write $v_k = v_{k,1} + v_{k,2}$ according to the splitting of $F_{(z_k,a_k)}$. Since $r_k \to \infty$, we have $|\overline{U}_{(z_k,a_k)} - U_{(z_k,a_k)}| \to 0$. Therefore it is easy to see that

$$\operatorname{dist}(F_{(z_k,a_k)}, \operatorname{span}\{T_{(z_k,a_k)}, U_{(z_k,a_k)}\}) \to 0.$$

Thus,

$$\lim_{k\to\infty} \operatorname{dist}(v_{k,1}, F^+_{(z_k,a_k)}) = 0$$

and, by using Lemma 7.1, we have for k big enough

$$\langle v_{k,1}, \Delta^2 v_{k,1} - p(w_k^+)^{p-1} v_{k,1}\rangle \ge \frac{\gamma}{2} \int_\Omega p(w_k^+)^{p-1} v_{k,1}^2.$$

Now let us assume for instance that $|v_{k,1}| > c$, for k big enough. Then there exists $\widetilde{c} > 0$, such that $\langle v_{k,1}, \Delta^2 v_{k,1} - p(w_k^+)^{p-1} v_{k,1}\rangle > \widetilde{c}$, and hence

$$\limsup \left\langle v_{k,1}, \Delta^2 v_{k,1} - p(w_k^+)^{p-1} v_{k,1}\right\rangle > \widetilde{c}.$$

By definition of $v_{k,2}$ we have

$$\langle v_{k,2}, \Delta^2 v_{k,2} - p(w_k^+)^{p-1} v_{k,2}\rangle \le |v_{k,2}|(1 - p).$$

Therefore, knowing also that

$$\lim_{k\to\infty}\operatorname{dist}(v_{k,2},F^-_{(z_k,a_k)})=0$$

we get that either $|v_{k,1}|=|v_{k,2}|=0$, that is $|v_k|=0$, or

$$\limsup\langle\Delta^2 v_k-p(w_k)^p_+v_k,Kv_k\rangle>0$$

which is a contradiction. Then the lemma holds. $\square$

Proposition 7.7. *There exist $r_2>0$ and $C_2>0$ satisfying: for every $f\in C(\overline{\Omega})$, $|f|_{C(\bar\Omega)}<\varepsilon_2$ and each $(z,a)\in O(r_2)$, there exists $w_{(a,z)}\in S\cap B_{\delta/2}(U_{(z,a)})$ such that*

$$|w_{(a,z)}-U_{(z,a)}|\le C_2|\nabla J(U_{(z,a)})| \tag{7.2}$$

and

$$J(w_{(a,z)})=\min_{u\in F^+_{(z,a)}\cap B_{\delta/2}(0)}\ \max_{v\in F^-_{(z,a)}\cap B_{\delta/2}(0)}J(U_{(z,a)}+u+v),$$

that is

$$J(w_{(a,z)}+v)\le J(w_{(a,z)})\le J(w_{(a,z)}+u),$$

for every $u\in F^+_{(z,a)}\cap B_\delta(0)$ and $v\in F^-_{(z,a)}\cap B_\delta(0)$.

Proof. The existence of $w_{(a,z)}$ follows from the fact that $|\nabla J(U_{(z,a)})|\to 0$ as $d(z)a\to\infty$ and (7.1): by Taylor expansion we see that the functional is convex in the direction of $F^+_{(z,a)}$ and concave in the direction of $F^-_{(z,a)}$. We have a saddle point, therefore $w(a,z)$ exists as in [2] and it is in $F_{(z,a)}$. Now we want to prove that

$$|w_{(a,z)}-U_{(z,a)}|\le C_2|\nabla J(U_{(z,a)})|.$$

We note first that since $w_{(a,z)}$ is a saddle point, we have $\langle\nabla J(w(a,z)),w(a,z)\rangle=0$, then $w(a,z)\in S$. Using again a Taylor expansion we have

$$\begin{aligned}\langle\nabla J(w_{(z,a)}),K(w_{(z,a)}-U_{(z,a)})\rangle\\ =\langle\nabla J(U_{(z,a)})+J''(U_{(z,a)})(w_{(z,a)}-U_{(z,a)}),K(w_{(z,a)}-U_{(z,a)})\rangle\\ +o(|w_{(a,z)}-U_{(z,a)}|^2).\end{aligned}$$

By noticing that $J''(U_{(z,a)})h=\Delta^2h-p|U_{(z,a)}|^{p-1}h$ and by using (7.1), we get

$$\begin{aligned}\langle\nabla J(w_{(z,a)}),K(w_{(z,a)}-U_{(z,a)})\rangle\ge\langle\nabla J(U_{(z,a)}),K(w_{(z,a)}-U_{(z,a)})\rangle\\ +C_1|w_{(a,z)}-U_{(z,a)}|^2+o(|w_{(a,z)}-U_{(z,a)}|^2).\end{aligned}$$

But $\langle\nabla J(w_{(z,a)}),K(w_{(z,a)}-U_{(z,a)})\rangle=0$ by construction of $w_{(z,a)}$, therefore we obtain the desired result by a simple application of Cauchy–Schwartz inequality. $\square$

Lemma 7.8. *Let $f = 0$. There exists $r_2 > 0$ such that for every $r > r_2$, there exists $c_r > C_n/n$ verifying*

$$J(w_{(z,a)}) > c_r, \quad \text{for every } (z,a) \in \mathcal{O}(r).$$

Proof. By using the expansion of $|U_{(z,a)}|^2$ (see Appendix), we have the existence of $m > 0$, such that $|U_{(z,a)}| > m$ for $(z,a) \in \overline{O}(r_2)$. Let now $r \geq r_2$. Since $f = 0$ and $w_{(z,a)} \in S$, then $J(w_{(z,a)}) > \frac{C_n}{n}$ for all $(z,a) \in O(r)$. So let us assume by contradiction that

$$\inf_{(z,a)\in O(r)} J(w_{(z,a)}) = \frac{C_n}{n}.$$

Then there exists a sequence $(z_k, a_k) \in O(r)$, such that

$$|w_{(z_k,a_k)} - \overline{U}_{(z_k',a_k')}| \to 0$$

where $(z_k', a_k') \in \Omega \times (1,\infty)$ is such that $d(z_k')a_k' \to \infty$. Thus

$$|w_{(z_k,a_k)} - U_{(z_k',a_k')}| \to 0.$$

Using (7.2), we have $|w_{(z_k,a_k)} - U_{(z_k,a_k)}| < m/4$, since $(z_k,a_k) \in \overline{O}(r_2)$. This leads to $|U_{(z_k,a_k)} - U_{(z_k',a_k')}| \leq m/4$. But we know that $d(z_k')a_k' \to \infty$ and $d(z_k)a_k = r$, therefore

$$\lim_{k\to\infty} |U_{(z_k,a_k)} - U_{(z_k',a_k')}| \geq 2m$$

which is a contradiction. □

Lemma 7.9. *Let $f \in C(\overline{\Omega})$, such that $|f|_{C(\overline{\Omega})} < \varepsilon_2$, then there exist $r_3 > 0$, $C_3, C_4 > 0$ such that*

$$J(w_{(z,a)}) \leq \frac{C_n}{n} + C_3(d(z)a)^{-(n-4)} - C_4 d(z)a^{(n-4)/2}$$

for every $(z,a) \in \overline{\mathcal{O}}(r_3)$.

Proof. For $(z,a) \in \overline{O}(r_2)$, we take $\widetilde{U}_{(z,a)} = t_{U_{(z,a)}} U_{(z,a)}$ as in [19]. So we have $J(\widetilde{U}_{(z,a)}) = \max_{t\geq 0}(tU_{(z,a)})$. Hence by construction of $w_{(z,a)}$, we have

$$J(w_{(z,a)}) \leq J(\widetilde{U}_{(z,a)}).$$

We see that in fact, $t_1 < t_{U_{(z,a)}} < t_2$ for every $(z,a) \in \overline{O}(r_2)$ with t_1 and t_2 two fixed real numbers. Now

$$J(\widetilde{U}_{(z,a)}) \leq \max_{t\geq 0} \left\{ \frac{1}{2} \int_\Omega t^2 (\Delta U_{(z,a)})^2 - \frac{1}{p+1} \int_\Omega t^{p+1} U_{(z,a)}^{p+1} \right\}$$
$$- \min_{t_1 \leq t \leq t_2} \left\{ \frac{1}{p+1} \int_\Omega ((u_0 + tU_{(z,a)})^+)^{p+1} - t^{p+1} U_{(z,a)}^{p+1} - (p+1) t u_0^p U_{(z,a)} - u_0^{p+1} \right\},$$

after studying the polynomial equation

$$\frac{1}{2} \int_\Omega t^2 (\Delta U_{(z,a)})^2 - \frac{1}{p+1} \int_\Omega t^{p+1} U_{(z,a)}^{p+1},$$

and using the estimate in the Appendix, one can see that

$$\max_{t\geq 0}\left\{\frac{1}{2}\int_\Omega t^2(\Delta U_{(z,a)})^2-\frac{1}{p+1}\int_\Omega t^{p+1}U_{(z,a)}^{p+1}\right\}\\ =\frac{C_n}{n}+O(a^{-(n-4)})\leq c+O((ad(z))^{-(n-4)}).$$

By using a Taylor expansion near zero and at infinity, we find that

$$\frac{1}{p+1}\int_\Omega((u_0+tU_{(z,a)})^+)^{p+1}-t^{p+1}U_{(z,a)}^{p+1}-(p+1)tu_0^pU_{(z,a)}-u_0^{p+1}\\ \geq\int_\Omega u_0t^pU_{(z,a)}^p-C\int_\Omega t^{n/(n-4)}u_0^{n/(n-4)}U_{(z,a)}^{n/(n-4)}.$$

Therefore

$$-\min_{t_1\leq t\leq t_2}\left\{\frac{1}{p+1}\int_\Omega((u_0+tU_{(z,a)})^+)^{p+1}-t^{p+1}U_{(z,a)}^{p+1}-(p+1)tu_0^pU_{(z,a)}-u_0^{p+1}\right\}\\ \leq C\int_\Omega t_2^{n/(n-4)}u_0^{n/(n-4)}U_{(z,a)}^{n/(n-4)}-\int_\Omega u_0t_1^pU_{(z,a)}^p.$$

By using the estimates in Lemma 7.5, we get

$$C\int_\Omega t_2^{n/(n-4)}u_0^{n/(n-4)}U_{(z,a)}^{n/(n-4)}-\int_\Omega u_0t_1^pU_{(z,a)}^p\\ \leq O(d(z)^{n/(n-4)}a^{-n/2}|\ln(a)|)-O(d(z)a^{-(n-4)/2}),$$

therefore

$$\begin{aligned}J(\widetilde{U}_{(z,a)})&\leq\frac{C_n}{n}+O((ad(z))^{-(n-4)})\\&\quad+O(d(z)^{n/(n-4)}a^{-n/2}|\ln(a)|)-O(d(z)a^{-(n-4)/2})\\&\leq\frac{C_n}{n}+O(ad(z))^{-(n-4)}+Ad(z)^{n/(n-4)}a^{-n/2}|\ln(a)|-Bd(z)a^{-(n-4)/2}\end{aligned}$$

for A and B two positive constants. The conclusion follows. □

Now we define the set:

$$\mathcal{R}=\{(z,a)\in\overline{\mathcal{O}}(r_3);\ C_3(d(z)a)^{-(n-4)}<C_4d(z)a^{(n-4)/2}\}.$$

In this set we have $J(w_{(z,a)})<C_n/n$ and thus Palais–Smale holds.

Proof of Theorem 1.3. Now the proof of the theorem follows straightforward. In fact, using a minmax argument on the homology classes of $\mathcal{R}$, we obtain critical points of $(z,a)\mapsto J(w_{(z,a)})$, namely for each $[\alpha]\in H_*(\mathcal{R})\cong H_*(\Omega)$, we have that the values c_α defined by

$$c_\alpha=\min_{\alpha\in[\alpha]}\max_{(z,a)\in\alpha}J(w_{(z,a)})$$

are critical values of the function defined before. Moreover, these critical values corresponds to critical points belonging to the inside of the set $\overline{\mathcal{O}}(r_3)$, by

Lemma 7.8. Now we use a transversality theorem (see Appendix) on the map defined by

$$\Psi(u,f)=\Delta^2 u-|u|^{p-1}u-f,$$

to show that these critical points are non-degenerate. This ends the proof. $\square$

8. Appendix

Here we will give a list of estimates that we used in some of the proofs. Here the O is for $d_i/\lambda_i\to\infty$ and $\varepsilon_{12}\to 0$. Let

$$\overline{U}_{(\xi,\lambda)}(x)=\bigg(\frac{\lambda}{1+\lambda^2|x-\xi|^2}\bigg)^{(n-4)/2},$$

and for $i=1,2$, we will set $\overline{U}_i=\overline{U}_{(\xi_i,\lambda_i)}$. By using the same notation as in Section 1, we set

$$U_i=P\overline{U}_i,\quad \varepsilon_{12}=\frac{1}{\lambda_2/\lambda_1+\lambda_1/\lambda_2+\lambda_1\lambda_2|\xi_1-\xi_2|^2}\quad\text{and}\quad d_i=\operatorname{dist}(\xi_i,\partial\Omega).$$

Lemma 8.1. *Let $\theta_1=\overline{U}_1-U_1$, then:*

(a) $0\le\theta_1\le\overline{U}_1$,
(b) $\theta_1(x)=H(\xi_1,x)\lambda_1^{(n-4)/2}+f_1(x)$,
(c) $f_1(x)=O\bigg(\dfrac{\lambda_1^{n/2}}{d_1^{n-2}}\bigg)$, $\dfrac{\partial}{\partial\lambda_1}f_1(x)=O\bigg(\dfrac{\lambda_1^{n/2+1}}{d_1^{n-2}}\bigg)$,
(d) $\dfrac{\partial}{\partial\xi_1}f_1(x)=O\bigg(\dfrac{\lambda_1^{n/2}}{d_1^{n-1}}\bigg)$.

Lemma 8.2. *It holds*

(a) $|U_1|^2=\langle U_1,U_1\rangle=C_n-c_1H(\xi_1,\xi_1)\lambda_1^{n-4}+O\bigg(\dfrac{\lambda_1^{n-2}}{d_1^{n-2}}\bigg)$,

(b) $\langle U_2,U_1\rangle=c_1(\varepsilon_{12}-H(\xi_1,\xi_2)\lambda_1^{(n-4)/2}\lambda_2^{(n-4)/2})$
$\quad+O\bigg(\varepsilon_{12}^{(n-2)/(n-4)}+\dfrac{\lambda_1^{n-2}}{d_1^{n-2}}+\dfrac{\lambda_2^{n-2}}{d_2^{n-2}}\bigg)$,

(c) $\displaystyle\int_{\Omega U_1^{2n/(n-4)}}=C_n-\frac{2n}{n-4}H(\xi_1,\xi_1)\lambda_1^{n-4}+O\bigg(\frac{\lambda_1^{n-2}}{d_1^{n-2}}\bigg)$,

(d) $\displaystyle\int_\Omega U_1^{(n+4)/(n-4)}U_2=\langle U_2,U_1\rangle$

$$+\begin{cases} O\bigg(\varepsilon_{12}^{n/(n-4)}\ln(\varepsilon_{12}^{-1})+\dfrac{\lambda_1^n}{d_1^n}\ln\bigg(\dfrac{d_1}{\lambda_1}\bigg)\bigg) & \text{if } n\ge 8,\\ O\bigg(\varepsilon_{12}\ln(\varepsilon_{12}^{-1})^{(n-4)/n}\dfrac{\lambda_1^{n-4}}{d_1^{n-4}}\bigg) & \text{if } n\le 7.\end{cases}$$

Lemma 8.3. *We have the following estimates on* $\frac{\partial}{\partial\lambda}U_1$*:*

(a) $$\left\langle U_1, \frac{1}{\lambda_1}\frac{\partial}{\partial\lambda}U_1\right\rangle = \frac{n-4}{2}c_1 H(\xi_1,\xi_1)\lambda_1^{n-4} + O\left(\frac{\lambda_1^{n-2}}{d_1^{n-2}}\right),$$

(b) $$\int_\Omega U_1^{(n+4)/(n-4)}\frac{1}{\lambda_1}\frac{\partial}{\partial\lambda}U_1 = 2\left\langle U_1, \frac{1}{\lambda_1}\frac{\partial}{\partial\lambda}U_1\right\rangle + O\left(\frac{\lambda_1^{n-2}}{d_1^{n-2}}\right),$$

(c) $$\left\langle U_2, \frac{1}{\lambda_1}\frac{\partial}{\partial\lambda}U_1\right\rangle = c_1\left(\frac{1}{\lambda_1}\frac{\partial}{\partial\lambda_1}\varepsilon_{12} + \frac{n-4}{2}H(\xi_1,\xi_2)\,\lambda_1^{(n-4)/2}\lambda_2^{(n-4)/2}\right) + O\left(\varepsilon_{12}^{(n-2)/(n-4)} + \frac{\lambda_1^{n-2}}{d_1^{n-2}} + \frac{\lambda_2^{n-2}}{d_2^{n-2}}\right),$$

(d) $$\int_\Omega U_2^{(n+4)/(n-4)}\frac{1}{\lambda_1}\frac{\partial}{\partial\lambda}U_1 = \left\langle U_2, \frac{1}{\lambda_1}\frac{\partial}{\partial\lambda}U_1\right\rangle + \begin{cases} O\left(\varepsilon_{12}^{n/(n-4)}\ln(\varepsilon_{12}^{-1}) + \frac{\lambda_1^n}{d_1^n}\ln\left(\frac{d_1}{\lambda_1}\right)\right) & \text{if } n \geq 8,\\ O\left(\varepsilon_{12}\ln(\varepsilon_{12}^{-1})^{(n-4)/n}\frac{\lambda_1^{n-4}}{d_1^{n-4}}\right) & \text{if } n \leq 7,\end{cases}$$

(e) $$\int_\Omega U_2\frac{1}{\lambda_1}\left(\frac{\partial}{\partial\lambda}U_1\right)^{(n+4)/(n-4)} = \left\langle U_2, \frac{1}{\lambda_1}\frac{\partial}{\partial\lambda}U_1\right\rangle + \begin{cases} O\left(\varepsilon_{12}^{n/(n-4)}\ln(\varepsilon_{12}^{-1}) + \frac{\lambda_1^n}{d_1^n}\ln\left(\frac{d_1}{\lambda_1}\right)\right) & \text{if } n \geq 8,\\ O\left(\varepsilon_{12}\ln(\varepsilon_{12}^{-1})^{(n-4)/n}\frac{\lambda_1^{n-4}}{d_1^{n-4}}\right) & \text{if } n \leq 7.\end{cases}$$

Lemma 8.4. *We have the following estimates on* $\frac{\partial}{\partial\xi}U_1$*:*

(a) $$\left\langle U_1, \frac{1}{\lambda_1}\frac{\partial}{\partial\xi_1}U_1\right\rangle = -\frac{1}{2}c_1 H(\xi_1,\xi_1)\lambda_1^{n-3} + O\left(\frac{\lambda_1^{n-2}}{d_1^{n-2}}\right),$$

(b) $$\int_\Omega U_1^{(n+4)/(n-4)}\frac{1}{\lambda_1}\frac{\partial}{\partial\xi_1}U_1 = 2\left\langle U_1, \frac{1}{\lambda_1}\frac{\partial}{\partial\xi_1}U_1\right\rangle + O\left(\frac{\lambda_1^{n-2}}{d_1^{n-2}}\right),$$

(c) $$\left\langle U_2, \frac{1}{\lambda_1}\frac{\partial}{\partial\xi_1}U_1\right\rangle = c_1\left(\frac{1}{\lambda_1}\frac{\partial}{\partial\xi_1}\varepsilon_{12} - \frac{\partial}{\partial\xi_1}H(\xi_1,\xi_2)\lambda_1^{(n-4)/2}\lambda_2^{(n-4)/2}\right) + O\left(\varepsilon_{12}^{(n-1)/(n-4)}\frac{|\xi_1-\xi_2|}{\lambda_2} + \frac{\lambda_1^{n-2}}{d_1^{n-2}} + \frac{\lambda_2^{n-2}}{d_2^{n-2}}\right),$$

(d) $$\int_\Omega U_2^{(n+4)/(n-4)}\frac{1}{\lambda_1}\frac{\partial}{\partial\xi_1}U_1 = \left\langle U_2, \frac{1}{\lambda_1}\frac{\partial}{\partial\xi_1}U_1\right\rangle + \begin{cases} O\left(\varepsilon_{12}^{n/(n-4)}\ln(\varepsilon_{12}^{-1}) + \frac{\lambda_1^n}{d_1^n}\ln\left(\frac{d_1}{\lambda_1}\right)\right) & \text{if } n \geq 8,\\ O\left(\varepsilon_{12}\ln(\varepsilon_{12}^{-1})^{(n-4)/n}\frac{\lambda_1^{n-4}}{d_1^{n-4}}\right) & \text{if } n \leq 7,\end{cases}$$

(e) $$\int_\Omega U_2 \frac{1}{\lambda_1}\bigg(\frac{\partial}{\partial \xi_1} U_1\bigg)^{(n+4)/(n-4)} = \bigg\langle U_2, \frac{1}{\lambda_1}\frac{\partial}{\partial \xi_1} U_1\bigg\rangle$$

$$+\begin{cases} O\bigg(\varepsilon_{12}^{n/(n-4)} \ln(\varepsilon_{12}^{-1}) + \dfrac{\lambda_1^n}{d_1^n}\ln\bigg(\dfrac{d_1}{\lambda_1}\bigg)\bigg) & \text{if } n \geq 8, \\ O\bigg(\varepsilon_{12}\ln(\varepsilon_{12}^{-1})^{(n-4)/n}\dfrac{\lambda_1^{n-4}}{d_1^{n-4}}\bigg) & \text{if } n \leq 7. \end{cases}$$

The proof of these estimates are similar to the ones in [3]. For more details we refer also to [7], [8] and [17].

Next we state a Transversality Theorem: see [19] for the proof.

THEOREM 8.5. *Let X, Y and Z be three Banach spaces, and $\Psi: X \times Y \longrightarrow Z$ be a C^1 map satisfying the following conditions for given $z \in Z$:*

(a) *for every $(x,y) \in \Psi^{-1}(z)$, the map $D_x\Psi(x,y): X \to Z$ is a Fredholm operator of index 0,*
(b) *for every $(x,y) \in \Psi^{-1}(z)$, the map $D\Psi(x,y): X \times Y \longrightarrow Z$ is surjective.*

Then the set of $y \in Y$, satisfying that z is a regular value of $\Psi(\,\cdot\,,y)$, is a residual set in Y.

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