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QUALITATIVE ANALYSIS FOR NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS VIA TOPOLOGICAL DEGREE METHOD

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Abstract. In this paper we study existence, uniqueness and data dependence for the solutions of some nonlinear fractional differential equations in Banach spaces. By means of topological degree method for condensing maps via a singular Gronwall inequality with mixed type integral terms, many new results are obtained.

1. Introduction

Recently, fractional differential equations have been proved to be valuable tools in the modeling of many phenomena in various fields of engineering, physics and economics. It draws a great application in nonlinear oscillations of earthquakes, many physical phenomena such as seepage flow in porous media and in fluid dynamic traffic model. Actually, they are considered as an alternative

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model to integer differential equations. For more details, one can see the monographs of Diethelm [16], Kilbas et al. [23], Lakshmikantham et al. [25], Miller and Ross [28], Podlubny [32] and Tarasov [33]. Recently, fractional differential systems and optimal controls in Banach spaces are studied by many researchers such as Agarwal et al. [1]–[3], Ahmad and Nieto [4], [5], Balachandran et al. [6], [7], Bai [8], Benchohra et al. [10]–[12], Chang and Nieto [13], Diagana et al. [15], El-Borai [17], Henderson and Ouahab [20], Hernández et al. [21], Li et al. [26], N'Guérékata [30], Mophou and N'Guérékata [29], Wang et al. [35]–[41], Zhang [42], [43] and Zhou et al. [41], [44]–[46].

It is well known that the topological methods proved to be a powerful tool in the study of various problems which appear in nonlinear analysis. Particularly, a priori estimate method has been often used together with the Brouwer degree, the Leray–Schauder degree or the coincidence degree in order to prove the existence of solutions for some boundary value problems and bifurcation problems for nonlinear differential equations or nonlinear partial differential equations. See, for example, Fečkan [18] and Mawhin [27].

In this paper, we are concerned with the existence of solutions to some classes of nonlocal Cauchy problems (NCP for short) and boundary value problems (BVP for short) for fractional differential equations, Cauchy problems for impulsive fractional differential equations (ICP for short) and Cauchy problems for fractional evolution equations (ECP for short) in Banach spaces via topology method which is different from the above quoted papers. More precisely we will consider the following problems via a coincidence degree for condensing maps in a Banach spaces X.

- Problem I: Fractional NCP

$$\begin{cases} {}^cD^q u(t) = f(t, u(t)) & \text{for } t \in J := [0, T], \\ u(0) + g(u) = u_0, \end{cases} \tag{1.1}$$

where ${}^cD^q$ is the Caputo fractional derivative of order $q \in (0, 1)$, u_0 is an element of X, $f\colon J \times X \to X$ is continuous. The nonlocal term $g : C(J, X) \to X$ is a given function, here $C(J, X)$ is the Banach space of all continuous functions from J into X with the norm $\|u\|_C := \sup_{t\in J} \|u(t)\|$ for $u \in C(J, X)$.

- Problem II: Fractional BVP

$$\begin{cases} {}^cD^q u(t) = f(t, u(t)) & \text{for } t \in J, \\ au(0) + bu(T) = c, \end{cases} \tag{1.2}$$

where f is as in fractional NCP (1.1), a, b, c are real constants with $a + b \neq 0$.

- Problem III: Fractional ICP

$$\begin{cases} {}^cD^q u(t) = f(t, u(t)) & \text{for } t \in J \setminus \widetilde{D} := \{t_1, \dots, t_m\}, \\ u(0) = u_0, \\ \Delta u(t_i) = I_i(u(t_i)) & \text{for } i = 1, \dots, m, \end{cases} \tag{1.3}$$

where f, u_0 are as in fractional NCP (1.1), $I_i\colon X \to X$ is a nonlinear map which determine the size of the jump at t_i, $0 = t_0 < t_1 < \dots < t_m < t_{m+1} = T$, $I_i(u(t_i)) = u(t_i^+) - u(t_i^-)$, $u(t_i^+) = \lim_{h \to 0^+} = u(t_i + h)$ and $u(t_i^-) = u(t_i)$ represent respectively the right and left limits of $u(t)$ at $t = t_i$.

- Problem IV: Fractional ECP

$$\begin{cases} {}^cD^q u(t) = -Au(t) + \overline{f}(t, u(t)) & \text{for } t \in J, \\ u(0) = u_0, \end{cases} \tag{1.4}$$

where $-A\colon D(A) \to X$ is the infinitesimal generator of a compact analytic semigroup of uniformly bounded linear operators $\{S(t),\ t \geq 0\}$, $\overline{f}\colon J \times X_\beta \to X$ is continuous, where $X_\beta = D(A^\beta)$ $(0 < \beta < 1)$ is a Banach space with the norm

$$\|u\|_\beta = \|A^\beta u\| \quad \text{for } u \in X_\beta.$$

We remark that existence results for solutions of fractional NCP (1.1), fractional BVP (1.2) and fractional ICP (1.3) in $\mathbb{R}$ have reported in Agarwal et al. [1] by utilizing all kinds of fixed point theorems such Banach contraction principle, Schaefer's fixed point theorem and nonlinear alternative of Leray–Schauder type via some stronger conditions on f, g and I_i. Moreover, existence result for mild solutions for fractional ECP (1.4) has been reported in Wang and Zhou [35] by virtue of Leray–Schauder fixed point theorem via standard singular Gronwall inequality.

In the present paper, we show the existence results for the above problems by virtue of topological degree for condensing maps via a priori estimate method for a coincidence degree introduced by Isaia [22]. Compared with the earlier existence results for fractional NCP (1.1), fractional BVP (1.2) and fractional ICP (1.3) appeared in Agarwal et al. [1], there are at least four differences:

(a) the growth conditions on f, g and I_i includes linear growth case and sublinear growth case;
(b) a new fixed point theorem with a priori estimate method via the topological degree for condensing maps is used;
(c) uniqueness and data dependence results are also deduced by using a useful singular Gronwall inequality with mixed type integral terms;
(d) the techniques of measure for noncompactness and topological degree for condensing maps are used.

Compared with the existence results obtained by Wang and Zhou [35] for fractional ECP (1.4), a new sufficient condition via a fixed point theorem for condensing maps for the existence, uniqueness and data dependence of mild solutions are established.

To end this section, we recall some basic definitions and properties of the fractional calculus theory which are used further in this paper. For more details, see Kilbas et al. [23].

Definition 1.1. The fractional order integral of the function $h \in L^1([a,b],\mathbb{R})$ of order $q \in \mathbb{R}_+$ is defined by

$$I_a^q h(t) = \int_a^t \frac{(t-s)^{q-1}}{\Gamma(q)} h(s)\,ds$$

where Γ is the Gamma function.

Definition 1.2. For a function h given on the interval $[a,b]$, the q-th Riemann–Liouville fractional order derivative of h, is defined by

$$(D_{a+}^q h)(t) = \frac{1}{\Gamma(n-q)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-q-1} h(s)\,ds,$$

here $n = [q]+1$ and $[q]$ denotes the integer part of q.

Definition 1.3. For a function h given on the interval $[a,b]$, the Caputo fractional order derivative of h, is defined by

$$({}^cD_{a+}^q h)(t) = \frac{1}{\Gamma(n-q)} \int_a^t (t-s)^{n-q-1} h^{(n)}(s)\,ds,$$

where $n = [q]+1$ and $[q]$ denotes the integer part of q.

Lemma 1.4. *Let $n > q > n-1$, then the differential equation ${}^cD^q h(t) = 0$ has solutions*

$$h(t) = c_0 + c_1 t + c_2 t^2 + \ldots + c_{n-1} t^{n-1},$$

where $c_i \in \mathbb{R}$, $i = 0, \ldots, n$, $n = [q]+1$.

Lemma 1.5. *Let $n > q > n-1$, then*

$$I^q({}^cD^q h)(t) = h(t) + c_0 + c_1 t + c_2 t^2 + \ldots + c_{n-1} t^{n-1},$$

for some $c_i \in \mathbb{R}$, $i = 0, \ldots, n$, $n = [q]+1$.

Remark 1.6. (a) If $h(t) \in C^n[0,\infty)$, then

$${}^cD^q h(t) = \frac{1}{\Gamma(n-q)} \int_0^t \frac{h^{(n)}(s)}{(t-s)^{q+1-n}}\,ds = I^{n-q} h^{(n)}(t), \quad t > 0,\ n-1 < q < n.$$

(b) The Caputo derivative of a constant is equal to zero.

(c) If h is an abstract function with values in X, then integrals which appear in Definitions 1.1–1.3 are taken in Bochner's sense.

2. Preliminaries

For a minute description of the following notions we refer the reader to Banas [9], Deimling [14], Heinz [19], Lakshmikantham and Leela [24].

In the following, $\mathcal{B} \in \mathcal{P}(X)$ will be the family of all its bounded sets.

Definition 2.1. The function $\alpha\colon \mathcal{B} \to \mathbb{R}_+$ defined by

$$\alpha(B) = \inf\{d > 0 : B \text{ admits a finite cover by sets of diameter } \leq d\},$$

$B \in \mathcal{B}$, is called the Kuratowski measure of noncompactness.

In the whole paper, the letter α will only be used in this context. We state without proof some properties of this measure.

Proposition 2.2. *The following assertions hold:*

(a) $\alpha(B) = 0$ *if and only if* B *is relatively compact.*
(b) α *is a seminorm, i.e.* $\alpha(\lambda B) = |\lambda|\alpha(B)$ *and* $\alpha(B_1 + B_2) \leq \alpha(B_1) + \alpha(B_2)$.
(c) $B_1 \subset B_2$ *implies* $\alpha(B_1) \leq \alpha(B_2)$; $\alpha(B_1 \cup B_2) = \max\{\alpha(B_1), \alpha(B_2)\}$.
(d) $\alpha(\operatorname{conv} B) = \alpha(B)$.
(e) $\alpha(\overline{B}) = \alpha(B)$.

For $\mathcal{M} \subset C(J, X)$, we define

$$\int_0^t \mathcal{M}(s)ds = \left\{ \int_0^t u(s)\, ds : u \in \mathcal{M} \right\}, \quad \text{for } t \in J,$$

where $\mathcal{M}(s) = \{u(s) \in X : u \in \mathcal{M}\}$.

Proposition 2.3. *If* $\mathcal{M} \subset C(J, X)$ *bounded and equicontinuous, then* $t \to \alpha(\mathcal{M}(t))$ *is continuous on* J, *and*

$$\alpha(\mathcal{M}) = \max_{t \in J} \alpha(\mathcal{M}(t)), \qquad \alpha\left(\int_0^t \mathcal{M}(s)ds \right) \leq \int_0^t \alpha(\mathcal{M}(s))\, ds, \quad \textit{for } t \in J.$$

Proposition 2.4. $\mathcal{M} \subset C(J, X)$ *is relatively compact if and only if* $\mathcal{M}$ *is equicontinuous and, for any* $t \in J$, $\mathcal{M}(t)$ *is a relatively compact set in* X.

Proposition 2.5. *Let* $\{u_n\}_{n \geq 1}$ *be a sequence of Bochner integrable functions from* J *into* X *with* $\|u_n(t)\| \leq m(t)$ *for almost all* $t \in J$ *and every* $n \geq 1$, *where* $m \in L^1(J, \mathbb{R}_+)$, *then the function* $\psi(t) = \alpha(\{u_n(t) : n \geq 1\})$ *belongs to* $L^1(J, \mathbb{R}_+)$ *and satisfies:*

$$\alpha\left(\left\{ \int_0^t u_n(s)\, ds : n \geq 1 \right\} \right) \leq 2 \int_0^t \psi(s)\, ds.$$

Definition 2.6. Consider $\Omega \subset X$ and $F\colon \Omega \to X$ a continuous bounded map. We say that F is α-Lipschitz if there exists $\kappa \geq 0$ such that

$$\alpha(F(B)) \leq \kappa\alpha(B) \quad \text{for all } B \subset \Omega \text{ bounded.}$$

If, in addition, $\kappa < 1$, then we say that F is a strict α-contraction.

We say that F is α-condensing if

$$\alpha(F(B)) < \alpha(B) \quad \text{for all } B \subset \Omega \text{ bounded with } \alpha(B) > 0.$$

In other words, $\alpha(F(B)) \geq \alpha(B)$ implies $\alpha(B) = 0$. The class of all strict α-contractions $F\colon \Omega \to X$ is denoted by $\mathcal{S}C_\alpha(\Omega)$ and the class of all α-condensing maps $F\colon \Omega \to X$ is denoted by $C_\alpha(\Omega)$.

We remark that $\mathcal{S}C_\alpha(\Omega) \subset C_\alpha(\Omega)$ and every $F \in C_\alpha(\Omega)$ is α-Lipschitz with constant $\kappa = 1$. We also recall that $F\colon \Omega \to X$ is Lipschitz if there exists $\kappa > 0$ such that

$$\|Fx - Fy\| \leq \kappa \|x - y\| \quad \text{for all } x, y \in \Omega$$

and that F is a strict contraction if $\kappa < 1$. Next, we collect some properties of the applications defined above.

Proposition 2.7. *If $F, G\colon \Omega \to X$ are α-Lipschitz maps with constants κ, respectively κ', then $F + G\colon \Omega \to X$ are α-Lipschitz with constants $\kappa + \kappa'$.*

Proposition 2.8. *If $F\colon \Omega \to X$ is compact, then F is α-Lipschitz with constant $\kappa = 0$.*

Proposition 2.9. *If $F\colon \Omega \to X$ is Lipschitz with constant κ, then F is α-Lipschitz with the same constant κ.*

The theorem below asserts the existence and the basic properties of the topological degree for α-condensing perturbations of the identity. For more details, see Isaia [22]. Let

$$\mathcal{T} = \{(I - F, \Omega, y) : \Omega \subset X \text{ open and bounded, } F \in C_\alpha(\overline{\Omega}),\ y \in X \setminus (I - F)(\partial\Omega)\}$$

be the family of the admissible triplets.

Theorem 2.10. *There exists one degree function $D\colon \mathcal{T} \to \mathbb{Z}$ which satisfies the properties:*

(D1) (Normalization) $D(I, \Omega, y) = 1$ *for every* $y \in \Omega$.

(D2) (Additivity on domain) *For every disjoint, open sets $\Omega_1, \Omega_2 \subset \Omega$ and every y is not belong to $(I - F)(\overline{\Omega} \setminus (\Omega_1 \cup \Omega_2))$ we have:*

$$D(I - F, \Omega, y) = D(I - F, \Omega_1, y) + D(I - F, \Omega_2, y).$$

(D3) (Invariance under homotopy) $D(I - H(t, \cdot), \Omega, y(t))$ *is independent of* $t \in [0,1]$ *for every continuous, bounded map* $H\colon [0,1] \times \overline{\Omega} \to X$ *which satisfies:*

$$\alpha(H([0,1] \times B)) < \alpha(B) \quad \text{for all } B \subset \overline{\Omega} \text{ with } \alpha(B) > 0$$

and every continuous function $y\colon [0,1] \to x$ *which satisfies:*

$$y(t) \neq x - H(t,x) \quad \text{for all } t \in [0,1], \ \text{for all } x \in \partial\Omega.$$

(D4) (Existence) $D(I - F, \Omega, y) \neq 0$ *implies* $y \in (I - F)(\Omega)$.

(D5) (Excision) $D(I - F, \Omega, y) = D(I - F, \Omega_1, y)$ *for every open set* $\Omega_1 \subset \Omega$ *and every* y *is not belong to* $(I - F)(\overline{\Omega} \setminus \Omega_1)$.

Having in hand a degree function defined on $\mathcal{T}$, we collect the usability of the a priori estimate method by means of this degree.

Theorem 2.11. *Let* $F\colon X \to X$ *be* α*-condensing and*

$$\mathcal{S} = \{x \in X : \text{ there exists } \lambda \in [0,1] \text{ such that } x = \lambda F x\}.$$

If $\mathcal{S}$ *is a bounded set in* X*, so there exists* $r > 0$ *such that* $\mathcal{S} \subset B_r(0)$*, then*

$$D(I - \lambda F, B_r(0), 0) = 1 \quad \text{for all } \lambda \in [0,1].$$

Consequently, F *has at least one fixed point and the set of the fixed points of* F *lies in* $B_r(0)$.

To end this section, we introduce the following singular type Gronwall inequality with mixed type integral terms. By using Lemma 3.2 of [34], one can obtain the following result immediately.

Lemma 2.12. *Let* $u \in C(J,X)$ *satisfy the following inequality:*

$$\|u(t)\| \leq a + b \int_0^t (t-s)^{q-1} \|u(s)\|^{\lambda}\, ds + c \int_0^T (T-s)^{q-1} \|u(s)\|^{\lambda}\, ds,$$

where $q \in (0,1)$, $\lambda \in [0, 1 - 1/p]$ *for some* $1 < p < 1/(1-q)$, $a, b, c \geq 0$ *are constants. Then we have*

$$\|u(t)\| \leq (a+1)e^{MT}.$$

3. Qualitative analysis for Problem I

This section deals with existence, uniqueness and data dependence of solutions for the fractional NCP (1.1). We make some following assumptions:

(H1) For arbitrary $u, v \in C(J,X)$, there exists a constant $K_g \in [0,1)$ such that

$$\|g(u) - g(v)\| \leq K_g \|u - v\|_C.$$

(H2) For arbitrary $u \in C(J,X)$, there exist $C_g, M_g > 0$, $q_1 \in [0,1)$ such that

$$\|g(u)\| \le C_g \|u\|_C^{q_1} + M_g.$$

(H3) For arbitrary $(t,u) \in J \times X$, there exist $C_f, M_f > 0$, $q_2 \in [0,1)$ such that

$$\|f(t,u)\| \le C_f \|u\|^{q_2} + M_f.$$

(H4) For any $r > 0$, there exists a constant $\beta_r > 0$ such that

$$\alpha(f(s,\mathcal{M})) \le \beta_r \alpha(\mathcal{M})$$

for all $t \in J$, $\mathcal{M} \subset \mathfrak{B}_r := \{\|u\|_C \le r : u \in C(J,X)\}$ and

$$\frac{2T^q \beta_r}{\Gamma(q+1)} < 1.$$

Let us recall the definition and lemma of a solution for fractional NCP (1.1). For more details, see Agarwal et al. [1].

Definition 3.1. A function $u \in C^1(J,X)$ is said to be a solution of the fractional NCP (1.1) if u satisfies the equation ${}^cD^q u(t) = f(t,u(t))$ almost everywhere on J, and the condition $u(0) + g(u) = u_0$.

Lemma 3.2. *A function $u \in C(J,X)$ is a solution of the fractional integral equation*

$$u(t) = u_0 - g(u) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s,u(s))\,ds, \tag{3.1}$$

if and only if u is a solution of the fractional NCP (1.1).

Under the assumptions (H1)–(H4), we will show that fractional integral equation (1.3) has at least one solution $u \in C(J,X)$.

Define operators:

$$\begin{aligned}
&F\colon C(J,X) \to C(J,X), \quad (Fu)(t) &&= u_0 - g(u), && t \in J,\\
&G\colon C(J,X) \to C(J,X), \quad (Gu)(t) &&= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s,u(s))\,ds, && t \in J,\\
&\mathbb{T}\colon C(J,X) \to C(J,X), \qquad\quad \mathbb{T}u &&= Fu + Gu.
\end{aligned}$$

It is obvious that $\mathbb{T}$ is well defined. Then, fractional integral equation (3.1) can be written as the following operator equation

$$u = \mathbb{T}u = Fu + Gu. \tag{3.2}$$

Thus, the existence of a solution for fractional NCP (1.1) is equivalent to the existence of a fixed point for operator $\mathbb{T}$ which satisfies operator equation (3.2).

Lemma 3.3. *The operator $F: C(J,X) \to C(J,X)$ is Lipschitz with constant K_g. Consequently F is α-Lipschitz with the same constant K_g. Moreover, F satisfies the following growth condition:*

$$\|Fu\|_C \leq \|u_0\| + C_g\|u\|_C^{q_1} + M_g, \tag{3.3}$$

for every $u \in C(J,X)$.

Proof. Using (H1), we have $\|Fu - Fv\|_C \leq \|g(u) - g(v)\| \leq K_g\|u - v\|_C$, for every $u, v \in C(J,X)$. By Proposition 2.9, F is α-Lipschitz with constant K_g. Relation (3.3) is a simple consequence of (H2). □

Lemma 3.4. *The operator $G: C(J,X) \to C(J,X)$ is continuous. Moreover, G satisfies the following growth condition:*

$$\|Gu\|_C \leq \frac{T^q(C_f\|u\|_C^{q_2} + M_f)}{\Gamma(q+1)}, \tag{3.4}$$

for every $u \in C(J,X)$.

Proof. For that, let $\{u_n\}$ be a sequence of a bounded set $\mathfrak{B}_K \subseteq C(J,X)$ such that $u_n \to u$ in $\mathfrak{B}_K(K > 0)$. We have to show that $\|Gu_n - Gu\|_C \to 0$.

It is easy to see that $f(s, u_n(s)) \to f(s, u(s))$ as $n \to \infty$ due to the continuity of f. On the one other hand using (H3), we get for each $t \in J$, $(t-s)^{q-1}\|f(s, u_n(s)) - f(s, u(s))\| \leq (t-s)^{q-1}2(C_gK + M_g)$. On the other hand, using the fact that the functions $s \to (t-s)^{q-1}2(C_gK + M_g)$ is integrable for $s \in [0,t]$, $t \in J$, by means of the Lebesgue Dominated Convergence Theorem yields

$$\int_0^t (t-s)^{q-1}\|f(s, u_n(s)) - f(s, u(s))\|\, ds \to 0 \quad \text{as } n \to \infty.$$

Then, for all $t \in J$,

$$\|(Gu_n)(t) - (Gu)(t)\| \leq \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}\|f(s, u_n(s)) - f(s, u(s))\|\, ds \to 0.$$

Therefore, $Gu_n \to Gu$ as $n \to \infty$ which implies that G is continuous. Relation (3.4) is a simple consequence of (H3). □

Lemma 3.5. *The operator $G: C(J,X) \to C(J,X)$ is compact. Consequently G is α-Lipschitz with zero constant.*

Proof. In order to prove the compactness of G, we consider a bounded set $\mathcal{M} \subseteq C(J,X)$ and the key step is to show that $G(\mathcal{M})$ is relatively compact in $C(J,X)$.

Let $\{u_n\}$ be a sequence on $\mathcal{M} \subset \mathfrak{B}_K$, for every $u_n \in \mathcal{M}$. By relation (3.4), we have

$$\|Gu_n\|_C \leq \frac{T^q(C_fK^{q_2} + M_f)}{\Gamma(q+1)} := r,$$

for every $u_n \in \mathcal{M}$, so $G(\mathcal{M})$ is bounded in $\mathfrak{B}_r$.

Now we prove that $\{Gu_n\}$ is is equicontinuous. For $0 \le t_1 < t_2 \le T$, we get

$$\begin{aligned}
\|(Gu_n)(t_1) &- (Gu_n)(t_2)\| \\
&\le \frac{1}{\Gamma(q)}\int_0^{t_1}((t_1-s)^{q-1}-(t_2-s)^{q-1})\|f(s,u_n(s))\|\,ds \\
&\quad + \frac{1}{\Gamma(q)}\int_{t_1}^{t_2}(t_2-s)^{q-1}\|f(s,u_n(s))\|\,ds \\
&\le \frac{1}{\Gamma(q)}\int_0^{t_1}((t_1-s)^{q-1}-(t_2-s)^{q-1})(C_f\|u_n(s)\|+M_f)\,ds \\
&\quad + \frac{1}{\Gamma(q)}\int_{t_1}^{t_2}(t_2-s)^{q-1}(C_f\|u_n(s)\|+M_f)\,ds \\
&\le \frac{(C_fK^{q_2}+M_f)}{\Gamma(q)}\left[\frac{t_1^q}{q}-\frac{t_2^q}{q}+\frac{(t_2-t_1)^q}{q}+\frac{(t_2-t_1)^q}{q}\right] \\
&\le \frac{2(C_fK^{q_2}+M_f)(t_2-t_1)^q}{\Gamma(q)}.
\end{aligned}$$

As $t_2 \to t_1$, the right-hand side of the above inequality tends to zero. Therefore $\{Gu_n\}$ is equicontinuous.

Consider a bounded set

$$\mathcal{M}(t) := \left\{v_n(t) : v_n(t) = \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s,v_n(s))\,ds\right\} \subset \mathfrak{B}_r.$$

Applying Proposition 2.4, we know that the function $t \to \alpha(\mathcal{M}(t))$ is continuous on J. Moreover,

$$(t-s)^{q-1}\|f(s,v_n(s))\| \le (t-s)^{q-1}(C_f r^{q_2}+M_f) \in L^1(J,\mathbb{R}_+),$$

for $s \in [0,t]$, $t \in J$. Using (H4) and Proposition 2.5, we have

$$\begin{aligned}
\alpha(\mathcal{M}(t)) &\le \alpha\bigg(\bigg\{\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s,\mathcal{M}(s))\,ds\bigg\}\bigg) \\
&\le \frac{2}{\Gamma(q)}\int_0^t (t-s)^{q-1}\alpha\Big(f(s,\mathcal{M}(s))\Big)\,ds \\
&\le \frac{2\beta_r}{\Gamma(q)}\int_0^t (t-s)^{q-1}\alpha(\mathcal{M}(s))\,ds,
\end{aligned}$$

which implies that

$$\alpha(\mathcal{M}) \le \left[\frac{2\beta_r}{\Gamma(q)}\int_0^t (t-s)^{q-1}\,ds\right]\alpha(\mathcal{M}) \le \frac{2T^q\beta_r}{\Gamma(q+1)}\,\alpha(\mathcal{M}) < \alpha(\mathcal{M}),$$

due to the condition

$$\frac{2T^q\beta_r}{\Gamma(q+1)} < 1.$$

Then we can deduce that $\alpha(\mathcal{M}) = 0$. Therefore, $G(\mathcal{M})$ is a relatively compact subset of $C(J, X)$, and so, there exists a subsequence v_n which converge uniformly on J to some $v_* \in C(J, X)$ together with the Arzela–Ascoli theorem. By Proposition 2.8, G is α-Lipschitz with zero constant. □

Now, we have the possibility to prove the main results for fractional NCP, see (1.1).

Theorem 3.6. *Assume that* (H1)–(H4) *hold, then fractional* NCP (1.1) *has at least one solution $u \in C(J, X)$ and the set of the solutions of system* (1.1) *is bounded in $C(J, X)$.*

Proof. Let $F, G, \mathbb{T}\colon C(J, X) \to C(J, X)$ be the operators defined in the beginning of this section. They are continuous and bounded. Moreover, F is α-Lipschitz with constant $K_g \in [0, 1)$ and G is α-Lipschitz with zero constant (see Lemmas 3.3–3.5). Proposition 2.7 shows us that $\mathbb{T}$ is a strict α-contraction with constant K_g. Set

$$S_0 = \{u \in C(J, X) : \text{there exists } \lambda \in [0, 1] \text{ such that } u = \lambda \mathbb{T} u\}.$$

Next, we prove that S_0 is bounded in $C(J, X)$. Consider $u \in S_0$ and $\lambda \in [0, 1]$ such that $u = \lambda \mathbb{T} u$. It follows from (3.3) and (3.4) that

$$\begin{aligned} \|u\|_C = \lambda \|\mathbb{T} u\|_C &\leq \lambda(\|Fu\|_C + \|Gu\|_C) \\ &\leq \|u_0\| + C_g \|u\|_C^{q_1} + M_g + \frac{T^q(C_f \|u\|_C^{q_2} + M_f)}{\Gamma(q+1)}. \end{aligned} \tag{3.5}$$

This inequality (3.5), together with $q_1 < 1$ and $q_2 < 1$, shows us that S_0 is bounded in $C(J, X)$. If not, we suppose by contradiction, $\rho := \|u\|_C \to \infty$. Dividing both sides of (3.5) by ρ, and taking $\rho \to \infty$, we have

$$1 \leq \lim_{\rho \to \infty} \frac{\|u_0\| + C_g \rho^{q_1} + M_g + \dfrac{T^q(C_f \rho^{q_2} + M_f)}{\Gamma(q+1)}}{\rho} = 0. \tag{3.6}$$

This is a contradiction. Consequently, by Theorem 2.11 we deduce that $\mathbb{T}$ has at least one fixed point and the set of the fixed points of $\mathbb{T}$ is bounded in $C(J, X)$. □

Remark 3.7. (a) If the growth condition (H2) is formulated for $q_1 = 1$, then the conclusions of Theorem 3.6 remain valid provided that $C_g < 1$;

(b) If the growth condition (H3) is formulated for $q_2 = 1$, then the conclusions of Theorem 3.6 remain valid provided that $T^q C_f / \Gamma(q+1) < 1$;

(c) If the growth conditions (H2) and (H3) are formulated for $q_1 = 1$ and $q_2 = 1$, then the conclusions of Theorem 3.6 remain valid provided that $C_g + T^q C_f / \Gamma(q+1) < 1$.

In order to obtain the uniqueness of solutions, we need the following assumptions:

(H3$^+$) There exist constants $L_f > 0$, $\lambda \in [0, 1-1/p]$ for some $p \in (1, 1/(1-q))$, such that

$$\|f(t,u) - f(t,v)\| \leq L_f \|u - v\|^\lambda, \quad \text{for each } t \in J \text{ and all } u, v \in X.$$

Theorem 3.8. *Assume that* (H1)–(H4) *and* (H3$^+$) *hold. Let* $u(\cdot)$ $(v(\cdot))$ *be the solution of the fractional* NCP (1.1) *with nonlocal condition* $u_0 - g(u)$ $(v_0 - g(v))$. *Then there exists a constant* $M^* > 0$ *such that*

$$\|u - v\|_C \leq M^* \left(\frac{1}{1 - L_g} \|u_0 - v_0\| + 1 \right).$$

Proof. Note that (H1) and (H3$^+$), we obtain

$$\|u(t) - v(t)\| \leq \|u_0 - v_0\| + L_g \|u - v\|_C + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} L_f \|u(s) - v(s)\|^\lambda \, ds$$

which implies that

$$(1 - L_g)\|u - v\|_C \leq \|u_0 - v_0\| + \frac{L_f}{\Gamma(q)} \int_0^t (t-s)^{q-1} \|u(s) - v(s)\|^\lambda \, ds.$$

Then, we have

$$\begin{aligned} \|u(t) - v(t)\| &\leq \|u - v\|_C \\ &\leq \frac{1}{1 - L_g} \|u_0 - v_0\| + \frac{L_f}{(1 - L_g)\Gamma(q)} \int_0^t (t-s)^{q-1} \|u(s) - v(s)\|^\lambda \, ds. \end{aligned}$$

By Lemma 2.12, one can obtain the following inequality

$$\|u(t) - v(t)\| \leq M^* \left(\frac{1}{1 - L_g} \|u_0 - v_0\| + 1 \right), \quad t \in J,$$

where

$$M^* = e^{MT}, \qquad M = \frac{L_f}{(1 - L_g)\Gamma(q)} \left[\frac{T^{p(q-1)+1}}{p(q-1)+1} \right]^{1/p}.$$

This completes the proof. □

In order to obtain the uniqueness of solutions, we revise (H3$^+$) to following assumption:

(H3^{++}) There exists a constant $L_f > 0$ such that

$$\|f(t,u) - f(t,v)\| \leq L_f \|u - v\|, \quad \text{for each } t \in J \text{ and all } u, v \in X.$$

THEOREM 3.9. *Assume that* (H1)–(H4) *and* (H3^{++}) *hold, then fractional* NCP (1.1) *has an unique solution* $u \in C(J, X)$.

PROOF. By Theorem 3.6, the fractional NCP (1.1) has a solution $u(\,\cdot\,)$ in $C(J, X)$. Let $v(\,\cdot\,)$ be another solution of fractional NCP (1.1) with nonlocal condition $v_0 - g(v)$. Note that (H1) and (H3^{++}), repeating the same process of Theorem 3.8, we have

$$\|u(t) - v(t)\| \leq \frac{1}{1 - L_g}\|u_0 - v_0\| + \frac{L_f}{(1 - L_g)\Gamma(q)} \int_0^t (t - s)^{q-1}\|u(s) - v(s)\|\, ds.$$

By the standard singular Gronwall inequality, we obtain there exists a constant $M^* > 0$ such that

$$\|u(t) - v(t)\| \leq \frac{M^*}{1 - L_g}\|u_0 - v_0\|, \quad t \in J,$$

which yields the uniqueness of $u(\,\cdot\,)$. □

4. Qualitative analysis for Problem II

For the existence and uniqueness of solutions for the fractional BVP (1.2), we need the following auxiliary definition and lemma. For more details, see Agarwal et al. [1].

DEFINITION 4.1. A function $u \in C^1(J, X)$ is said to be a solution of the fractional BVP (1.2) if u satisfies the equation ${}^cD^q u(t) = f(t, u(t))$ almost everywhere on J, and the condition $au(0) + bu(T) = c$.

LEMMA 4.2. *A function* $u \in C(J, X)$ *is a solution of the fractional integral equation*

$$u(t) = \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} f(s, u(s))\, ds - \frac{1}{a + b}\left[\frac{b}{\Gamma(q)} \int_0^T (T - s)^{q-1} f(s, u(s))\, ds - c\right], \tag{4.1}$$

if and only if u *is a solution of the fractional* BVP (1.2).

Define operators

$$F_1 \colon C(J, X) \to C(J, X), \qquad (F_1 u)(t) = \frac{c}{a + b}, \qquad t \in J,$$

$$G_{11} \colon C(J, X) \to C(J, X), \qquad (G_{11} u)(t) = \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} f(s, u(s))\, ds, \quad t \in J,$$

$$G_{12} \colon C(J, X) \to C(J, X),$$

$$(G_{12} u)(t) = -\frac{b}{(a + b)\Gamma(q)} \int_0^T (T - s)^{q-1} f(s, u(s))\, ds, \quad t \in J,$$

$$\mathbb{T}_1 \colon C(J, X) \to C(J, X), \qquad \mathbb{T}_1 u = F_1 u + G_{11} u + G_{12} u.$$

It is obvious that $\mathbb{T}_1$ is well defined. Then, fractional integral equation (4.1) can be written as the following operator equation

$$u = \mathbb{T}_1 u = F_1 u + G_{11} u + G_{12} u. \tag{4.2}$$

Thus, the existence of a solution for fractional BVP (1.2) is equivalent to the existence of a fixed point for operator $\mathbb{T}_1$ given by (4.2).

Compared with the definition of operators F, G and repeat the process of Lemmas 3.3–3.5 to deal with F_1, G_{11}, G_{12}, one can obtain the following results. We omit the proof here.

LEMMA 4.3. *The operator $F_1: C(J,X) \to C(J,X)$ is Lipschitz with zero constant. Consequently F_1 is α-Lipschitz with zero constant. Moreover, F_1 satisfies the following estimation:*

$$\|F_1 u\|_C \le \frac{|c|}{|a+b|}, \tag{4.3}$$

for every $u \in C(J,X)$.

LEMMA 4.4. *The operators $G_{11}, G_{12}: C(J,X) \to C(J,X)$ is continuous. Consequently $G_{11} + G_{12}$ are continuous. Moreover, $G_{11} + G_{12}$ satisfies the following growth condition:*

$$\|G_{11} u\|_C + \|G_{12} u\|_C \le \left(1 + \frac{|b|}{|a+b|}\right) \frac{T^q (C_f \|u\|_C^{q_2} + M_f)}{\Gamma(q+1)}, \tag{4.4}$$

for every $u \in C(J,X)$.

LEMMA 4.5. *The operators $G_{11}, G_{12}: C(J,X) \to C(J,X)$ are compact. Consequently G_{11}, G_{12} are α-Lipschitz with zero constant. Further, $G_{11} + G_{12}$ is α-Lipschitz with zero constant.*

Now, we have the possibility to prove the main results of this section.

THEOREM 4.6. *Assume that* (H3)–(H4) *hold. Then the fractional* BVP (1.2) *has at least one solution $u \in C(J,X)$ and the set of the solutions of the fractional* BVP (1.2) *is bounded in $C(J,X)$.*

PROOF. Let $F_1, G_{11}, G_{12}, \mathbb{T}_1: C(J,X) \to C(J,X)$ be the operators defined in the beginning of this section. They are continuous and bounded. Moreover, F_1 is α-Lipschitz with zero constant and $G_{11} + G_{12}$ is α-Lipschitz with zero constant. Proposition 2.7 shows us that $\mathbb{T}_1$ is a strict α-contraction with zero constant.

Set $S_1 = \{u \in C(J,X) :$ there exists $\lambda \in [0,1]$ such that $u = \lambda \mathbb{T}_1 u\}$. Next, we prove that S_1 is bounded in $C(J,X)$. Consider $u \in S_1$ and $\lambda \in [0,1]$ such that $u = \lambda \mathbb{T}_1 u$. It follows from (4.3) and (4.4) that

$$\begin{aligned}\|u\|_C = \lambda\|\mathbb{T}_1 u\|_C &\leq \lambda(\|F_1 u\|_C + \|G_{11}u\|_C + \|G_{12}u\|_C) \\ &\leq \frac{|c|}{|a+b|} + \left(1 + \frac{|b|}{|a+b|}\right)\frac{T^q(C_f\|u\|_C^{q_2} + M_f)}{\Gamma(q+1)}.\end{aligned} \tag{4.5}$$

This inequality (4.5), together with $q_2 < 1$, shows us that S_1 is bounded in $C(J,X)$. Consequently, by Theorem 2.11 we deduce that $\mathbb{T}_1$ has at least one fixed point and the set of the fixed points of $\mathbb{T}_1$ is bounded in $C(J,X)$. $\square$

REMARK 4.7. If the growth condition (H3) is formulated for $q_2 = 1$, then the conclusions of Theorem 3.6 remain valid provided that

$$\left(1 + \frac{|b|}{|a+b|}\right)\frac{T^q C_f}{\Gamma(q+1)} < 1.$$

Further, we give the following uniqueness result.

THEOREM 4.8. *Assume that* (H3)–($\mathrm{H3}^+$) *hold, then the fractional* BVP (1.2) *has an unique solution* $u \in C(J,X)$.

PROOF. By Theorem 4.6, the fractional BVP (1.2) has a solution $u(\,\cdot\,)$ in $C(J,X)$. Let $v(\,\cdot\,)$ be another solution of the fractional BVP (1.2) with boundary conditions $av(0) + bv(T) = c$. Note that ($\mathrm{H3}^+$), we obtain

$$\begin{aligned}\|u(t) - v(t)\| \leq{}& \frac{L_f}{\Gamma(q)}\int_0^t (t-s)^{q-1}\|u(s) - v(s)\|^{\lambda}\,ds \\ &+ \frac{|b|L_f}{|a+b|\Gamma(q)}\int_0^T (T-s)^{q-1}\|u(s) - v(s)\|^{\lambda}\,ds.\end{aligned}$$

By Lemma 2.12, we obtain $\|u(\,\cdot\,) - v(\,\cdot\,)\| \leq 0$, which yields the uniqueness of $u(\,\cdot\,)$. $\square$

5. Qualitative analysis for Problem III

Set $\mathrm{PC}(J,X) \equiv \{x\colon J \to X \mid x$ is continuous at $t \in J\setminus\widetilde{D}$, and x is continuous from left and has right hand limits at $t\in\widetilde{D}\}$. It is obvious that $(\mathrm{PC}(J,X), \|\cdot\|_{\mathrm{PC}})$ is a Banach space endowed with the norm $\|x\|_{\mathrm{PC}} = \sup_{t\in J}\|x(t)\|$.

In addition to the assumptions (H3)–(H4), we need the following assumptions on impulsive terms I_i, $i = 1,\ldots,m$.

- (H5) $I_i\colon X \to X$ and there exists a constant $K_I \in [0, 1/m)$ such that $\|I_i(u) - I_i(v)\| \leq K_I\|u - v\|$, for all $u, v \in X$.
- (H6) For arbitrary $u \in X$, there exist $C_I, M_I > 0$, $q_3 \in [0,1)$ such that

$$\|I_i(u)\| \leq C_I\|u\|^{q_3} + M_I.$$

We also recall the definition and lemma of a solution of the fractional ICP (1.3). For more details, see [1].

Definition 5.1. A function $u \in \mathrm{PC}(J,X)$ with its q-derivative exists on $J \setminus \widetilde{D}$ is said to be a solution of fractional ICP (1.3) if u satisfies the equation ${}^cD^q u(t) = f(t,u(t))$ on $J \setminus \widetilde{D}$, and conditions $\Delta u(t_i) = I_i(u(t_i))$, $i = 1,\ldots,m$, and $u(0) = u_0$.

For the existence of solutions for the fractional ICP (1.3), we need the following auxiliary lemma.

Lemma 5.2. *A function $u \in \mathrm{PC}(J,X)$ is a solution of the fractional integral equation*

$$
u(t) = \begin{cases} u_0 + \dfrac{1}{\Gamma(q)} \displaystyle\int_0^t (t-s)^{q-1} f(s,u(s))\,ds & \text{for } t \in [0,t_1], \\ u_0 + \dfrac{1}{\Gamma(q)} \displaystyle\sum_{i=1}^{m} \int_{t_{i-1}}^{t_i} (t-s)^{q-1} f(s,u(s))\,ds & \\ \quad + \dfrac{1}{\Gamma(q)} \displaystyle\int_{t_m}^{t} (t-s)^{q-1} f(s,u(s))\,ds & \\ \quad + \displaystyle\sum_{0<t_i<t} I_i(u(t_i)) & \text{for } t \in (t_i,t_{i+1}], \\ & i = 1,\ldots,m, \end{cases} \tag{5.1}
$$

if and only if u is a solution of the fractional ICP (1.3).

Define operators as follows:

- $F_2\colon \mathrm{PC}(J,X) \to \mathrm{PC}(J,X)$ given by

$$
(F_2u)(t) = \begin{cases} (F_2^0u)(t) & \text{for } t \in [0,t_1], \\ (F_2^0u)(t) + \displaystyle\sum_{0<t_i<t} (F_2^iu)(t) & \text{for } t \in (t_i,t_{i+1}],\ i = 1,\ldots,m, \end{cases}
$$

where

$$
(F_2^0u)(t) = u_0, \qquad (F_2^iu)(t) = I_i(u(t_i)).
$$

- $G_2\colon \mathrm{PC}(J,X) \to \mathrm{PC}(J,X)$ given by

$$
(G_2u)(t) = \begin{cases} (G_2^0u)(t) & \text{for } t \in [0,t_1], \\ \displaystyle\sum_{i=1}^{m} (G_2^iu)(t) + (G_2^{m+1}u)(t) & \text{for } t \in (t_i,t_{i+1}],\ i = 1,\ldots,m, \end{cases}
$$

where

$$
\begin{aligned}
(G_2^0u)(t) &= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s,u(s))\,ds, && t \in [0,t_1], \\
(G_2^iu)(t) &= \frac{1}{\Gamma(q)} \int_{t_{i-1}}^{t_i} (t-s)^{q-1} f(s,u(s))\,ds, && i = 1,\ldots,m, \\
(G_2^{m+1}u)(t) &= \frac{1}{\Gamma(q)} \int_{t_m}^{t} (t-s)^{q-1} f(s,u(s))\,ds, && t \in (t_m,T].
\end{aligned}
$$

• $\mathbb{T}_2\colon \mathrm{PC}(J,X) \to \mathrm{PC}(J,X)$ given by $\mathbb{T}_2 u = F_2 u + G_2 u$.

It is not difficult to verify that $\mathbb{T}_2$ is well defined. Thus, the existence of a solution for fractional ICP (1.3) is equivalent to the existence of a fixed point for operator $\mathbb{T}_2$.

Lemma 5.3. *The operator $F_2\colon \mathrm{PC}(J,X) \to \mathrm{PC}(J,X)$ is Lipschitz with constant $K_I = mK_I \in [0,1)$. Consequently F_2 is α-Lipschitz with the same constant $K_I \in [0,1)$. Moreover, F_2 satisfies the following growth condition:*

$$\|F_2 u\|_{\mathrm{PC}} \le \|u_0\| + m(C_I \|u\|_C^{q_3} + M_I), \tag{5.2}$$

for every $u \in \mathrm{PC}(J,X)$.

Proof. For $t \in [0,t_1]$, it is obvious that

$$\|F_2 u - F_2 v\|_{C([0,t_1],X)} = \|F_2^0 u - F_2^0 v\|_{C([0,t_1],X)} = 0,$$

for every $u, v \in X$. By Proposition 2.9, F_2 is α-Lipschitz with constant $K_I = 0$ for $t \in [0,t_1]$. For $t \in (t_i, t_{i+1}]$, using (H5), F_2^i is α-Lipschitz with constant $K_I = [0, 1/m)$. Using Proposition 2.7, F_2 is α-Lipschitz with constant $K_I = mK_I \in [0,1)$ for $t \in (t_i, t_{i+1}]$. Relation (5.2) is a simple consequence of (H6). □

Repeat the same process of Lemma 3.4, one can obtain the continuity of the operators G_2^0 on $C([0,t_1],X)$, G_2^i on $C((t_i,t_{i+1}],X)$, G_2^{m+1} on $C((t_m,T],X)$, using Proposition 2.7 again and again, one can obtain the following results.

Lemma 5.4. *The operator $G_2\colon \mathrm{PC}(J,X) \to \mathrm{PC}(J,X)$ is continuous. Moreover, G_2 satisfies the following growth condition:*

$$\|G_2 u\|_{PC} \le \frac{(m+1)T^q (C_f \|u\|_{PC}^{q_2} + M_f)}{\Gamma(q+1)}, \tag{5.3}$$

for every $u \in C(J,X)$.

It is obvious that the relation (5.3) is a simple consequence of (H3).

Repeating the same process of Lemma 3.5, one can obtain the compactness of the operators:

$$G_2^0 \text{ on } C([0,t_1],X), \quad G_2^i \text{ on } C((t_i,t_{i+1}],X), \quad G_2^{m+1} \text{ on } C((t_m,T],X),$$

using Proposition 2.7 again and again, one can obtain the following results.

Lemma 5.5. *The operator $G_2\colon \mathrm{PC}(J,X) \to \mathrm{PC}(J,X)$ is compact. Consequently G_2 is α-Lipschitz with zero constant.*

Now, we have the possibility to prove the main results of this section.

Theorem 5.6. *Assume that* (H3)–(H6) *hold, then the fractional* ICP (1.3) *has at least one solution* $u \in \mathrm{PC}(J,X)$ *and the set of the solutions of the fractional* ICP (1.3) *is bounded in* $\mathrm{PC}(J,X)$.

Proof. Let $F_2, G_2, \mathbb{T}_2\colon \mathrm{PC}(J,X) \to \mathrm{PC}(J,X)$ be the operators defined in the beginning of this section. They are continuous and bounded. Moreover, F_2 is α-Lipschitz with constant $K_I \in [0,1)$ and G_2 is α-Lipschitz with zero constant (see Lemmas 5.3–5.5). Proposition 2.7 shows us that $\mathbb{T}_2$ is a strict α-contraction with constant K_I.

Set $S_2 = \{u \in \mathrm{PC}(J,X) : \text{there exists } \lambda \in [0,1] \text{ such that } u = \lambda \mathbb{T}_2 u\}$. Next, we prove that S_2 is bounded in $\mathrm{PC}(J,X)$. Consider $u \in S_2$ and $\lambda \in [0,1]$ such that $u = \lambda \mathbb{T}_2 u$. It follows from (5.2) and (5.3) that

$$
\begin{aligned}
(5.4)\quad \|u\|_{\mathrm{PC}} &\leq \lambda(\|F_2 u\|_{\mathrm{PC}} + \|G_2 u\|_{\mathrm{PC}}) \\
&\leq \|u_0\| + m(C_I \|u\|_{PC}^{q_3} + M_I) + \frac{(m+1)T^q(C_f\|u\|_{\mathrm{PC}}^{q_2} + M_f)}{\Gamma(q+1)}.
\end{aligned}
$$

This inequality (5.4), together with $q_2 < 1$ and $q_3 < 1$, shows us that S_2 is bounded in $\mathrm{PC}(J,X)$. Consequently, by Theorem 2.11 we deduce that $\mathbb{T}_2$ has at least one fixed point and the set of the fixed points of $\mathbb{T}_2$ is bounded in $\mathrm{PC}(J,X)$. □

Remark 5.7. (a) If the growth condition (H3) is formulated for $q_2 = 1$, then the conclusions of Theorem 5.6 remain valid provided that

$$\frac{(m+1)T^q C_f}{\Gamma(q+1)} < 1;$$

(b) If the growth condition (H6) is formulated for $q_3 = 1$, then the conclusions of Theorem 5.6 remain valid provided that $mC_I < 1$;

(c) If the growth conditions (H3) and (H6) are formulated for $q_2 = 1$ and $q_3 = 1$, then the conclusions of Theorem 5.6 remain valid provided that

$$mC_I + \frac{(m+1)T^q C_f}{\Gamma(q+1)} < 1.$$

Now, we give the following data dependence result.

Theorem 5.8. *Assume that* (H3)–(H6) *and* ($\mathrm{H3}^+$) *hold. Let* $u(\,\cdot\,)$ ($v(\,\cdot\,)$) *be the solution of fractional* ICP (1.3) *with initial value* $u_0(v_0)$. *Then there exists a constant* $M^{**} > 0$ *such that*

$$\|u - v\|_{PC} \leq M^{**}\bigg(\frac{1}{1-K_I}\|u_0 - v_0\| + 1\bigg).$$

Proof. Note that (H3$^+$) and (H5), we obtain

$$\begin{aligned}\|u(t)-v(t)\| \le \|u_0-v_0\| &+ \sum_{0<t_i<t} K_I\|u-v\|_{PC}\\ &+\frac{L_f}{\Gamma(q)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}(t-s)^{q-1}\|u(s)-v(s)\|^{\lambda}\,ds\\ &+\frac{L_f}{\Gamma(q)}\int_{t_m}^{t}(t-s)^{q-1}\|u(s)-v(s)\|^{\lambda}\,ds,\end{aligned}$$

which implies that

$$\begin{aligned}(1-K_I)\|u(t)-v(t)\| &\le (1-K_I)\|u-v\|_{PC}\\ &\le \|u_0-v_0\|+\frac{L_f}{\Gamma(q)}\int_0^t(t-s)^{q-1}\|u(s)-v(s)\|^{\lambda}\,ds.\end{aligned}$$

Thus,

$$\|u(t)-v(t)\| \le \frac{1}{1-K_I}\|u_0-v_0\|+\frac{L_f}{(1-K_I)\Gamma(q)}\int_0^t(t-s)^{q-1}\|u(s)-v(s)\|^{\lambda}\,ds.$$

By Lemma 2.12, we have the following inequality

$$\|u(t)-v(t)\| \le M^{**}\bigg(\frac{1}{1-K_I}\|u_0-v_0\|+1\bigg),\quad t\in J,$$

where $M^{**}=e^{MT}$ and

$$M=\frac{L_f}{(1-K_I)\Gamma(q)}\bigg[\frac{T^{p(q-1)+1}}{p(q-1)+1}\bigg]^{1/p}.$$

This completes the proof. □

Further, we can give the following uniqueness result.

Theorem 5.9. *Assume that* (H3)–(H6) *and* (H3^{++}) *hold, then the fractional* ICP (1.3) *has an unique solution* $u\in \mathrm{PC}(J,X)$.

Proof. The fractional ICP (1.3) has a solution $u(\,\cdot\,)$ in $\mathrm{PC}(J,X)$ by Theorem 5.6. Let $v(\,\cdot\,)$ be another solution of the fractional ICP (1.3) with initial value v_0. Note that (H3^{++}) and repeating the same process of Theorem 5.8, we obtain

$$\|u(t)-v(t)\| \le \frac{1}{1-K_I}\|u_0-v_0\|+\frac{L_f}{(1-K_I)\Gamma(q)}\int_0^t(t-s)^{q-1}\|u(s)-v(s)\|ds.$$

By the standard singular Gronwall inequality, one can deduce the uniqueness of $u(\,\cdot\,)$. □

6. Qualitative analysis for Problem IV

Denote the Banach space $C(J, X_\beta)$ endowed with supnorm given by

$$\|u\|_\infty := \sup_{t\in J} \|u\|_\beta, \quad \text{for } x \in C(J, X_\beta).$$

Suppose that $-A: D(A) \to X$ is the infinitesimal generator of a compact analytic semigroup of uniformly bounded linear operators $\{S(t),\, t \geq 0\}$. This means that there exists $M > 1$ such that $\|S(t)\| \leq M$. We assume without loss of generality that $0 \in \rho(A)$. This allows us to define the fractional power A^β for $0 < \beta < 1$, as a closed linear operator on its domain $D(A^\beta)$ with inverse $A^{-\beta}$ (see Pazy [31]).

We make the following assumptions.

(H7) $\overline{f}: J \times X_\beta \to X$ is continuous.

(H8) For arbitrary $(t,u) \in J \times X_\beta$, there exist a function $m_{\overline{f}}(t) \in L^{1/q_4}(J, \mathbb{R}_+)$, $q_4 \in [0, q(1-\beta))$ such that $\|\overline{f}(t,u)\| \leq m_{\overline{f}}(t)$ where the norm of m is defined by

$$\|m_{\overline{f}}\|_{L^p(J,\mathbb{R}_+)} = \begin{cases} \bigg(\displaystyle\int_J |m_{\overline{f}}(t)|^{1/p}\, dt\bigg)^p & \text{if } 1 < p < \infty, \\ \displaystyle\inf_{\mu(\overline{J})=0} \Big\{ \sup_{t\in J-\overline{J}} |m_{\overline{f}}(t)| \Big\} & \text{if } p = \infty, \end{cases}$$

where $\mu(\overline{J})$ is the Lebesgue measure on $\overline{J}$.

Let us recall the definition and lemma of mild solutions for fractional ECP (1.4). For more details, see Wang and Zhou [35], Zhou and Jiao [41], [44].

Definition 6.1. By a mild solution of fractional ECP (1.4), we mean that the function $u \in C(J, X_\beta)$ which satisfies

$$u(t) = \mathscr{T}(t)u_0 + \int_0^t (t-s)^{q-1} \mathscr{S}(t-s) \overline{f}(s, u(s))\, ds, \quad t \in J,$$

where

$$\mathscr{T}(t) = \int_0^\infty \xi_q(\theta) S(t^q\theta)\, d\theta, \qquad \mathscr{S}(t) = q\int_0^\infty \theta\xi_q(\theta) S(t^q\theta)\, d\theta,$$

and

$$\begin{aligned} \xi_q(\theta) &= \frac{1}{q}\theta^{-1-1/q}\varpi_q(\theta^{-1/q}) \geq 0, \\ \varpi_q(\theta) &= \frac{1}{\pi}\sum_{n=1}^\infty (-1)^{n-1}\theta^{-qn-1}\frac{\Gamma(nq+1)}{n!}\sin(n\pi q), \end{aligned}$$

for $\theta \in (0,\infty)$, ξ_q is a probability density function defined on $(0,\infty)$, that is, $\xi_q(\theta) \geq 0$, $\theta \in (0,\infty)$ and $\int_0^\infty \xi_q(\theta)\, d\theta = 1$.

LEMMA 6.2. *The operators $\mathscr{T}$ and $\mathscr{S}$ have the following properties:*

(a) *For any fixed $t \geq 0$, $\mathscr{T}(t)$ and $\mathscr{S}(t)$ are linear and bounded operators, i.e. for any $u \in X$,*

$$\|\mathscr{T}(t)u\| \leq M\|u\| \quad and \quad \|\mathscr{S}(t)u\| \leq \frac{qM}{\Gamma(1+q)}\|u\|.$$

(b) *$\{\mathscr{T}(t), t \geq 0\}$ and $\{\mathscr{S}(t), t \geq 0\}$ are strongly continuous.*

(c) *For every $t > 0$, $\mathscr{T}(t)$ and $\mathscr{S}(t)$ are also compact operators.*

(d) *For any $x \in X$, $\eta \in (0,1)$ and $\gamma \in (0,1)$, we have*

$$\begin{aligned} A\mathscr{S}(t)x &= A^{1-\eta}\mathscr{S}(t)A^{\eta}x, && t \in J, \\ \|A^{\gamma}\mathscr{S}(t)\| &\leq \frac{M_{\gamma}q\Gamma(2-\gamma)}{\Gamma(1+q(1-\gamma))}\,t^{-\gamma q}, && 0 < t \leq T. \end{aligned}$$

(e) *For fixed $t \geq 0$ and any $x \in X_{\beta}$,*

$$\|\mathscr{T}(t)u\|_{\beta} \leq M\|u\|_{\beta} \quad and \quad \|\mathscr{S}(t)u\|_{\beta} \leq \frac{qM}{\Gamma(1+q)}\|u\|_{\beta}.$$

(f) *$\mathscr{T}_{\beta}(t)$ and $\mathscr{S}_{\beta}(t)$, $t > 0$ are uniformly continuous, that is for each fixed $t > 0$, and $\epsilon > 0$, there exists $h > 0$ such that*

$$\|\mathscr{T}_{\beta}(t+\epsilon) - \mathscr{T}_{\beta}(t)\|_{\alpha} < \varepsilon, \qquad \|\mathscr{S}_{\beta}(t+\epsilon) - \mathscr{S}_{\beta}(t)\|_{\alpha} < \varepsilon,$$

for $t + \epsilon \geq 0$ and $|\epsilon| < h$ where

$$\mathscr{T}_{\beta}(t) = \int_0^{\infty} \xi_q(\theta) S_{\beta}(t^q\theta)\,d\theta, \qquad \mathscr{S}_{\beta}(t) = q\int_0^{\infty} \theta\xi_q(\theta) S_{\beta}(t^q\theta)\,d\theta,$$

and the restriction $S_{\beta}(t)$ of $S(t)$ to X_{β} is exactly the part of $S(t)$ in X_{β}.

Define operators:

$$\begin{aligned} &F_3\colon C(J,X_{\beta}) \to C(J,X_{\beta}), \quad (F_3u)(t) = \mathscr{T}(t)u_0, \quad t \in J. \\ &G_3\colon C(J,X_{\beta}) \to C(J,X_{\beta}), \\ &\qquad (G_3u)(t) = \int_0^t (t-s)^{q-1}\mathscr{S}(t-s)\overline{f}(s,u(s))\,ds, \quad t \in J, \\ &\mathbb{T}_3\colon C(J,X_{\beta}) \to C(J,X_{\beta}), \qquad \mathbb{T}_3u = F_3u + G_3u. \end{aligned}$$

It is easy to see that $\mathbb{T}_3$ is well defined due to (H7). Thus, the existence of a solution for fractional ECP (1.4) is equivalent to the existence of a fixed point for operator $\mathbb{T}_3$.

Lemma 6.3. *The operator $F_3\colon C(J,X_\beta)\to C(J,X_\beta)$ is Lipschitz with zero constant. Consequently F_3 is α-Lipschitz with zero constant. Moreover, F satisfies the following growth estimation:*

$$\|F_3u\|_\infty \le M\|u_0\|_\beta, \tag{6.1}$$

for every $u\in C(J,X_\beta)$.

Proof. For every $u,v\in C(J,X_\beta)$,

$$\|F_3u-F_3v\|_\infty \le \|\mathscr{T}(t)u_0-\mathscr{T}(t)u_0\|_\beta = 0.$$

By Proposition 2.9, F_3 is α-Lipschitz with zero constant. Relation (6.1) is a simple consequence of (e) of Lemma 6.2. $\square$

Lemma 6.4. *The operator $G_3\colon C(J,X_\beta)\to C(J,X_\beta)$ is continuous. Moreover, G_3 satisfies the following growth condition:*

$$\|G_3u\|_\infty \le \frac{M_\beta q\Gamma(2-\beta)}{\Gamma(1+q(1-\beta))}\times\frac{T^{q(1-\beta)-q_4}}{\left[\dfrac{q(1-\beta)-q_4}{1-q_4}\right]^{1-q_4}}\times\|m_{\overline{f}}\|_{L^{1/q_4}(J,\mathbb{R}_+)} \tag{6.2}$$

for every $u\in C(J,X_\beta)$.

Proof. Choose $\widetilde{K}>0$ such that

$$\widetilde{K}\ge\frac{M_\beta q\Gamma(2-\beta)}{\Gamma(1+q(1-\beta))}\times\frac{T^{q(1-\beta)-q_4}}{\left[\dfrac{q(1-\beta)-q_4}{1-q_4}\right]^{1-q_4}}\times\|m_{\overline{f}}\|_{L^{1/q_4}(J,\mathbb{R}_+)},$$

and define

$$\mathfrak{B}_{\widetilde{K}} := \{\|u\|_\infty\le\widetilde{K} : u\in C(J,X_\beta)\}.$$

Let $\{u_n\}$ be a sequence of a bounded set $\mathfrak{B}_{\widetilde{K}}\subset C(J,X_\beta)$ such that $u_n\to u$ in $\mathfrak{B}_{\widetilde{K}}$. We have to show that $\|G_3u_n-G_3u\|_\infty\to 0$.

It is easy to see that $\overline{f}(s,u_n(s))\to\overline{f}(s,u(s))$ as $n\to\infty$ due to the continuity of f. On the one other hand using (H8), we get for each $t\in J$,

$$(t-s)^{q(1-\beta)-1}\|\overline{f}(s,u_n(s))-f(s,u(s))\|\le(t-s)^{q(1-\beta)-1}2m_{\overline{f}}(s)\in L^1(J,\mathbb{R}_+).$$

By means of the Lebesgue Dominated Convergence Theorem yields

$$\int_0^t(t-s)^{q(1-\beta)-1}\|f(s,u_n(s))-f(s,u(s))\|\,ds\to 0,\quad\text{as } n\to\infty.$$

Then, for all $t\in J$,

$$\begin{aligned}&\|G_3u_n-G_3u\|_\infty\\ &\qquad\le\frac{M_\beta q\Gamma(2-\beta)}{\Gamma(1+q(1-\beta))}\int_0^t(t-s)^{q(1-\beta)-1}\|f(s,u_n(s))-f(s,u(s))\|\,ds\to 0.\end{aligned}$$

Therefore, $G_3 u_n \to G_3 u$ as $n \to \infty$ which implies that G_3 is continuous. Relation (6.2) is a simple consequence of (H8) and (d) of Lemma 6.2. □

Now repeat the same process of Step 3 of Theorem 3.2 in Diagana et al. [15], or Theorem 3.1 in Wang and Zhou [35] or Theorem 3.1 in Zhou and Jiao [44], one can obtain the compactness of G_3 by virtue of Lemma 6.2.

Lemma 6.5. *The operator* $G_3\colon C(J, X_\beta) \to C(J, X_\beta)$ *is compact. Consequently* G_3 *is* α*-Lipschitz with zero constant.*

Now, we have the possibility to prove the main results of this section.

Theorem 6.6. *Assume that* (H7)–(H8) *hold. Then the fractional* ECP (1.4) *has at least one solution* $u \in C(J, X_\beta)$ *and the set of the solutions of the fractional* ECP (1.4) *is bounded in* $C(J, X_\beta)$.

Proof. Let $F_3, G_3, \mathbb{T}_3\colon C(J, X_\beta) \to C(J, X_\beta)$ be the operators defined in the beginning of this section. They are continuous and bounded. Moreover, F_3 is α-Lipschitz with zero constant, G_3 is also α-Lipschitz with zero constant. Proposition 2.7 shows us that $\mathbb{T}_3$ is a strict α-contraction with zero constant.

Set $S_3 = \{u \in C(J, X_\beta) : \text{there exists } \lambda \in [0,1] \text{ such that } u = \lambda \mathbb{T}_3 u\}$. Next, we prove that S_3 is bounded in $C(J, X_\beta)$. Consider $u \in S_3$ and $\lambda \in [0,1]$ such that $u = \lambda \mathbb{T}_3 u$. It follows from (6.1) and (6.2) that

$$\begin{aligned}(6.3)\quad \|u\|_\infty \le M\|u_0\|_\beta + \frac{M_\beta q \Gamma(2-\beta)}{\Gamma(1+q(1-\beta))} \\ \times \frac{T^{q(1-\beta)-q_4}}{\left[\dfrac{q(1-\beta)-q_4}{1-q_4}\right]^{1-q_4}} \times \|m_{\overline{f}}\|_{L^{1/q_4}(J,\mathbb{R}_+)},\end{aligned}$$

which shows us that S_3 is bounded in $C(J, X_\beta)$, Consequently, by Theorem 2.11 we deduce that $\mathbb{T}_3$ has at least one fixed point and the set of the fixed points of $\mathbb{T}_3$ is bounded in $C(J, X_\beta)$. □

Remark 6.7. (a) Compared with the results in Diagana et al. [15], our condition (H8) on $\overline{f}$ is more general.

(b) Compared with the results in Wang and Zhou [35] and Zhou and Jiao [44], a different sufficient condition to guarantee the existence results for mild solutions is given.

Now, we give the following data dependence result.

Theorem 6.8. *Assume that* (H7)–(H8) *and* (H3$^+$) *hold. Let* $u(\,\cdot\,)$ $(v(\,\cdot\,))$ *be the solution of fractional* ECP (1.4) *with initial value* $u_0 (v_0)$. *Then there exists a constant* $M^{***} > 0$ *such that*

$$\|u - v\|_\infty \le M^{***}(M\|u_0 - v_0\|_\beta + 1).$$

Proof. Note that (H3$^+$) and (d), (e) of Lemma 6.2, we obtain

$$\|u(t)-v(t)\|_\beta \le M\|u_0-v_0\|_\beta + \frac{L_f M_\beta q\Gamma(2-\beta)}{\Gamma(1+q(1-\beta))}\int_0^t (t-s)^{q(1-\beta)-1}\|u(s)-v(s)\|_\beta^\lambda\,ds.$$

By Lemma 2.12, we obtain

$$\|u(t)-v(t)\|_\beta \le e^{M^{***}T}(M\|u_0-v_0\|_\beta+1).$$

where

$$M^{***}=\frac{L_f M_\beta q\Gamma(2-\beta)}{\Gamma(1+q(1-\beta))}\bigg[\frac{T^{p(q-1)+1}}{p(q-1)+1}\bigg]^{1/p}.$$

This completes the proof. □

Last, we give the following uniqueness result.

Theorem 6.9. *Assume that* (H7)–(H8) *and* (H3^{++}) *hold. Then fractional* ECP (1.4) *has an unique solution* $u\in C(J,X_\beta)$.

Proof. By Theorem 6.6, the fractional ECP (1.4) has a solution $u(\,\cdot\,)$ in $C(J,X_\beta)$. Let $v(\,\cdot\,)$ be another solution of fractional ECP (1.4) with initial value v_0. Note that (H3^{++}), repeating the same process of Theorem 6.8, we obtain

$$\|u(t)-v(t)\|_\beta \le M\|u_0-v_0\|_\beta + \frac{L_f M_\beta q\Gamma(2-\beta)}{\Gamma(1+q(1-\beta))}\int_0^t (t-s)^{q(1-\beta)-1}\|u(s)-v(s)\|_\beta\,ds.$$

This yields that the uniqueness of $u(\,\cdot\,)$ due to the standard singular Gronwall inequality. □

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