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PERIODIC SOLUTIONS TO NONLINEAR EQUATIONS WITH OBLIQUE BOUNDARY CONDITIONS

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Abstract. We study the existence of positive periodic solutions to nonlinear elliptic and parabolic equations with oblique and dynamical boundary conditions and non-local terms. The results are obtained through fixed point theory, topological degree methods and properties of related linear elliptic problems with natural boundary conditions and possibly non-symmetric principal part. As immediate consequences, we also obtain estimates on the principal eigenvalue for non-symmetric elliptic eigenvalue problems.

1. Introduction

In this paper we consider the existence of positive periodic solutions to nonlinear elliptic/parabolic equations subject to oblique natural boundary conditions. We first consider an elliptic problem in Section 2 and then apply these results to parabolic problems that, in particular, involve situations with dynamic boundary conditions. For the sake of simplicity we assume that the right hand side is described by a standard logistic formula to which we have added a nonlocal term. This has been previously done for various biological problems ([2], [6], [7], [27]). It invalidates the use of order methods. For a reference to these we direct the reader to [24], [29]. We thus proceed with topological methods (for a detailed reference see the book [3]).

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We observe that the oblique boundary conditions problems we consider would arise in situations where the motion due to diffusion induced an effect in a different direction, for example in the situation of charged bacteria [23] moving in a magnetic field. On the other hand, the dynamic boundary condition could be used to model situations where the biological species was stored and released depending on conditions at the boundary. To give the flavor of our results we state as an example the following:

Lemma 1.1. *Let $M, h, e \geq 0$ (possibly $e \equiv 0$) and $P > 0$. Assume that there exists a periodic function $c(t) > 0$ such that*

$$\int_0^T \int_\Omega \frac{M}{c} > \int_0^T \int_{\partial\Omega} \frac{h}{c}.$$

Then the problem

$$\begin{cases} e(x)c(t)u_t - \Delta u = [M(x,t) - P(x,t)u]u & \text{in } \Omega \times (0,T), \\ \dfrac{\partial u}{\partial \nu} + c(t)u_t + h(x,t)u = 0 & \text{on } \partial\Omega \times (0,T), \\ e(x)u(x,0) = e(x)u(x,T) & \text{for } x \in \Omega, \\ c(0)u(x,0) = c(T)u(x,T) & \text{for } x \in \partial\Omega, \end{cases}$$

has a positive generalized solution. Here we assume all problem data regular.

The proof of Lemma 1.1 is given in Theorem 3.4 below. We remark that the existence results of Sections 3 and 4 are obtained via fixed point theorems and topological degree arguments. In this respect the use of estimates in Hölder spaces $C^{\alpha,\alpha/2}$ will ensure the compactness of the maps that are involved in the arguments. Furthermore, solution bounds in these spaces depend only on coefficients estimates, not on the specific coefficients themselves.

The history of problems with oblique boundary conditions is vast, but we were unable to find our results in previous work. Problems with dynamic boundary conditions have somewhat fewer results. However we were only able to find [1] that deals with the periodic case. There the model is a degenerate parabolic equation and the existence and asymptotic stability of periodic solutions are proved. We note that in [1] the existence of a positive solution in cases where there is also the identically zero solution was not considered, nor were the effects on the solution existence of changing c.

Other references deal with the initial value problem and other questions. For example, in [22] dynamical boundary conditions are considered for the Laplace and heat equations with semi-linear forcing terms. Existence and uniqueness of initial value problems are obtained via semigroup theory. See also [13]–[15], [21], [31] for analogous results. [38] studies a non-symmetric elliptic equation with respect to global existence for initial value problems. In [35] and [36] the problems of global existence and blow-up in finite time are tackled for elliptic or parabolic equations with a nonlinear dynamical boundary condition. The blow-up phenomenon is considered also in [8] for the Laplace equation and conditions for the continuability after the blow-up are given. Well- or ill-posedness of the initial value problem for linear heat and Laplace equations with dynamical and reactive boundary conditions are studied in [33], [34]. The paper [16] deals with reaction-diffusion equation from the point of view of global existence for initial value problems and global attractor. [30] considers an analogous problem but it is mainly concerned with quenching solutions, that is: bounded solutions with a bounded maximal time-interval of existence. In [18] a distributed model for the ecology of mangroves featuring dynamic boundary conditions is considered; existence and uniqueness of solutions of initial value problems and convergence to steady state are proved. [4] proves existence and uniqueness of initial value problems for degenerate elliptic-parabolic equation with nonlinear diffusion and nonlinear dynamical boundary condition. [17] also deals with a degenerate parabolic equation with p-Laplacean and nonlinear dynamic boundary conditions and shows the existence of a global attractor. In [12] a Hamilton–Jacobi equation with dynamic boundary condition is studied: in order to prove existence of a viscosity solution of the initial value problem, an approximating parabolic problem with dynamic boundary conditions is solved. The papers [37] and [11] deal with global existence and convergence to steady states for Cahn–Hilliard and Caginalp equations with dynamic boundary conditions and regular potentials. On the other hand, [19], [28], [9], [10], [20] considers different assumptions on the potentials for Cahn–Hilliard and Caginalp phase-field systems.

2. Oblique elliptic problems

We consider in this section the elliptic problem:

$$-\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\,\frac{\partial u}{\partial x_j}\right) + \sum_{j=1}^{n} \beta_j(x)\,\frac{\partial u}{\partial x_j} + l(x)\,u = f(x) \tag{2.1}$$

for $x \in \Omega \subset \mathbb{R}^n$ with $n \geq 3$ and Ω a smooth bounded domain, subject to the natural boundary condition:

$$\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial u}{\partial x_j}\,\nu_i(x) + h(x)\,u = g(x), \tag{2.2}$$

where $h \geq 0$ and $\overrightarrow{\nu} = (\nu_1, \ldots, \nu_n)$ is the outward normal to $\partial\Omega$. We assume all data is regular and set $A = (a_{ij})$, $\overrightarrow{\beta} = (\beta_1, \ldots, \beta_n)$. We also assume that (2.1)–(2.2) is elliptic, i.e. $\langle A\overrightarrow{\xi}, \overrightarrow{\xi}\rangle > \delta|\overrightarrow{\xi}|^2$ for some $\delta > 0$, but do not require that A be symmetric. Consequently, condition (2.2) becomes:

$$\langle (A_{\rm s} + A_{\rm a})\nabla u, \overrightarrow{\nu}\rangle + hu = g$$

where $A_{\mathrm{s}} = (A + A^\top)/2$, $A_{\mathrm{a}} = (A - A^\top)/2$. Since $\langle A_{\mathrm{a}}\overrightarrow{\nu}, \overrightarrow{\nu}\rangle = 0$, we recover in this way oblique derivative problems. We remark that if an oblique condition is given a priori, then the form associated with (2.1) can be modified so that the given condition becomes "natural". We will do this later explicitly for a special case, and the general process may be found in detail in the book by Troianiello [32]. We also note that it will be convenient for us to consider a related T-periodic elliptic problem in $Q_T \triangleq \Omega \times (0,T)$: now (2.2) is to apply only to $\partial\Omega \times (0,T)$, and we add periodic conditions on the problem data and u:

$$u(x,0) = u(x,T) \quad \text{for } x \in \Omega.$$

We observe that if u solves either (2.1)–(2.2) or the periodic problem, then u is a classical solution (see, e.g. [32]. For the periodic problem extend u to $\Omega \times (-T, 2T)$ by periodicity).

Lemma 2.1. *Let $u \geq 0$, nontrivial, solve (2.1)–(2.2) then*

$$(2.3)\quad 0 \leq \int_\Omega \bigg\{ \langle A A_{\mathrm{s}}^{-1} A^\top \nabla\phi, \nabla\phi\rangle + \Big\langle \nabla\phi + A_{\mathrm{s}}^{-1} A_{\mathrm{a}}^\top \nabla\phi, \overrightarrow{\beta} \Big\rangle\, \phi + \Big\langle A^{-1}\overrightarrow{\beta}, \overrightarrow{\beta}\Big\rangle \frac{\phi^2}{4} + l\,\frac{u}{u+\eta}\,\phi^2 \bigg\} + \int_{\partial\Omega} \frac{hu}{u+\eta}\,\phi^2 - \int_\Omega \frac{\phi^2}{u+\eta}\, f - \int_{\partial\Omega} \frac{\phi^2}{u+\eta}\, g$$

for all $\phi \in H^1(\Omega)$ and $\eta > 0$. If, moreover, f, g are nonnegative and $R(\phi) \leq 0$ for some nontrivial $\phi \in H^1(\Omega)$, where

$$R(\phi) \triangleq \int_\Omega \bigg\{ \langle A A_{\mathrm{s}}^{-1} A^\top \nabla\phi, \nabla\phi\rangle + \Big\langle \nabla\phi + A_{\mathrm{s}}^{-1} A_{\mathrm{a}}^\top \nabla\phi, \overrightarrow{\beta}\Big\rangle\, \phi + \Big\langle A_{\mathrm{s}}^{-1}\overrightarrow{\beta}, \overrightarrow{\beta}\Big\rangle \frac{\phi^2}{4} + l\phi^2 \bigg\} + \int_{\partial\Omega} h\phi^2,$$

then either $\mu\{x \in \Omega \mid u(x) = 0\} + \mu'\{x \in \partial\Omega \mid u(x) = 0\} > 0$ (where μ and μ' denote the measures in $\mathbb{R}^n$ and in $\partial\Omega$ respectively), or $R(\phi) = 0$, $\phi^2 f = \phi^2 g = 0$ and

$$\nabla\bigg(\frac{\phi}{u}\bigg) = (A^\top)^{-1}\bigg(A_{\mathrm{a}}\frac{\nabla u}{u} - \frac{\overrightarrow{\beta}}{2}\bigg)\frac{\phi}{u} \quad \textit{wherever } u > 0.$$

Proof. Assume first that $\phi(x) > 0$, $u(x) > 0$. We observe by direct calculation:

$$\bigg\langle A_{\mathrm{s}}\nabla\bigg(\frac{\phi}{u}\bigg), \nabla\bigg(\frac{\phi}{u}\bigg)\bigg\rangle u^2 = \langle A_{\mathrm{s}}\nabla\phi, \nabla\phi\rangle + 2\langle A_{\mathrm{a}}^\top\nabla\phi, \nabla u\rangle \frac{\phi}{u} - \bigg\langle A^\top\nabla\bigg(\frac{\phi^2}{u}\bigg), \nabla u\bigg\rangle.$$

Put $\overrightarrow{b} = 2(A_{\mathrm{a}}^\top\nabla\phi)/\phi$, whence

$$2\,\langle A_{\mathrm{a}}^\top\nabla\phi, \nabla u\rangle\, \frac{\phi}{u} = \Big\langle \overrightarrow{b}, \nabla u\Big\rangle \frac{\phi^2}{u} = -\bigg\langle \overrightarrow{b}, \nabla\bigg(\frac{\phi}{u}\bigg)\bigg\rangle u\phi + \Big\langle \overrightarrow{b}, \nabla\phi\Big\rangle\, \phi.$$

We note that $\left\langle \overrightarrow{b}, \nabla\phi \right\rangle \phi = 0$, whence

$$
\begin{aligned}
&u^2\left\langle A_{\mathrm{s}}\nabla\left(\frac{\phi}{u}\right), \nabla\left(\frac{\phi}{u}\right)\right\rangle + \left\langle \overrightarrow{b} + \overrightarrow{\beta}, \nabla\left(\frac{\phi}{u}\right)\right\rangle u\phi \\
&\qquad = \langle A_{\mathrm{s}}\nabla\phi, \nabla\phi\rangle - \left\langle A^{\top}\nabla\left(\frac{\phi^2}{u}\right), \nabla u\right\rangle + \left\langle \overrightarrow{\beta}, \nabla\phi\right\rangle \phi - \left\langle \overrightarrow{\beta}, \nabla u\right\rangle \frac{\phi^2}{u}.
\end{aligned}
$$

We add the term $\langle (A_{\mathrm{s}})^{-1}(\overrightarrow{b} + \overrightarrow{\beta}), \overrightarrow{b} + \overrightarrow{\beta}\rangle \phi^2/4$ to both sides, thus completing the square on the left hand side and obtaining

$$
\begin{aligned}
0 &\leq \left\langle A_{\mathrm{s}}^{-1}\left[uA_{\mathrm{s}}\nabla\left(\frac{\phi}{u}\right) + \frac{\phi}{2}\left(\overrightarrow{b} + \overrightarrow{\beta}\right)\right], \left[uA_{\mathrm{s}}\nabla\left(\frac{\phi}{u}\right) + \frac{\phi}{2}\left(\overrightarrow{b} + \overrightarrow{\beta}\right)\right]\right\rangle \\
&= \langle A_{\mathrm{s}}\nabla\phi, \nabla\phi\rangle + \left\langle \overrightarrow{\beta}, \nabla\phi\right\rangle \phi - \left\langle A^{\top}\nabla\left(\frac{\phi^2}{u}\right), \nabla u\right\rangle \\
&\quad - \left\langle \overrightarrow{\beta}, \nabla u\right\rangle \frac{\phi^2}{u} + \left\langle A_{\mathrm{s}}^{-1}\left(\overrightarrow{b} + \overrightarrow{\beta}\right), \overrightarrow{b} + \overrightarrow{\beta}\right\rangle \frac{\phi^2}{4}.
\end{aligned}
$$

We expand the last term on the right hand side and obtain:

$$
\begin{aligned}
&\left\langle A_{\mathrm{s}}^{-1}\left(\overrightarrow{b} + \overrightarrow{\beta}\right), \overrightarrow{b} + \overrightarrow{\beta}\right\rangle \frac{\phi^2}{4} \\
&\qquad = \langle A_{\mathrm{a}}A_{\mathrm{s}}^{-1}A_{\mathrm{a}}^{\top}\nabla\phi, \nabla\phi\rangle + \left\langle A_{\mathrm{s}}^{-1}A_{\mathrm{a}}^{\top}\nabla\phi, \overrightarrow{\beta}\right\rangle \phi + \left\langle A_{\mathrm{s}}^{-1}\overrightarrow{\beta}, \overrightarrow{\beta}\right\rangle \frac{\phi^2}{4}.
\end{aligned}
$$

We thus obtain for x such that $\phi(x) > 0$:

$$
\begin{aligned}
0 &\leq \left\langle A_{\mathrm{s}}^{-1}\left[uA_{\mathrm{s}}\nabla\left(\frac{\phi}{u}\right) + \frac{\phi}{2}\left(\overrightarrow{b} + \overrightarrow{\beta}\right)\right], \left[uA_{\mathrm{s}}\nabla\left(\frac{\phi}{u}\right) + \frac{\phi}{2}\left(\overrightarrow{b} + \overrightarrow{\beta}\right)\right]\right\rangle \\
&= \langle AA_{\mathrm{s}}^{-1}A^{\top}\nabla\phi, \nabla\phi\rangle + \left\langle \nabla\phi + A_{\mathrm{s}}^{-1}A_{\mathrm{a}}^{\top}\nabla\phi, \overrightarrow{\beta}\right\rangle \phi + \left\langle A_{\mathrm{s}}^{-1}\overrightarrow{\beta}, \overrightarrow{\beta}\right\rangle \frac{\phi^2}{4} \\
&\quad - \left\langle A^{\top}\nabla\left(\frac{\phi^2}{u}\right), \nabla u\right\rangle - \left\langle \overrightarrow{\beta}, \nabla u\right\rangle \frac{\phi^2}{u},
\end{aligned}
$$

where we used the identity $A_{\mathrm{s}} + A_{\mathrm{a}}A_{\mathrm{s}}^{-1}A_{\mathrm{a}}^{\top} = (A_{\mathrm{s}} + A_{\mathrm{a}})A_{\mathrm{s}}^{-1}(A_{\mathrm{s}} - A_{\mathrm{a}}) = AA_{\mathrm{s}}^{-1}A^{\top}$. Since on the set $\{x : \phi(x) = 0\}$ we have $\nabla\phi = 0$ almost everywhere, the above inequality holds for almost all $x \in \Omega$. Integrating, we obtain:

$$
\begin{aligned}
0 &\leq \int_{\Omega} \left\langle A_{\mathrm{s}}^{-1}\left[uA_{\mathrm{s}}\nabla\left(\frac{\phi}{u}\right) + \frac{\phi}{2}\left(\overrightarrow{b} + \overrightarrow{\beta}\right)\right], \left[uA_{\mathrm{s}}\nabla\left(\frac{\phi}{u}\right) + \frac{\phi}{2}\left(\overrightarrow{b} + \overrightarrow{\beta}\right)\right]\right\rangle \\
&= \int_{\Omega} \Bigg[\langle AA_{\mathrm{s}}^{-1}A^{\top}\nabla\phi, \nabla\phi\rangle + \left\langle \overrightarrow{\beta}, \nabla\phi\right\rangle \phi \\
&\quad + \left\langle A_{\mathrm{s}}^{-1}A_{\mathrm{a}}^{\top}\nabla\phi, \overrightarrow{\beta}\right\rangle \phi + \left\langle A_{\mathrm{s}}^{-1}\overrightarrow{\beta}, \overrightarrow{\beta}\right\rangle \frac{\phi^2}{4} + l\phi^2\Bigg] \\
&\quad + \int_{\partial\Omega} h\phi^2 - \int_{\Omega} \frac{\phi^2}{u} f - \int_{\partial\Omega} \frac{\phi^2}{u} g.
\end{aligned}
$$

If $u \geq 0$, we repeat the argument, replacing u by $u+\eta$, for $\eta > 0$, in the inequality. We obtain the same estimate with

$$\int_\Omega l\phi^2 + \int_{\partial\Omega} h\phi^2 - \int_\Omega \frac{\phi^2}{u} f - \int_{\partial\Omega} \frac{\phi^2}{u} g$$

replaced by

$$\int_\Omega l \frac{u}{u+\eta}\,\phi^2 + \int_{\partial\Omega} \frac{hu}{u+\eta}\,\phi^2 - \int_\Omega \frac{\phi^2}{u+\eta}\, f - \int_{\partial\Omega} \frac{\phi^2}{u+\eta}\, g$$

That is exactly (2.3).

If moreover $R(\phi) \leq 0$, f, g are non-negative and

$$\mu\{x \in \Omega \mid u(x) = 0\} + \mu'\{x \in \partial\Omega \mid u(x) = 0\} = 0,$$

then from (2.3), as $\eta \to 0$, we have that $R(\phi) = 0$, $\phi^2 f = \phi^2 g = 0$ and at any x with $u(x) > 0$:

$$\begin{aligned} 0 &= A_{\mathrm{s}} \nabla\left(\frac{\phi}{u}\right) + \frac{\phi}{2u}\left(\overrightarrow{b} + \overrightarrow{\beta}\right) \\ &= A_{\mathrm{s}} \nabla\left(\frac{\phi}{u}\right) - A_{\mathrm{a}} \nabla\left(\frac{\phi}{u} u\right)\frac{1}{u} + \frac{\phi}{u}\frac{\overrightarrow{\beta}}{2} = A^\top \nabla\left(\frac{\phi}{u}\right) - \left(A_{\mathrm{a}}\frac{\nabla u}{u} - \frac{\overrightarrow{\beta}}{2}\right)\frac{\phi}{u} \end{aligned}$$

and the result follows. $\square$

As immediate applications of Lemma 2.1 we obtain estimates of the principal eigenvalue of non-symmetric elliptic operators, by letting $\eta \to 0$.

Corollary 2.2. *Let λ denote the principal eigenvalue for* (2.1)–(2.2)*, where $f \triangleq \lambda u$ and $g \equiv 0$, with eigenvector $u > 0$. Then:*

$$\lambda \leq \inf_{\phi \in H^1(\Omega)} \left[\frac{R(\phi)}{\int_\Omega \phi^2}\right].$$

The choice $\phi \equiv 1$ gives:

Corollary 2.3.

$$\lambda \leq \frac{1}{|\Omega|}\left\{ \int_\Omega \left(\frac{\langle A_{\mathrm{s}}^{-1}\overrightarrow{\beta}, \overrightarrow{\beta}\rangle}{4} + l\right) + \int_{\partial\Omega} h\right\}.$$

The following result is an application of Lemma 2.1 to the principal Steklov eigenvalue. We refer to [5] for recent results on Steklov eigenvalues in the symmetric case.

Corollary 2.4. *Let λ denote the principal eigenvalue for the following Steklov eigenvalue problem:*

$$\begin{cases} -\nabla \cdot [A\nabla u] + \overrightarrow{\beta} \cdot \nabla u + lu = 0 & \text{in } \Omega, \\ \langle A\nabla u, \overrightarrow{\nu}\rangle + hu = \lambda u & \text{on } \partial\Omega. \end{cases}$$

Then

$$\lambda \leq \frac{R(\phi)}{\int_{\partial\Omega}\phi^2} \quad \textit{for all } \phi \in H^1(\Omega).$$

We comment on the analogous situation for the periodic-elliptic problem in $Q_T \triangleq \Omega \times (0,T)$. Observe that in this case similar results hold if all data is periodic and now

$$\phi \in H^{1,\mathrm{per}}(Q_T) = \{\phi \mid \phi \in H^1(Q_T),\ \phi \text{ is periodic in } x_{n+1}\},$$

once we observe that a solution u must also have $\partial u/\partial x_{n+1}$ periodic, while on $\partial\Omega \times (0,T)$ the outward normal $\overrightarrow{n} = \left(\overrightarrow{\nu}^\top, 0\right)^\top$ must be perpendicular to the x_{n+1}-axis. In particular it is also convenient to observe for the periodic-elliptic problem we consider next, that in the preceding argument the variables can be treated differently in the case of a cylindrical domain as follows: set $x_{n+1} = t$, $\nabla u = \left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}\right)^\top$, $\delta u = (\nabla^\top u, u_t)^\top$. Consider the problem:

$$\mathcal{L}u \triangleq -\nabla\cdot\left[A\nabla u + \overrightarrow{b}\, u_t\right] - \frac{\partial}{\partial t}\left[-\overrightarrow{b}\cdot\nabla u + \varepsilon u_t\right] + \overrightarrow{z}\cdot\nabla u + e u_t + r u = f \quad \text{in } Q_T$$

with smooth periodic data, and

$$\begin{aligned} &\left(A\nabla u + \overrightarrow{b}\, u_t\right)\cdot\overrightarrow{\nu} + hu = 0 && \text{on } \partial\Omega \times [0,T],\\ &u(x,0) = u(x,T) && \text{for } x \in \Omega, \end{aligned}$$

where $\overrightarrow{\nu}$ is the outward normal to $\partial\Omega$. Assume further that $A = (a_{ij})$ is a symmetric positive definite $n \times n$ matrix, $\varepsilon > 0$. We then have:

COROLLARY 2.5. *Let $k > 1$, $u > 0$, and set*

$$\mathfrak{A} = \begin{pmatrix} \left(1 - \frac{1}{k}\right) A & \overrightarrow{b} \\ -\overrightarrow{b}^\top & \varepsilon \end{pmatrix}$$

with $\mathfrak{A}_{\mathrm{s}} = (\mathfrak{A} + \mathfrak{A}^\top)/2$, $\mathfrak{A}_{\mathrm{a}} = (\mathfrak{A} - \mathfrak{A}^\top)/2$. Let ϕ be smooth, periodic in t. Then:

$$\begin{aligned} 0 \leq &\int_{Q_T}\left\langle \frac{A}{k}\nabla\phi, \nabla\phi\right\rangle + \int_{Q_T}\langle\overrightarrow{z}, \nabla\phi\rangle\,\phi + \int_{Q_T}\frac{k}{4}\left\langle A^{-1}\overrightarrow{z}, \overrightarrow{z}\right\rangle\phi^2 \\ &+ \int_{Q_T}\left\langle \mathfrak{A}\,\mathfrak{A}_{\mathrm{s}}^{-1}\mathfrak{A}^\top\delta\phi, \delta\phi\right\rangle + \int_{Q_T} r\phi^2 + \int_0^T\int_{\partial\Omega}\phi^2 h - \int_{Q_T}\frac{\phi^2}{u}(f - e u_t). \end{aligned}$$

PROOF. Choose $k > 1$ and set

$$\begin{aligned} \mathcal{L}_1 u &= -\nabla\cdot\left[\frac{A}{k}\nabla u\right] + \overrightarrow{z}\cdot\nabla u, \\ \mathcal{L}_2 u &= -\nabla\cdot\left[\left(1 - \frac{1}{k}\right)A\nabla u + \overrightarrow{b}\, u_t\right] - \frac{\partial}{\partial t}\left[\overrightarrow{b}\cdot\nabla u + \varepsilon u_t\right] + ru. \end{aligned}$$

We then observe that $\mathcal{L}(u) = f$ implies $\mathcal{L}_1(u) + \mathcal{L}_2(u) = f - eu_t$. We basically repeat the calculation of Lemma 2.1 for this case and obtain for any $t \in (0,T)$:

$$
\begin{aligned}
(2.4)\quad 0 \le \int_\Omega \left\langle \frac{A}{k}\nabla\phi, \nabla\phi \right\rangle &+ \int_\Omega \langle \overrightarrow{z}, \nabla\phi \rangle\ \phi \\
&+ \int_\Omega \frac{k}{4}\ \langle A^{-1}\overrightarrow{z}, \overrightarrow{z} \rangle\ \phi^2 - \int_{\partial\Omega} \left\langle \frac{A}{k}\nabla u, \overrightarrow{\nu} \right\rangle \frac{\phi^2}{u} - \int_\Omega \frac{\phi^2}{u}\,\mathcal{L}_1(u).
\end{aligned}
$$

Next observe that once again repeating the calculations of Lemma 2.1 yields, with $\mathfrak{A}$ replacing A and $\overrightarrow{\beta} = 0$ in (2.3), and recalling $\delta\phi = (\nabla\phi, \phi_t)$

$$
(2.5)\ \ 0 \le \int_{Q_T} \langle \mathfrak{A}\,\mathfrak{A}_{\mathrm{s}}^{-1}\mathfrak{A}^\top \delta\phi, \delta\phi\rangle + \int_{Q_T} r\phi^2 - \int_{\partial Q_T} \frac{\phi^2}{u}\,\langle \mathfrak{A}\,\delta u, \overrightarrow{n}\rangle - \int_{Q_T} \frac{\phi^2}{u}\,\mathcal{L}_2(u).
$$

Integrating (2.4) with respect to t and adding to (2.5) yield, noting that on $\partial\Omega \times (0,T)$ the normal $\overrightarrow{n}$ is perpendicular to the t-axis,

$$
\begin{aligned}
(2.6)\quad 0 \le \int_{Q_T} \left\langle \frac{A}{k}\nabla\phi, \nabla\phi \right\rangle &+ \int_{Q_T} \langle \overrightarrow{z}, \nabla\phi \rangle\ \phi \\
&+ \int_{Q_T} \frac{k}{4}\ \langle A^{-1}\overrightarrow{z}, \overrightarrow{z} \rangle\ \phi^2 + \int_{Q_T} \langle \mathfrak{A}\,\mathfrak{A}_{\mathrm{s}}^{-1}\mathfrak{A}^\top \delta\phi, \delta\phi\rangle \\
&+ \int_{Q_T} r\phi^2 - \int_0^T \int_{\partial\Omega} \frac{\phi^2}{u}\ \left\langle A\nabla u + \overrightarrow{b}\,u_t, \overrightarrow{\nu} \right\rangle - \int_{Q_T} \frac{\phi^2}{u}\,\mathcal{L}(u).
\end{aligned}
$$

We observe that the last two terms of (2.6) are:

$$
\int_0^T \int_{\partial\Omega} \phi^2 h - \int_{Q_T} \frac{\phi^2}{u}\,(f - eu_t). \qquad \square
$$

3. The periodic parabolic problem

We now consider, as an application of the results in Section 2, the following periodic parabolic problem:

$$
(3.1)\quad
\begin{cases}
d(x,t)u_t - \Delta u = \left(M(x,t) - P(x,t)u - \displaystyle\int_\Omega S(\xi,t)u\,d\xi \right) u \\
\qquad\qquad\qquad\qquad\qquad\qquad \text{in } Q_T \triangleq \Omega \times (0,T) \subset \mathbb{R}^{n+1}, \\
\dfrac{\partial u}{\partial \nu} + c(x,t)\,\dfrac{\partial u}{\partial t} + h(x,t)u = 0 \quad \text{on } \partial\Omega \times (0,T), \\
d(x,0)\,u(x,0) = d(x,T)u(x,T) \quad \text{for } x \in \Omega, \\
c(x,0)\,u(x,0) = c(x,T)u(x,T) \quad \text{for } x \in \partial\Omega,
\end{cases}
$$

with $d, M, S, h \ge 0$, $P, c > 0$ and all periodic. Specifically, we perturb the problem to an elliptic equation as follows: for $0 < \varepsilon < 1$ and a suitable smooth function $a(x,t) \geq a_0 > 0$

$$
\begin{aligned}
(3.2) \qquad -\frac{\varepsilon}{a(x,t)}\, u_{tt} + d(x,t)\, u_t - \Delta u& \\
= \left(M(x,t) - P(x,t)u - \int_\Omega S(\xi,t)u\, d\xi \right) u^+ + \frac{\varepsilon}{a(x,t)}&
\end{aligned}
$$

subject to the (dynamic) boundary conditions:

$$
\begin{aligned}
(3.3) \qquad & \frac{\partial u}{\partial \nu} + c(x,t)\, \frac{\partial u}{\partial t} + h(x,t)\, u = 0 \quad && (x,t) \in \partial\Omega \times (0,T), \\
(3.4) \qquad & u(x,0) = u(x,T) && x \in \Omega.
\end{aligned}
$$

We can incorporate (3.3) as a natural condition in (3.2) by dividing (3.3) by c and rewriting (3.2) in the form:

$$
\begin{aligned}
(3.5) \qquad \mathcal{L}_1(u) &\triangleq -\nabla \cdot \left(a\nabla u + \overrightarrow{b}\, u_t \right) - \frac{\partial}{\partial t} \left(-\overrightarrow{b} \cdot \nabla u + \varepsilon u_t \right) \\
&\quad + \nabla a \cdot \nabla u + adu_t + \left(\nabla \cdot \overrightarrow{b} \right) u_t - \frac{\partial \overrightarrow{b}}{\partial t} \cdot \nabla u \\
&= a\left(M - Pu - \int_\Omega Su \right) u^+ + \varepsilon
\end{aligned}
$$

with the following choices: $a(x,t) = 1/c(x,t)$ and $\overrightarrow{b} : \Omega \to \mathbb{R}^n$ such that $\overrightarrow{b} \cdot \overrightarrow{\nu} = 1$ on $\partial\Omega$, where c is extended to a positive smooth function on $\overline{Q}_T$ and $\overrightarrow{\nu}$ denotes the outward normal to $\partial\Omega$). We recall that we are interested in the solution of (3.3)–(3.5) in the limit as $\varepsilon \to 0$, and that all data is assumed smooth, periodic.

THEOREM 3.1. *Problem* (3.3)–(3.5) *has a positive classical solution* u_ε *for any* $\varepsilon > 0$. *This* u_ε *also solves* (3.2)–(3.4).

PROOF. First of all add a linear term aRu to both sides of (3.5) so that the left hand side is coercive, and such that any regular solution of (3.3)–(3.5) is positive in $\Omega \times (0,T)$ by the maximum principle.

Secondly, any solution of (3.3)–(3.5) is bounded above uniformly with respect to $\varepsilon \in (0,1)$. Indeed, since $u > 0$, we have

$$
\left(M + R - Pu - \int_\Omega Su \right) u + \frac{\varepsilon}{a} \leq (M + R - Pu)\, u + \frac{1}{a_0} \leq \frac{\|M+R\|_\infty^2}{4 \min P} + \frac{1}{a_0} \triangleq K.
$$

Now let $z = z(x) > 0$ be the solution of

$$
\begin{cases}
-\Delta z + Rz = K & \text{in } \Omega, \\
\dfrac{\partial z}{\partial \nu} = 0 & \text{on } \partial\Omega,
\end{cases}
$$

and observe that $\mathcal{L}_1(z-u) + aR(z-u) \geq 0$, therefore $u(x,t) \leq z(x)$ again by the maximum principle. In particular, the regularity estimates in [25] show that any solution of (3.2)–(3.4) is bounded in $C^\alpha(\overline{Q}_T)$ with α and the bound depending only on ε. For the reader's convenience, we recall that, if $\partial\Omega$ is smooth and

$0 < \alpha < 1$, $C^\alpha(\overline{Q}_T)$ is the Banach space of continuous functions $u: \overline{Q}_T \to \mathbb{R}$ such that

$$\langle u \rangle_{Q_T}^{(\alpha)} \triangleq \sup \frac{|u(x,t) - u(x',t')|}{|(x-x', t-t')|^\alpha} < +\infty \tag{3.6}$$

where the supremum is taken over all $(x,t), (x',t') \in \overline{Q}_T$ such that $|x-x'| + |t-t'| \le \rho_0$ for some fixed $\rho_0 > 0$. Therefore we can consider the compact operator $\mathcal{T}: C^{\alpha/2}(\overline{Q}_T) \to C^{\alpha/2}(\overline{Q}_T)$ such that $\mathcal{T}(\xi)$ is the solution of $\mathcal{L}_1(v) + aRv = a(M + R - P\xi - \int_\Omega S\xi)\xi^+ + \varepsilon$ with (3.3)–(3.4). The existence of a solution u_ε follows by the Schauder Fixed Point Theorem. Regularity is also immediate from local elliptic estimates (see [25]) after we extend u to $t \in [-T, 2T]$ by periodicity. Finally the equivalence of (3.3)–(3.5) to (3.2)–(3.4) is by direct calculation, since we observe that (3.3) can be recovered as the natural boundary condition associated with (3.5). □

We remark that during the preceding proof we obtained also:

Lemma 3.2. *The solutions u_ε of* (3.2)–(3.4) *are bounded above uniformly with respect to $\varepsilon \in [0,1]$.*

We employ the results of Section 2 to obtain conditions to ensure that $u_\varepsilon \not\to 0$. Specifically:

Lemma 3.3. *If one of the following three conditions is satisfied:*

(a) *$d = d(x)$ and $M \ge 0$, $c = c(x)$ and*

$$\int_{Q_T} M > \int_0^T \int_{\partial\Omega} h;$$

(b) *$d = d(x)$ and $M \ge 0$ and the Dirichlet problem:*

$$\begin{cases} -\Delta w - \overline{M}(x) w = \lambda_1 w & \textit{in } \Omega, \\ w = 0 & \textit{on } \partial\Omega, \end{cases}$$

has least eigenvalue $\lambda_1 < 0$ with eigenvector ϕ_1, where

$$\overline{M}(x) = \frac{1}{T} \int_0^T M(x,t)\, dt \quad \textit{and} \quad \|\phi\|_{L^2(\Omega)} = 1;$$

(c) *The quotient d/c is a function of x (we allow $d \equiv 0$) and*

$$\int_{Q_T} \frac{M}{c} > \int_{Q_T} \frac{c}{4} \left| \nabla \left(\frac{1}{c} \right) \right|^2 + \int_0^T \int_{\partial\Omega} \frac{h}{c} \tag{3.7}$$

then $\{u_\varepsilon\}$ *are bounded away from zero.*

Proof. We first deal with cases (a) and (b). Since $c = c(x)$ and $a(x) = 1/c(x)$ in (3.2), we observe that u_ε satisfies:

$$\begin{cases} -\Delta u = \left(M(x,t) - P(x,t)u - \displaystyle\int_\Omega Su\right)u^+ + \dfrac{\varepsilon}{a(x)} + \dfrac{\varepsilon u_{tt}}{a(x)} - d(x)u_t \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } \Omega,\ t \in (0,T), \\ \dfrac{\partial u}{\partial \nu} + h(x,t)u = -c(x)u_t \qquad\qquad\qquad\qquad\quad \text{on } \partial\Omega, t \in (0,T), \end{cases}$$

thus we can apply (2.3) in Lemma 2.1 with the choices $A \equiv \operatorname{diag}(1,\dots,1)$, $\overrightarrow{\beta} \equiv 0$, $l \equiv 0$, $g = -c(x)\frac{\partial u_\varepsilon}{\partial t}$ and

$$f(x,t) = \left(M(x,t) - P(x,t)u_\varepsilon - \int_\Omega Su_\varepsilon\right)u_\varepsilon^+ + \frac{\varepsilon}{a(x)} + \frac{\varepsilon}{a(x)}\frac{\partial^2 u_\varepsilon}{\partial t^2} - d(x)\frac{\partial u_\varepsilon}{\partial t}$$

and, after integrating on $(0,T)$, we obtain by T-periodicity:

$$\begin{aligned} 0 \le &\int_{Q_T} |\nabla\phi|^2 + \int_0^T \int_{\partial\Omega} \frac{hu_\varepsilon\phi^2}{u_\varepsilon + \eta} \\ &- \int_{Q_T} \phi^2 \left(M - Pu_\varepsilon - \int_\Omega Su_\varepsilon\right)\frac{u_\varepsilon}{u_\varepsilon + \eta} - \varepsilon\int_{Q_T} \frac{\phi^2}{a(u_\varepsilon+\eta)}\left(\frac{\partial^2 u_\varepsilon}{\partial t^2} + 1\right), \end{aligned}$$

where the last integral can be dropped since:

$$\int_{Q_T} \frac{\phi^2}{a(u_\varepsilon+\eta)}\frac{\partial^2 u_\epsilon}{\partial t^2} = \int_{Q_T} \frac{\phi^2}{a(u_\varepsilon+\eta)^2}\left(\frac{\partial u_\varepsilon}{\partial t}\right)^2 \ge 0.$$

In case (a) we choose $\phi \equiv 1$ and let $\eta \to 0$ to obtain

$$0 \le \int_0^T \int_{\partial\Omega} h - \int_{Q_T}\left(M - Pu_\varepsilon - \int_\Omega Su_\varepsilon\right)$$

and we observe that $\|u_\varepsilon\|_{L^1}$ cannot tend to zero.

In case (b) we choose $\phi = \phi_1$ and note that the Dirichlet condition eliminates the boundary integrals. Hence we obtain, as $\eta \to 0$,

$$\begin{aligned} 0 &\le \int_{Q_T} |\nabla\phi_1|^2 - \int_{Q_T}\left(M - Pu_\varepsilon - \int_\Omega Su_\varepsilon\right)\phi_1^2 \\ &= \int_{Q_T} (\overline{M} - \lambda_1)\,\phi_1^2 - \int_{Q_T}\left(M - Pu_\varepsilon - \int_\Omega Su_\varepsilon\right)\phi_1^2 \end{aligned}$$

and $\|u_\varepsilon\|_{L^1} \not\to 0$ since $\lambda_1 < 0$.

Finally, to deal with case (c) we choose $a(x,t) = 1/c(x,t)$ in (3.5) or, equivalently, in (3.2) and we apply Corollary 2.5 with

$$\begin{aligned} A &= \operatorname{diag}(a,\dots,a), & a &= \frac{1}{c}, & \overrightarrow{z} &= \nabla\left(\frac{1}{c}\right), \\ r &= \frac{-M + Pu_\varepsilon + \int_\Omega Su_\varepsilon}{c}, & e &= \frac{d}{c} + \nabla\cdot\overrightarrow{b}, & f &= \varepsilon, \quad k > 1 \end{aligned}$$

and $\phi \equiv 1$ to obtain

$$0 \le \int_{Q_T} \frac{kc}{4} \left|\nabla\left(\frac{1}{c}\right)\right|^2 + \int_0^T \int_{\partial\Omega} \frac{h}{c} \\ - \int_{Q_T} \frac{1}{c}\left(M - Pu_\varepsilon - \int_\Omega Su_\varepsilon\right) + \int_{Q_T} \left(\frac{d}{c} + \nabla \cdot \overrightarrow{b}\right) \frac{(u_\varepsilon)_t}{u_\varepsilon}.$$

Once again, since $d/c + \nabla \cdot \overrightarrow{b}$ are functions purely of x, integration with respect to t shows that the last term vanishes by periodicity. We thus conclude that $\|u_\varepsilon\|_{L^1} \not\to 0$ if (3.7) holds and the result follows. □

We note that since u_ε are bounded above in L^∞, then there is a subsequence that converges strongly in $L^2(Q_T)$ [26], while (3.2)–(3.4) indicate that $\{u_\varepsilon\}$ are also bounded in $V_2^{1,0}$ (see [26]) by integration. We recall that $V_2^{1,0}(Q_T)$ can be obtained by completing the Sobolev space $W_2^{1,1}(Q_T)$ with respect to the norm ess $\sup_{t\in[0,T]} \|u(\cdot,t)\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(Q_T)}$. It follows that without loss of generality we may assume the existence of a nontrivial $u \ge 0$ such that $u_\varepsilon \to u$ strongly in $L^2(Q_T)$ and weakly in $V_2^{1,0}$.

Let $\phi\colon Q_T \to \mathbb{R}$ be a smooth function such that $\phi(x,0) = \phi(x,T)$, $\phi_t(x,0) = \phi_t(x,T)$. We recall that $\overrightarrow{b} \cdot \overrightarrow{\nu} = 1$ and integrate (3.2)–(3.4) to obtain:

$$- \int_{Q_T} u_\varepsilon \frac{\partial}{\partial t}(d\phi) + \int_{Q_T} \nabla u_\varepsilon \cdot \nabla\phi + \int_{Q_T} u_\varepsilon \nabla \cdot \left\{\overrightarrow{b}\left[h\phi - \frac{\partial}{\partial t}(c\phi)\right]\right\} \\ - \varepsilon \int_{Q_T} u_\varepsilon \frac{\partial^2}{\partial t^2}\left(\frac{\phi}{a}\right) + \int_{Q_T} \nabla u_\varepsilon \cdot \overrightarrow{b}\left[h\phi - \frac{\partial}{\partial t}(c\phi)\right] \\ = \int_{Q_T} \left[\left(M - Pu_\varepsilon - \int_\Omega Su_\varepsilon\right)u_\varepsilon + \frac{\varepsilon}{a}\right]\phi.$$

If one of the conditions of Lemma 3.3 holds, we pass to the limit as $\varepsilon \to 0$ and find the existence of a weak V_2 solution to (3.1) after noting that functions that are periodic as well as their derivatives are dense in the space of periodic functions.

We have thus obtained:

Theorem 3.4. *If one of the conditions of Lemma* 3.3 *holds, then there exists a positive weak solution of problem* (3.1).

We note some of the consequences of condition (c) of Lemma 3.3, and in particular that if $c = c(t)$, then (3.7) reduces to

$$\int_0^T \frac{1}{c}\left(\int_\Omega M\,dx\right)dt > \int_0^T \frac{1}{c}\left(\int_{\partial\Omega} h\,dx\right)dt.$$

Whence we have

Corollary 3.5. *If*

$$\int_{\Omega} M(x,t_0)\,dx > \int_{\partial\Omega} h(x,t_0)$$

for some $t_0 \in (0,T)$ and recalling that the coefficients are smooth, then there exists a positive function $c = c(t)$ such that problem (3.1) *has a positive solution with $d(x,t) = c(t)\,p(x)$ and any non-negative function p.*

Observe that the condition on d/c will always hold if $d \equiv 0$.

4. Solution of the nonlinear periodic parabolic problem

We now consider the existence of a solution of the nonlinear version of problem (3.1) given by

$$\text{(4.1)}\quad \begin{cases} d(x,t)u_t - \nabla\cdot[A(x,t,u)\nabla u] = \left(M - Pu - \displaystyle\int_{\Omega} Su\right)u & \text{in } Q_T,\\ \langle A(x,t,u)\nabla u, \overrightarrow{\nu}\rangle + c(x,t)u_t + h(x,t)u = 0 & \text{on } \partial\Omega\times(0,T),\\ u(x,0) = u(x,T) & x\in\Omega. \end{cases}$$

We recall that all data is smooth, $c, d, P > 0$, $M, S, h \geq 0$ and that A is uniformly elliptic and bounded:

$$a_0\left|\overrightarrow{\xi}\right|^2 \leq \left\langle A(x,t,u)\overrightarrow{\xi}, \overrightarrow{\xi}\right\rangle \leq A_0\left|\overrightarrow{\xi}\right|^2 \quad \text{for all suitable } x,\ t,\ u,\ \overrightarrow{\xi}$$

for some positive constants a_0, A_0. We do not require A to be symmetric, although we believe the results to be new even in this case. We explicitly observe that henceforth we assume $d > 0$.

We proceed by observing the following regularity results which will be useful in the next section. Specifically consider the linear parabolic problem:

$$\text{(4.2)}\quad \begin{cases} \mathcal{L}(w) \triangleq d(x,t)w_t - \nabla\cdot[B(x,t)\nabla w] + N(x,t)w = f(x,t) & \text{in } Q_T,\\ \langle B\nabla w, \overrightarrow{\nu}\rangle + c(x,t)w_t + h(x,t)w = 0 & \text{on } \partial\Omega\times(0,T) \end{cases}$$

with B, d, N, c, h smooth, $d, c > 0$, $h \geq 0$, $f \in L^\infty$ and $\langle B(x,t)\overrightarrow{\xi}, \overrightarrow{\xi}\rangle \geq a_0|\overrightarrow{\xi}|^2$ for all suitable x, t, $\overrightarrow{\xi}$ and a positive constant a_0. B is not necessarily symmetric. Existence, uniqueness and suitable regularity of the solution of (4.2) follow from [31] and its references and can also be obtained by adaptations of the techniques described in the book [26]. However, for the reader's convenience we list here the properties that we need. For $0 < \alpha < 1$, let $C^{\alpha,\alpha/2}(\overline{Q}_T)$ denote the Hölder space of continuous functions $u\colon \overline{Q}_T \to \mathbb{R}$ such that

$$\sup_{t\in[0,T]} \langle u(\,\cdot\,,t)\rangle_{\Omega}^{(\alpha)} + \sup_{x\in\overline{\Omega}} \langle u(x,\,\cdot\,)\rangle_{(0,T)}^{(\alpha/2)} < +\infty$$

(see (3.6) and [26]). In the sequel α will denote a generic positive constant that may change from proof to proof or even within the same proof. Let $N > 0$ be sufficiently large.

Lemma 4.1. *If the initial data w_0 is smooth, the Initial Value Problem associated with* (4.2) *has a weak solution $w \in C^{\alpha,\alpha/2}(\overline{Q}_T) \cap V_2^{1,0}(Q_T)$. If the initial data satisfies $w_0 \geq 0$ and $f \geq 0$, then $w \geq 0$.*

We then note that w is defined in Q_T for any $T > 0$ and furthermore:

Lemma 4.2. *Let w be the solution of Lemma* 4.1. *Then*

$$\|w\|_{L^\infty(\Omega\times[T/2,3T/2])} \leq K[\|w\|_{L^2(\Omega\times[T/4,7T/4])} + \|f\|_{L^\infty(Q_T)}].$$

Lemma 4.3. *There exist constants $K_0 > 0$, $\alpha > 0$ such that*

$$\|w\|_{C^{\alpha,\alpha/2}(\overline{\Omega}\times[T/2,3T/2])} \leq K_0[\|w\|_{L^2(\Omega\times(T/4,7T/4))} + \|f\|_{L^\infty(Q_T)}] \tag{4.3}$$

with K_0 independent of the coefficient N of (4.2). *If $w \geq 0$ and $f \leq 0$ then the dependence on $\|f\|_{L^\infty(Q_T)}$ may be dropped.*

Consider now the periodic problem associated with (4.2). For any $w_0 \in C^\alpha(\overline{\Omega})$ (α small) we put $\mathcal{T}$ to be the Poincaré map: $\mathcal{T}(w_0) = w(\,\cdot\,, T)$, where w is the (generalized) solution in $C^{\alpha,\alpha/2}(\overline{Q}_T) \cap V_2^{1,0}(Q_T)$ of the initial value problem. We then have:

Theorem 4.4. *Let N be large enough. Then the Poincaré map has a fixed point, i.e. problem* (4.2) *has a unique solution in $C^{\alpha,\alpha/2}(\overline{Q}_T)$. The coefficient α only depends on the estimates for $B(x,t)$, not on B itself.*

Without loss of generality, suppose $N > 0$ and let $w_0 \in C^\alpha(\overline{\Omega})$. We recall that the coefficient K_0 in (4.3) is independent of N. Assume $\|w_0\|_{C^\alpha} \leq C_0\|f\|_{L^\infty}$ for some C_0 to be chosen below. The energy inequality yields

$$\|w\|_{L^2(\Omega\times[T/4,7T/4])} \leq \frac{B(1+C_0)}{\inf N}\,\|f\|_{L^\infty}$$

for some constant B independent of w, f. Choosing N shows that $\mathcal{T}$ maps a ball in $C^\alpha(\overline{\Omega})$ to itself. It is easy to see that $\mathcal{T}$ is continuous and completely continuous. The fixed point of $\mathcal{T}$ yields the desired solution, whose uniqueness follows in the usual way by taking differences of two possible solutions.

Theorem 4.5. *If one of the following three conditions is satisfied:*

(a) *$d = d(x)$ and $M \geq 0$, $c = c(x)$ and*

$$\int_{Q_T} M > \int_0^T \int_{\partial\Omega} h;$$

(b) *$d = d(x)$ and $M \geq 0$ and the Dirichlet problem:*

$$\begin{cases} -\dfrac{A_0^2}{a_0}\Delta w - \overline{M}(x)w = \lambda_1 w & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases}$$

has least eigenvalue $\lambda_1 < 0$ *with eigenfunction* ϕ_1*, where* $\overline{M}(x) = (1/T)\int_0^T M(x,t)\,dt$ *and* $\|\phi_1\|_{L^2(\Omega)} = 1$*;*

(c) *The quotient* d/c *is a function of* x *and*

$$\int_{Q_T} \frac{M}{c} > \frac{A_0^2}{a_0}\int_{Q_T} \frac{c}{4}\left|\nabla\left(\frac{1}{c}\right)\right|^2 + \int_0^T\int_{\partial\Omega} \frac{h}{c}.$$

Then there exists a non-negative solution to problem (4.1)*.*

PROOF. We add the linear term $+Nu$ to both sides of the equation of (4.1), with $N > 0$ to be chosen later sufficiently large depending only on data, and for $v \in C^{\alpha,\alpha/2}(\overline{Q}_T)$ put $u = Z(v)$ if and only if

$$du_t - \nabla\cdot[A(x,t,J_\eta(v))\nabla u] + Nu = \left(M + N - Pv - \int_\Omega Sv\right)v^+$$

subject to the same boundary conditions of (4.1). Here J_η denotes a map: $C^{\alpha,\alpha/2}(\overline{Q}_T) \to C^\infty(\overline{Q}_T)$ such that $J_\eta(v) \to v$ in $C^{\alpha,\alpha/2}(\overline{Q}_T)$ as $\eta \to 0$. Using the previous regularity results, we view Z as a map $C^{\alpha,\alpha/2}(\overline{Q}_T) \to C^{\alpha,\alpha/2}(\overline{Q}_T)$, for some small positive α, whose fixed points are the nonnegative solutions of (4.1).

Under the assumptions of the theorem, no solution u of

$$du_t - \nabla\cdot[A(x,t,J_\eta(u))\nabla u] = \left(M - Pu - \int_\Omega Su\right)u + \varepsilon, \tag{4.4}$$

subject to the boundary conditions of (4.1) can have a small $C^{\alpha,\alpha/2}$-norm for a fixed $\varepsilon \geq 0$. In particular we show that the norm of the solutions u to (4.4) are bounded from below independently of ε small. Indeed, in case (a) we apply Lemma 2.1 with the choices $\phi \equiv 1$, $\overrightarrow{\beta} \equiv 0$, $l \equiv 0$, $g = -cu_t$ and

$$f = \left(M - Pu\int_\Omega Su\right)u + \varepsilon - du_t,$$

we integrate (2.3) on $(0,T)$, use T-periodicity, let $\eta \to 0$ and obtain

$$\int_{Q_T} M - \int_0^T\int_{\partial\Omega} h \leq \int_{Q_T}\left(Pu_\epsilon + \int_\Omega Su_\varepsilon\right) \leq (\|P\|_\infty + |\Omega|\|S\|_\infty)\|u_\varepsilon\|_{L^1}.$$

In case (b) we make the same choices as in case (a) except for $\phi = \phi_1$ and we get

$$\begin{aligned}
\|\phi_1\|_\infty^2(\|P\|_\infty + |\Omega|\|S\|_\infty)\|u_\varepsilon\|_{L^1} &\geq \int_{Q_T}\left(Pu_\varepsilon + \int_\Omega Su_\varepsilon\right)\phi_1^2 \\
&\geq -\int_{Q_T}\langle AA_s^{-1}A^\top\nabla\phi_1, \nabla\phi_1\rangle + \int_{Q_T} M\phi_1^2 \\
&\geq -\int_{Q_T}\frac{A_0^2}{a_0}|\nabla\phi_1|^2 + \int_{Q_T} M\phi_1^2 = -T\lambda_1.
\end{aligned}$$

In case (c) we write (4.4) in the following way:

$$-\nabla\cdot\left[\frac{A}{c}\nabla u\right]+\left[A^{\top}\nabla\left(\frac{1}{c}\right)\right]\cdot\nabla u=\left(M-Pu-\int_\Omega Su\right)\frac{u}{c}+\frac{\varepsilon}{c}-\frac{d}{c}u_t \tag{4.5}$$

and the boundary condition on $\partial\Omega\times(0,T)$ as

$$\left\langle\frac{A}{c}\nabla u,\overrightarrow{\nu}\right\rangle+\frac{h}{c}u=-u_t,$$

therefore we can apply Lemma 2.1 with the choices $\phi\equiv 1$, A/c in place of A, $\overrightarrow{\beta}=A^{\top}\nabla(1/c)$, $l\equiv 0$, h/c in place of h, $g=-u_t$ and f equal to the right hand side of (4.5). Recalling that now d/c does not depend on t, the usual computations with (2.3) lead to

$$\begin{aligned}\frac{\|P\|_\infty+|\Omega|\|S\|_\infty}{\min c}\|u_\varepsilon\|_{L^1}&\geq\int_{Q_T}\frac{Pu_\varepsilon+\int_\Omega Su_\varepsilon}{c}\\ &\geq\int_{Q_T}\frac{M}{c}-\int_{Q_T}\frac{c}{4}\left\langle A_{\mathrm{s}}^{-1}A^{\top}\nabla\left(\frac{1}{c}\right),A^{\top}\nabla\left(\frac{1}{c}\right)\right\rangle-\int_0^T\int_{\partial\Omega}\frac{h}{c}\\ &\geq\int_{Q_T}\frac{M}{c}-\frac{A_0^2}{a_0}\int_{Q_T}\frac{c}{4}\left|\nabla\left(\frac{1}{c}\right)\right|^2-\int_0^T\int_{\partial\Omega}\frac{h}{c}.\end{aligned}$$

In all three cases the norm $\|u_\varepsilon\|_{L^1}$ is bounded away from zero uniformly with respect to ε (and η), therefore, the same holds for the stronger norm $\|u_\varepsilon\|_{C^{\alpha,\alpha/2}(\overline{Q}_T)}$. By the continuity of the Leray-Schauder degree, we conclude that $\deg(u-Z(u),B_r,0)=0$ where B_r is the ball of radius r in $C^{\alpha,\alpha/2}(\overline{Q}_T)$ for some small $r>0$ independent of η.

In the same way, if

$$du_t-\nabla\cdot[A(x,t,J_\eta(u))\nabla u]+Nu=\lambda\left[\left(M+N-Pu-\int_\Omega Su\right)u^+\right] \tag{4.6}$$

for some λ, $0\leq\lambda\leq 1$, then we show that $\|u\|_{L^2}$ is bounded uniformly with respect to λ (and η). Indeed, we multiply both sides of equation (4.6) by u and integrate over Q_T using the boundary conditions and obtain

$$\begin{aligned}&\int_{Q_T}\left(M+N-Pu-\int_\Omega Su\right)u^2\\ &\qquad\geq a_0\int_{Q_T}|\nabla u|^2+\int_0^T\int_{\partial\Omega}hu^2+\int_{Q_T}\left(N-\frac{d_t}{2}\right)u^2-\int_0^T\int_{\partial\Omega}\frac{c_tu^2}{2}.\end{aligned}$$

Now, let the function c be extended to a smooth T-periodic function on $\overline{Q}_T$ and let $\overrightarrow{b}:\overline{\Omega}\to\mathbb{R}^n$ be a smooth vector field such that $\overrightarrow{b}\cdot\overrightarrow{\nu}=1$ on $\partial\Omega$. We can estimate the last integral in the preceding inequality as follows:

$$\begin{aligned}\int_{\partial\Omega}\frac{c_t u^2}{2} &= \int_{\partial\Omega}\frac{c_t u^2}{2}\overrightarrow{b}\cdot\overrightarrow{\nu} = \int_\Omega u^2\nabla\cdot\left(\frac{c_t}{2}\overrightarrow{b}\right)+\int_\Omega c_t u\overrightarrow{b}\cdot\nabla u\\ &\le \int_\Omega u^2\nabla\cdot\left(\frac{c_t}{2}\overrightarrow{b}\right)+\eta\int_\Omega|\nabla u|^2+\int_\Omega\frac{c_t^2|\overrightarrow{b}|^2}{4\eta}u^2\end{aligned}$$

for any $\eta > 0$. Therefore, if we choose $\eta < a_0$ and

$$N > \sup_{\overline{Q}_T}\left[\frac{d_t}{2}+\nabla\cdot\left(\frac{c_t}{2}\overrightarrow{b}\right)+\frac{c_t^2|\overrightarrow{b}|^2}{4\eta}\right],$$

we have that

$$0<\int_{Q_T}\left(M+N-Pu-\int_\Omega Su\right)u^2\le(\|M\|_\infty+N)\|u\|_{L^2}^2-\frac{\min P}{|Q_T|^{1/2}}\|u\|_{L^2}^3$$

by Hölder's inequality and, hence, $\|u\|_{L^2}$ is bounded uniformly with respect to $\lambda\in[0,1]$.

We conclude from (4.3) by periodicity that $\|u\|_{C^{\alpha,\alpha/2}(\overline{Q}_T)}$ is bounded uniformly with respect to $\lambda\in[0,1]$. It follows that $\deg(u-Z(u),B_R\setminus\overline{B}_r,0)=1$ and the existence of a nontrivial nonnegative solution to (4.1) is immediate by the properties of the Leray-Schauder degree and a limit argument as $\eta\to 0$ (we recall that the obtained bounds on u are uniform with respect to η). □

Remark 4.6. If $A(x,t,u)$ is symmetric then the constant A_0^2/a_0 in conditions (b) and (c) can be replaced by A_0.

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