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ON THE KURATOWSKI MEASURE OF NONCOMPACTNESS FOR DUALITY MAPPINGS

George Dinca

Abstract. Let $(X, \|\cdot\|)$ be an infinite dimensional real Banach space having a Fréchet differentiable norm and $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a gauge function. Denote by $J_\varphi\colon X \to X^*$ the duality mapping on X corresponding to φ.

Then, for the Kuratowski measure of noncompactness of J_φ, the following estimate holds:

$$\alpha(J_\varphi) \geq \sup\bigg\{\frac{\varphi(r)}{r} \;\bigg|\; r > 0\bigg\}.$$

In particular, for $-\Delta_p\colon W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$, $1 < p < \infty$, $1/p + 1/p' = 1$, viewed as duality mapping on $W_0^{1,p}(\Omega)$, corresponding to the gauge function $\varphi(t) = t^{p-1}$, one has

$$\alpha(-\Delta_p) = \begin{cases} 1 & \text{for } p = 2, \\ \infty & \text{for } p \in (1,2) \cup (2,\infty). \end{cases}$$

1. Introduction

The first part of this introduction deals with the definition and some fundamental properties of duality mappings, which are needed in what follows.

Being introduced by A. Beurling and A.E. Livingston [3], duality mappings on Banach spaces were intensively studied by many authors. A rather complete list of references may be found in I. Cioranescu [4]. We only note here

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the remarkable contributions of E. Asplund and F. E. Browder and the deep connection between the denseness of the range of a duality mapping and the Bishop–Phelps theorem.

Let $\varphi: \mathbb{R}_+ \to \mathbb{R}_+$ be a gauge function, that is: φ is continuous, strictly increasing, $\varphi(0) = 0$ and $\varphi(t) \to \infty$ as $t \to \infty$.

Let X be a real Banach space, X^* its dual and $\mathcal{P}(X^*)$ the set of all parts of X^*.

By definition, the duality mapping on X, corresponding to the gauge function φ, is the set valued mapping $J_\varphi: X \to \mathcal{P}(X^*)$, defined by:

$$
\begin{aligned}
&J_\varphi 0 = \{0\}, \\
&J_\varphi x = \varphi(\|x\|)\{x^* \in X^* \mid \langle x^*, x\rangle = \|x\|,\ \|x^*\| = 1\}, \quad \text{if } x \neq 0.
\end{aligned}
\tag{1.1}
$$

Due to the Hahn–Banach theorem

$$D(J_\varphi) = \{x \in X \mid J_\varphi x \neq \emptyset\} = X.$$

We also note that, according to the well-known Bishop–Phelps theorem (see R.R. Phelps [11, Theorem 3.19]), the set of all functionals x^* in X^* which attain their norms on the unit ball, that is, which satisfy

$$\langle x^*, x\rangle = \|x^*\| \quad \text{for some } x \in X \text{ with } \|x\| = 1$$

is norm dense in X^*.

Equivalently, the duality map corresponding to the identity gauge function $\varphi(t) = t$, denoted by J, has dense range:

$$
\begin{aligned}
\overline{J(X)} = X^*, \quad \text{where } J(X) &= \bigcup_{x \in X} Jx \\
&= \bigcup_{x \in X} \{u^* \in X^* \mid \langle u^*, x\rangle = \|u^*\|\|x\|, \|u^*\| = \|x\|\}.
\end{aligned}
$$

It follows from this that $\overline{J_\varphi(X)} = X^*$ for any gauge function φ. Indeed, let $x^* \in X^*$. According to Bishop–Phelps theorem, there exists a sequence (x_n^*) such that $\langle x_n^*, x_n\rangle = \|x_n^*\|$ for some x_n with $\|x_n\| = 1$ and $x_n^* \to x^*$. It suffices to prove that there exists a sequence (y_n) such that $x_n^* \in J_\varphi y_n$. It suffices to take $y_n = r_n x_n$ with $\varphi(r_n) = \|x_n^*\|$.

By definition of J_φ, it is clear that J_φ is single valued if and only if X is smooth (at any $x \neq 0$ there is an unique $x^* \in X^*$ which satisfies $\langle x^*, x\rangle = \|x\|$ and $\|x^*\| = 1$) which, in turn, is the same as saying that the norm of X is Gâteaux differentiable at every nonzero point of X (see J. Diestel [5, Theorem 1 in Chapter 2]).

Let us assume that X is smooth and denote by $(\mathrm{grad}\|\cdot\|)\colon X\setminus\{0\}\to X^*$ the map defined by

$$\langle(\mathrm{grad}\,\|\cdot\|)(x),h\rangle=\lim_{t\to 0}\frac{\|x+th\|-\|x\|}{t}\quad\text{for all } h\in X.$$

It is easily seen (M.M. Vajnberg [12, Lemma 2.5]) that this map has the following metric properties:

$$\|(\mathrm{grad}\,\|\cdot\|)(x)\|=1;\quad \langle(\mathrm{grad}\,\|\cdot\|)(x),x\rangle=\|x\| \tag{1.2}$$

for any nonzero x, and

$$(\mathrm{grad}\,\|\cdot\|)(\alpha x)=\mathrm{sign}\alpha(\mathrm{grad}\,\|\cdot\|)(x)$$

for any nonzero x and $\alpha\neq 0$.

By comparing (1.1) and (1.2) we deduce that, on a smooth Banach space, any duality mapping J_φ is a single-valued map from X into X^* defined as follows

$$\begin{aligned} J_\varphi 0 &= 0\\ J_\varphi x &= \varphi(\|x\|)(\mathrm{grad}\,\|\cdot\|)(x)\quad \text{if } x\neq 0. \end{aligned}\tag{1.3}$$

The second part of this introduction deals with the Kuratowski measure of noncompactness for continuous and bounded operators in Banach spaces.

Recall that, given any bounded subset B of a Banach space X, the Kuratowski measure of noncompactness of B, $\alpha(B)$, is defined as the infimum of those $\varepsilon>0$ such that B can be covered with a finite number of subsets of B having diameter less or equal to ε. For the properties of α see M. Furi, M. Martelli and A. Vignoli [8], J.M. Ayerbe Toledano, T. Dominguez Benavides and G. López Acedo [2]. We only recall here the following two properties which are needed for the purpose of this paper: for any bounded set $B\subset X$, $\alpha(B)=0$ if and only if $\overline{B}$ is compact and $\alpha(B)=\alpha(\overline{B})$. We also recall the Nussbaum's nice result concerning the Kuratowski measure of noncompactness of the sphere in an infinite dimensional Banach space X: for any $r>0$, denote by $S_{X,r}=\{x\in X\mid \|x\|=r\}$. Then $\alpha(S_{X,r})=2r$. For the proof, cf. R.D. Nussbaum [10] (see also M. Furi and A. Vignoli [7]).

Let X and Y be Banach spaces and $F\colon X\to Y$ be a continuous and bounded operator. By definition, the Kuratowski measure of noncompactness of F is

$$\alpha(F)=\inf\{k\geq 0\mid \alpha[F(B)]\leq k\cdot\alpha(B),\ B\subset X \text{ bounded}\}.$$

If $\dim X=\infty$, $\alpha(F)$ may be equivalently defined as

$$\alpha(F)=\sup\left\{\frac{\alpha[F(B)]}{\alpha(B)}\ \middle|\ B\subset X\text{ bounded}, \alpha(B)>0\right\}. \tag{1.4}$$

By definition of $\alpha(F)$, it easily follows that $\alpha(F)=0$ if and only if F is compact. For many other properties of $\alpha(F)$, cf. M. Furi, M. Martelli and

A. Vignoli [8], J.M. Ayerbe Toledano, T. Dominguez Benavides and G. López Acedo [2].

2. Statement and proof of the main result

We can now prove the main theorem of this paper.

Theorem 2.1. *Assume that $(X, \|\cdot\|)$ is a Banach space with Fréchet differentiable norm. Then one has:*

(a) *Any duality mapping $J_\varphi : X \to X^*$ is norm-to-norm continuous and bounded;*
(b) $\alpha(J_\varphi) = 0$ *if and only if* $\dim X < \infty$*;*
(c)

$$\alpha(J_\varphi) \geq \sup \left\{ \frac{\varphi(r)}{r} \;\middle|\; r > 0 \right\} \quad \text{if } \dim X = \infty. \tag{2.1}$$

For the proof we need the following lemma:

Lemma 2.2. *Let $(X, \|\cdot\|)$ be a real Banach space with Gâteaux differentiable norm. Then one has:*

(a) *For any gauge function φ and any $r > 0$, $J_\varphi(S_{X,r})$ is dense in $S_{X^*,\varphi(r)}$;*
(b) *$J_\varphi(X)$ is dense in X^*.*

Proof. (a) Clearly, J_φ is defined by (1.3) and acts from $S_{X,r}$ into $S_{X^*,\varphi(r)}$.

Let $x^* \in S_{X^*,\varphi(r)}$. According to Bishop–Phelps theorem, there is a sequence $(x_n) \subset X$ such that

$$Jx_n = \|x_n\|(\operatorname{grad} \|\cdot\|)(x_n) \to x^*. \tag{2.2}$$

We deduce from this that $\|x_n\| \to \|x^*\| = \varphi(r)$ such that from (2.2) we infer that

$$\varphi(r)\operatorname{grad} \|\cdot\|(x_n) \to x^*. \tag{2.3}$$

Setting $y_n = rx_n/\|x_n\|$, (2.3) reads as $\varphi(\|y_n\|)(\operatorname{grad} \|\cdot\|)(y_n) \to x^*$, that is $J_\varphi y_n \to x^*$ with $y_n \in S_{X,r}$.

(b) Clearly, if $x^* = 0_{X^*}$, $J_\varphi x_n \to 0_{X^*}$ for any $(x_n) \subset X$ with $x_n \to 0_X$.

Let $x^* \in X^* \setminus \{0\}$ and $r > 0$ be such that $\|x^*\| = \varphi(r)$. According to the preceding point in lemma, there is a sequence $(y_n) \subset S_{X,r}$ such that $J_\varphi y_n \to x^*$.□

Proof of Theorem 2.1. (a) Since the norm of X is Fréchet differentiable, J_φ is single valued and defined by (1.3).

Accordingly, if $\|x\| \leq C$ then $\|J_\varphi x\| \leq \varphi(C)$, thus J_φ is bounded. (Notice that the boundedness is a property that any duality mapping possesses, even in case that J_φ would be a multivalued map. This easily follows by the definition of J_φ).

On the other hand, the Fréchet differentiability of the norm of X implies that the map

$$x \in X \setminus \{0\} \mapsto (\mathrm{grad}\,\|\cdot\|)(x) \in S_{X^*,1}$$

is norm-to-norm continuous (see M.I. Kadeč [9, Lemma 2]) and then by (1.3) again, the continuity of J_φ follows.

(b) Assume that $\alpha(J_\varphi) = 0$. Then J_φ is compact. Accordingly, $\overline{J_\varphi(S_{X,r})}$ is compact. But, according to Lemma 2.2, $\overline{J_\varphi(S_{X,r})} = S_{X^*,\varphi(r)}$ and the compactness of $S_{X^*,\varphi(r)}$ implies that X is finite dimensional.

Conversely, if $\dim X < \infty$, any continuous and bounded operator from X into X^* is compact. In particular, $J_\varphi : X \to X^*$ is compact thus, $\alpha(J_\varphi) = 0$.

(c) Since $\dim X = \infty$, $\alpha(J_\varphi)$ is given by (see (1.4))

$$\alpha(J_\varphi) = \sup\left\{ \frac{\alpha[J_\varphi(B)]}{\alpha(B)} \;\middle|\; B \subset X \text{ bounded}, \ \alpha(B) > 0 \right\}.$$

Take $B = S_{X,r}$, for any $r > 0$. The above quoted properties of α, Lemma 2.2 and Nussbaum's result allow us to write

$$\frac{\alpha[J_\varphi(S_{X,r})]}{\alpha(S_{X,r})} = \frac{\alpha[\overline{J_\varphi(S_{X,r})}]}{\alpha(S_{X,r})} = \frac{\alpha(S_{X^*,\varphi(r)})}{\alpha(S_{X,r})} = \frac{\varphi(r)}{r}.$$

Consequently

$$\left\{ \frac{\varphi(r)}{r} \;\middle|\; r > 0 \right\} \subset \left\{ \frac{\alpha[J_\varphi(B)]}{\alpha(B)} \;\middle|\; B \subset X \text{ bounded}, \ \alpha(B) > 0 \right\}$$

and, from this, estimate (2.1) follows. □

3. Application: the Kuratowski measure of noncompactness for p-Laplacian

In what follows, Ω designates a bounded domain in $\mathbb{R}^N$, $N \geq 2$, and p is a real number such that $1 < p < \infty$. We shall denote by $(W^{1,p}(\Omega), \|\cdot\|)$ the classical Sobolev space

$$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega) \;\middle|\; \frac{\partial u}{\partial x_i} \in L^p(\Omega) \text{ for } i = 1, \dots, N, \right. \\ \left. \text{where } \frac{\partial u}{\partial x_i} \text{ is the distributional derivative} \right\},$$

endowed with the norm

$$\|u\|^p = \|u\|^p_{L^p(\Omega)} + \sum_{i=1}^N \left\| \frac{\partial u}{\partial x_i} \right\|^p_{L^p(\Omega)}, \tag{3.1}$$

and by $W_0^{1,p}(\Omega)$, the closure of $\mathcal{C}_0^\infty(\Omega)$ in the space $(W^{1,p}(\Omega), \|\cdot\|)$.

Due to Poincaré's inequality

$$\|u\|_{L^p(\Omega)} \le C\||\nabla u|\|_{L^p(\Omega)} \quad \text{for all } u \in W_0^{1,p}(\Omega),$$

the norm (3.1) is equivalent on $W_0^{1,p}(\Omega)$ to the norm

$$\|u\|_{1,p} = \||\nabla u|\|_{L^p(\Omega)}, \tag{3.2}$$

where $|\nabla u|$ stands for the Euclidean norm of ∇u.

Starting from now, $W_0^{1,p}(\Omega)$ will be always considered as endowed with the norm (3.2).

The space $(W_0^{1,p}(\Omega), \|\cdot\|_{1,p})$ is an infinite dimensional, separable, reflexive and uniformly convex Banach space (see R.A. Adams [1]), with Fréchet differentiable norm (see, for example [6]).

Consider the operator (also known as the *minus p-Laplacian*)

$$-\Delta_p\colon (W_0^{1,p}(\Omega), \|\cdot\|_{1,p}) \to (W_0^{1,p}(\Omega), \|\cdot\|_{1,p})^*,$$

defined by

$$-\Delta_p u = -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = -\frac{\partial}{\partial x_i}\left(|\nabla u|^{p-2}\frac{\partial u}{\partial x_i}\right)$$

for all $u \in W_0^{1,p}(\Omega)$ or, equivalently,

$$\langle -\Delta_p u, v\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v\, dx \tag{3.3}$$

for all $u \in W_0^{1,p}(\Omega)$.

Defined in this manner, $-\Delta_p$ is nothing else but the duality mapping on $W_0^{1,p}(\Omega)$ corresponding to the gauge function $\varphi(t) = t^{p-1}$.

According to (2.1), one has

$$\alpha(-\Delta_p) \ge \sup\{r^{p-2} \mid r > 0\}. \tag{3.4}$$

It follows that, for $p \in (1,2) \cup (2,\infty)$, $\alpha(-\Delta_p) = \infty$.

For $p = 2$, it follows from (3.3) that

$$-\Delta_2\colon W_0^{1,2}(\Omega) = H_0^1(\Omega) \to (H_0^1(\Omega))^* = H^{-1}(\Omega)$$

is defined by

$$-\Delta_2 u = -\frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial x_i}\right) \quad \text{for any } u \in H_0^1(\Omega)$$

or, equivalently

$$\langle -\Delta_2 u, v\rangle_{H_0^1(\Omega), H^{-1}(\Omega)} = (u,v)_{H_0^1(\Omega)}, \quad \text{for all } u, v \in H_0^1(\Omega),$$

where $\langle \cdot\,,\cdot\rangle_{H_0^1(\Omega),H^{-1}(\Omega)}$ stands for the duality pairing between $H_0^1(\Omega)$ and its dual $H^{-1}(\Omega)$ and $(\cdot\,,\cdot)_{H_0^1(\Omega)}$ designates the inner product on $H_0^1(\Omega)$.

In other words, $-\Delta_2 = -\Delta$ viewed as the canonical isomorphism between $H_0^1(\Omega)$ and its dual $H^{-1}(\Omega)$ given by Riesz theorem. Consequently, we have (see M. Furi, M. Martelli and A. Vignoli [8]) $\alpha(-\Delta_2) = \alpha(-\Delta) \leq \| - \Delta\| = 1$. On the other hand, the estimation (3.4) gives us $\alpha(-\Delta_2) \geq 1$. Thus $\alpha(-\Delta_2) = 1$ and the proof is complete.

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GEORGE DINCA
University of Bucharest
Faculty of Mathematics and Computer Science
14 Academiei Street
010014 Bucharest, ROMANIA

E-mail address: dinca@fmi.unibuc.ro, dinca2003@yahoo.fr