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GLOBAL EXISTENCE OF SOLUTIONS TO THE NONLINEAR THERMOVISCOELASTICITY SYSTEM WITH SMALL DATA

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Abstract. We consider the nonlinear system of partial differential equations describing the thermoviscoelastic medium ocupied a bounded domain $\Omega \subset \mathbb{R}^3$. We proved the global existence (in time) of solution for the nonlinear thermoviscoelasticity system for the initial-boundary value problem with the Dirichlet boundary conditions for the displacement vector and the heat flux at the boundary. In the proof we assume some growth conditions on nonlinearity and some smallness conditions on data in some norms.

1. Introduction

We consider the following thermoviscoelasticity system

$$u_{tt} = \operatorname{div}\sigma \quad \text{in } \Omega^T = \Omega \times (0,T), \tag{1.1}$$

$$c_v\theta_t - \varkappa\Delta\theta = \theta\frac{\partial F_1}{\partial\varepsilon}\cdot\varepsilon_t + \mathbb{A}\varepsilon_t\cdot\varepsilon_t, \tag{1.2}$$

where $u = u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t)) \in \mathbb{R}^3$ is the displacement vector, $x = (x_1, x_2, x_3)$ the Cartesian system of coordinates in $\mathbb{R}^3$, $t \in \mathbb{R}_+ \cup \{0\}$, and $\mathbb{A} = \{A_{ijkl}\}_{i,j,k,l=1,2,3}$ is the fourth order tensor such that

$$\varepsilon \to \mathbb{A}\varepsilon = \nu tr\varepsilon I + 2\mu\varepsilon = \{\nu tr\varepsilon\delta_{ij} + 2\mu\varepsilon_{ij}\}_{i,j=1,2,3}$$

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and ν, μ are the constant Lame coefficients with values within the elasticity range $\mu > 0$, $3\nu + 2\mu > 0$. Then

$$\mathbb{A}\varepsilon_t \cdot \varepsilon_t = 3\nu(tr\varepsilon_t)^2 + 2\mu\varepsilon_t \cdot \varepsilon_t,$$

where $tr\varepsilon = \varepsilon_{i,i}$ with the summation convention. Next $\sigma = \{\sigma_{ij}\}_{i,j=1,2,3}$ is the stress tensor of the form

$$\sigma_{ij} = \frac{\partial\psi}{\partial\varepsilon_{ij}} + \mathbb{A}\varepsilon_{ij,t} = \theta\frac{\partial F_1}{\partial\varepsilon_{ij}} + \frac{\partial F_2}{\partial\varepsilon_{ij}} + \mathbb{A}\varepsilon_{ij,t}, \tag{1.3}$$

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,x_j} + u_{j,x_i}), \quad \frac{\partial F_k}{\partial\varepsilon} \cdot \varepsilon_t = \frac{\partial F_k}{\partial\varepsilon_{ij}}\varepsilon_{ij,t}, \quad \varepsilon_t \cdot \varepsilon_t = \varepsilon_{ij,t} \cdot \varepsilon_{ij,t}, \quad k = 1, 2,$$

where the summation convention over the repeated indices is assumed.

$\varepsilon = \varepsilon(u)$ is the linearized strain tensor, function $F_k = F_k(\varepsilon)$, $k = 1, 2$, will be specified later.

Using (1.3) in (1.1) we obtain the equation

$$u_{tt} - \operatorname{div}(A\varepsilon_t) = \operatorname{div}(\theta F_{1,\varepsilon} + F_{2,\varepsilon}) \tag{1.4}$$

which can be written in the more explicit form

$$u_{tt} - (\mu\Delta u_t + \nu\nabla\operatorname{div} u_t) = \nabla\theta F_{1,\varepsilon} + \theta F_{1,\varepsilon\varepsilon}\nabla\varepsilon + F_{2,\varepsilon\varepsilon}\nabla\varepsilon, \tag{1.5}$$

where

$$(\nabla\theta F_{1,\varepsilon})_i = \theta_{x_j} F_{1,\varepsilon_{ij}}, \quad (F_{s,\varepsilon\varepsilon}\nabla\varepsilon)_i = F_{s,\varepsilon_{ij}\varepsilon_{kl}}\nabla_j\varepsilon_{kl},$$

for $s = 1, 2$ and we add the initial conditions to the system (1.1), (1.2)

$$u|_{t=0} = u_0, \quad u_t|_{t=0} = u_1, \quad \theta|_{t=0} = \theta_0, \tag{1.6}$$

and the boundary conditions

$$u|_S = 0, \quad \overline{n} \cdot \nabla\theta|_S = 0, \tag{1.7}$$

where $S = \partial\Omega$, Ω is bounded domain in $\mathbb{R}^3$.

Before starting the proof we recall the related results in the literature: C.M. Dafermos (cf. [6]), C.M. Dafermos and L. Hsiao (cf. [7]) and S. Jang (cf. [14]) proved the global existence of a classical solution to the system (1.1), (1.2) in the one dimensional case with a stress free boundary conditions at least at one end of the rod. The asymptotic behaviour of smooth solutions as time tends to infinity has been investigated in T. Luo's thesis (cf. [16]) for a special class of solidlike materials in which $e = c_v\theta$, where e is the internal energy, and $F_2 = 0$. R. Racke and S. Zheng (cf. [21]) investigated the global existence, uniqueness and asymptotic behaviour of weak solutions for the model in shape memory alloys also with a stress-free boundary condition at least at one end of the rod. In all of these papers, a variant of Andrew's technique used by C.M. Dafermos (cf. [6])

and R.L. Pego (cf. [20]) was crucial to obtain an uniform a priori estimate on L_∞ norm of u.

However, this technique does not apply to the case where both ends of the rod are damped. Thus the problem of global existence and uniqueness of classical solutions for the case in which both ends of the rod are changed remained open for about 10 years until the paper by Z. Chen and K.H. Hoffman (cf. [5]). We should mention that in the paper the estimates for solutions depend crucially on at least H^1-norm of the initial data of u.

As a result their techniques is not applicable to the problem with $u_0 \in L_\infty$. So, Z. Chen and K.H. Hoffman (cf. [5]) applied new techniques and more delicate estimates to obtain the global existence and uniqueness of the solution of (1.1)–(1.2) in the one-dimensional space with boundary conditions $\theta_x = 0$ for $x = 0, 1$ and $u = 0$ for $x = 0, 1$ and initial conditions $u|_{t=0} = u_0$, $v|_{t=0} = v_0$, $\theta|_{t=0} = \theta_0$ with assumptions $u_0 \in L_\infty$, $u_0 \in H^1$, $\theta_0 \in H^1$, with $\theta > 0$ for $x \in [0, 1]$.

Concerning the model in shape memory alloys, we refer to M. Niezgódka and J. Sprekels [18] for the local existence of solutions in the weak sense. After that paper uniqueness and global existence where proved by K.H. Hoffman and S. Zheng (cf. [12]) and by M. Niezgódka at al. (cf. [19]). We also refer to M. Sprekels, S. Zheng and P. Zhu (cf. [25]), and K.H. Hoffmann and A. Żochowski (cf. [13]) for the global existence results for the model with shape memory alloys with the Helmholtz free energy density as the potential of Landau–Ginzburg form. Sprekels et al. (cf. [25]) obtained results on asymptotic behaviour of solutions for the Landau–Ginzburg model in shape memory alloys.

We should also mention G. Andrews (cf. [1]), G. Andrews and J.M. Ball (cf. [2]) and R.L. Pego (cf. [20]) for the purely viscoelastic case.

It should be emphasized that in all those papers the considered system of equations (1.1)–(1.2) is *one-dimensional*.

Among the paper devoted to nonlinear thermoviscoelasticity in the three-dimensional space we mention some of them below.

The global in time existence of small solution of non-linear thermoviscoelastic equation was proved by Y. Shibata (cf. [23]) under special assumptions about nonlinearity. J.A. Gawinecki proved (cf. [10]) the global existence of solutions for *non-small data* to non-linear spherically symmetric thermoviscoelasticity in the three-dimensional space. In the proof he used the method of Sobolev spaces, method of successive approximations and new techniques implying approximate estimates enough to deduce global existence.

The aim of our paper is to prove global existence of solution to the thermoviscoelasticity system (1.1)–(1.2) with small data in the fixed domain $\Omega \subset \mathbb{R}^3$ by the method of succesive approximations and the method of energy estimates which give us a possibility to extend the solution to the interval $[0,+\infty]$. Since equations (1.2) and (1.4) are parabolic we use the theory of parabolic initial-boundary value problems developed in the anisotropic Sobolev spaces with a mixed norm (see [8], [15], [24]). The spaces with a mixed norm imply more possibilities in deriving necessary estimates guaranteeing existence of solutions to the considered problem. The time traces of elements of such spaces belong to some Besov spaces (see Lemma 1.3). Below, we describe more precisely our result.

We are interested to prove the existence of global solutions for small displacements and variation of temperature. Therefore we introduce some equilibrium temperature $\theta_e = \text{const.}$ and examine the variations

$$\widetilde{\theta} = \theta - \theta_e. \tag{1.8}$$

Then $\widetilde{\theta}$ is a solution to the problem

$$c_v\widetilde{\theta}_t - \varkappa\Delta\widetilde{\theta} = \theta F_{1,\varepsilon}\cdot\varepsilon_t + \mathbb{A}\varepsilon_t\cdot\varepsilon_t \tag{1.9}$$

and

$$\widetilde{\theta}|_{t=0} = \widetilde{\theta}_0, \qquad \overline{n}\cdot\nabla\widetilde{\theta}|_S = 0. \tag{1.10}$$

For solutions to the above problem we have the conservation of energy

$$\frac{d}{dt}\int_\Omega [u_t^2 + c_v\widetilde{\theta} + F_2(\varepsilon)]\,dx = 0. \tag{1.11}$$

Integrating with respect to time yields

$$\int_\Omega \Big[u_t^2 + c_v\widetilde{\theta} + F_2(\varepsilon)\Big]\,dx = \int_\Omega \Big[u_1^2 + c_v\widetilde{\theta}_0 + F_2(\varepsilon_0)\Big]\,dx \equiv a_1, \tag{1.12}$$

where $\varepsilon_0 = \varepsilon(u_0)$.

We assume the following growth conditions:

$$c_1|\varepsilon|^\sigma \le F_2(\varepsilon) \le c_2|\varepsilon|^\sigma, \qquad F_1(\varepsilon) \le c_3|\varepsilon|^{\sigma_0}, \tag{1.13}$$

where $\sigma > 1$, $\sigma_0 \ge 1$, c_i, $i = 1,2,3$, are some positive constants.

Now we formulate the main results.

Theorem A (local existence). *Assume that $u_0, u_1 \in B^{2-2/p_0}_{p,p_0}(\Omega)$, $\widetilde{\theta} = \theta - \theta_e$ in $B^{2-2/q_0}_{q,q_0}(\Omega)$, $u_0 \in W^2_p(\Omega)$, where θ_e is a constant which should correspond to some equilibrium temperature and*

$$\frac{3}{p} + \frac{2}{p_0} < 1, \quad \frac{3}{q} + \frac{2}{q_0} < 1.$$

Let

$$D = \|u_0\|_{B^{2-2/p_0}_{p,p_0}(\Omega)} + \|u_1\|_{B^{2-2/p_0}_{p,p_0}(\Omega)} + \|\widetilde{\theta}_0\|_{B^{2-2/q_0}_{q,q_0}(\Omega)} + \|u_0\|_{W^2_p(\Omega)}.$$

Then there exist a constant c_1 *and* T *sufficiently small* (*see* (2.3)) *that there exists a solution to problem* (1.1)–(1.5) *such that* $u, u_t \in W^{2,1}_{p,p_0}(\Omega^T)$, $\widetilde{\theta} \in W^{2,1}_{q,q_0}(\Omega^T)$ *and*

$$X(0,T) \equiv \|u\|_{W^{2,1}_{p,p_0}(\Omega^T)} + \|u_t\|_{W^{2,1}_{p,p_0}(\Omega^T)} + \|\widetilde{\theta}\|_{W^{2,1}_{q,q_0}(\Omega^T)} \le 2c_1 D \equiv A.$$

Theorem B (global existence). *Let the assumptions of Theorem* A *hold. Let* D *be sufficiently small. Let* $|F_i(\varepsilon)| \sim |\varepsilon|^{2+\delta}$, $\delta > 0$. *Let* $F_{i,\varepsilon\varepsilon}$, $i = 1,2$, *be Lipschitz continuous. Then there exists* T *sufficiently large such that*

$$\|u(kT)\|_{W^2_p(\Omega)} + \|u(kT)\|_{B^{2-2/p_0}_{p,p_0}(\Omega)} + \|u_t(kT)\|_{B^{2-2/p_0}_{p,p_0}(\Omega)} + \|\widetilde{\theta}(kT)\|_{B^{2-2/q_0}_{q,q_0}(\Omega)} \le D$$

for any $k \in \mathbb{N}$ *where* $v(kT) = v|_{t=kT}$. *Moreover, there exists a global solution to problem* (1.1)–(1.5) *such that*

$$u, u_t \in W^{2,1}_{p,p_0}(\Omega \times (kT,(k+1)T)), \qquad \widetilde{\theta} \in W^{2,1}_{q,q_0}(\Omega \times (kT,(k+1)T))$$

for any $k \in \mathbb{N}_0 \equiv \mathbb{N} \cup \{0\}$ *and, for any* $k \in \mathbb{N}_0$,

$$X(kT,(k+1)T) \equiv \|u\|_{W^{2,1}_{p,p_0}(\Omega\times(kT,(k+1)T))} + \|u_t\|_{W^{2,1}_{p,p_0}(\Omega\times(kT,(k+1)T))} + \|\widetilde{\theta}\|_{W^{2,1}_{q,q_0}(\Omega\times(kT,(k+1)T))} \le A.$$

In this paper we prove the existence of global regular solutions to problem (1.1)–(1.3), (1.6), (1.7). We prove Theorem B under strong nonlinearity of the stress tensor with respect to strain. We need that

$$F_i(\varepsilon) \sim |\varepsilon|^{2+\delta}, \quad \delta > 0,\ i = 1,2. \tag{1.14}$$

Then we show the existence of solutions with small displacement and small variation of temperature near some given constant equilibrium temperature. The proof is divided into two steps. First we prove the existence of local solutions by the method of successive approximations. In the second step we assume that D (see Theorem A) is sufficiently small. Then by D small we can choose that T (the time of local existence) is sufficiently large (see (3.2)). For D small, T large and thanks to (1.14) we are able to prolong the local solution step by step in time up to infinity (see Theorem B).

The proof of global existence of solutions in the case $|F_1(\varepsilon)| \le c|\varepsilon|^\sigma$, $\sigma \le 2$ must be performed in a different way because some decay type estimates must be proved. This will be a topic of the next paper. Moreover, the case of nonvanishing stress tensor with vanishing strain needs also another approach.

Finally, we introduce some notation and recall some auxiliary problems.

Definition 1.1 (see [3]). By $W^{2k,k}_{p,p_0}(\Omega^T)$, $k\in\mathbb{N}\cup\{0\}$, $p,p_0\in[1,\infty]$, we denote a closure of $C^\infty(\Omega^T)$ functions in the norm

$$\|u\|_{W^{2k,k}_{p,p_0}(\Omega^T)}=\sum_{|\alpha|+2a\le 2k}\bigg[\int_0^T\bigg(\int_\Omega|D^\alpha_x\partial^a_t u|^p\,dx\bigg)^{p_0/p}dt\bigg]^{1/p_0}.$$

Definition 1.2 (see [3]). By Besov space $B^\lambda_{p,p_0}(\Omega)$, $\lambda\in\mathbb{R}_+$, $p,p_0\in[1,\infty]$, we denote a set of functions with the finite norm

$$\|u\|_{B^\lambda_{p,p_0}(\Omega)}=\|u\|_{L_p(\Omega)}+\bigg(\sum_{i=1}^n\int_{\mathbb{R}_+}\frac{\|\Delta^m_i(h)\partial^l_{x_i}u\|^{p_0}_{L_p(\Omega)}}{h^{1+(\lambda-l)p_0}}dh\bigg)^{1/p_0},$$

where $m>\lambda-l>0$, $l\le[\lambda]$, $m,l\in\mathbb{N}\cup\{0\}$, $\Delta^k_i(h)u(x)$ is a finite difference of the function $u=u(x)$ of the order k with respect to x_i:

$$\begin{aligned}\Delta^1_i(h)u&=\Delta_i(h)u=u(x_1,\dots,x_i+h,x_{i+1},\dots,x_n)-u(x_1,\dots,x_n),\\ \Delta^k_i(h)u&=\Delta_i(h)\Delta^{k-1}_i(h)u,\quad k\in\mathbb{N},\end{aligned}$$

and $x+h\in\Omega$.

The norms of Besov space $B^\lambda_{p,p_0}(\Omega)$ are equivalent for all $m,l\in\mathbb{N}\cup\{0\}$ satisfying $m>\lambda-l>0$.

We need

Lemma 1.3 (see [4], [17], [24]). *Let $u\in W^{k,k/2}_{p,p_0}(\Omega\times\mathbb{R}_+)$, $p,p_0\in(1,\infty)$. Then $u(x,t_0)=u(x,t)|_{t=0}\in B^{k-2/p_0}_{p,p_0}(\Omega)$ and*

$$\|u(\,\cdot\,,t_0)\|_{B^{k-2/p_0}_{p,p_0}(\Omega)}\le c\|u\|_{W^{k,k/2}_{p,p_0}(\Omega\times\mathbb{R}_+)},\tag{1.15}$$

where c does not depend on u. Moreover, for a given $v\in B^{k-2/p_0}_{p,p_0}(\Omega)$ there exists a function $\widetilde v\in W^{k,k/2}_{p,p_0}(\Omega\times\mathbb{R}_+)$ such that $\widetilde v(x,t)|_{t=t_0}=v(x)$ and

$$\|\widetilde v\|_{W^{k,k/2}_{p,p_0}(\Omega\times\mathbb{R}_+)}\le c\|v\|_{B^{k-2/p_0}_{p,p_0}(\Omega)},\tag{1.16}$$

where c does not depend on v.

Let us consider the problem

$$\begin{aligned}u_t-Qu&=f&&\text{in }\Omega^T,\\ u&=0&&\text{on }S^T,\\ u|_{t=0}&=u_0&&\text{in }\Omega,\end{aligned}\tag{1.17}$$

where $S=\partial\Omega$ and $Q=\mu\Delta u+\nu\nabla\operatorname{div}u$ where $\mu>0$, and $2\mu+3\nu>0$ are numbers.

LEMMA 1.4 (see [8], [15]). *Assume that* $S \in C^2$, $f \in L_{p,p_0}(\Omega^T)$, $u_0 \in B^{2-2/p_0}_{p,p_0}(\Omega)$, $p, p_0 \in (1,\infty)$. *Then there exists a solution to problem* (1.17) *such that* $u \in W^{2,1}_{p,p_0}(\Omega^T)$ *and*

$$\|u\|_{W^{2,1}_{p,p_0}(\Omega^T)} \le c\Big(\|f\|_{L_{p,p_0}(\Omega^T)} + \|u_0\|_{B^{2-2/p_0}_{p,p_0}(\Omega)}\Big), \tag{1.18}$$

where c does not depend on u, f, u_0.

The constant c in (1.18) does not depend on T. For T large we use the results of V.A. Solonnikov [24] and N.V. Krylov [15] and obtain estimate (1.18) with a constant $c = c(T)$ increasing with T. But for solutions to problem (1.17) we have the energy type estimate

$$\|u(t)\|^2_{L_2(\Omega)} + \int_0^t \|u(t')\|^2_{H^1(\Omega)}\,dt' \le c\Big(\|f\|^2_{L_2(\Omega^T)} + \|u_0\|^2_{L_2(\Omega)}\Big),$$

where $t \le T$ and with the constant c independent of T. Then applying the W. von Wahl [26, Chapter 3, Theorem 3.1.1] technique we show that c in (1.15) is independent of T for $T \ge 1$.

For $T = T_0$ small we extend f by zero for $t > T_0$ up to $t \le T_*$. Then we obtain (1.15) for $T = T_*$. Restricting (1.18) to interval $(0, T_0)$ we obtain (1.18) with c independent of T.

Now we recall necessary for us theorems of imbedding for anisotropic Sobolev with a mixed norm and Besov spaces.

LEMMA 1.5 (see [3, Chapter 3, Section 10]). *Assume that* $u \in W^{2,1}_{p,p_0}(\Omega^T)$, $\Omega \subset \mathbb{R}^3$ *satisfies the cone condition,* $p, p_0, q, q_0 \in [1,\infty]$,

$$\varkappa = \left(|\alpha| + \frac{3}{p} + \frac{2}{p_0} - \frac{3}{q} - \frac{2}{q_0}\right)\frac{1}{2} \le 1$$

and for $\varkappa = 1$ *either* $1 < p = q < \infty$ *or* $1 < p_0 = q_0 < \infty$, *and either* $1 < p < q < \infty$ *or* $1 < p_0 < q_0 < \infty$, *and either* $1 = p < q = \infty$ *or* $1 = p_0 < q_0 = \infty$. *Then* $D^\alpha_x u \in L_{q,q_0}(\Omega^T)$ *and there exist numbers* h_0 *and* c *such that*

$$\|D^\alpha_x u\|_{L_{q,q_0}(\Omega^T)} \le ch^{1-\varkappa}\Big(\|\partial^2_x u\|_{L_{p,p_0}(\Omega^T)} + \|\partial_t u\|_{L_{p,p_0}(\Omega^T)}\Big) + ch^{-\varkappa}\|u\|_{L_{p,p_0}(\Omega^T)}, \tag{1.19}$$

where $h \in (0, h_0)$ *and* $D^\alpha_x = \partial^{\alpha_1}_{x_1}\partial^{\alpha_2}_{x_2}\partial^{\alpha_3}_{x_3}$, $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ *is a multiindex,* $\alpha_i \in \mathbb{N} \cup \{0\}$, $i = 1, 2, 3$, $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$.

LEMMA 1.6 (see [3, Chapter 4, Section 18]). *Let* $u \in B^\sigma_{p,\theta}(\Omega)$, $\sigma \in \mathbb{R}_+$, $p, \theta \in [1,\infty]$, $\Omega \subset \mathbb{R}^3$. *Let* $3/p - 3/q + |\alpha| \le \sigma$ *and let either* $\theta = 1$, $1 \le p \le q \le \infty$ *or* $1 \le \theta \le q$, $1 \le p < q < \infty$. *Then* $D^\alpha_x u \in L_q(\Omega)$ *and there exists a constant c independent of u such that*

$$\|D^\alpha_x u\|_{L_q(\Omega)} \le c\|u\|_{B^\sigma_{p,\theta}(\Omega)}. \tag{1.20}$$

The constant c in the first part of Lemma 1.3 might depend on T if $u \in W^{2,1}_{p,p_0}(\Omega^T)$. In this case (1.15) takes the form

$$\sup_{t\in(0,T)} \|u(t)\|_{B^{2-2/p_0}_{p,p_0}(\Omega)} \le c\|u\|_{W^{2,1}_{p,p_0}(\Omega^T)}. \tag{1.21}$$

Passing with $T \to 0$ in (1.21) we see that the constant c should blow up.

To omit the difficulty we assume that u is a solution to the problem

$$\begin{aligned} u_t - \Delta u &= f && \text{in } \Omega^T,\\ u|_S &= 0 && \text{on } S^T,\\ u|_{t=0} &= u_0 && \text{in } \Omega. \end{aligned} \tag{1.22}$$

Problem (1.22) is a model problem so it could replace any parabolic initial-boundary value problem appearing in this paper.

Lemma 1.7. *Assume that* $f \in L_{p,p_0}(\Omega^T)$, $u_0 \in B^{2-2/p_0}_{p,p_0}(\Omega)$, $p, p_0 \in (1,\infty)$. *Then for solutions to problem* (1.22) *the following inequality holds:*

$$\sup_{t\in(0,T]} \|u(t)\|_{B^{2-2/p_0}_{p,p_0}(\Omega)} \le c\Big(\|u\|_{W^{2,1}_{p,p_0}(\Omega^T)} + \|u_0\|_{B^{2-2/p_0}_{p,p_0}(\Omega)}\Big) \tag{1.23}$$

where the constant c does not depend on T.

Proof. Extending f by zero for $t > T$ we can consider problem (1.22) in $\Omega \times \mathbb{R}_+$. Applying Lemma 1.4 to problem (1.22) we get

$$\begin{aligned} \|u\|_{W^{2,1}_{p,p_0}(\Omega\times\mathbb{R}_+)} &\le c\Big(\|f\|_{L_{p,p_0}(\Omega\times\mathbb{R}_+)} + \|u_0\|_{B^{2-2/p_0}_{p,p_0}(\Omega)}\Big)\\ &\le c\Big(\|f\|_{L_{p,p_0}(\Omega^T)} + \|u_0\|_{B^{2-2/p_0}_{p,p_0}(\Omega)}\Big), \end{aligned} \tag{1.24}$$

where c do not depend on T.

In view of the first part of Lemma 1.3 we have

$$\begin{aligned} \|u(t)\|_{B^{2-2/p_0}_{p,p_0}(\Omega)} &\le c\|u\|_{W^{2,1}_{p,p_0}(\Omega\times\mathbb{R}_+)}\\ &\le c\Big(\|f\|_{L_{p,p_0}(\Omega^T)} + \|u_0\|_{B^{2-2/p_0}_{p,p_0}(\Omega)}\Big), \end{aligned} \tag{1.25}$$

where $t \in (0,T]$ and c do not depend on T. Continuing, (1.25) yields

$$\sup_{t\in(0,T]} \|u(t)\|_{B^{2-2/p_0}_{p,p_0}(\Omega)} \le c\Big(\|f\|_{L_{p,p_0}(\Omega^T)} + \|u_0\|_{B^{2-2/p_0}_{p_0}(\Omega)}\Big), \tag{1.26}$$

where c does not depend on T.

From $(1.22)_1$ we have

$$\|f\|_{L_{p,p_0}(\Omega^T)} \le \|u\|_{W^{2,1}_{p,p_0}(\Omega^T)}. \tag{1.27}$$

Hence (1.26) and (1.27) yield (1.23) and proves the lemma. □

By c we denote the generic constant which changes its value from formula to formula. By φ we denote the generic function which is always positive and increasing function of its arguments which might change its form from formula to formula.

2. Local existence

To prove the existence of local solutions we use the method of successive approximations described by the following system of problems:

$$
\begin{aligned}
u_{tt}^{n+1} - Qu_t^{n+1} &= \nabla\theta^n F_{1,\varepsilon^n} + (\theta^n F_{1,\varepsilon^n\varepsilon^n} + F_{2,\varepsilon^n\varepsilon^n})\nabla\varepsilon^n,\\
u^{n+1}|_{t=0} &= u_0,\\
u_t^{n+1}|_{t=0} &= u_1,\\
u^{n+1}|_S &= 0,\\
u^0 &= u_0, \quad \varepsilon^n = \varepsilon(u^n), \quad Q = \mu\Delta + \nu\nabla\mathrm{divv}
\end{aligned}
\tag{2.1}
$$

and

$$
\begin{aligned}
c_v\widetilde{\theta}_t^{n+1} - \varkappa\Delta\widetilde{\theta}^{n+1} &= \theta^n F_{1,\varepsilon^n}\varepsilon_t^n + \mathbb{A}\varepsilon_t^n\cdot\varepsilon_t^n,\\
\widetilde{\theta}^{n+1}|_{t=0} &= \widetilde{\theta}_0,\\
\overline{n}\cdot\nabla\widetilde{\theta}^{n+1}|_S &= 0,\\
\widetilde{\theta}^0 &= \widetilde{\theta}_0.
\end{aligned}
\tag{2.2}
$$

First we obtain an uniform estimate for the sequence $\{u^n,\widetilde{\theta}^n\}$.

Lemma 2.1. *Let us assume that*

$$
u_0\in B_{p,p_0}^{2-2/p_0}(\Omega),\quad u_1\in B_{p,p_0}^{2-2/p_0}(\Omega),\quad \widetilde{\theta}_0\in B_{q,q_0}^{2-2/q_0}(\Omega),\quad u_0\in W_p^2(\Omega),
$$

and that u^0, u_1^0, $\widetilde{\theta}^0$ *are extensions of* u_0, u_1, $\widetilde{\theta}_0$ *such that* (*see Lemma* 1.8)

$$
\begin{aligned}
\|u^0\|_{W_{p,p_0}^{2,1}(\Omega^T)} &\le c\|u_0\|_{B_{p,p_0}^{2-2/p_0}(\Omega)},\\
\|u_1^0\|_{W_{p,p_0}^{2,1}(\Omega^T)} &\le c\|u_1\|_{B_{p,p_0}^{2-2/p_0}(\Omega)},\\
\|\widetilde{\theta}^0\|_{W_{q,q_0}^{2,1}(\Omega^T)} &\le c\|\widetilde{\theta}_0\|_{B_{q,q_0}^{2-2/q_0}(\Omega)}.
\end{aligned}
$$

Let us assume additionally that $3/q+2/q_0<2$, $3/p+2/p_0<1$ *and* σ,σ_0 *in* (1.13) *are not less than* $2+\delta$, $\delta>0$. *Then there exists a constant* A *such that*

$$
\begin{aligned}
&\|u^0\|_{W_{p,p_0}^{2,1}(\Omega^T)} + \|u_1^0\|_{W_{p,p_0}^{2,1}(\Omega^T)} + \|\widetilde{\theta}^0\|_{W_{q,q_0}^{2,1}(\Omega^T)} < A,\\
&c_1(D_0+D_1) \equiv c_1\Big(\|u_0\|_{B_{p,p_0}^{2-2/p_0}(\Omega)} + \|u_1\|_{B_{p,p_0}^{2-2/p_0}(\Omega}\\
&\qquad\qquad + \|\widetilde{\theta}_0\|_{B_{q,q_0}^{2-2/q_0}(\Omega)} + \|u_0\|_{W_p^2(\Omega)}\Big) < A,
\end{aligned}
$$

where c_1 *is some constant,* $D_1=\|u_0\|_{W_p^2(\Omega)}$, D_0 *is defined from the above identity and the time* T *is such that*

$$
\varphi(T^aA,D_0)T^aA + \varphi(T^aA,D_0)T^a(D_0+D_1) + c_1(D_0+D_1) \le A,
\tag{2.3}
$$

where $a > 0$ and φ is a generic function which is an increasing positive function. Then, for any $n \in \mathbb{N}$,

$$\|u^n\|_{W^{2,1}_{p,p_0}(\Omega^T)} + \|u^n_t\|_{W^{2,1}_{p,p_0}(\Omega^T)} + \|\widetilde{\theta}^n\|_{W^{2,1}_{q,q_0}(\Omega^T)} \le A. \tag{2.4}$$

We have to mention that $D = D_0 + D_1$ is introduced in Theorem A.

Proof. Applying Lemma 1.4 to problem (2.1) yields

$$\begin{aligned}\|u^{n+1}_t\|_{W^{2,1}_{p,p_0}(\Omega^T)} \le c\Big(&\|\widetilde{\theta}^n_x F_{1,\varepsilon^n}\|_{L_{p,p_0}(\Omega^T)} \\ &+ \|(\theta^n F_{1,\varepsilon^n\varepsilon^n} + F_{2,\varepsilon^n\varepsilon^n})\nabla\varepsilon^n\|_{L_{p,p_0}(\Omega^T)} + \|u_1\|_{B^{2-2/p_0}_{p,p_0}(\Omega)}\Big)\end{aligned} \tag{2.5}$$

and to problem (2.2) we get

$$\begin{aligned}\|\widetilde{\theta}^{n+1}\|_{W^{2,1}_{q,q_0}(\Omega^T)} \le c\Big(&\|\theta^n F_{1,\varepsilon^n}\varepsilon^n_t\|_{L_{q,q_0}(\Omega^T)} \\ &+ \|\varepsilon^n_t\|^2_{L_{2q,2q_0}(\Omega^T)} + \|\widetilde{\theta}_0\|_{B^{2-2/q_0}_{q,q_0}(\Omega)}\Big).\end{aligned} \tag{2.6}$$

In view of the growth conditions (1.13) inequalities (2.5) and (2.6) assume the form

$$\begin{aligned}\|u^{n+1}_t\|_{W^{2,1}_{p,p_0}(\Omega^T)} \le c\Big(&\|\widetilde{\theta}^n_x |u^n_x|^{\sigma_0-1}\|_{L_{p,p_0}(\Omega^T)} \\ &+ \big\|(\theta^n|u^n_x|^{\sigma_0-2} + |u^n_x|^{\sigma-2})|u^n_{xx}|\big\|_{L_{p,p_0}(\Omega^T)} + \|u_1\|_{B^{2-2/p_0}_{p,p_0}(\Omega)}\Big),\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}\|\widetilde{\theta}^{n+1}\|_{W^{2,1}_{q,q_0}(\Omega^T)} \le c\Big(&\|\theta^n|u^n_x|^{\sigma_0-1}|u^n_{xt}|\|_{L_{q,q_0}(\Omega^T)} \\ &+ \|u^n_{xt}\|^2_{L_{2q,2q_0}(\Omega^T)} + \|\widetilde{\theta}_0\|_{B^{2-2/q_0}_{q,q_0}(\Omega)}\Big).\end{aligned} \tag{2.8}$$

Estimating the r.h.s. of (2.7) we get

$$\begin{aligned}\|u^{n+1}_t\|_{W^{2,1}_{p,p_0}(\Omega^T)} &\le c\|u^n_x\|^{\sigma_0-1}_{L_\infty(\Omega^T)}\|\widetilde{\theta}^n_x\|_{L_{p,p_0}(\Omega^T)} \\ &+ c\Big(\|\theta^n\|_{L_\infty(\Omega^T)}\|u^n_x\|^{\sigma_0-2}_{L_\infty(\Omega^T)} + \|u^n_x\|^{\sigma-2}_{L_\infty(\Omega^T)}\Big)\|u^n_{xx}\|_{L_{p,p_0}(\Omega^T)} \\ &+ c\|u_1\|_{B^{2-2/p_0}_{p,p_0}(\Omega)},\end{aligned} \tag{2.9}$$

and estimating (2.8) yields

$$\begin{aligned}\|\widetilde{\theta}^{n+1}\|_{W^{2,1}_{q,q_0}(\Omega^T)} \le\, &c\|\theta^n\|_{L_\infty(\Omega^T)}\|u^n_x\|^{\sigma_0-1}_{L_\infty(\Omega^T)}\|u^n_{xt}\|_{L_{q,q_0}(\Omega^T)} \\ &+ c\|u^n_{xt}\|^2_{L_{2q,2q_0}(\Omega^T)} + c\|\widetilde{\theta}_0\|_{B^{2-2/q_0}_{q,q_0}(\Omega)}.\end{aligned} \tag{2.10}$$

From the definition of $\widetilde{\theta}^n$ we have

$$\theta^n = \widetilde{\theta}^n + \theta_e$$

which, by Lemma 1.6 for $3/q < 2 - 2/q_0$ and Lemma 1.7 applied to problem (2.2) for n, implies

$$\|\theta^n\|_{L_\infty(\Omega^T)} \le \sup_t \|\widetilde{\theta}^n\|_{B^{2-2/q_0}_{q,q_0}(\Omega)} + |\theta_e| \le c\Big(\|\widetilde{\theta}^n\|_{W^{2,1}_{q,q_0}(\Omega^T)} + \|\widetilde{\theta}_0\|_{B^{2-2/q_0}_{q,q_0}(\Omega)}\Big) + |\theta_e|, \tag{2.11}$$

where c does not depend on T and

$$\frac{3}{q} + \frac{2}{q_0} < 2. \tag{2.12}$$

Moreover, we need

$$\begin{aligned}\|u^n_x\|_{L_\infty(\Omega^T)} &\le \sup_t \|u^n_x\|_{L_\infty(\Omega)} \le \sup_t \left\| \int_0^t u^n_{xt'}\,dt' + u_x(0)\right\|_{L_\infty(\Omega)} \\ &\le \sup_{t\le T}\int_0^t \|u^n_{xt'}\|_{L_\infty(\Omega)}\,dt' + \|u_0\|_{W^1_\infty(\Omega)} \\ &\le c\bigg(\sup_{t\le T}\int_0^t \|u^n_{t'}\|_{W^2_p(\Omega)}\,dt' + \|u_0\|_{B^{2-2/p_0}_{p,p_0}(\Omega)}\bigg) \equiv I,\end{aligned} \tag{2.13}$$

where we used the imbedding $D^\alpha_x W^2_p(\Omega) \subset L_\infty(\Omega)$, $|\alpha| = 1$, and Lemma 1.6 with

$$\frac{3}{p} < 1 - \frac{2}{p_0} < 1. \tag{2.14}$$

By the Hölder inequality we have

$$I \le c\Big(T^{1-1/p_0}\|u^n_t\|_{W^{2,1}_{p,p_0}(\Omega^T)} + \|u_0\|_{B^{2-2/p_0}_{p,p_0}(\Omega)}\Big).$$

By Lemma 1.5 and the Hölder inequality we obtain

$$\|u^n_{xt}\|_{L_{q,q_0}(\Omega^T)} \le c\|u^n_t\|_{W^{2,1}_{p,p'_0}(\Omega^T)} \le cT^{1/p'_0 - 1/p_0}\|u^n_t\|_{W^{2,1}_{p,p_0}(\Omega^T)}, \tag{2.15}$$

where

$$\frac{3}{p} + \frac{2}{p'_0} - \frac{3}{q} - \frac{2}{q_0} \le 1, \quad p'_0 < p_0 \tag{2.16}$$

and

$$\|u^n_{xt}\|_{L_{2q,2q_0}(\Omega^T)} \le c\|u^n_t\|_{W^{2,1}_{p,p'_0}(\Omega^T)} \le cT^{1/p'_0 - 1/p_0}\|u^n_t\|_{W^{2,1}_{p,p_0}(\Omega^T)} \tag{2.17}$$

where Lemma 1.5 was employed with the restriction

$$\frac{3}{p} + \frac{2}{p'_0} - \frac{3}{2q} - \frac{2}{2q_0} \le 1. \tag{2.18}$$

In view of the above estimates and since T is assumed to be considered small we choose a as a smallnest exponent of T and obtain from (2.9) the inequality

$$\begin{aligned}(2.19)\quad &\|\widetilde{\theta}^{n+1}\|_{W^{2,1}_{q,q_0}(\Omega^T)} \\ &\le \varphi\Big(T^a\|\widetilde{\theta}^n\|_{W^{2,1}_{q,q_0}(\Omega^T)}, T^a\|u^n_t\|_{W^{2,1}_{p,p_0}(\Omega^T)}, D_0\Big)T^a\|u^n_t\|_{W^{2,1}_{p,p_0}(\Omega)} \\ &\qquad + c\|\widetilde{\theta}_0\|_{B^{2-2/q_0}_{q,q_0}(\Omega)},\end{aligned}$$

where φ is an increasing positive function of its arguments which is a generic function and

$$(2.20)\qquad D_0 = \|u_0\|_{B^{2-2/p_0}_{p,p_0}(\Omega)} + \|u_1\|_{B^{2-2/p_0}_{p,p_0}(\Omega)} + \|\widetilde{\theta}_0\|_{B^{2-2/q_0}_{q,q_0}(\Omega)}.$$

Introducing the quantity

$$(2.21)\qquad A_n(T) = \|u^n\|_{W^{2,1}_{p,p_0}(\Omega^T)} + \|u^n_t\|_{W^{2,1}_{p,p_0}(\Omega^T)} + \|\widetilde{\theta}^n\|_{W^{2,1}_{q,q_0}(\Omega^T)},$$

we express (2.19) in the form

$$(2.22)\qquad \|\widetilde{\theta}^{n+1}\|_{W^{2,1}_{q,q_0}(\Omega^T)} \le \varphi(T^aA_n, D_0)T^aA_n + cD_0,$$

where $\varphi(X,Y) \sim a_1(X+Y)^{\sigma-2} + a_2(X+Y)^{\sigma_0-2}$ and a_i, $i=1,2$, are positive constants.

Using (2.11), (2.13) and (2.22) in (2.9) yields

$$\begin{aligned}(2.23)\quad \|u^{n+1}_t\|_{W^{2,1}_{p,p_0}(\Omega^T)} \le{}& \varphi\Big(T^aA_n, D_0)T^a(\|u^n_t\|_{W^{2,1}_{p,p_0}(\Omega^T)} + \|\widetilde{\theta}_0\|_{B^{2-2/q_0}_{q,q_0}(\Omega)}\Big) \\ &+ \varphi(T^a_nA_n, D_0)\|u^n_{xx}\|_{L_{p,p_0}(\Omega^T)} + c\|u_1\|_{B^{2-2/p_0}_{p,p_0}(\Omega)},\end{aligned}$$

where $a>0$ and φ is the generic function. Moreover, we have

$$\begin{aligned}(2.24)\quad \|u^{n+1}\|_{W^{2,1}_{p,p_0}(\Omega^T)} \le{}& T\|u^{n+1}_t\|_{W^{2,1}_{p,p_0}(\Omega^T)} + T^{1/p_0}\|u_0\|_{W^2_p(\Omega)} \\ \le{}& \varphi(T^aA_n, D_0)T^{a+1}\Big(\|u^n_t\|_{W^{2,1}_{p,p_0}(\Omega^T)} + \|\widetilde{\theta}_0\|_{B^{2-2/q_0}_{q,q_0}(\Omega)}\Big) \\ &+ T\varphi(T^aA_n, D_0)\|u^n\|_{W^{2,1}_{p,p_0}(\Omega^T)} \\ &+ cD_0 + cT^{1/p_0}\|u_0\|_{W^2_p(\Omega)}.\end{aligned}$$

Now, we examine the second term on the r.h.s. of (2.23). We have

$$\|u^n_{xx}\|_{L_{p,p_0}(\Omega^T)} = \bigg(\int_0^T \|u^n_{xx}\|^{p_0}_{L_p(\Omega)}\,dt\bigg)^{1/p_0}.$$

But $u^n(t) = \int_0^t u^n_{t'} dt' + u_0$, so $u^n_{xx}(t) = \int_0^t u^n_{xxt'}\, dt' + u_{0xx}$. Then

$$\begin{aligned}(2.25)\quad \|u^n_{xx}\|_{L_{p,p_0}(\Omega^T)} &= \left(\int_0^T \left\| \int_0^t u^n_{xxt'}\, dt' + u_{0xx}\right\|^{p_0}_{L_p(\Omega)} dt\right)^{1/p_0} \\ &\le \left[\int_0^T \left(\int_0^t \|u^n_{xxt'}\|^{p_0}_{L_p(\Omega)}\, dt' + \|u_{0xx}\|^{p_0}_{L_p(\Omega)}\right) dt\right]^{1/p_0} \\ &\le T^{1/p_0}\Big(\|u^n_{xxt'}\|_{L_{p,p_0}(\Omega^T)} + \|u_0\|_{W^2_p(\Omega)}\Big).\end{aligned}$$

Using (2.25) in (2.23) yields

$$\begin{aligned}(2.26)\quad \|u^{n+1}_t\|_{W^{2,1}_{p,p_0}(\Omega^T)} &\le \varphi(T^a A_n, D_0) T^a \Big(\|u^n_t\|_{W^{2,1}_{p,p_0}(\Omega^T)} + D_0\Big) \\ &\quad + \varphi(T^a A_n, D_0) T^{1/p_0}\Big(\|u^n_t\|_{W^{2,1}_{p,p_0}(\Omega^T)} + D_1\Big),\end{aligned}$$

where $a > 0$ and

$$(2.27)\qquad D_1 = \|u_0\|_{W^2_p(\Omega)}.$$

From (2.24) and (2.26) we have

$$\begin{aligned}(2.28)\quad &\|u^{n+1}\|_{W^{2,1}_{p,p_0}(\Omega^T)} + \|u^{n+1}_t\|_{W^{2,1}_{p,p_0}(\Omega^T)} \\ &\quad\le \varphi(T^a A_n, D_0) T^a \Big(\|u^n\|_{W^{2,1}_{p,p_0}(\Omega^T)} + \|u^n_t\|_{W^{2,1}_{p,p_0}(\Omega^T)}\Big) \\ &\qquad + \varphi(T^a A_n, D_0) T^a (D_0 + D_1) + c(D_0 + D_1).\end{aligned}$$

Using notation (2.21) in (2.28) yields

$$(2.29)\quad A_{n+1} \le \varphi(T^a A_n, D_0) T^a A_n + \varphi(T^a A_n, D_0)(D_0 + D_1) T^a + c_1(D_0 + D_1).$$

By the zero approximation we denote extensions of initial data such as (see Lemma 1.3 part 2)

$$(2.30)\qquad \begin{aligned}\|u^0\|_{W^{2,1}_{p,p_0}(\Omega^T)} &\le c\|u_0\|_{B^{2-2/p_0}_{p,p_0}(\Omega)}, \\ \|u^0_1\|_{W^{2,1}_{p,p_0}(\Omega^T)} &\le c\|u_1\|_{B^{2-2/p_0}_{p,p_0}(\Omega)}, \\ \|\widetilde{\theta}^0\|_{W^{2,1}_{q,q_0}(\Omega^T)} &\le c\|\widetilde{\theta}_0\|_{B^{2-2/q_0}_{q,q_0}(\Omega)}.\end{aligned}$$

Then

$$(2.31)\qquad A_0 = \|u^0\|_{W^{2,1}_{p,p_0}(\Omega^T)} + \|u^0_1\|_{W^{2,1}_{p,p_0}(\Omega^T)} + \|\widetilde{\theta}^0\|_{W^{2,1}_{q,q_0}(\Omega^T)}.$$

Let A be a constant such that

$$(2.32)\qquad A_0 < A, \qquad c_1(D_0 + D_1) < A.$$

Let

$$(2.33)\qquad A_n \le A.$$

Then there exists T sufficiently small such that

$$\varphi(T^aA,D_0)T^aA+\varphi(T^aA,D_0)T^a(D_0+D_1)+c_1(D+D_1)\le A. \tag{2.34}$$

Hence, (2.34) implies

$$A_{n+1}\le A. \tag{2.35}$$

This concludes the proof. □

To show convergence of the sequence $\{u^n,\widetilde\theta^n\}$ we introduce the differences

$$U^n=u^n-u^{n-1},\quad \vartheta^n=\widetilde\theta^n-\widetilde\theta^{n-1}=\theta^n-\theta^{n-1},\quad \mathcal{E}^n=\varepsilon^n-\varepsilon^{n-1} \tag{2.36}$$

which are solutions to the problems:

$$\begin{aligned} U^{n+1}_{tt}-QU^{n+1}_t&=\theta^n_xF_{1,\varepsilon^n}-\theta^{n-1}_xF_{1,\varepsilon^{n-1}}\\ &\quad+(\theta^nF_{1,\varepsilon^n\varepsilon^n}+F_{2,\varepsilon^n\varepsilon^n})\nabla\varepsilon^n\\ &\quad-(\theta^{n-1}F_{1,\varepsilon^{n-1}\varepsilon^{n-1}}+F_{2,\varepsilon^{n-1}\varepsilon^{n-1}})\nabla\varepsilon^{n-1},\\ U^{n+1}|_{t=0}=0,&\quad U^{n+1}_t|_{t=0}=0,\quad U^{n+1}|_S=0, \end{aligned} \tag{2.37}$$

and

$$\begin{aligned} c_v\vartheta^{n+1}_t-\varkappa\Delta\vartheta^{n+1}&=\theta^nF_{1,\varepsilon^n}\varepsilon^n_t-\theta^{n-1}F_{1,\varepsilon^{n-1}}\varepsilon^{n-1}_t\\ &\quad+(\mathbb{A}\varepsilon^n_t\cdot\varepsilon^n_t-\mathbb{A}\varepsilon^{n-1}_t\cdot\varepsilon^{n-1}_t),\\ \vartheta^{n+1}|_{t=0}=0,&\quad \overline{n}\cdot\nabla\vartheta^{n+1}|_S=0. \end{aligned} \tag{2.38}$$

Lemma 2.2. *Let the assumptions of Lemma 2.1 be satisfied. Let $3/p+2/p_0<1$, $3/q+2/q_0<1$, $q_0>p_0$. Let $F_{i,\varepsilon\varepsilon}$, $i=1,2$, be Lipschitz continuous. Then*

$$Y_{n+1}\le\varphi(A,D_0,D_1)T^aY_n,\quad a>0, \tag{2.39}$$

where

$$Y_n=\|U^n\|_{W^{2,1}_{p,p_0}(\Omega^T)}+\|U^n_t\|_{W^{2,1}_{p,p_0}(\Omega^T)}+\|\vartheta^n\|_{W^{2,1}_{q,q_0}(\Omega^T)}.$$

Proof. To prove the lemma we express the r.h.s. of $(2.37)_1$ and $(2.38)_1$ in terms of differences U^n,ϑ^n. Then problem (2.37) takes the form

$$\begin{aligned} U^{n+1}_{tt}-QU^{n+1}_t&=\vartheta^n_xF_{1,\varepsilon^n}+\theta^{n-1}_x\widetilde F_{1,\varepsilon^n\varepsilon^n}\mathcal{E}^n+\vartheta^nF_{1,\varepsilon^n\varepsilon^n}\varepsilon^n_x\\ &\quad+\theta^{n-1}(F_{1,\varepsilon^n\varepsilon^n}-F_{1,\varepsilon^{n-1}\varepsilon^{n-1}})\varepsilon^n_x+\theta^{n-1}F_{1,\varepsilon^{n-1}\varepsilon^{n-1}}\mathcal{E}^n_x\\ &\quad+(F_{2,\varepsilon^n\varepsilon^n}-F_{2,\varepsilon^{n-1}\varepsilon^{n-1}})\varepsilon^n_x+F_{2,\varepsilon^{n-1}\varepsilon^{n-1}}\mathcal{E}^n_x,\\ U^{n+1}|_{t=0}=0,&\quad U^{n+1}_t|_{t=0}=0,\quad U^{n+1}|_S=0, \end{aligned} \tag{2.40}$$

where $F_{i,\varepsilon\varepsilon}$, $i=1,2$, are Lipschitz continuous. Similarly,

$$\begin{aligned} c_1\vartheta^{n+1}_t-\varkappa\Delta\vartheta^{n+1}&=\vartheta^nF_{1,\varepsilon^n}\cdot\varepsilon^n_t+\theta^{n-1}\widetilde F_{1,\varepsilon^n\varepsilon^n}\mathcal{E}^n_t\\ &\quad+\theta^{n-1}F_{1,\varepsilon^{n-1}\varepsilon^{n-1}}\mathcal{E}^n_t+(\mathbb{A}\mathcal{E}^n_t\varepsilon^n_t+\mathbb{A}\varepsilon^{n-1}_t\mathcal{E}^n_t),\\ \vartheta^{n+1}|_{t=0}=0,&\quad \overline{n}\cdot\nabla\vartheta^{n+1}|_S=0, \end{aligned} \tag{2.41}$$

where $\widetilde{F}_{1,\varepsilon\varepsilon}(\widetilde{\varepsilon})$, $\widetilde{\varepsilon} \in (\varepsilon^{n-1}, \varepsilon^n)$ is calculated by the mean value theorem.

In view of Lemma 1.4 solutions to problem (2.40) satisfy the inequality

$$\begin{aligned}(2.42)\quad \|U_t^{n+1}\|_{W^{2,1}_{p,p_0}(\Omega^T)} \le \varphi(A)\Big[&\|\vartheta_x^n\|_{L_{p,p_0}(\Omega^T)} + \|U_x^n\|_{L_{p,p_0}(\Omega^T)} \\ &+ \|\,|\vartheta^n|\,|u_{xx}^n|\,\|_{L_{p,p_0}(\Omega^T)} + \|\,|U_x^n|\,|u_{xx}^n|\,\|_{L_{p,p_0}(\Omega^T)} + \|U_{xx}^n\|_{L_{p,p_0}(\Omega^T)}\Big],\end{aligned}$$

where we used that

$$\frac{3}{q} + \frac{2}{q_0} < 1.$$

To estimate the r.h.s. of (2.42) we need the following imbeddings (see Lemma 1.5)

$$\begin{aligned}&\|\vartheta_x^n\|_{L_{p,p_0}(\Omega^T)} \le cT^a\|\vartheta^n\|_{W^{2,1}_{q,q_0}(\Omega^T)}, \quad \frac{3}{q} + \frac{2}{q_0} - \frac{3}{p} - \frac{2}{p_0} < 1,\\ &\|U_x^n\|_{L_{p,p_0}(\Omega^T)} \le cT^a\|U^n\|_{W^{2,1}_{p,p_0}(\Omega^T)}, \quad \frac{3}{p} + \frac{2}{p_0} - \frac{3}{p} - \frac{2}{p_0'} < 1,\ p_0' > p_0.\end{aligned}$$

The third integral on the r.h.s. of (2.42) we treat in the following way

$$\begin{aligned}\|\,|\vartheta^n|\,|u_{xx}^n|\,\|_{L_{p,p_0}(\Omega^T)} &\le \Big\||\vartheta^n|\Big(\int_0^t |u_{xxt'}^n|\,dt' + |u_{0xx}|\Big)\Big\|_{L_{p,p_0}(\Omega^T)}\\ &\le \Big\||\vartheta^n|\int_0^t |u_{xxt'}^n|\,dt'\Big\|_{L_{p,p_0}(\Omega^T)} + \|\,|\vartheta^n|\,|u_{0xx}|\,\|_{L_{p,p_0}(\Omega^T)}\\ &\le \bigg(\int_0^T \|\vartheta^n\|^{p_0}_{L_\infty(\Omega)}\Big|\int_0^t \|u_{xxt'}^n\|_{L_p(\Omega)}\,dt'\Big|^{p_0}dt\bigg)^{1/p_0}\\ &\quad + \bigg(\int_0^T \|\vartheta^n\|^{p_0}_{L_\infty(\Omega)}\|u_{0xx}\|^{p_0}_{L_p(\Omega)}\,dt\bigg)^{1/p_0}\\ &\le \bigg(\int_0^T \|\vartheta^n\|^{p_0}_{L_\infty(\Omega)}t^{p_0-1}\int_0^t \|u_{xxt'}^n\|^{p_0}_{L_p(\Omega)}\,dt'\,dt\bigg)^{1/p_0}\\ &\quad + \|u_{0xx}\|_{L_p(\Omega)}\bigg(\int_0^T \|\vartheta^n\|^{p_0}_{L_\infty(\Omega)}dt\bigg)^{1/p_0}\\ &\le \Big(T^{1-1/p_0}\|u_t^n\|_{W^{2,1}_{p,p_0}(\Omega^T)} + \|u_0\|_{W^2_p(\Omega)}\Big)\|\vartheta^n\|_{L_{p_0}(0,T;L_\infty(\Omega))}\\ &\le T^{1/p_0-1/q_0}(T^{1-1/p_0}A + D_1)\|\vartheta^n\|_{L_{q_0}(0,T;W^2_q(\Omega))},\end{aligned}$$

where we assumed that $q_0 > p_0$, $3/q < 2$ because the imbedding $\|\vartheta^n\|_{L_\infty(\Omega)} \le \|\vartheta^n\|_{W^2_q(\Omega)}$ is used.

The fourth integral on the r.h.s. of (2.42) we estimate as follows

$$\begin{aligned}&\|\,|U_x^n|\,|u_{xx}^n|\,\|_{L_{p,p_0}(\Omega^T)} \le \Big\||U_x^n|\Big(\int_0^t |u_{xxt'}^n|\,dt' + |u_{0xx}|\Big)\Big\|_{L_{p,p_0}(\Omega^T)}\\ &\quad\le \bigg(\int_0^T \|U_x^n\|^{p_0}_{L_\infty(\Omega)}t^{p_0-1}\int_0^t \|u_{xxt'}^n\|^{p_0}_{L_p(\Omega)}\,dt'\bigg)^{1/p_0}\end{aligned}$$

$$
\begin{aligned}
&+ \|u_{0xx}\|_{L_p(\Omega)}\bigg(\int_0^T \|U_x^n\|_{L_\infty(\Omega)}^{p_0}\,dt\bigg)^{1/p_0}\\
&\le \Big(T^{1-1/p_0}\|u_t^n\|_{W^{2,1}_{p,p_0}(\Omega^T)} + \|u_0\|_{W^2_p(\Omega)}\Big)\bigg(\int_0^T \|U_x^n\|_{L_\infty(\Omega)}^{p_0}\,dt\bigg)^{1/p_0}\\
&\le c(A+D_1)\bigg[\int_0^T \Big(\varepsilon^{p_0}\|U_{xx}^n\|_{L_p(\Omega)}^{p_0} + c^{p_0}(1/\varepsilon)\|U^n\|_{L_p(\Omega)}^{p_0}\Big)\,dt\bigg]^{1/p_0}\\
&\le c(A+D_1)\bigg[\int_0^T \bigg(\varepsilon^{p_0}\|U_{xx}^n\|_{L_p(\Omega)}^{p_0} + c^{p_0}(1/\varepsilon)\bigg\|\int_0^t U_{t'}^n\,dt'\bigg\|_{L_p(\Omega)}^{p_0}\bigg)\,dt\bigg]^{1/p_0}\\
&\le c(A+D_1)\bigg[\int_0^T\bigg(\varepsilon^{p_0}\|U_{xx}^n\|_{L_p(\Omega)}^{p_0} + c^{p_0}(1/\varepsilon)t^{p_0-1}\int_0^t \|U_{t'}^n\|_{L_p(\Omega)}^{p_0}\,dt'\bigg)\,dt\bigg]^{1/p_0}\\
&\le c(A+D_1)(\varepsilon + c(1/\varepsilon)T^{1-1/p_0})\|U^n\|_{W^{2,1}_{p,p_0}(\Omega^T)}\\
&\le c(A+D_1)T^a\|U^n\|_{W^{2,1}_{p,p_0}(\Omega^T)},
\end{aligned}
$$

where $a>0$, $3/p<1$ because we used the interpolation

$$
\|u_x\|_{L_\infty(\Omega)} \le \varepsilon\|u\|_{W^2_p(\Omega)} + c(1/\varepsilon)\|u\|_{L_p(\Omega)}
$$

and $c(1/\varepsilon)\sim\varepsilon^{-b}$, $b>0$.

Finally, we examine the last norm on the r.h.s. of (2.42),

$$
\|U_{xx}^n\|_{L_{p,p_0}(\Omega^T)} = \bigg\|\int_0^t U_{xxt}^n\,dt'\bigg\|_{L_{p,p_0}(\Omega^T)} \le T\|U_t^n\|_{W^{2,1}_{p,p_0}(\Omega^T)}.
$$

Finally, from (2.42) we obtain

$$
\begin{aligned}
\|U_t^{n+1}\|_{W^{2,1}_{p,p_0}(\Omega^T)} \le{}& \varphi(A,D_1)T^a\Big[\|\vartheta^n\|_{W^{2,1}_{q,q_0}(\Omega^T)}\\
&+ \|U^n\|_{W^{2,1}_{p,p_0}(\Omega^T)} + \|U_t^n\|_{W^{2,1}_{p,p_0}(\Omega^T)}\Big],
\end{aligned}
\tag{2.43}
$$

where we assumed $3/q+2/q_0<1$, $p>3$, $q_0>p_0$, $q>3/2$.

For solutions to problem (2.41) we get

$$
\begin{aligned}
\|\vartheta^{n+1}\|_{W^{2,1}_{q,q_0}(\Omega^T)} \le{}& \varphi(A,D,D_1)\Big[\|\,|\vartheta^n|\,|u_{xt}^n|\,\|_{L_{q,q_0}(\Omega^T)}\\
&+ \|U_{xt}^n\|_{L_{q,q_0}(\Omega^T)} + \|\,|u_{xt}^n|\,|U_{xt}^n|\,\|_{L_{q,q_0}(\Omega^T)}\Big].
\end{aligned}
\tag{2.44}
$$

We estimate the first norm under the square bracket by

$$
\begin{aligned}
\|\,|\vartheta^n|\,|u_{xt}^n|\,\|_{L_{q,q_0}(\Omega^T)} &\le \|\vartheta^n\|_{L_{q,q_0}(\Omega^T)}\sup_t\|u_{xt}^n\|_{L_\infty(\Omega)}\\
&\le \sup_t\|u_t^n\|_{B^{2-2/p_0}_{p,p_0}(\Omega)}\bigg\|\int_0^t \vartheta_t^n\,dt'\bigg\|_{L_{q,q_0}(\Omega^T)}\\
&\le \varphi(A,D_0,D_1)T^a\|\vartheta^n\|_{W^{2,1}_{q,q_0}(\Omega^T)}, \quad a>0,
\end{aligned}
$$

where we used that $\|u_x\|_{L_\infty(\Omega)}\le c\|u\|_{B^{2-2/p_0}_{p,p_0}(\Omega)}$ which holds for $3/p+2/p_0<1$.

The second norm under the square bracket in (2.44) we estimate by

$$\|U^n_{xt}\|_{L_{q,q_0}(\Omega^T)} \le T^{1/p_0'-1/p_0}\|U^n_t\|_{W^{2,1}_{p,p_0}(\Omega^T)},$$

where $3/p+2/p_0'-3/q-2/q_0 \le 1$, $p_0' < p_0$.

Finally the last norm under the square bracket on the r.h.s. of (2.44) we estimate by

$$\begin{aligned}\|\,|u^n_{xt}|\,|U^n_{xt}|\,\|_{L_{q,q_0}(\Omega^T)} &\le \sup_t \|u^n_{xt}\|_{L_\infty(\Omega)}\|U^n_{xt}\|_{L_{q,q_0}(\Omega^T)}\\ &\le \varphi(A,D_0,D_1)T^{1/p_0'-1/p_0}\|U^n\|_{W^{2,1}_{p,p_0}(\Omega^T)},\end{aligned}$$

where Lemma 1.6 for the first factor and Lemma 1.5 for the second factor were used.

Using the above estimates in (2.44) yields

$$\|\vartheta^{n+1}\|_{W^{2,1}_{q,q_0}(\Omega^T)} \le \varphi(A,D_0,D_1)T^a\Big(\|\vartheta^n\|_{W^{2,1}_{q,q_0}(\Omega^T)} + \|U^n\|_{W^{2,1}_{p,p_0}(\Omega^T)}\Big). \tag{2.45}$$

Finally,

$$\|U^n\|_{W^{2,1}_{p,p_0}(\Omega^T)} \le \bigg(\int_0^T dt\bigg|\int_0^t \|U^n_{t'}\|_{L_p(\Omega)}\,dt'\bigg|^{p_0}\bigg)^{1/p_0} \le T\|U^n_t\|_{W^{2,1}_{p,p_0}(\Omega^T)}. \tag{2.46}$$

From (2.43), (2.45) and (2.46) we obtain (2.39). □

From Lemmas 2.1 and 2.2 we have

LEMMA 2.3. *Let the assumptions of Lemmas* 2.1, 2.2 *hold. Then there exists a solution to problem* (1.1)–(1.5) *such that*

$$u \in W^{2,1}_{p,p_0}(\Omega^T), \quad u_t \in W^{2,1}_{p,p_0}(\Omega^T), \quad \theta \in W^{2,1}_{q,q_0}(\Omega^T)$$

and

$$X \le A \tag{2.47}$$

where $X = \lim_{n\to\infty} A_n$ *and* A_n *is defined by* (2.21).

3. Global existence

First we prove a long time existence of solutions to problem (1.1)–(1.5). For this purpose we have to obtain the estimate in Lemma 2.3 for large existence time T. Let us consider inequality (2.3). Let T be fixed.

Setting $A = 2c_1(D_0+D_1)$ and using that $\varphi(X,Y) \ge |X|+|Y|$ we see that (2.3) is satisfied if

$$T^{2a}(D_0+D_1)^{1+\delta} \le (D_0+D_1) \tag{3.1}$$

Hence

$$T^a \le \sqrt{\frac{1}{(D_0+D_1)^\delta}}, \quad a>0. \tag{3.2}$$

For D_0+D_1 small the existence time T can be chosen large.

We have

Lemma 3.1. *Let the assumptions of Lemmas* 2.1 *and* 2.2 *hold. Then there exists a solution to problem* (1.1)–(1.5) *such that*

$$X(0,T) \equiv \|u\|_{W^{2,1}_{p,p_0}(\Omega^T)} + \|u_t\|_{W^{2,1}_{p,p_0}(\Omega^T)} + \|\widetilde{\theta}\|_{W^{2,1}_{q,q_0}(\Omega^T)} \le c_1(D_0+D_1) \equiv c_1 D, \tag{3.3}$$

where T is described by (3.2).

To prove global existence we obtain estimate (3.3) for $X(kT,(k+1)T)$, $k\in\mathbb{N}$, where

$$X(kT,(k+1)T) = \|u\|_{W^{2,1}_{p,p_0}(\Omega\times(kT,(k+1)T))} + \|u_t\|_{W^{2,1}_{p,p_0}(\Omega\times(kT,(k+1)T))} + \|\widetilde{\theta}\|_{W^{2,1}_{q,q_0}(\Omega\times(kT,(k+1)T))}. \tag{3.4}$$

Lemma 3.2. *Let the assumptions of Lemmas* 2.1 *and* 2.2 *hold. Assume that D is sufficiently small and T is sufficiently large* (*see* (3.2)). *Then*

$$X(kT,(k+1)T) \le c_1 D \quad \textit{for any } k\in\mathbb{N}\cup\{0\}.$$

Proof. For this purpose we introduce the cut-off smooth function such that

$$\zeta(t) = \begin{cases} 1 & \text{for } t \ge 2t_0, \\ 0 & \text{for } t \le t_0. \end{cases}$$

Introducing the functions

$$\overline{u} = u\zeta, \quad \overline{\overline{\theta}} = \widetilde{\theta}\zeta, \quad \overline{u}_t = (u\zeta)_{,t}, \tag{3.5}$$

we see that they are solutions to the problems

$$\begin{gathered} \overline{u}_{tt} - Q\overline{u}_t = \overline{\overline{\theta}}_x F_{1,\varepsilon} + (\theta F_{1,\varepsilon\varepsilon} + F_{2,\varepsilon\varepsilon})\nabla\varepsilon(\overline{u}) + 2u_t\zeta_t + u\zeta_{,tt} - Qu\zeta_t, \\ \overline{u}|_{t=0} = 0, \quad \overline{u}_t|_{t=0} = 0, \quad \overline{u}|_S = 0 \end{gathered} \tag{3.6}$$

and

$$\begin{gathered} c_0\overline{\overline{\theta}}_t - \varkappa\Delta\overline{\overline{\theta}} = (\theta F_{1,\varepsilon}\cdot\varepsilon_{,t} + \mathbb{A}\varepsilon_t\cdot\varepsilon_t)\zeta + c_v\widetilde{\theta}\zeta_t, \\ \overline{\overline{\theta}}|_{t=0} = 0, \quad \overline{n}\cdot\nabla\overline{\overline{\theta}}|_S = 0. \end{gathered} \tag{3.7}$$

Now, we obtain estimates for solutions to problems (3.6) and (3.7). For solutions of (3.6) we have

$$\begin{aligned}(3.8)\quad \|\overline{u}_t\|_{W^{2,1}_{p,p_0}(\Omega\times\mathbb{R}_{t_0,3t_0})} &\le c\|\overline{\overline{\theta}}_x F_{1,\varepsilon}\|_{L_{p,p_0}(\Omega\times\mathbb{R}_{t_0,3t_0})}\\ &+ c\|(\theta F_{1,\varepsilon\varepsilon}+F_{2,\varepsilon\varepsilon})\overline{u}_{xx}\|_{L_{p,p_0}(\Omega\times\mathbb{R}_{t_0,3t_0})} + c\|u_t\zeta_t\|_{L_{p,p_0}(\Omega\times\mathbb{R}_{t_0,3t_0})}\\ &+ c\|u\zeta_{tt}\|_{L_{p,p_0}(\Omega\times\mathbb{R}_{t_0,3t_0})} + c\|Qu\zeta_t\|_{L_{p,p_0}(\Omega\times\mathbb{R}_{t_0,3t_0})},\end{aligned}$$

where $\mathbb{R}_{t_1,t_2}=\{t\in\mathbb{R}: t_1\le t\le t_2\}$.

The first term on the r.h.s. of (3.8) is estimated by

$$c\|u_x\|^{\sigma_0-1}_{L_\infty(\Omega\times\mathbb{R}_{t_0,3t_0})}\|\overline{\overline{\theta}}_x\|_{L_{p,p_0}(\Omega\times\mathbb{R}_{t_0,3t_0})} \le c\|u\|^{\sigma_0-1}_{W^{2,1}_{p,p_0}(\Omega\times\mathbb{R}_{t_0,3t_0})}\|\overline{\overline{\theta}}\|_{W^{2,1}_{q,q_0}(\Omega\times\mathbb{R}_{t_0,3t_0})},$$

where to use Lemma 1.5 we assumed that

$$(3.9)\qquad \frac{3}{p}+\frac{2}{p_0}<1,\quad \frac{3}{q}+\frac{2}{q_0}<1.$$

The second term by

$$\begin{aligned}&\Big(\|\theta\|_{L_\infty(\Omega\times\mathbb{R}_{t_0,3t_0})}\|u_x\|^{\sigma_0-2}_{L_\infty(\Omega\times\mathbb{R}_{t_0,3t_0})}+\|u_x\|^{\sigma-2}_{L_\infty(\Omega\times\mathbb{R}_{t_0,3t_0})}\Big)\|\overline{u}_{xx}\|_{L_{p,p_0}(\Omega\times\mathbb{R}_{t_0,3t_0})}\\ &\quad\le\Big(\|\theta\|_{W^{2,1}_{q,q_0}(\Omega\times\mathbb{R}_{t_0,3t_0})}\|u\|^{\sigma_0-2}_{W^{2,1}_{p,p_0}(\Omega\times\mathbb{R}_{t_0,3t_0})}+\|u\|^{\sigma-2}_{W^{2,1}_{p,p_0}(\Omega\times\mathbb{R}_{t_0,3t_0})}\Big)\\ &\qquad\cdot\|\overline{u}\|_{W^{2,1}_{p,p_0}(\Omega\times\mathbb{R}_{t_0,3t_0})},\end{aligned}$$

where Lemma 1.5 with restrictions (3.9) was again used.

Finally, the last three norms on the r.h.s. of (3.8) we estimate by

$$\frac{c}{t_0}\Big(\|u_t\|_{L_{p,p_0}(\Omega\times\mathbb{R}_{t_0,2t_0})}+\|u\|_{L_{p_0}(\mathbb{R}_{t_0,2t_0};W^2_p(\Omega))}\Big)\le\frac{c}{t_0}\|u\|_{W^{2,1}_{p,p_0}(\Omega\times\mathbb{R}_{t_0,2t_0})},$$

where we assumed that $t_0\ge 1$.

Summarizing, we obtain from (3.8) the inequality

$$\begin{aligned}(3.10)\quad \|\overline{u}_t\|_{W^{2,1}_{p,p_0}(\Omega\times\mathbb{R}_{t_0,3t_0})}&\le\varphi\Big(\|u\|_{W^{2,1}_{p,p_0}(\Omega\times\mathbb{R}_{t_0,3t_0})}\Big)\\ &\cdot\Big(\|\overline{\overline{\theta}}\|_{W^{2,1}_{q,q_0}(\Omega\times\mathbb{R}_{t_0,3t_0})}+\|\overline{u}\|_{W^{2,1}_{p,p_0}(\Omega\times\mathbb{R}_{t_0,3t_0})}\Big)+\frac{c}{t_0}\|u\|_{W^{2,1}_{p,p_0}(\Omega\times\mathbb{R}_{t_0,2t_0})},\end{aligned}$$

where φ is a generic function such that $\varphi(Z)\le c|Z|^\delta$, where $\delta>0$.

For solutions to problem (3.7) we obtain

$$\begin{aligned}(3.11)\quad &\|\overline{\overline{\theta}}\|_{W^{2,1}_{q,q_0}(\Omega\times\mathbb{R}_{t_0,3t_0})}\\ &\le c\|\theta\|_{L_\infty(\Omega\times\mathbb{R}_{t_0,3t_0})}\|u\|^{\sigma_0-1}_{W^{2,1}_{p,p_0}(\Omega\times\mathbb{R}_{t_0,3t_0})}\|\overline{\varepsilon}_t\|_{L_{q,q_0}(\Omega\times\mathbb{R}_{t_0,3t_0})}\\ &\quad+c\|\varepsilon_t\|_{L_{2q,2q_0}(\Omega\times\mathbb{R}_{t_0,3t_0})}\|\overline{\varepsilon}_t\|_{L_{2q,2q_0}(\Omega\times\mathbb{R}_{t_0,3t_0})}+\frac{c}{t_0}\|\widetilde{\theta}\|_{L_{q,q_0}(\Omega\times\mathbb{R}_{t_0,2t_0})}\\ &\le c\Big(\|\theta\|_{W^{2,1}_{q,q_0}(\Omega\times\mathbb{R}_{t_0,3t_0})}\|u\|^{\sigma_0-1}_{W^{2,1}_{p,p_0}(\Omega\times\mathbb{R}_{t_0,3t_0})}\\ &\quad+\|u_t\|_{W^{2,1}_{p,p_0}(\Omega\times\mathbb{R}_{t_0,3t_0})}\Big)\|\overline{u}_t\|_{W^{2,1}_{p,p_0}(\Omega\times\mathbb{R}_{t_0,3t_0})}+\frac{c}{t_0}\|\widetilde{\theta}\|_{L_{q,q_0}(\Omega\times\mathbb{R}_{t_0,2t_0})}.\end{aligned}$$

From (3.10) and (3.11) we have

$$
\begin{aligned}
(3.12)\quad & \|\overline{u}_t\|_{W^{2,1}_{p,p_0}(\Omega\times\mathbb{R}_{t_0,3t_0})} + \|\overline{\widetilde{\theta}}\|_{W^{2,1}_{q,q_0}(\Omega\times\mathbb{R}_{t_0,3t_0})} \\
&\le \varphi\Big(\|u\|_{W^{2,1}_{p,p_0}(\Omega\times\mathbb{R}_{t_0,3t_0})}, \|u_t\|_{W^{2,1}_{p,p_0}(\Omega\times\mathbb{R}_{t_0,3t_0})}\Big) \\
&\quad\cdot \Big(\|\overline{\widetilde{\theta}}\|_{W^{2,1}_{q,q_0}(\Omega\times\mathbb{R}_{t_0,3t_0})} + \|\overline{u}\|_{W^{2,1}_{p,p_0}(\Omega\times\mathbb{R}_{t_0,3t_0})} + \|\overline{u}_t\|_{W^{2,1}_{p,p_0}(\Omega\times\mathbb{R}_{t_0,3t_0})}\Big) \\
&\quad + \frac{c}{t_0}\Big(\|u\|_{W^{2,1}_{p,p_0}(\Omega\times\mathbb{R}_{t_0,2t_0})} + \|\widetilde{\theta}\|_{W^{2,1}_{q,q_0}(\Omega\times\mathbb{R}_{t_0,2t_0})}\Big)
\end{aligned}
$$

where φ is the generic function such that

$$
(3.13)\qquad \varphi(Z_1, Z_2) \le c(|Z_1| + |Z_2|)^\delta, \quad \text{where } \delta > 0.
$$

Finally, we prove the global existence. Let $T = 3t_0$. Let (3.3) hold. Then (3.12) and (3.13) imply

$$
(3.14)\qquad \|\overline{u}_t\|_{W^{2,1}_{p,p_0}(\Omega\times\mathbb{R}_{t_0,3t_0})} + \|\overline{\widetilde{\theta}}\|_{W^{2,1}_{q,q_0}(\Omega\times\mathbb{R}_{t_0,3t_0})} \le cD^{1+\delta} + \frac{c}{T}D \le D,
$$

for D sufficiently small and T sufficiently large.

From (3.14) we obtain

$$
(3.15)\qquad \|u_t(T)\|_{B^{2-2/p_0}_{p,p_0}(\Omega)} + \|\widetilde{\theta}(T)\|_{B^{2-2/q_0}_{q,q_0}(\Omega)} \le D.
$$

Let $\zeta_1(t)$ be a smooth cut-off function such that $\zeta_1(t) = 1$ for $t > T - \varepsilon$, and $\zeta_1(t) = 0$ for $t \le t_1$, where $t_1 < T - \varepsilon$ and $|T - t_1| \le 1$.

Let $\widetilde{u} = u\zeta_1$. Then

$$
\|\widetilde{u}(T)\|_{W^2_p(\Omega)} \le \int_{t_1}^T \|\widetilde{u}_{t'}(t')\|_{W^2_p(\Omega)}\,dt' \le \bigg(\int_{t_1}^T \|\widetilde{u}_{t'}(t')\|^{p_0}_{W^2_p(\Omega)}dt'\bigg)^{1/p_0} \le D.
$$

Hence

$$
(3.16)\qquad \|u(T)\|_{B^{2-2/p_0}_{p,p_0}(\Omega)} \le \|u(T)\|_{W^2_p(\Omega)} \le D.
$$

From (3.15) and (3.16) we obtain that $X(T, 2T) \le c_1 D$ so we proved the existence of solutions in the interval $[T, 2T]$.

Continuing the considerations step by step we prove Lemma 3.2. □

From Lemmas 3.1 and 3.2 Theorem B follows.

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