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INVERSES, POWERS AND CARTESIAN PRODUCTS OF TOPOLOGICALLY DETERMINISTIC MAPS

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Abstract. We show that if (X,T) is a topological dynamical system which is deterministic in the sense of Kamiński, Siemaszko and Szymański then (X,T^{-1}) and $(X\times X,T\times T)$ need not be deterministic in this sense. However if $(X\times X,T\times T)$ is deterministic then (X,T^n) is deterministic for all $n\in\mathbb{N}\setminus\{0\}$.

1. Introduction

By a topological dynamical system we mean a pair (X,T), where X is a compact metric space, and $T\colon X\to X$ an onto continuous map. A factor map between systems (X,T) and (Y,S) is a continuous onto map $\pi\colon X\to Y$ satisfying $S\pi=\pi T$.

This note concerns systems (X,T) which are topologically deterministic (TD), i.e. whenever (Y,S) is a factor of (X,T), the map S is invertable. This notion was introduced by Kamiński, Siemaszko and Szymański in [3] as a natural topological analogue of determinism in ergodic theory, which can be defined similarly. Most work to date has focused on the relation of TD and topological entropy, see [3] and [2]. A relative version, analogous to the relative entropy theory, was introduced in [4]. Our purpose here is to study some other basic properties of TD

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systems, namely, the relation between determinism of (X,T) and determinism of the systems (X,T^n) and $(X\times X, T\times T)$.

In the ergodic category, i.e. for measurable transformations T preserving a probability measure μ, the analogous notion of determinism is that every measurable factor is invertible, and this is well-known to be equivalent to the vanishing of the Kolmogorov–Sinai entropy. Since $h(T^n,\mu)=|n|h(T,\mu)$, $n\in\mathbb{Z}\setminus\{0\}$, and $h(T\times T,\mu\times\mu)=2h(T,\mu)$, the vanishing of any one of these implies the same for the others, and hence determinism of T, T^n and $T\times T$ are equivalent. In the topological category, determinism is not equivalent to zero topological entropy, and, as it turns out, the relation between determinism of powers and products is more tenuous.

Theorem 1.1. *There exist* TD *systems* (X,T) *such that* (X,T^{-1}) *is not* TD.

Theorem 1.2. *There exists* TD *systems* (X,T) *such that* $(X\times X, T\times T)$ *is not* TD.

On the other hand,

Proposition 1.3. *If* $(X\times X, T\times T)$ *is TD then* (X,T^n) *is* TD *for all* $n\geq 1$.

It is not clear as yet whether determinism of (X,T) implies the same for (X,T^n), $n\geq 1$, although the converse is trivially true, i.e. determinism of (X,T^n) for any $n>1$ implies it for (X,T).

In the next section we prove the proposition. In Sections 3 and 4 we give the constructions which prove Theorems 1.1, 1.2, respectively.

2. Basic properties of TD systems

For general background on topological dynamics see e.g. [5]. Given a system (X,T) and $x\in X$ we write

$$\omega_T(x)=\bigcap_{n=1}^{\infty}\overline{\bigcup_{k\geq n}T^kx}.$$

Let $T\times T$ denote the diagonal map on $X\times X$: i.e. $T\times T(x',x'')=(Tx',Tx'')$. Let $\mathrm{CER}(X)$ denote the space of closed equivalence relations on X, and $\mathrm{ICER}(X)$ for the invariant ones, i.e.

$$\mathrm{ICER}(X)=\{R\in\mathrm{CER}(X):T\times T(R)=R\}.$$

Also write $\mathrm{ICER}^+(X)$ for the forward invariance equivalence relations:

$$\mathrm{ICER}^+(X)=\{R\in\mathrm{CER}(X):T\times T(R)\subseteq R\}.$$

There is a bijection between factors of (X,T) and members of $\mathrm{ICER}^+(X)$, given by the partition induced by the factor map. The image system is invertable if and only if the corresponding relation is in $\mathrm{ICER}(X)$. It follows that (see in [3]):

PROPOSITION 2.1. *(X,T) is* TD *if and only if* $\mathrm{ICER}^+(X) = \mathrm{ICER}(X)$.

A point $x \in X$ is forward recurrent if there is a sequence $n_k \to \infty$ such that $T^{n_k}x \to x$. Clearly if every point in $X \times X$ is $T \times T$ forward-recurrent then every forward invariant subset of $X \times X$ is invariant, and in particular $\mathrm{ICER}^+(X) = \mathrm{ICER}(X)$. This implies:

LEMMA 2.2. *Let (X,T) be a topological dynamical system. If every point of $X \times X$ is forward-recurrent for $T \times T$ then (X,T) is* TD.

This is the main condition used to establish that a system is TD. We shall see that it is not in fact equivalent to TD, see Section 4. However, there is a partial converse:

LEMMA 2.3. *If (X,T) is deterministic then every point in X is forward recurrent for T.*

PROOF. Suppose $x \in X$ is not forward recurrent. Set

$$X_0 = \{T^n x : n \geq 0\} \cup \omega_T(x).$$

It is easily checked that X_0 is a closed and forward-invariant but not invariant subset of X. Let

$$R = \{(x', x'') : x',\ x'' \in X_0\} \cup \{(x,x) : x \in X\}.$$

Then $R \in \mathrm{ICER}^+$ but $R \notin \mathrm{ICER}$. Hence (X,T) is not TD.

LEMMA 2.4. *If x is forward recurrent for T then x is forward recurrent for T^n for every $n \geq 0$.*

PROOF. Denote by $\omega_f(y)$ the ω-limit set of a point y under a map f. Assuming the contrary, let N be the least natural number such that x is not forward recurrent for (X, T^N), i.e. $x \notin \omega_{T^N}(x)$ but $x \in \omega_{T^n}(x)$ for all $1 \leq n < N$. Since

$$\omega_T(x) = \bigcup_{k=0}^{N-1} \omega_{T^N}(T^k x)$$

there is some $0 < r < N$ for which $x \in \omega_{T^N}(T^r x)$, or equivalently $T^M x \in \omega_{T^N}(x)$, where $M = N - r$. Hence $\omega_{T^N}(T^M x) \subseteq \omega_{T^N}(x)$. Since T^M is an endomorphism of (X,T), it follows from $T^M x \in \omega_{T^N}(x)$ that

$$T^{2M}x = T^M(T^M x) \in \omega_{T^N}(T^M x) \subseteq \omega_{T^N}(x)$$

and by induction $T^{kM}x \in \omega_{T^N}(x)$ for every $k \geq 0$, so $\omega_{T^M}(x) \subseteq \omega_{T^N}(x)$. Hence $x \notin \omega_{T^M}(x)$. But $0 < M < N$, contradicting the definition of N. □

PROOF OF PROPOSITION 1.3. Suppose $(X \times X, T \times T)$ is TD; we wish to show that (X, T^n) is TD for all $n \geq 1$.

If $(X\times X, T\times T)$ is TD then, by Lemma 2.3, every point in $X\times X$ is forward recurrent for $T\times T$. Hence, by the last lemma, for every $n\geq 1$, every point in $X\times X$ is forward recurrent for $(T\times T)^n$. Thus by Lemma 2.2, (X,T^n) is deterministic. $\square$

3. Proof of Theorem 1.1

We construct a deterministic system (X,T) such that (X,T^{-1}) is not deterministic.

A system (X,T) is pointwise rigid if there exists a sequence $(n_k)_{k=1}^{\infty}\subseteq\mathbb{N}$ such that $T^{n_k}x\to x$ for every $x\in X$. Clearly this implies that $(X\times X, T\times T)$ is also pointwise rigid and that every point in $X\times X$ is forward recurrent, so by Lemma 2.2 (X,T) is TD. We shall construct a pointwise rigid system such that (X,T^{-1}) contains a fixed point x_0 and a point $x_0\neq x\in X$ such that $T^{-n}x\to x_0$; thus x is not forward recurrent for T^{-1} so (X,T^{-1}) is not deterministic. Note that this also shows that (X,T^{-1}) is not pointwise rigid, even though (X,T) is. A similar construction appears in [1].

Write $I=[0,1]$. Let $\mathbb{N}=\{1,2,\dots\}$ and endow $I^{\mathbb{N}}$ with the product topology. Write $x(i)$ for the i-th coordinate of $x\in I^{\mathbb{N}}$ and let T denote the shift map on $I^{\mathbb{N}}$, i.e. $(Tx)(i)=x(i+1)$.

We aim to construct a point $x\in I^{\mathbb{N}}$ and a sequence $(n_k)_{k=1}^{\infty}$, $n_k\to\infty$, such that

(1) 0^k1 appears in x for arbitrarily large k,
(2) If $ab_1\dots b_k$ appears in x for some symbols $a,b_i\in[0,1]$, and $b_i\leq\varepsilon$ for $i=1,\dots,k$ then $a\leq\varepsilon+1/u(k)$, where $u(k)$ is a sequence tending to ∞ as $k\to\infty$.
(3) If $y=T^mx$ and $y(1)\dots y(k)\neq 0\dots 0$ then $|T^{n_k}y(i)-y(i)|<1/k$ for $i=1,\dots,k$.

Assuming we have constructed such a point x, take $X\subseteq[0,1]^{\mathbb{Z}}$ to be the bilateral extension of the orbit closure of x, that is, the set of $y\in I^{\mathbb{Z}}$ such that every finite subword of y appears in some accumulation point of $\{T^kx\}_{k=1}^{\infty}$. Condition (1) implies that the fixed point $\overline{0}=\dots000\dots$ is in X and that there is a point $y=\dots0001y'$ in X for some $y'\in I^{\mathbb{N}}$. Clearly the backward orbit of y under the shift converges to $\overline{0}$. Condition (3) implies that if $z\in X$ is not forward-asymptotic to $\overline{0}$ then $T^{n(k)}z\to z$. Finally, (2) guarantees that the only point which is forward asymptotic to $\overline{0}$ is $\overline{0}$ itself: indeed, if z is asymptotic to $\underline{0}$ then, for every $\varepsilon>0$, there is an i_0 such that $z(i)<\varepsilon$ for every $i>i_0$, and it follows from this that $z(i)\leq\varepsilon$ for every $i\leq i_0$ as well, and consequently $z=\overline{0}$. Since $\overline{0}$ is a fixed point, (1)–(3) imply that (X,T) is pointwise rigid.

The definition of x is by induction. Start the induction with $n_1=3$ and $x^1=100$.

At the m-th stage of the construction we will have defined $n_1, \dots, n_m \in \mathbb{N}$ and $x^m = x(1)\dots x(n_m)$ and the final $m+1$ letters of x^m will be 0.

Suppose this is the case; we must define n_{m+1} and x^{m+1}. For $t \in [0,1]$ let $t \cdot x^m$ for the pointwise product, i.e. $(t \cdot x^m)(i) = t \cdot x^m(i)$. Note that $0 \cdot x^m = 00\dots0$. Also write ab for the concatenation of a and b. Define

$$x^{m+1} = x^m x^m \left(\frac{m}{m+1} \cdot x^m\right)\left(\frac{m-1}{m+1} \cdot x^m\right)\dots\left(\frac{1}{m+1} \cdot x^m\right)\left(0 \cdot x^m\right)$$

and let n_{m+1} be the length of x^{m+1} (so by induction $n_{m+1} = (m+3)n_m$ and, in particular, $n_m \geq m$).

Each x^m thus begins with a 1 and ends with 0^{n_m}, and since x^{m+1} begins with $x^m x^m$ condition (1) of the construction holds.

To verify (2), define $u = u(k)$ to be the least integer u such that the length of x^u is at least k. Note that all x^n for $n \geq u$, and therefore all $x \in X$, are concatenations of blocks of the form $t \cdot x^{u-1}$. Therefore our given block $c = ab_1 \dots b_k$ is contained in a concatenation $c_0 c_1 \dots c_{p+1}$, where $p \geq 2$ is some integer, c_i are of the form $t_i \cdot x^{u-1}$ for $1 \leq i \leq p$, c_0 is $t_0 \cdot x^{u-1}$ or a non-empty terminal sub-block of it, and c_{p+1} is $t_{p+1} \cdot x^{u-1}$ or an initial sub-block of it. Notice that, by choice of u, the sequence t_i is either decreasing (if c is actually contained in a block of the form $t \cdot x^u$), or is the concatenation of two decreasing sequences. In the latter case, the second decreasing sequence must begin with a 1, so one of the b_j's is the first symbol of x^{u-1}, which is a 1, and so $\varepsilon \geq 1$ and the conclusion $a \leq \varepsilon$ is trivial. In the former case, there exists an $r \geq i$ such that $t_i = (r-i)/u$ for all i. Then once of the b_j's is the first symbol of $(r-1)/u \cdot x^u$, so $\varepsilon \geq (r-1)/u$. But a belongs to $r/u \cdot x^u$ so $a \leq r/u \leq \varepsilon + 1/u$, which is the desired inequality.

For (3), we claim that for each m and $k < m$ if $0 \leq i < n_m - n_k$ and $x^m(i), \dots, x^m(i+k-1) \neq 0$ then $|x^m(i) - x^m(i+n_k-1)| < 1/k$. The proof is by induction on m, using the fact that if y satisfies this condition then so does $t \cdot y$ for $t \in [0,1]$. Specifically, let m, k, i as above. If $k = m-1$ the proof is immediate from the construction. Otherwise write $x^m = y_1 \dots y_{m+2}$ with $y_j = t_j x^{m-1}$ as in the definition. Let $i = s \cdot n_{m-1} + i'$ for $s \in \{0, \dots, m+1\}$, $i' \in \{0, \dots, n_{m-1}-1\}$. If $0 \leq i' \leq n_{m-1} - n_k$ we can apply the induction hypothesis. Otherwise, i' is in the final $0^{n_{m-2}}$-block of y_s so the assumption that $x^m(i), \dots, x^m(i+k-1) \neq 0$ implies that $i' > n_{m-1} - k$. But now note that $y_{s+1} = x^k z$ for some z, so $y_{s+1}(n_k - i') = 0$ because the final k letters of x^k are 0. So $x^m(i) = x^m(i+n_k) = 0$ and we are done. $\square$

4. Proof of Theorem 1.2

We shall construct a system (X,T) which is TD, but $(X \times X, T \times T)$ is not TD. To establish the first property, we rely on the following result:

Lemma 4.1. *Suppose (X,T) has the property that for every $(x',x'')\in X\times X$, either (x',x'') is forward recurrent for $T\times T$ or else there is a $p\in X$ such that $(x',p),(p,x'')\in\omega_{T\times T}(x',x'')$. Then (X,T) is deterministic.*

Proof. It suffices to show that $\mathrm{ICER}^+=\mathrm{ICER}$. Let $R\in\mathrm{ICER}^+$ and let $(x',x'')\in R$. Since $\omega_{T\times T}(x',x'')\subseteq\mathrm{TR}$, if the first condition holds (i.e. if $(x',x'')\in\omega_{T\times T}(x',x''))$ then $(x',x'')\in\mathrm{TR}$. Otherwise there is a $p\in X$ so that $(x',p),(p,x'')\in\omega_{T\times T}(x',x'')\subseteq\mathrm{TR}$, and since TR is an equivalence relation, this means $(x',x'')\in\mathrm{TR}$. We have shown that $(x',x'')\in\mathrm{TR}$ whenever $(x',x'')\in R$, so $R\subseteq\mathrm{TR}$. The reverse containment holds by assumption so $R\in\mathrm{ICER}$, and the lemma follows. □

We shall construct a system containing a fixed point which will play the role of the point p in the lemma, i.e. every pair (x',x'') in the system which is not forward recurrent will have $(x',p),(p,x'')\in\omega_{T\times T}(x',x'')$. For simplicity we describe a non-transitive example, and then explain how to modify it to get a transitive one.

Let T be the shift on $[0,1]^{\mathbb{Z}}$. A block is a finite subsequence $x\in[0,1]^{\{1,\dots,n\}}$; here n is the length of the block. If x,y are blocks of length m,n respectively their concatenation is written xy and is the block $x(1)\dots x(m)y(1)\dots y(n)$ of length $m+n$. For $x\in[0,1]^{\mathbb{Z}}$ a sub-block is a block of the form $x(i),x(i+1),\dots,x(j)$; this is the block of length $j-i+1$ appearing in x at i. We denote this sub-block by $x(i;j)$. We say that blocks x_1,x_2 occur consecutively in x if $x_1=x(i,j)$ and $x_2=x(j+1,k)$ for some $i\le j<k$.

To construct the example we define two points $x^*,y^*\in[0,1]^{\mathbb{Z}}$ with $x^*(0)=y^*(0)=1$, and take X,Y to be their orbit closure, respectively. We also define sequences $m_k\to\infty$ and $n_k\to\infty$ so that the following conditions are satisfied:

(i) $\|x^*-T^{m_k}x^*\|_\infty\le 1/k$ for $k\ge 1$.
(ii) $\|y^*-T^{n_k}y^*\|_\infty\le 1/k$ for $k\ge 1$.
(iii) For $k\ge 1$, out of every three consecutive blocks in x^* of length n_k at least two are identically 0.
(iv) For $k\ge 1$, out of every three consecutive blocks in y^* of length m_k at least two are identically 0.
(v) For every $k\ne 0$, at least one of the symbols $x^*(k)$ or $y^*(k)$ is equal to 0.

Let X be the orbit closure of x^* and Y the orbit closure of y^*. We claim that given such points x^*,y^* the system $Z=X\cup Y$ is deterministic, but $Z\times Z$ is not. Indeed, the latter statement follows from the observation that by condition (v) and the fact that $x^*(0)=y^*(0)=1$, the pair $(x^*,y^*)\in Z\times Z$ is not forward recurrent for $T\times T$, so $Z\times Z$ is not deterministic.

To see that Z is deterministic, note that the properties (i)–(iv) above hold when x^*,y^* is replaced by any pair $x\in X$, $y\in Y$. Condition (i) now implies that

$T^{m_k}|_X \to \mathrm{id}_X$ uniformly, and similarly (ii) implies that $T^{n_k}|_Y \to \mathrm{id}_Y$ uniformly, and in particular every pair in X is forward recurrent for $T \times T$ and so is every pair from Y. For $x \in X$, $y \in Y$, conditions (i) and (iv) imply that there is a choice of $r(k) \in \{1, 2, 3\}$ so that $T^{r(k)m_k}x \to x$ but $T^{r(k)m_k}y \to \overline{0}$, and hence $(x, \overline{0}) \in \omega_{T\times T}(x, y)$. Similarly (ii) and (iii) imply that there is a choice $s(k) \in \{1, 2, 3\}$ so that $T^{s(k)n_k}x \to \overline{0}$ but $T^{s(k)n_k}y \to y$, so also $(\overline{0}, y) \in \omega_{T\times T}(x, y)$. From the lemma it now follows that $Z = X \cup Y$ is deterministic.

Here are the details of the construction. We proceed by induction on r. At the r-th stage we will be given an integer $L(r) \geq r - 1$ and finite sequences $x_r, y_r \in [0, 1]^{\{-L(r),-L(r)+1,\dots,L(r)\}}$, and if $r \geq 2$ we are also given integers m_{r-1}, n_{r-1}. We extend x_r to x_{r+1} and $y_r$to y_{r+1} without changing the symbols already defined. The blocks x_r, y_r will satisfy the following versions of the conditions above, and an additional condition which is required for the induction:

- (I) $\|x_r(i; i + k) - x_r(i + m_k; i + m_k + k)\|_\infty \leq 1/k$ for $1 \leq k \leq r - 1$ and $-L(r) \leq i \leq L(r) - m_k - k$.
- (II) $\|y_r(i; i + k) - y_r(i + n_k; i + n_k + k)\|_\infty \leq 1/k$ for $1 \leq k \leq r - 1$ and $-L(r) \leq i \leq L(r) - n_k - k$.
- (III) For $1 \leq k \leq r - 1$, out of every three consecutive blocks in x_r of length n_k at least two are identically 0.
- (IV) For $1 \leq k \leq r - 1$, out of every three consecutive blocks in y_r of length m_k at least two are identically 0.
- (V) For every $k \neq 0$ between $-L(r)$ and $L(r)$, at least one of the symbols $x_r(k)$ or $y_r(k)$ are equal to 0.
- (VI) $m_k, n_k \leq L(r-1)$ for each $1 \leq k \leq r-1$, and the first and last $2L(r-1)$ symbols of x_r and y_r are 0.

Assuming that such a sequence x_r, y_r exists, define $x^*, y^* \in [0, 1]^{\mathbb{Z}}$ by $x^*(i) = x_{i+1}(i)$ and $y^*(i) = y_{i+1}(i)$. It is straightforward to verify that these conditions guarantee that x^*, y^* have the desired properties.

We start the induction by $L(1) = 0$ and $x_1(0) = y_1(0) = 1$; all conditions are satisfied trivially.

For some $r \geq 2$ suppose we are given x_r, y_r, $L(r)$ and also m_k, n_k for $1 \leq k < r$, such that (I)–(VI) are satisfied. For a block z and $\alpha \in [0, 1]$, denote by $\alpha \cdot z$ the block with $(\alpha z)(i) = \alpha \cdot z(i)$.

Let s, t, s', t' be integers which we shall specify later. Let u and v be blocks of 0's of length s, t, respectively, and set

$$x_{r+1} = v\left(\frac{1}{r+1}\cdot x_r\right)u \,\dots\, u\left(\frac{r}{r+1}\cdot x_r\right)u\, x_r\, u\left(\frac{r}{r+1}\cdot x_r\right)u\left(\frac{r-1}{r+1}\cdot x_r\right)u \\ \dots\, u\left(\frac{1}{r+1}\cdot x_r\right)v.$$

Let u', v' to be blocks of 0's of length s', t', respectively, and set

$$y_{r+1} = v' \left(\frac{1}{r+1}\cdot y_r\right) u' \ldots u' \left(\frac{r}{r+1}\cdot y_r\right) u'\, y_r\, u' \left(\frac{r}{r+1}\cdot y_r\right) u' \left(\frac{r-1}{r+1}\cdot y_r\right) u' \\ \ldots u \left(\frac{1}{r+1}\cdot y_r\right) v'.$$

Note that in defining x_{r+1}, y_{r+1} we have added blocks to the left and right of the central copy of x_r, y_r, respectively, without changing the central blocks. We will assume that s, t, s', t' are chosen so that the lengths of x_{r+1}, y_{r+1} are equal. We define $L(r+1)$ to be their common length (see Figure 1).

By condition (VI), x_{r+1} and y_{r+1} satisfy (I) and (II) for $r+1$ and $1 \le k < r$. More precisely, suppose that $1 \le k < r$ and $-L(r+1) \le i \le L(r+1)-m_k+1-k$, and consider the blocks of length k in x_{r+1} at positions i and $i+m_k$. There are two possibilities. Either both blocks are located inside the same copy of $t\cdot x_r$ for some t, in which case $\|x_r(i;i+k)-x_r(i+m_k;i+m_k+k)\|_\infty \le 1/k$ by the induction hypothesis, or else at least one is located in an u and the other either in the first or last m_r symbols of a block of the form $t\cdot x_r$. In both of the last possibilities, the blocks are blocks of 0's (because u is all 0's and because of condition (VI) of the induction hypothesis) so $\|x_r(i;i+k) - x_r(i+m_k;i+m_k+k)\|_\infty \le 1/k$ is satisfied trivially. The analysis for y_{r+1} is similar.

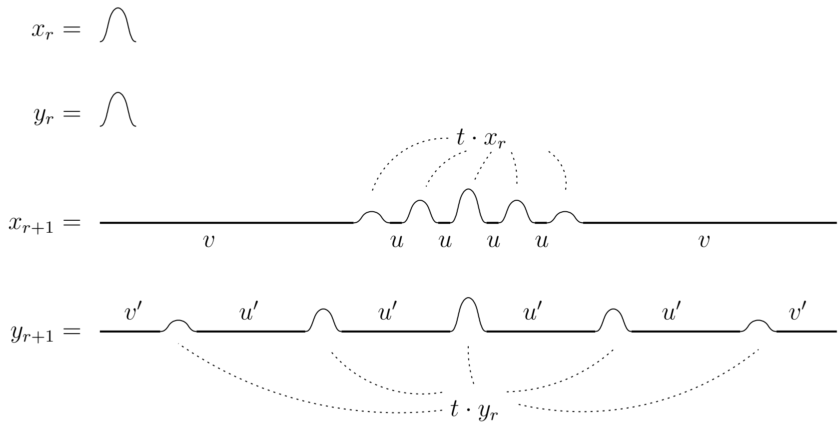

Figure 1. The construction of x_{r+1}, y_{r+1} form x_r, y_r (schematic)

Define $m_r = L(r) + s$. Then x_{r+1} also satisfies condition (I) for $k = r$, because every two symbols in x_{r+1} whose distance is $L(r)+s$ belong to blocks of the form $(i/r+1)\cdot x_r$ and $((i\pm 1)/(r+1))\cdot x_r$, and so differ in value by at most $1/(r+1)$. Similarly, if we define $n_r = L(r)+s'$ then y_{r+1} satisfies (II) for $k = r$.

If we choose s, t, s', t' large enough, conditions (III), (IV) hold for x_{r+1}, y_{r+1}. The same is true also for (VI).

It remains to obtain (V). We still have freedom to choose s, s', t, t' subject to the restriction that x_{r+1}, y_{r+1} have the same length, and as long as they are large enough. We first fix s some arbitrarily sufficiently large number (this determines the value of m_k). Next, we select s' large enough so that each non-zero component of x_{r+1} is opposite the central block $0^{s'}\, y_r\, 0^{s'}$ in y_{r+1} (here 0^m is the word consisting of m zeros); this implies also that each non-zero symbol in y_{r+1} outside of the central block y_r is opposite a 0 in x_{r+1}. See Figure 1. This and the induction hypothesis guarantees that (V) holds. It remains only to note that although t determines t', we can still make each as large as we want. This completes the construction.

Note that not every point in $X \times X$ is forward recurrent but X is TD. This shows that Lemma 2.2 is only a sufficient condition for TD, not a necessary condition.

Finally, in order to give a transitive example, one adds an intermediate step between each step of the construction above. Given x_r, y_r one forms the blocks

$$x'_r = b y_r a x_r a y_r b, \quad y'_r = d x_r c y_r c x_r d,$$

where a, b, c, d are sufficiently long blocks of 0's chosen (in a manner depending on r) so that x'_r, y'_r have the same length $L'(r)$, and condition (V) holds as before. The resulting limiting points x^*, y^* will have the same orbit closure Z. The previous analysis no longer holds, but instead one can show that locally something similar is true. For each $z \in Z$, if one looks at a sub-block of length of order $L(k)$ in the sequences x_r or y_r for some $r > k$, then one will see a block which, for m_k, n_k, satisfies conditions similar to the ones we had previously. Thus the same holds for any block of this length in Z. Therefore for any $z, z' \in Z$, we will either have that $T^{u_k} z \to z$ and $T^{u_k} z' \to z'$ for some squence $u_k \in \{m_k, n_k\}$, or else $(z', \bar{0})$ is in the forward orbit closure of the pair (z, z) and $(\bar{0}, z')$ is in the forward orbit closure of (z', z'), and as before we conclude that Z is TD but $Z \times Z$ is not.

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