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STATIONARY STATES FOR NONLINEAR DIRAC EQUATIONS WITH SUPERLINEAR NONLINEARITIES

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Abstract. In this paper we consider the nonlinear Dirac equation

$$-i\partial_t \psi = ic\hbar \sum_{k=1}^{3} \alpha_k \partial_k \psi - mc^2 \beta \psi + G_\psi(x, \psi).$$

Under suitable superlinear assumptions on the nonlinearities we can obtain the existence of at least one stationary state for the equation by applying a generalized linking theorem.

1. Introduction and main results

In this paper we are going to investigate the following nonlinear Dirac equation

$$-i\hbar\partial_t \psi = ic\hbar \sum_{k=1}^{3} \alpha_k \partial_k \psi - mc^2 \beta \psi + G_\psi(x, \psi), \tag{1.1}$$

where $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $\partial_k = \partial/\partial x_k$, $G\colon \mathbb{R}^3 \times \mathbb{C}^4 \to \mathbb{R}$ satisfies $G(x, e^{i\theta}\psi) = G(x, \psi)$ for all $\theta \in \mathbb{R}$, $\psi\colon \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}^4$ represents the wave function of the state

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of an electron, c denotes the speed of light, $m > 0$, the mass of the electron, $\hbar$ is Planck's constant, α_1, α_2, α_3 and β are the 4×4 complex matrices:

$$\beta = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \qquad \alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix} \qquad (k = 1, 2, 3)$$

with

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Such equations have been widely used to build relativistic models of extended particles in relativistic quantum mechanics (cf. [6], [12]). We are going to look for stationary solutions of (1.1), i.e. solutions of the form $\psi(t,x) = e^{i\theta t/\hbar} z(x)$. Then $\psi(t,x)$ is a solution of (1.1) if and only if z satisfies

$$-ic\hbar \sum_{k=1}^{3} \alpha_k \partial_k z + mc^2 \beta z = G_z(x,z) - \theta z. \tag{1.2}$$

For simplicity, we rewrite (1.2) as

$$-i \sum_{k=1}^{3} \alpha_k \partial_k z + a\beta z + \omega z = F_z(x,z) \tag{D}$$

where $a > 0$ and $\omega \in \mathbb{R}$, various assumptions on F have been used to model various types of self-couplings. Existence and multiplicity of stationary solutions of several models of particle physics have been established. In [2], [3], [7] and [18] the so-called Soler model (with F independent of x):

$$F(z) = \frac{1}{2} H(\widetilde{z}z), \quad H \in \mathcal{C}^2(\mathbb{R},\mathbb{R}), \quad H(0) = 0 \quad \text{where } \widetilde{u}u := (\beta z, z)_{\mathbb{C}^4} \tag{1.3}$$

was investigated. There if $\omega \in (-a, 0)$, by setting $r = |x|$ and some suitable assumptions on H, the authors obtained by a shooting method the solutions of (D) of the type

$$u(x) = \begin{pmatrix} v(r) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ iw(r) \begin{pmatrix} \cos\theta \\ \sin\theta e^{i\phi} \end{pmatrix} \end{pmatrix}. \tag{1.4}$$

In [13], the authors also dealt with more general nonlinearities $F(z)$ where (1.3) does not hold and the ansatz (1.4) does not apply. The authors show the existence of one (nontrivial) solution provided $F \in C^2(\mathbb{C}^4, \mathbb{R})$ satisfies various growth and sign conditions. Variational methods were also used in [11] to study the Soler model (1.2) with the classical Ambrosetti–Rabinowitz condition

$$H'(s) \cdot s \ge \theta H(s), \quad \theta > 2, \ s \in \mathbb{R}.$$

In this paper we are going to consider the nonlinearities of the type

$$F(x,z) = -\frac{1}{2}V(x)\widetilde{z}z + H(x,z).$$

Then the equation (D) reads as

$$(\mathrm{D})_V \qquad -i\sum_{k=1}^{3}\alpha_k\partial_k z + (V(x)+a)\beta z + \omega z = H_z(x,z).$$

If the potential is non-periodic (typically, the Coulomb-type potential), in [9] Y.H. Ding and B. Ruf considered some asymptotically quadratic nonlinearities, and in [10] Y.H. Ding and J.C. Wei treated the super-quadratic subcritical nonlinearities with asymptotically periodic condition, there the Ambrosetti–Rabinowitz condition also plays an important role.

In a recent paper [5], T. Bartsch and Y.H. Ding studied the nonlinear Dirac equations $(\mathrm{D})_V$ by critical point theory for strongly indefinite problems. By assuming $V(x)$ and $F(x,z)$ were periodic in x and some conditions weaker than the Ambrosetti–Rabinowitz condition, the authors first established the analytic setting for the problem and then obtained the existence of stationary solutions. When $F(x,z)$ was even in z, then $(D)_V$ possesses infinitely many geometrically different solutions. For other results about nonlinear Dirac equation, we refer readers to [8] and references therein.

The purpose of this paper is to establish the existence results for periodic Dirac equation without the Ambrosetti–Rabinowitz condition. We make the following hypothesis:

- (V_1) $\omega \in (-a,a)$, $V \in C^1(\mathbb{R}^3,\mathbb{R})$, $V \geq 0$, and $V(x)$ is 1-periodic in x_k for $k = 1,2,3$.
- (H_1) $H_z(x,z)$ is 1-periodic in x and there exist $p \in (2,3)$, $c_1 > 0$ such that $|H_z(x,z)| \leq c_1(1+|z|^{p-1})$.
- (H_2) $|H_z(x,z)| = o(|z|)$ as $|z| \to 0$ and $H(x,z)/|z|^2 \to \infty$ as $|z| \to \infty$ uniformly in x.
- (H_3) $H_z(x,z)\cdot z > 2H(x,z) > 0$, for all $z \neq 0$.
- (H_4) $H_z(x,z)\cdot(u+z) \leq 0$ for all (z,u) satisfying $H_z(x,z)\cdot u = H_z(x,u)\cdot z$.

Set $A := -i\sum_{k=1}^{3}\alpha_k\partial_k + (V(x)+a)\beta$ and $L = A+\omega$. We define the functional

$$\Phi(z) = \frac{1}{2}((A+\omega)z,z) - \int_{\mathbb{R}^3} H(x,z) = \frac{1}{2}(Lz,z) - \int_{\mathbb{R}^3} H(x,z).$$

The hypotheses on $H(x,z)$ imply that $\Phi \in C^1(E,\mathbb{R})$ and a standard argument shows that critical points of Φ are solutions of $(\mathrm{D})_V$. Let $\widehat{\mathcal{K}} := \{z \in E : \Phi'(z) = 0,\ z \neq 0\}$ be the critical set of Φ and

$$\widehat{C} := \inf\{\Phi(z) : z \in \widehat{\mathcal{K}} \setminus \{0\}\}.$$

The main result of this paper is the following theorem.

Theorem 1.1. *If assumptions* (V_1) *and* (H_1)–(H_4) *are satisfied, then problem* $(\mathrm{D})_V$ *has at least one least energy solution with* $\widehat{C} > 0$.

It is well known that without Ambrosetti-Rabinowitz condition, such problems become quite difficult and complex. Ground states for periodic Schrödinger equation under Nehari type monotone condition have received much attention, see [15], [16], [25]. For the Dirac equation, because 0 lies in a gap of the spectrum of Dirac operator $\sigma(L)$, the action functional is strongly indefinite and it is not easy to obtain the boundedness of the Palais–Smale sequence. Motivated by the above works, the aim of this paper is to find at least one ground state solution under (V_1) and (H_1)–(H_4). We will first find a special bounded Palais–Smale sequence for problem $(\mathrm{D})_V$ and then prove the existence of the solution by concentration compactness arguments. The main idea here lies in an application of a variant generalized weak linking theorem for strongly indefinite problems developed by Schechter and Zou [20], see also [23].

2. Variational tools

We will denote by $|\cdot|_p$ the usual L^p norm for $p \geq 1$. Under (V_1), $(V+a) \in L^2_{\mathrm{loc}}(\mathbb{R}^3,\mathbb{R})$, A is a selfadjoint operator in $L^2 = L^2(\mathbb{R}^3,\mathbb{C}^4)$. Let $\sigma(A)$, $\sigma_e(A)$ and $\sigma_c(A)$ denote, respectively, the spectrum, essential spectrum and continuous spectrum of the self-adjoint operator A on L^2. The following spectrum property of the operator A is established in [5].

Lemma 2.1 ([5]). *If* (V_1) *holds, then* $\sigma(A) \subset \mathbb{R} \setminus (-a,a)$, $\sigma(A) = \sigma_c(A)$.

It follows that the space L^2 possesses the orthogonal decomposition:

$$L^2 = L^- \oplus L^+, \quad u = u^- + u^+$$

so that A is negative definite (resp. positive definite) in L^- (resp. L^+). Let $|A|$ denote the absolute value, $|A|^{1/2}$ the squared root, and take $E = \mathcal{D}(|A|^{1/2})$. E is a Hilbert space equipped with the inner product

$$(u,v) = \mathcal{R}(|A|^{1/2}u, |A|^{1/2}v)_2$$

and the induced norm $\|z\| = (z,z)^{1/2}$. E possesses the following decomposition

$$E = E^- \oplus E^+ \quad \text{with } E^{\pm} = E \cap L^{\pm},$$

orthogonal with respect to both $(\,\cdot\,,\,\cdot\,)_2$ and $(\,\cdot\,,\,\cdot\,)$ inner products.

LEMMA 2.2 ([5]). *If* (V_1) *holds, then* $E = H^{1/2}$ *and* E *embeds continuously into* L^q *for* $q \in [2,3]$ *and compactly into* L^q_{loc} *for* $q \in [1,3)$.

The solutions of equation $(\mathrm{D})_V$ will be obtained as critical points of the functional defined on E:

$$\Phi(z) = \frac{1}{2}(\|z^+\|^2 - \|z^-\|^2 + \omega|z|_2^2) - \int_{\mathbb{R}^3} H(x,z). \tag{2.1}$$

The hypotheses on $H(x,z)$ imply that $\Phi \in C^1(E,\mathbb{R})$ and a standard argument shows that critical points of Φ are solutions of $(\mathrm{D})_V$. By (V_1) and Lemma 2.2,

$$\frac{a-|\omega|}{a}\|z^+\|^2 \le (\|z^+\|^2 \pm \omega|z^+|_2^2) \le \frac{a+|\omega|}{a}\|z^+\|^2 \tag{2.2}$$

and

$$\frac{a-|\omega|}{a}\|z^-\|^2 \le (\|z^-\|^2 \pm \omega|z^-|_2^2) \le \frac{a+|\omega|}{a}\|z^-\|^2. \tag{2.3}$$

Now we can introduce the norm on E

$$\|z\|_\omega = (\|z\|^2 + \omega(|z^+|_2^2 - |z^-|_2^2))^{1/2}.$$

Using (2.2)–(2.3) one has

$$\omega_0|z|_2^2 \le \|z\|_\omega^2 \quad \text{and} \quad \frac{a-|\omega|}{a}\|z\|^2 \le \|z\|_\omega^2 \le \frac{a+|\omega|}{a}\|z\|^2. \tag{2.4}$$

The following abstract critical point theorem plays an important role in proving our main result.

Let E be a Hilbert space with norm $\|\cdot\|$ and have an orthogonal decomposition $E = N \oplus N^\perp$, $N \subset E$ is a closed and separable subspace. There exists a norm $|v|_w$ satisfying $|v|_w \le \|v\|$ for all $v \in N$ and inducing a topology equivalent to the weak topology of N on a bounded subset of N. For $u = v+w \in E = N \oplus N^\perp$ with $v \in N$, $w \in N^\perp$, we define $|u|_w^2 = |v|_w^2 + \|w\|^2$. In particular, if $(u_n = v_n + w_n)$ is $|\cdot|_w$-bounded and $u_n \xrightarrow{|\cdot|_w} u$, then $v_n \rightharpoonup v$ weakly in N, $w_n \to w$ strongly in $N^\perp$, $u_n \rightharpoonup v + w$ weakly in E (cf. [20]).

Let $E = E^- \oplus E^+$, $z_0 \in E^+$ with $\|z_0\| = 1$. Let $N := E^- \oplus \mathbb{R}z_0$ and $E_1^+ := N^\perp = (E^- \oplus \mathbb{R}z_0)^\perp$. For $R > 0$, let

$$Q := \{u := u^- + sz_0 : s > 0,\ u^- \in E^-,\ \|u\| < R\}.$$

For $0 < s_0 < R$,we define

$$B := \{u := sz_0 + w^+ : s \ge 0,\ w^+ \in E_1^+,\ \|sz_0 + w^+\| = s_0\}.$$

For $\Phi \in C^1(E,\mathbb{R})$, define $\Gamma := \{h \mid h\colon [0,1] \times \overline{Q} \mapsto E$ is $|\cdot|_w$-continuous, $h(0,u) = u$, $\Phi(h(s,u)) \le \Phi(u)$, for all $u \in \overline{Q}\}$. For any $(s_0,u_0) \in [0,1] \times \overline{Q}$, there is a $|\cdot|_w$-neighbourhood $U_{(s_0,u_0)}$, such that

$$\{u - h(t,u) : (t,u) \in U_{(s_0,u_0)} \cap ([0,1] \times \overline{Q})\} \subset E_{\mathrm{fin}}\}.$$

where $E_{\rm fin}$ denotes various finite-dimensional subspaces of E, $\Gamma \neq \emptyset$ since $\mathrm{id} \in \Gamma$.

The variant weak linking theorem is:

THEOREM 2.3 [20]. *Let $\{\Phi_\lambda\}$ be a family of C^1-functionals of the form*

$$\Phi_\lambda(u) := I(u) - \lambda J(u), \quad \textit{for all } \lambda \in [1,2].$$

Assume:

(a) *$J(u) \geq 0$ for all $u \in E$, $\Phi_1 = \Phi$;*
(b) *$I(u) \to \infty$ or $J(u) \to \infty$ as $\|u\| \to \infty$;*
(c) *Φ_λ is $|\cdot|_w$-upper semicontinuous, Φ'_λ is weakly sequentially continuous on E. Moreover, Φ_λ maps bounded sets to bounded sets;*
(d) *$\sup_{\partial Q} \Phi_\lambda < \inf_B \Phi_\lambda$, for all $\lambda \in [1,2]$.*

Then for almost all $\lambda \in [1,2]$, there exists a sequence $\{u_n\}$ such that

$$\sup_n \|u_n\| < \infty, \quad \Phi'_\lambda(u_n) \to 0, \quad \Phi_\lambda(u_n) \to C_\lambda;$$

where

$$C_\lambda := \inf_{h\in\Gamma} \sup_{u\in\overline{Q}} \Phi_\lambda(h(1,u)) \in [\inf_B \Phi_\lambda, \sup_{\overline{Q}} \Phi].$$

3. Proof of the main results

In order to use Theorem 2.3, we consider

$$\Phi_\lambda(z) := \frac{1}{2}\|z^+\|_\omega^2 - \lambda\Big(\frac{1}{2}\|z^-\|_\omega^2 + \Psi(z)\Big). \tag{3.1}$$

It is easy to see that Φ_λ verifies conditions (a), (b) in Theorem 2.3. To see (c), if $z_n \xrightarrow{|\cdot|_w} z$ and $\Phi_\lambda(z_n) \geq a$, then $z_n^+ \to z^+$ and $z_n^- \rightharpoonup z^-$ in E, going to a subsequence if necessary, $z_n(x) \to z(x)$ almost everywhere on $\mathbb{R}^3$. Using Fatou's lemma, we know $\Phi_\lambda(z) \geq a$, this means that Φ_λ is $|\cdot|_w$-upper semicontinuous. Φ'_λ is weakly sequentially continuous on E. To continue the discussion, we still need to verify condition (d). Indeed, we have

LEMMA 3.1. *Under assumptions* (V_1) *and* (H_1)–(H_3)*, there hold:*

(a) *There exists $\rho > 0$ independent of $\lambda \in [1,2]$ such that $\kappa := \inf \Phi_\lambda(S_\rho E^+) > 0$ where $S_\rho E^+ := \{z \in E^+ : \|z\|_\omega = \rho\}$.*
(b) *For fixed $z_0 \in E^+$ with $\|z_0\|_\omega = 1$ and any $\lambda \in [1,2]$, there is $R > \rho > 0$ such that $\sup \Phi_\lambda(\partial Q) \leq 0$ where $Q = \{z = v + sz_0 : v \in E^-,\ s \geq 0,\ \|z\|_\omega < R\}$.*

PROOF. (a) From assumptions (H_1), (H_2), we know that, for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that $|H(x,z)| \leq \varepsilon|z|^2 + C_\varepsilon|z|^p$. Hence, for any $z \in E^+$,

$$\Phi_\lambda(z) \geq \frac{1}{2}\|z\|_\omega^2 - \lambda\varepsilon\|z\|_\omega^2 - C'_\varepsilon\|z\|_\omega^p.$$

The conclusion follows.

(b) Suppose by contradiction that there exists a sequence $z_n \in E^- \oplus \mathbb{R}^+ z_0$ such that $\Phi(z_n) > 0$ for all n and $\|z_n\|_\omega \to \infty$ as $n \to \infty$. Set $w_n = z_n/\|z_n\|_\omega = s_n z_0 + w_n^-$, then $1 = \|w_n\|_\omega^2 = s_n^2 + \|w_n^-\|_\omega^2$ and

$$0 < \frac{\Phi(z_n)}{\|z_n\|_\omega^2} = \frac{1}{2}(s_n^2 - \|w_n^-\|_\omega^2) - \int_{\mathbb{R}^3} \frac{H(x,z_n)}{|z_n|^2}|w_n|^2.$$

From (H_3), we know $H(x,z) \ge 0$ and have

$$\|w_n^-\|_\omega^2 < s_n^2 = 1 - \|w_n^-\|_\omega^2,$$

therefore

$$\|w_n^-\|_\omega \le \frac{1}{\sqrt{2}} \quad \text{and} \quad \frac{1}{\sqrt{2}} \le s_n \le 1.$$

Going to a subsequence if necessary, we may assume $s_n \to s > 0$, $w_n \rightharpoonup w$ and $w_n^-(x) \to w^-(x)$ almost everywhere in $\mathbb{R}^3$. Hence $w = sz_0 + w^-(x) \ne 0$ and therefore $|z_n| = \|z_n\|_\omega |w_n| \to \infty$. From (H_2) and Fatou's lemma, we obtain

$$\int_{\mathbb{R}^3} \frac{H(x,z_n)}{|z_n|^2}|w_n|^2 \to \infty \quad \text{as } n \to \infty.$$

This is a contradiction. □

Applying Theorem 2.3, we obtain the following lemma.

LEMMA 3.2. *Under assumptions* (V_1) *and* (H_1)–(H_3), *for almost every* $\lambda \in [1,2]$, *there exists a sequence* $\{z_n\}$ *such that*

$$\sup_n \|z_n\|_\omega < \infty, \quad \Phi'_\lambda(z_n) \to 0, \quad \Phi_\lambda(z_n) \to C_\lambda \in \Big[\kappa, \sup_{\overline{Q}} \Phi\Big], \tag{3.2}$$

as $n \to \infty$.

LEMMA 3.3. *Under assumptions* (V_1) *and* (H_1)–(H_3), *for almost every* $\lambda \in [1,2]$, *there exists a* z_λ *such that*

$$\Phi'_\lambda(z_\lambda) = 0, \quad \Phi_\lambda(z_\lambda) \le \sup_{\overline{Q}} \Phi.$$

PROOF. Let $\{z_n\}$ be the sequence obtained in Lemma 3.2, write $z_n = z_n^+ + z_n^-$ with $z_n^\pm \in E^\pm$. Since $\{z_n\}$ is bounded, we have either $\{z_n^+\}$ is vanishing, i.e.

$$\lim_{n\to\infty} \sup_{y \in \mathbb{R}^3} \int_{B_1(y)} |z_n^+|^2 = 0$$

or non-vanishing, i.e. there exist $r, \delta > 0$ and a sequence $\{y_n\} \in \mathbb{Z}^3$ such that

$$\lim_{n\to\infty} \int_{B_r(y_n)} |z_n^+|^2 \ge \delta.$$

If $\{z_n^+\}$ is vanishing, by Lions' concentration compactness principle [17], we have that $z_n^+ \to 0$ in $L^s(\mathbb{R}^3, \mathbb{C}^4)$ for all $s \in (2,3)$. Since for any $\varepsilon > 0$ there exists C_ε such that $|H_z(x,z)| \le \varepsilon |z| + C_\varepsilon |z|^{p-1}$, by Hölder's inequality, we know

$$\left| \int_{\mathbb{R}^3} H_z(x, z_n) \cdot z_n^+ \right| \le \varepsilon \int_{\mathbb{R}^3} |z_n||z_n^+| + C_\varepsilon \int_{\mathbb{R}^3} |z_n^+|^{p-1}|z_n^+| \to 0, \quad \text{as } n \to \infty$$

therefore

$$\Phi_\lambda(z_n) \le \frac{1}{2}\|z_n^+\|_\omega^2 = \langle \Phi'_\lambda(z_n), z_n^+ \rangle + \int_{\mathbb{R}^3} H_z(x, z_n) \cdot z_n^+ \to 0, \quad \text{as } n \to \infty$$

this contradicts with the fact that $C_\lambda \ge \kappa$. Hence $\{z_n^+\}$ must be non-vanishing. Let us define $v_n = z_n(\,\cdot\, - y_n)$, then

$$\int_{B(r,0)} |v_n^+|^2 \, dx \ge \frac{\delta}{2}.$$

We know

$$\Phi'_\lambda(v_n) \to 0 \quad \text{and} \quad \Phi_\lambda(v_n) \to C_\lambda, \quad \text{as } n \to \infty.$$

Since $\{v_n\}$ is still bounded, we may assume $v_n^+ \rightharpoonup z_\lambda^+$, $v_n^- \rightharpoonup z_\lambda^-$. Since $v_n^+ \to z_\lambda^+$ in $L^2_{\rm loc}(\mathbb{R}^3, \mathbb{C}^4)$, we have $z_\lambda^+ \ne 0$. Moreover,

$$< \Phi'_\lambda(z_\lambda), \varphi >= \lim_{n\to\infty} < \Phi'_\lambda(v_n), \varphi >= 0, \quad \text{for all } \varphi \in E,$$

we know $\Phi'_\lambda(z_\lambda) = 0$. Since

$$\frac{1}{2} H_z(x,z)z - H(x,z) > 0, \quad \text{if } z \ne 0,$$

by Fatou's lemma, we have

$$\begin{aligned} C_\lambda &= \lim_{n\to\infty} \Phi_\lambda(z_n) - \frac{1}{2}\langle \Phi'_\lambda(z_n), z_n \rangle \\ &= \lim_{n\to\infty} \int_{\mathbb{R}^3} \left(\frac{1}{2} H_z(x, z_n) z_n - H(x, z_n) \right) \\ &\ge \int_{\mathbb{R}^3} \left(\frac{1}{2} H_z(x, z_\lambda) z_\lambda - H(x, z_\lambda) \right) = \Phi_\lambda(z_\lambda), \end{aligned}$$

we know $\Phi_\lambda(z_\lambda) \le C_\lambda$. □

Similar to Lemma 2.2 of [22], we need to establish the following lemma for the nonlinear function $H(x,z)$. As we know, Lemma 2.2 of [22] plays an important role in proving the existence of standing waves for Schrödinger equations. Here we want to establish similar results for Dirac equation without the Nehari type monotone condition.

Lemma 3.4. *Under assumptions* (H_1)–(H_4). *Let* $s \in \mathbb{R}, s \geq 1$ *and* $z, u \in \mathbb{C}^4$ *with* $w := sz + u \neq 0$. *Then*

$$H_z(x,z)\cdot\left(s\left(\frac{s}{2}-1\right)z+(s-1)u\right)+H(x,z)-H(x,w-z)<0, \quad \textit{for all } x \in \mathbb{R}^3.$$

Proof. We fix $x \in \mathbb{R}^3$ and $z, u \in \mathbb{C}^4$. Let $s \geq 1$ and $\zeta(s) = (s-1)z + u$,

$$h(s) := H_z(x,z)\cdot\left(s(\frac{s}{2}-1)z+(s-1)u\right)+H(x,z)-H(x,\zeta(s)).$$

We need to show $h(s) < 0$ whenever $s \geq 1$.

In fact, $u = \zeta - (s-1)z$, we know

$$\begin{aligned} h(s) = {}& H_z(x,z)\cdot\left(s\left(\frac{s}{2}-1\right)z+(s-1)(\zeta(s)-(s-1)z)\right) \\ & + H(x,z)-H(x,\zeta(s)) \\ = {}& -\left(\frac{s^2}{2}-\frac{1}{2}\right)H_z(x,z)\cdot z-\frac{1}{2}H_z(x,z)\cdot z+H(x,z) \\ & +(s-1)H_z(x,z)\cdot(\zeta(s)+z)-H(x,\zeta(s)). \end{aligned} \tag{3.3}$$

From the assumption that $H(x,z)/|z|^2 \to \infty$ as $|z| \to \infty$ uniformly in x, we have

$$\lim_{s\to\infty} h(s) = -\infty.$$

Thus $h(s)$ must attain its maximum on $[1,\infty)$ at some point $s_0 \geq 1$. A direct computation shows that

$$0 = h^{'}(s_0) = H_z(x,z)\cdot\zeta(s_0) - H_z(x,\zeta(s_0))\cdot z,$$

then (H_4) implies that

$$H_z(x,z)\cdot(\zeta(s_0)+z) \leq 0. \tag{3.4}$$

From (3.3), by (H_4) and (3.4), we know $h(s) \leq h(s_0) < 0$. □

Lemma 3.5. *Let* $z_\lambda \neq 0$ *be any critical point of* Φ_λ, *we have*

$$\Phi_\lambda(w-z_\lambda) < \Phi_\lambda(z_\lambda) \quad \textit{for any } w \in \Sigma := \{sz_\lambda + \psi : s \geq 1,\ \psi \in E^-\}, \quad w \neq 0.$$

Proof. We rewrite Φ_λ as

$$\Phi_\lambda(z) = \frac{1}{2}(Lz^+, z^+)_{L^2} + \frac{\lambda}{2}(Lz^-, z^-)_{L^2} - \lambda\int_{\mathbb{R}^3} H(x,z).$$

Since $\Phi'_\lambda(z_\lambda)=0$, for $\psi\in E^-$, we have

$$
\begin{aligned}
(3.5)\quad 0&=\Big\langle \Phi'_\lambda(z_\lambda),\frac{s^2-2s}{2}z_\lambda+(1+s)\psi\Big\rangle\\
&=\frac{s^2-2s}{2}(Lz_\lambda^+,z_\lambda^+)_{L^2}+\lambda\frac{s^2-2s}{2}(Lz_\lambda^-,z_\lambda^-)_{L^2}\\
&\quad+\lambda(s-1)(Lz_\lambda^-,\psi)_{L^2}-\lambda\int_{\mathbb{R}^3}H_z(x,z_\lambda)\cdot\Big(\frac{s^2-2s}{2}z_\lambda+(s-1)\psi\Big)
\end{aligned}
$$

Therefore, for $w=sz_\lambda+\psi$, by Lemma 3.4 and (3.5), we know

$$
\begin{aligned}
\Phi_\lambda&(w-z_\lambda)-\Phi_\lambda(z_\lambda)\\
&=\frac12\{(L(s-1)z_\lambda^+,(s-1)z_\lambda^+)_{L^2}-(Lz_\lambda^+,z_\lambda^+)_{L^2}\}\\
&\quad+\frac{\lambda}{2}\{(L((s-1)z_\lambda^-+\psi),(s-1)z_\lambda^-+\psi)_{L^2}-(Lz_\lambda^-,z_\lambda^-)_{L^2}\}\\
&\quad+\lambda\Big\{\int_{\mathbb{R}^3}H(x,z_\lambda)-\int_{\mathbb{R}^3}H(x,w-z_\lambda)\Big\}\\
&=\frac{s^2-2s}{2}(Lz_\lambda^+,z_\lambda^+)_{L^2}+\lambda\frac{s^2-2s}{2}(Lz_\lambda^-,z_\lambda^-)_{L^2}\\
&\quad+\frac{\lambda}{2}(L\psi,\psi)_{L^2}+\lambda(s-1)(Lz_\lambda^-,\psi)_{L^2}\\
&\quad+\lambda\Big\{\int_{\mathbb{R}^3}H(x,z_\lambda)-\int_{\mathbb{R}^3}H(x,w-z_\lambda)\Big\}\\
&=\frac{\lambda}{2}(L\psi,\psi)_{L^2}+\lambda\int_{\mathbb{R}^3}\Big(H_z(x,z_\lambda)\cdot\Big(\frac{s^2-2s}{2}z_\lambda+(s-1)\psi\Big)\\
&\quad+H(x,z_\lambda)-H(x,w-z_\lambda)\Big)<0. \qquad\square
\end{aligned}
$$

By using the above two lemmas, we are able to show the existence of bounded (PS) sequence.

Lemma 3.6. *Under assumptions* (V_1) *and* (H_1)–(H_4), *there exist* $\lambda_n\to1$ *and sequence* $\{z_{\lambda_n}\}$ *such that*

$$\Phi'_{\lambda_n}(z_{\lambda_n})=0,\quad \Phi_{\lambda_n}(z_{\lambda_n})\le\sup_{\overline{Q}}\Phi.$$

Moreover, $\{z_{\lambda_n}\}$ *is bounded.*

Proof. The existence of $\{z_{\lambda_n}\}$ such that there

$$\Phi'_{\lambda_n}(z_{\lambda_n})=0,\quad \Phi_{\lambda_n}(z_{\lambda_n})\le\sup_{\overline{Q}}\Phi$$

is the direct consequence of Lemma 3.3. To show the boundedness, we argue by contradiction, suppose that $\|z_{\lambda_n}\|_\omega\to\infty$. Since $\Phi_{\lambda_n}(z_{\lambda_n})\ge0$, we know $\|z_{\lambda_n}^+\|_\omega\ge\|z_{\lambda_n}^-\|_\omega$.

Let $v_{\lambda_n} := z_{\lambda_n}/\|z_{\lambda_n}\|_\omega$. Then $\|v^+_{\lambda_n}\|^2 \geq 1/2$ and $v_{\lambda_n}(x) \rightharpoonup v(x)$ almost everywhere in $\mathbb{R}^3$ after passing to a subsequence. We have either $\{v^+_n\}$ is vanishing, i.e.

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B(y,1)} |v^+_{\lambda_n}|^2 = 0$$

or non-vanishing, i.e. there exist $r, \delta > 0$ and a sequence $\{y_n\} \in \mathbb{Z}^3$ such that

$$\lim_{n\to\infty} \int_{B(y_n,r)} |v^+_{\lambda_n}|^2 \geq \delta.$$

If $\{v^+_n\}$ is vanishing, Lion's concentration compactness principle implies $v^+_{\lambda_n} \to 0$ in $L^r(\mathbb{R}^3)$ for $r \in (2, 2^*)$. Therefore assumption (H_1) and Lebesgue Dominated Convergence Theorem imply that $\int_{\mathbb{R}^3} H(x, Tv^+_{\lambda_n}) \to 0$ for any $T \in \mathbb{R}^+$. From Lemma 3.5, we know that

$$\begin{aligned}\sup_{\overline{Q}} \Phi &\geq \Phi_{\lambda_n}(z_{\lambda_n}) \geq \Phi_{\lambda_n}(Tv^+_{\lambda_n}) \\ &= \frac{T^2}{2}\|v^+_n\|^2_\omega - \lambda_n \int_{\mathbb{R}^3} H(x, Tv^+_{\lambda_n}) \geq \frac{T^2}{4} - 2\int_{\mathbb{R}^3} H(x, Tv^+_{\lambda_n}) \to \frac{T^2}{4},\end{aligned}$$

we arrive at a contradiction if T is large enough. Hence non-vanishing must hold. The invariance of Φ_{λ_n} under translation implies the sequence $\{y_n\}$ can be selected to be bounded. Then $v^+_{\lambda_n} \to v^+$ in $L^2_{\mathrm{loc}}(\mathbb{R}^3)$ with $v^+ \neq 0$ and therefore $|z_{\lambda_n}(x)| \to \infty$. It follows again from (H_3) and Fatou's lemma that

$$\int_{\mathbb{R}^3} \frac{H(x, z_{\lambda_n})}{|z_{\lambda_n}|^2} |v_{\lambda_n}|^2 \to \infty, \quad \text{as } n \to \infty$$

and therefore

$$0 \leq \frac{\Phi(z_{\lambda_n})}{\|z_{\lambda_n}\|^2_\omega} = \frac{1}{2}\|v^+_{\lambda_n}\|^2_\omega - \lambda\biggl(\frac{1}{2}\|v^-_{\lambda_n}\|^2_\omega + \int_{\mathbb{R}^3} \frac{H(x, z_{\lambda_n})}{|z_{\lambda_n}|^2}|v_{\lambda_n}|^2\biggr) \to -\infty,$$

as $n \to \infty$, a contradiction. Thus we have the conclusion. □

Corollary 3.7. *If $\{z_{\lambda_n}\}$ is the sequence obtained in Lemma 3.6, then it is also a (PS) sequence for Φ satisfying*

$$\lim_{n\to\infty} \Phi'(z_{\lambda_n}) = 0, \quad \lim_{n\to\infty} \Phi(z_{\lambda_n}) \leq \sup_{\overline{Q}} \Phi.$$

Proof. Since z_{λ_n} is bounded, from

$$\lim_{n\to\infty} \Phi(z_{\lambda_n}) = \lim_{n\to\infty} \biggl(\Phi_{\lambda_n}(z_{\lambda_n}) + (\lambda_n - 1)\biggl(\frac{1}{2}\|z^-_{\lambda_n}\|^2_\omega - \int_{\mathbb{R}^3} H(x, z_{\lambda_n})\biggr)\biggr)$$

and

$$\lim_{n\to\infty} \langle \Phi'(z_{\lambda_n}), \varphi\rangle = \lim_{n\to\infty} \langle \Phi'_{\lambda_n}(z_{\lambda_n}), \varphi\rangle + (\lambda_n - 1)\biggl((z^-_{\lambda_n}, \varphi^-) - \int_{\mathbb{R}^3} H_z(x, z_{\lambda_n}) \cdot \varphi\biggr)$$

uniformly in $\|\varphi\|_\omega \leq 1$, we obtain the conclusions. □

Proof of Theorem 1.1. Since $\{z_n\}$ is bounded, we have either $\{z_n^+\}$ is vanishing, i.e.

$$\lim_{n\to\infty}\sup_{y\in\mathbb{R}^3}\int_{B(y,1)}|z_{\lambda_n}^+|^2=0$$

or non-vanishing, i.e. there exist $r,\delta>0$ and a sequence $y_n\in\mathbb{Z}^3$ such that

$$\lim_{n\to\infty}\int_{B(y_n,r)}|z_{\lambda_n}^+|^2\ge\delta.$$

If $\{z_{\lambda_n}\}$ is vanishing, then by Lion's concentration compactness principle again, we have that $z_{\lambda_n}\to 0$ in $L^s(\mathbb{R}^3)$ for $2<s<3$. However, for any $\varepsilon>0$ there exists C_ε such that $|H_z(x,z)|\le\varepsilon|z|+C_\varepsilon|z|^{p-1}$. From the fact that $\langle\Phi'_{\lambda_n}(z_{\lambda_n}),z_{\lambda_n}^+\rangle=0$ and Hölder's inequality, we know

$$\begin{aligned}\|z_{\lambda_n}^+\|_\omega^2&=\lambda_n\int_{\mathbb{R}^3}H_z(x,z_{\lambda_n})\cdot z_{\lambda_n}^+\\&\le\varepsilon\int_{\mathbb{R}^3}|z_{\lambda_n}||z_{\lambda_n}^+|+C_\varepsilon\int_{\mathbb{R}^3}|z_{\lambda_n}^+|^{p-1}|z_{\lambda_n}^+|\le\varepsilon\|z_{\lambda_n}^+\|_\omega^2+C'_\varepsilon\|z_{\lambda_n}^+\|_\omega^p.\end{aligned}\tag{3.6}$$

Similarly, we have

$$\|z_{\lambda_n}^-\|_\omega^2\le\varepsilon\|z_{\lambda_n}^-\|_\omega^2+C'_\varepsilon\|z_{\lambda_n}^-\|_\omega^p.\tag{3.7}$$

From (3.6) and (3.7), we have

$$\|z_{\lambda_n}\|_\omega^2\le\varepsilon\|z_{\lambda_n}\|_\omega^2+C'_\varepsilon\|z_{\lambda_n}\|_\omega^p.$$

Which means $\|z_{\lambda_n}\|_\omega\ge c$ for some constant c, hence the vanishing case does not hold. Let us now define $v_{\lambda_n}=z_{\lambda_n}(\,\cdot\,-y_n)$, then

$$\int_{B(r,0)}|v_{\lambda_n}^+|^2\,dx\ge\frac{\delta}{2}.$$

Φ and Φ' are both invariant by translation, we know

$$\Phi'(v_{\lambda_n})\to 0,\quad\text{as } n\to\infty.$$

Since $\{v_{\lambda_n}\}$ is also bounded, we may assume $v_{\lambda_n}\rightharpoonup v$. Since $v_{\lambda_n}\to v$ in $L^2_{\mathrm{loc}}(\mathbb{R}^3)$, we have $v\ne 0$ and $\Phi'(v)=0$.

Let $\widehat{\mathcal{K}}:=\{u\in E:\Phi'(u)=0,\ u\ne 0\}$ be the critical set of Φ and

$$\widehat{C}:=\inf\{\Phi(z):z\in\widehat{\mathcal{K}}\setminus\{0\}\}.$$

For any critical point z of Φ_λ,

$$\Phi_\lambda(z)=\Phi_\lambda(z)-\frac12\langle\Phi'_\lambda(z),z\rangle=\int_{\mathbb{R}^3}\bigg(\frac12H_z(x,z)\cdot z-H(x,z)\bigg)>0,\quad\text{if } z\ne 0.$$

Therefore $\widehat{C}_\lambda\ge 0$. We prove that $\widehat{C}>0$ and there is $z\in\widehat{\mathcal{K}}$ satisfying $\Phi(z)=\widehat{C}$. Let $z_j\in\widehat{\mathcal{K}}\setminus\{0\}$ be such that $\Phi(z_j)\to\widehat{C}$. Then, similar to the proof of

Lemma 3.6 $\{z_j\}$ is bounded, and by the concentration compactness principle discussion above $z_j \rightharpoonup z \in \widehat{\mathcal{K}} \setminus \{0\}$. Then

$$\begin{aligned}\widehat{C} = \lim_{j\to\infty} \Phi(z_j) &= \lim_{j\to\infty} \int_{\mathbb{R}^3} \left(\frac{1}{2} H_z(x, z_j) \cdot z_j - H(x, z_j)\right) \\ &\geq \int_{\mathbb{R}^3} \left(\frac{1}{2} H_z(x, z) \cdot z - H(x, z)\right) = \Phi(z) \geq \widehat{C}\end{aligned}$$

that is, $\Phi(z) = \widehat{C}$ and $\widehat{C} > 0$ because $z \neq 0$. □

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