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POSITIVE SOLUTIONS FOR A $2n$TH-ORDER p-LAPLACIAN BOUNDARY VALUE PROBLEM INVOLVING ALL EVEN DERIVATIVES

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Abstract. In this paper, we investigate the existence and multiplicity of positive solutions for the following $2n$th-order p-Laplacian boundary value problem

$$\begin{cases} -(((-1)^{n-1}x^{(2n-1)})^{p-1})' \\ \qquad = f(t,x,-x'',\dots,(-1)^{n-1}x^{(2n-2)}) & \text{for } t\in[0,1], \\ x^{(2i)}(0)=x^{(2i+1)}(1)=0 & \text{for } i=0,\dots,n-1, \end{cases}$$

where $n \geq 1$ and $f \in C([0,1]\times\mathbb{R}_+^n,\mathbb{R}_+)(\mathbb{R}_+ := [0,\infty))$ depends on x and all derivatives of even orders. Based on a priori estimates achieved by utilizing properties of concave functions and Jensen's integral inequalities, we use fixed point index theory to establish our main results. Moreover, our nonlinearity f is allowed to grow superlinearly and sublinearly.

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1. Introduction

The paper mainly concerns with the existence and multiplicity of positive solutions for the following $2n$th-order p-Laplacian boundary value problem involving all even derivatives

$$
(1.1)\qquad \begin{cases} -(((-1)^{n-1}x^{(2n-1)})^{p-1})' \\ \qquad\qquad = f(t,x,-x'',\dots,(-1)^{n-1}x^{(2n-2)}) & \text{for } t\in[0,1], \\ x^{(2i)}(0)=x^{(2i+1)}(1)=0 & \text{for } i=0,\dots,n-1, \end{cases}
$$

where $f\in C([0,1]\times\mathbb{R}_+^n,\mathbb{R}_+)$. Here, by a positive solution of (1.1) we mean a function $u\in C^{2n-1}[0,1]$ such that $(u^{(2n-1)})^{p-1}\in C[0,1]$, $u(t)>0$ for $t\in(0,1]$ and u solves (1.1).

The so-called Lidstone problem

$$
\begin{cases} (-1)^n u^{(2n)} = f(t,u,-u'',\dots,(-1)^{n-1}u^{(2n-2)}), \\ u^{(2i)}(0)=u^{(2i)}(1)=0, & i=0,\dots,n-1, \end{cases}
$$

which f involves all even derivatives explicitly, arises in many different areas of applied mathematics and physics, and has been extensively studied in recent years, for more details, the reader is referred to [1]–[5], [8], [10]–[13], [22], [24], and references cited therein.

In [28], Z. Yang studied the existence and uniqueness of positive solutions for the generalized Lidstone problem

$$
\begin{cases} (-1)^n u^{(2n)} = f(t,u,-u'',\dots,(-1)^{n-1}u^{(2n-2)}), \\ au^{(2i)}(0)-bu^{(2i+1)}(0)=0 & \text{for } i=0,\dots,n-1, \\ cu^{(2i)}(1)+du^{(2i+1)}(1)=0 & \text{for } i=0,\dots,n-1. \end{cases}
$$

where $a,b,c,d\geq 0$ with $ac+ad+bc>0$ and $f\in C([0,1]\times\mathbb{R}_+^n,\mathbb{R}_+)$. The main results obtained in [28] are formulated in terms of spectral radii of some associated linear integral operators and thus can be viewed as extensions of corresponding *sharp* results for the case $n=1$ due to Z. Liu et al. [21].

Meanwhile, because equations of the p-Laplacian form occur in the study of non-Newtonian fluid theory and the turbulent flow of a gas in a porous medium. Since 1980s, there exist a very large number of papers devoted to the existence of solutions for differential equations with p-Laplacian, see, for instance, [6], [7], [9], [14], [15], [18]–[20], [25], [27], [29]–[31] and references therein.

However, the existence of positive solutions for p-Laplacian equation with Lidstone boundary value problems has not been extensively studied yet. In [16], Y. Guo and W. Ge considered the following boundary value problems

$$
\begin{cases} (\Phi(y^{(2n-1)}))' = f(t,y,y'',\dots,y^{(2n-2)}) & \text{for } 0\leq t\leq 1, \\ y^{(2i)}(0)=y^{(2i)}(1)=0 & \text{for } i=0,\dots,n-1, \end{cases}
$$

where $f \in C([0,1]\times\mathbb{R}^n, \mathbb{R})(\mathbb{R} := (-\infty,\infty))$. Some growth conditions are imposed on f which yield the existence of at least two symmetric positive solutions by using a fixed point theorem on cones. An interesting feature in [16] is that the nonlinearity f may be sign-changing.

In [23], Wei et al. considered the existence of positive solutions for the following four-point boundary value problems with higher-order p-Laplacian operator with the nonlinear f involving all derivatives

$$\text{(BVP)} \qquad \begin{cases} (\phi_p(u^{(n-1)}))' + g(t)f(u(t), u'(t), \dots, u^{(n-2)}(t)) = 0, \\ \hfill t \in (0,1),\ n \geq 3, \\ \text{(BC)} \quad \begin{cases} u^{(i)}(0) = 0, & 0 \leq i \leq n-3, \\ \alpha u^{(n-1)}(0) - \beta u^{(n-2)}(\xi) = 0, & n \geq 3, \\ \gamma u^{(n-1)}(1) + \delta u^{(n-2)}(\eta) = 0, & n \geq 3, \end{cases} \end{cases}$$

where $0 < \xi < \eta < 1$, $\alpha > 0$, $\beta \geq 0$, $\gamma > 0$, $\delta \geq 0$. By using fixed-point index theory, they obtained the existence of positive solution and many positive solutions for the above problem. J. Zhao and W. Ge [32] reconsidered the above problem and improved the corresponding results in [23] to some extend.

Motivated by the works mentioned above, in this paper, we discuss the positive solutions for (1.1). To overcome the difficulty resulting from even derivatives, we first transform (1.1) into a boundary value problem for an associated second-order integro-ordinary differential equation. Then, using fixed point index theory, combined with a priori estimates achieved by utilizing some properties of concave functions, properties including Jensen's inequalities, we obtain some results on the existence and multiplicity of positive solutions for (1.1). Nevertheless, our methodology and results in this paper are entirely different from those in the papers cited above. We observe that if $p = 2$, then (1.1) reduces to the following Lidstone problem

$$(1.2) \quad \begin{cases} (-1)^n x^{(2n)} = f(t, x, -x'', \dots, (-1)^{n-1}x^{(2n-2)}) & \text{for } t \in [0,1], \\ x^{(2i)}(0) = x^{(2i+1)}(1) = 0 & \text{for } i = 0, \dots, n-1. \end{cases}$$

It is of interest to note that we obtain some connections between (1.1) and (1.2) by repeatedly invoking Jensen's integral inequalities in our proofs. This methodology is initially put forward by Jiafa Xu and Zhilin Yang in [26]. However, this paper deals with the nonlinearity involving all even derivatives, while the nonlinearity of [26] is independent of derivatives. Thus our main results extend and improve the corresponding ones in [26].

This paper is organized as follows. Section 2 contains some preliminary results. Section 3 is devoted to the existence and multiplicity of positive solutions for (1.1).

2. Preliminaries

The basic space used in this paper is $E := C[0,1]$. It is well known that E is a real Banach space with the norm $\|\cdot\|$ defined by $\|u\| := \max_{t\in[0,1]} |u(t)|$. Put $P := \{u \in E : u(t) \geq 0,$ for all $t \in [0,1]\}$, then P is a cone on E. We denote $B_\rho := \{u \in E : \|u\| < \rho\}$ for $\rho > 0$ in the sequel.

Let

$$u := (-1)^{n-1}x^{(2n-2)}, \quad (B_n u)(t) := \int_0^1 G_n(t,s)u(s)\,ds,$$

where

$$G_1(t,s) = \min\{t,s\}, \quad G_n(t,s) = \int_0^1 G_1(t,\tau)G_{n-1}(\tau,s)\,d\tau, \quad t,s \in [0,1],\ n \geq 2.$$

It is easy to see that problem (1.1) is equivalent to

$$\begin{cases} -((u')^{p-1})' = f\Big(t, (B_{n-1}u)(t), (B_{n-2}u)(t), \ldots, (B_1u)(t), u(t)\Big), \\ u(0) = u'(1) = 0, \end{cases}$$

which can be written in the form

$$u(t) = \int_0^t \bigg(\int_s^1 f(\tau, (B_{n-1}u)(\tau), \ldots, (B_1u)(\tau), u(\tau))\,d\tau\bigg)^{1/(p-1)} ds \\ := (Au)(t).$$

Clearly, u is increasing on $[0,1]$. Note that if $f \in C([0,1]\times\mathbb{R}_+^n, \mathbb{R}_+)$, then $A\colon P \to P$ is a completely continuous operator. Moreover, we have by the definition of G_n, $0 \leq G_n(t,s) \leq G_{n-1}(t,s) \leq \ldots \leq G_1(t,s) \leq 1$.

LEMMA 2.1. *Let $\psi(t) := \sin \pi t/2$ and $\lambda_1 = \pi^2/4$. Then $\psi \in P\setminus\{0\}$ and*

$$\psi(s) = \lambda_1^i \int_0^1 G_i(t,s)\psi(t)\,dt = \lambda_1^i (B_i\psi)(s) \quad \text{for } i = 1,2,\ldots \tag{2.1}$$

LEMMA 2.2. *Let $u \in C[0,1]$ is concave and increasing on [0,1] and $u(0) = 0$. Then*

$$\int_0^1 u(t)\psi(t)\,dt \geq \frac{4}{\pi^2}\|u\|.$$

PROOF. Since $\max_{t\in[0,1]} u(t) = u(1) = \|u\|$, then

$$\int_0^1 u(t)\psi(t)\,dt = \int_0^1 u(t\cdot 1 + (1-t)\cdot 0)\psi(t)\,dt \geq u(1)\int_0^1 t\psi(t)\,dt = \frac{4}{\pi^2}\|u\|. \qquad \square$$

LEMMA 2.3 ([17]). *Let $\Omega \subset E$ be a bounded open set and $A\colon \overline{\Omega}\cap P \to P$ is a completely continuous operator. If there exists $v_0 \in P\setminus\{0\}$ such that $v - Av \neq \lambda v_0$ for all $v \in \partial\Omega \cap P$ and $\lambda \geq 0$, then $i(A, \Omega\cap P, P) = 0$.*

Lemma 2.4 ([17]). *Let $\Omega \subset E$ be a bounded open set with $0 \in \Omega$. Suppose $A\colon \overline{\Omega} \cap P \to P$ is a completely continuous operator. If $v \neq \lambda Av$ for all $v \in \partial\Omega \cap P$ and $0 \le \lambda \le 1$, then $i(A, \Omega \cap P, P) = 1$.*

Lemma 2.5. *Let $\Omega \subset E$ be a bounded open set. Suppose $A\colon \overline{\Omega} \cap P \to P$ is a completely continuous operator. If $\|Au\| < \|u\|$ for all $u \in \partial\Omega \cap P$, then $i(A, \Omega \cap P, P) = 1$.*

Proof. We first claim $u \neq \lambda Au$, for all $u \in \partial\Omega \cap P$, $\lambda \in [0,1]$ is satisfied. If the claim is false, there exist $u_0 \in \partial\Omega \cap P$ and $\lambda_0 \in [0,1]$ such that $u_0 = \lambda_0 Au_0$. This implies $\|u_0\| = \|\lambda_0 Au_0\| \le \|Au_0\|$, contradicting $\|Au\| < \|u\|$ for all $u \in \partial\Omega \cap P$ and $0 \le \lambda \le 1$. Consequently, the above claim holds. By Lemma 2.4, we find $i(A, \Omega \cap P, P) = 1$. $\square$

Lemma 2.6 (Jensen's inequalities). *Let $\theta > 0$, $x, y \ge 0$ and $\varphi \in C([0,1], \mathbb{R}^+)$. Then*

$$\bigg(\int_0^1 \varphi(t)\,dt\bigg)^\theta \le \int_0^1 (\varphi(t))^\theta\,dt$$
$$\text{and} \quad x^\theta + y^\theta \le (x+y)^\theta \le 2^{\theta-1}x^\theta + 2^{\theta-1}y^\theta, \quad \text{for all } \theta \ge 1,$$

and

$$\bigg(\int_0^1 \varphi(t)\,dt\bigg)^\theta \ge \int_0^1 (\varphi(t))^\theta\,dt$$
$$\text{and} \quad 2^{\theta-1}x^\theta + 2^{\theta-1}y^\theta \le (x+y)^\theta \le x^\theta + y^\theta, \quad \text{for all } 0 < \theta \le 1.$$

3. Main results

For the reason of notational brevity, we denote by $y = (y_1, \dots, y_n) \in \mathbb{R}^n_+$, $p_* := \min\{1, p-1\}$, $p^* := \max\{1, p-1\}$,

$$\beta_p := \bigg[1 \Big/ 2^{p_*-1}\bigg((n-1)^{p_*-1}\sum_{i=2}^n \Big(\frac{4}{\pi^2}\Big)^i + \frac{4}{\pi^2}\bigg)\bigg]^{(p-1)/p_*},$$
$$\alpha_p := \bigg[1 \Big/ 2^{p^*-1}\bigg((n-1)^{p^*-1}\sum_{i=2}^n \Big(\frac{4}{\pi^2}\Big)^i + \frac{4}{\pi^2}\bigg)\bigg]^{(p-1)/p^*}.$$

We now list our hypotheses.

(H1) $f \in C([0,1] \times \mathbb{R}^n_+, \mathbb{R}_+)$.

(H2) There exist $a_1 > \beta_p$ and $c > 0$ such that

$$f(t,y) \ge a_1\bigg(\sum_{i=1}^n y_i\bigg)^{p-1} - c \quad \text{for all } y \in \mathbb{R}^n_+ \text{ and } t \in [0,1].$$

(H3) There exist $b_1 \in (0, \alpha_p)$ and $r > 0$ such that

$$f(t,y) \le b_1 \bigg(\sum_{i=1}^{n} y_i \bigg)^{p-1} \quad \text{for all } y \in [0,r]^n \text{ and } t \in [0,1].$$

(H4) There exist $a_2 > \beta_p$ and $r > 0$ such that

$$f(t,y) \ge a_2 \bigg(\sum_{i=1}^{n} y_i \bigg)^{p-1} \quad \text{for all } y \in [0,r]^n \text{ and } t \in [0,1].$$

(H5) There exist $b_2 \in (0, \alpha_p)$ and $c > 0$ such that

$$f(t,y) \le b_2 \bigg(\sum_{i=1}^{n} y_i \bigg)^{p-1} + c \quad \text{for all } y \in \mathbb{R}_+^n \text{ and } t \in [0,1].$$

(H6) There are $\zeta > 0$ and $\omega \in (0, p/(p-1))$ such that the inequality $f(t,y) < \omega^{p-1}\zeta^{p-1}$ holds for all $y \in [0,\zeta]^n$ and $t \in [0,1]$.

Theorem 3.1. *Suppose that* (H1)–(H3) *are satisfied. Then* (1.1) *has at least one positive solution.*

Proof. Let $\mathcal{M}_1 := \{u \in P : u = Au + \lambda\psi \text{ for some } \lambda \ge 0\}$, where $\psi(t)$ is determined by Lemma 2.1. We claim $\mathcal{M}_1$ is bounded. Indeed, $u \in \mathcal{M}_1$ implies u is concave and $u(t) \ge (Au)(t)$. For any $u \in \mathcal{M}_1$, by definition we obtain

$$u(t) \ge \int_0^t \bigg(\int_s^1 f(\tau, (B_{n-1}u)(\tau), \ldots, (B_1 u)(\tau), u(\tau))\, d\tau \bigg)^{1/(p-1)} ds.$$

Notice that $p_*, p_*/(p-1) \in (0,1]$. Now, by Jensen's inequality and (H2), we find

$$\begin{aligned}
u^{p_*}(t) &\ge \bigg[\int_0^t \bigg(\int_s^1 f(\tau, (B_{n-1}u)(\tau), \ldots, (B_1 u)(\tau), u(\tau))\, d\tau \bigg)^{1/(p-1)} ds \bigg]^{p_*} \\
&\ge \int_0^t \int_s^1 f^{p_*/(p-1)}(\tau, (B_{n-1}u)(\tau), \ldots, (B_1 u)(\tau), u(\tau))\, d\tau\, ds \\
&= \int_0^1 G_1(t,s) f^{p_*/(p-1)}(s, (B_{n-1}u)(s), \ldots, (B_1 u)(s), u(s))\, ds \\
&\ge \int_0^1 G_1(t,s) \bigg[a_1 \bigg(\sum_{i=1}^{n-1} (B_i u)(s) + u(s) \bigg)^{p-1} - c \bigg]^{p_*/(p-1)} ds \\
&\ge a_1^{p_*/(p-1)} \int_0^1 G_1(t,s) \bigg(\sum_{i=1}^{n-1} (B_i u)(s) + u(s) \bigg)^{p_*} ds \\
&\quad - c^{p_*/(p-1)} \int_0^1 G_1(t,s)\, ds \\
&\ge 2^{p_*-1} a_1^{p_*/(p-1)} \int_0^1 G_1(t,s) \bigg(\bigg(\sum_{i=1}^{n-1} (B_i u)(s) \bigg)^{p_*} + u^{p_*}(s) \bigg) ds
\end{aligned} \tag{3.1}$$

$$- c^{p_*/(p-1)} \int_0^1 G_1(t,s)\, ds.$$

Let

$$\mathcal{G}(s,\tau) := \frac{G_{n-1}(s,\tau) + \ldots + G_1(s,\tau)}{n-1} \in [0,1]. \tag{3.2}$$

Combining this and (3.1) together Jensen's inequality, we obtain

$$\begin{aligned}
u^{p_*}(t) &\geq 2^{p_*-1} a_1^{p_*/(p-1)} \int_0^1 G_1(t,s) \bigg(\Big(\int_0^1 (n-1)\mathcal{G}(s,\tau) u(\tau)\, d\tau \Big)^{p_*} + u^{p_*}(s) \bigg) ds \\
&\quad - c^{p_*/(p-1)} \int_0^1 G_1(t,s)\, ds \\
&\geq 2^{p_*-1} a_1^{p_*/(p-1)} \int_0^1 G_1(t,s) \bigg((n-1)^{p_*} \int_0^1 \mathcal{G}(s,\tau) u^{p_*}(\tau)\, d\tau + u^{p_*}(s) \bigg) ds \\
&\quad - c^{p_*/(p-1)} \int_0^1 G_1(t,s)\, ds \\
&= (2n-2)^{p_*-1} a_1^{p_*/(p-1)} \sum_{i=2}^{n} \int_0^1 G_i(t,s) u^{p_*}(s)\, ds \\
&\quad + 2^{p_*-1} a_1^{p_*/(p-1)} \int_0^1 G_1(t,s) u^{p_*}(s)\, ds - c^{p_*/(p-1)} \int_0^1 G_1(t,s)\, ds.
\end{aligned}$$

Multiply the both sides of the above by $\psi(t)$ and integrate over $[0,1]$ and use (2.1) to obtain

$$\begin{aligned}
\int_0^1 u^{p_*}(t)\psi(t)\, dt &\geq (2n-2)^{p_*-1} a_1^{p_*/(p-1)} \sum_{i=2}^{n} \lambda_1^{-i} \int_0^1 u^{p_*}(t)\psi(t)\, dt \\
&\quad + 2^{p_*-1} a_1^{p_*/(p-1)} \lambda_1^{-1} \int_0^1 u^{p_*}(t)\psi(t)\, dt - \frac{8c^{p_*/(p-1)}}{\pi^3}
\end{aligned}$$

and thus

$$\begin{aligned}
&\int_0^1 u^{p_*}(t)\psi(t)\, dt \\
&\leq 8c^{p_*/(p-1)} \Big/ \pi^3 \left[2^{p_*-1} a_1^{p_*/(p-1)} \left((n-1)^{p_*-1} \sum_{i=2}^{n} \left(\frac{4}{\pi^2} \right)^i + \frac{4}{\pi^2} \right) - 1 \right] := N_1.
\end{aligned}$$

Recall that every $u \in \mathcal{M}_1$ is concave and increasing on $[0,1]$. So is u^{p_*} with $p_* \in (0,1]$. Now Lemma 2.2 yields

$$\|u^{p_*}\| \leq \frac{\pi^2}{4} N_1 \quad \text{for all } u \in \mathcal{M}_1,$$

which implies the boundedness of $\mathcal{M}_1$, as claimed. Taking $R > \sup\{\|u\| : u \in \mathcal{M}_1\}$ and $R > r$ (r is defined by (H3)), we have $u - Au \neq \lambda\psi$, for all $u \in \partial B_R \cap P$, $\lambda \geq 0$. Now, by virtue of Lemma 2.3, we obtain

$$i(A, B_R \cap P, P) = 0. \tag{3.3}$$

Let $\mathcal{M}_2 := \{u \in \overline{B}_r \cap P : u = \lambda Au \text{ for some } \lambda \in [0,1]\}$. We shall prove $\mathcal{M}_2 = \{0\}$.

Indeed, if $u \in \mathcal{M}_2$, we have for any $u \in \overline{B}_r \cap P$

$$u(t) \leq \int_0^t \bigg(\int_s^1 f(\tau, (B_{n-1}u)(\tau), \dots, (B_1u)(\tau), u(\tau))\, d\tau\bigg)^{1/(p-1)} ds.$$

Note that $p^*, p^*/(p-1) \geq 1$. Now by (H3), (3.2) and Jensen's inequality, we obtain

$$\begin{aligned}
u^{p^*}(t) &\leq \bigg[\int_0^t \bigg(\int_s^1 f(\tau, (B_{n-1}u)(\tau), \dots, (B_1u)(\tau), u(\tau))\, d\tau\bigg)^{1/(p-1)} ds\bigg]^{p^*} \\
&\leq \int_0^t \int_s^1 f^{p^*/(p-1)}(\tau, (B_{n-1}u)(\tau), \dots, (B_1u)(\tau), u(\tau))\, d\tau\, ds \\
&= \int_0^1 G_1(t,s) f^{p^*/(p-1)}(s, (B_{n-1}u)(s), \dots, (B_1u)(s), u(s))\, ds \\
&\leq \int_0^1 G_1(t,s) b_1^{p^*/(p-1)} \bigg[\sum_{i=1}^{n-1}(B_iu)(s) + u(s)\bigg]^{p^*} ds \\
&\leq 2^{p^*-1} b_1^{p^*/(p-1)} \int_0^1 G_1(t,s)\bigg[\bigg(\sum_{i=1}^{n-1}(B_iu)(s)\bigg)^{p^*} + u^{p^*}(s)\bigg] ds \\
&= 2^{p^*-1} b_1^{p^*/(p-1)} \int_0^1 G_1(t,s)\bigg[\bigg(\int_0^1 (n-1)\mathcal{G}(s,\tau)u(\tau)\, d\tau\bigg)^{p^*} + u^{p^*}(s)\bigg] ds \\
&\leq 2^{p^*-1} b_1^{p^*/(p-1)} \int_0^1 G_1(t,s)\bigg[(n-1)^{p^*}\int_0^1 \mathcal{G}(s,\tau)u^{p^*}(\tau)\, d\tau + u^{p^*}(s)\bigg] ds \\
&= (2n-2)^{p^*-1} b_1^{p^*/(p-1)} \sum_{i=2}^{n} \int_0^1 G_i(t,s) u^{p^*}(s)\, ds \\
&\quad + 2^{p^*-1} b_1^{p^*/(p-1)} \int_0^1 G_1(t,s) u^{p^*}(s)\, ds.
\end{aligned}$$

Multiply the both sides of the above by $\psi(t)$ and integrate over $[0,1]$ and use (2.1) to obtain

$$\begin{aligned}
\int_0^1 u^{p^*}(t)\psi(t)\, dt \leq (2n-2)^{p^*-1} b_1^{p^*/(p-1)} \sum_{i=2}^{n} \lambda_1^{-i} \int_0^1 u^{p^*}(t)\psi(t)\, dt \\
+ 2^{p^*-1} b_1^{p^*/(p-1)} \lambda_1^{-1} \int_0^1 u^{p^*}(t)\psi(t)\, dt.
\end{aligned}$$

Therefore, $\int_0^1 u^{p^*}(t)\psi(t)\, dt = 0$, whence $u(t) \equiv 0$, for all $u \in \mathcal{M}_2$. As a result, $\mathcal{M}_2 = \{0\}$, as claimed. Consequently,

$$u \neq \lambda Au \quad \text{for all } u \in \partial B_r \cap P,\ \lambda \in [0,1].$$

Now Lemma 2.4 yields $i(A, B_r \cap P, P) = 1$. Combining this with (3.3) gives

$$i(A, (B_R \setminus \overline{B}_r) \cap P, P) = 0 - 1 = -1.$$

Hence the operator A has at least one fixed point on $(B_R \setminus \overline{B}_r) \cap P$ and therefore (1.1) has at least one positive solution. □

THEOREM 3.2. *Suppose that* (H1), (H4) *and* (H5) *are satisfied. Then* (1.1) *has at least one positive solution.*

PROOF. Let $\mathcal{M}_3 := \{u \in \overline{B}_r \cap P : u = Au + \lambda\psi \text{ for some } \lambda \geq 0\}$, where $\psi(t)$ is determined by Lemma 2.1. We claim $\mathcal{M}_3 \subset \{0\}$(this indicates $\mathcal{M}_3$ is either $\mathcal{M}_3 = \emptyset$ or $\mathcal{M}_3 = \{0\}$). Indeed, if $u \in \mathcal{M}_3$, then we have $u \geq Au$ by definition. Consequently, for all $u \in \mathcal{M}_3$,

$$u(t) \geq \int_0^t \left(\int_s^1 f(\tau, (B_{n-1}u)(\tau), \dots, (B_1u)(\tau), u(\tau))\, d\tau \right)^{1/(p-1)} ds.$$

Recall that $p_*, p_*/(p-1) \in (0, 1]$. Now by (H4), (3.2) and Jensen's inequality, we obtain

$$\begin{aligned}
u^{p_*}(t) &\geq \left[\int_0^t \left(\int_s^1 f(\tau, (B_{n-1}u)(\tau), \dots, (B_1u)(\tau), u(\tau))\, d\tau \right)^{1/(p-1)} ds \right]^{p_*} \\
&\geq \int_0^t \int_s^1 f^{p_*/(p-1)}(\tau, (B_{n-1}u)(\tau), \dots, (B_1u)(\tau), u(\tau))\, d\tau\, ds \\
&= \int_0^1 G_1(t,s) f^{p_*/(p-1)}(s, (B_{n-1}u)(s), \dots, (B_1u)(s), u(s))\, ds \\
&\geq \int_0^1 G_1(t,s) a_2^{p_*/(p-1)} \left[\sum_{i=1}^{n-1} (B_iu)(s) + u(s) \right]^{p_*} ds \\
&\geq 2^{p_*-1} a_2^{p_*/(p-1)} \int_0^1 G_1(t,s) \left[\left(\int_0^1 (n-1)\mathcal{G}(s,\tau)u(\tau)\, d\tau \right)^{p_*} + u^{p_*}(s) \right] ds \\
&\geq (2n-2)^{p_*-1} a_2^{p_*/(p-1)} \sum_{i=2}^{n} \int_0^1 G_i(t,s) u^{p_*}(s)\, ds \\
&\quad + 2^{p_*-1} a_2^{p_*/(p-1)} \int_0^1 G_1(t,s) u^{p_*}(s)\, ds.
\end{aligned}$$

Multiply the both sides of the above by $\psi(t)$ and integrate over $[0, 1]$ and use (2.1) to obtain

$$\begin{aligned}
\int_0^1 u^{p_*}(t)\psi(t)\, dt \geq (2n-2)^{p_*-1} a_2^{p_*/(p-1)} \sum_{i=2}^{n} \lambda_1^{-i} \int_0^1 u^{p_*}(t)\psi(t)\, dt \\
+ 2^{p_*-1} a_2^{p_*/(p-1)} \lambda_1^{-1} \int_0^1 u^{p_*}(t)\psi(t)\, dt,
\end{aligned}$$

so that $\int_0^1 u^{p_*}(t)\psi(t)\,dt = 0$, whence $u(t) \equiv 0$, for all $u \in \mathcal{M}_3$. Therefore, we claim $\mathcal{M}_3 \subset \{0\}$. As a result of this, we have

$$u - Au \neq \lambda\psi, \quad \text{for all } u \in \partial B_r \cap P,\ \lambda \geq 0.$$

Now Lemma 2.3 gives

$$i(A, B_r \cap P, P) = 0. \tag{3.4}$$

Let $\mathcal{M}_4 := \{u \in P : u = \lambda Au \text{ for some } \lambda \in [0,1]\}$. We assert $\mathcal{M}_4$ is bounded. Indeed, if $u \in \mathcal{M}_4$, then u is concave and $u \leq Au$, which can be written in the form

$$u(t) \leq \int_0^t \bigg(\int_s^1 f(\tau, (B_{n-1}u)(\tau), \ldots, (B_1u)(\tau), u(\tau))\,d\tau\bigg)^{1/(p-1)} ds,$$

for all $u \in \mathcal{M}_4$. Recall that $p^*, p^*/(p-1) \geq 1$. Now by (H5), (3.2) and Jensen's inequality, we obtain

$$\begin{aligned}
u^{p^*}(t) &\leq \bigg[\int_0^t \bigg(\int_s^1 f(\tau, (B_{n-1}u)(\tau), \ldots, (B_1u)(\tau), u(\tau))\,d\tau\bigg)^{1/(p-1)} ds\bigg]^{p^*} \\
&\leq \int_0^t \int_s^1 f^{p^*/(p-1)}(\tau, (B_{n-1}u)(\tau), \ldots, (B_1u)(\tau), u(\tau))\,d\tau\,ds \\
&= \int_0^1 G_1(t,s) f^{p^*/(p-1)}(s, (B_{n-1}u)(s), \ldots, (B_1u)(s), u(s))\,ds \\
&\leq \int_0^1 G_1(t,s)\bigg[b_2\bigg(\sum_{i=1}^{n-1}(B_iu)(s) + u(s)\bigg)^{p-1} + c\bigg]^{p^*/(p-1)} ds \\
&\leq b_3^{p^*/(p-1)} \int_0^1 G_1(t,s)\bigg(\sum_{i=1}^{n-1}(B_iu)(s) + u(s)\bigg)^{p^*} ds \\
&\quad + c_1^{p^*/(p-1)} \int_0^1 G_1(t,s)\,ds \\
&\leq 2^{p^*-1} b_3^{p^*/(p-1)} \int_0^1 G_1(t,s)\bigg[\bigg(\sum_{i=1}^{n-1}(B_iu)(s)\bigg)^{p^*} + u^{p^*}(s)\bigg] ds \\
&\quad + c_1^{p^*/(p-1)} \int_0^1 G_1(t,s)\,ds \\
&\leq (2n-2)^{p^*-1} b_3^{p^*/(p-1)} \sum_{i=2}^{n} \int_0^1 G_i(t,s) u^{p^*}(s)\,ds \\
&\quad + 2^{p^*-1} b_3^{p^*/(p-1)} \int_0^1 G_1(t,s) u^{p^*}(s)\,ds \\
&\quad + c_1^{p^*/(p-1)} \int_0^1 G_1(t,s)\,ds
\end{aligned} \tag{3.5}$$

for all $u \in \mathcal{M}_4$, $b_3 \in (b_2, \alpha_p)$ and $c_1 > 0$ being chosen so that

$$(b_2 z + c)^{p^*/(p-1)} \le b_3^{p^*/(p-1)} z^{p^*/(p-1)} + c_1^{p^*/(p-1)}, \quad \text{for all } z \ge 0.$$

Multiply the both sides of (3.5) by $\psi(t)$ and integrate over $[0,1]$ and use (2.1) to obtain

$$\int_0^1 u^{p^*}(t)\psi(t)\,dt \le (2n-2)^{p^*-1} b_3^{p^*/(p-1)} \sum_{i=2}^n \lambda_1^{-i} \int_0^1 u^{p^*}(t)\psi(t)\,dt + 2^{p^*-1} b_3^{p^*/(p-1)} \lambda_1^{-1} \int_0^1 u^{p^*}(t)\psi(t)\,dt + \frac{8c_1^{p^*/(p-1)}}{\pi^3}$$

and thus

$$\int_0^1 u^{p^*}(t)\psi(t)\,dt \le 8c_1^{p^*/(p-1)} \Big/ \pi^3 \left[1 - 2^{p^*-1} b_3^{p^*/(p-1)} \left((n-1)^{p^*-1} \sum_{i=2}^n \left(\frac{4}{\pi^2}\right)^i + \frac{4}{\pi^2}\right)\right] := N_2.$$

This, together with Jensen's inequality and $\psi(t) \in [0,1]$, leads to

$$\int_0^1 u(t)\psi(t)\,dt \le \left(\int_0^1 u^{p^*}(t)\psi^{p^*}(t)\,dt\right)^{1/p^*} \le N_2^{1/p^*}$$

for all $u \in \mathcal{M}_4$. From Lemma 2.2, we find

$$\|u\| \le \frac{\pi^2}{4} N_2^{1/p^*}, \quad \text{for all } u \in \mathcal{M}_4.$$

Now the boundedness of $\mathcal{M}_4$, as asserted. Taking $R > \sup\{\|u\| : u \in \mathcal{M}_4\}$ and $R > r$ (r is defined by (H4)), we have

$$u \ne \lambda A u, \quad \text{for all } u \in \partial B_R \cap P,\ \lambda \in [0,1].$$

Now Lemma 2.4 yields

$$i(A, B_R \cap P, P) = 1.$$

Combining this with (3.4) gives

$$i(A, (B_R \setminus \overline{B}_r) \cap P, P) = 1 - 0 = 1.$$

Hence the operator A has at least one fixed point on $(B_R \setminus \overline{B}_r) \cap P$ and therefore (1.1) has at least one positive solution. $\square$

Theorem 3.3. *Suppose that* (H1), (H2), (H4) *and* (H6) *are satisfied. Then* (1.1) *has at least two positive solutions.*

Proof. By (H6), we have

$$\begin{aligned}\|Au\| = (Au)(1) &= \int_0^1 \bigg(\int_s^1 f(\tau, (B_{n-1}u)(\tau), \dots, (B_1u)(\tau), u(\tau))\, d\tau\bigg)^{1/(p-1)} ds \\ &\le \int_0^1 \bigg(\int_s^1 \omega^{p-1}\zeta^{p-1}\, d\tau\bigg)^{1/(p-1)} ds \\ &= \omega\zeta \int_0^1 (1-s)^{1/(p-1)}\, ds = \frac{\omega\zeta p}{p-1} < \zeta\end{aligned}$$

and thus $\|Au\| < \|u\|$ for all $u \in B_\zeta \cap P$. Now Lemma 2.5 yields

$$i(A, B_\zeta \cap P, P) = 1. \tag{3.6}$$

On the other hand, in view of (H2) and (H4), we may choose $R > \zeta$ and $r \in (0, \zeta)$ so that (3.3) and (3.4) hold (see the proofs of Theorems 3.1 and 3.2). Combining (3.3), (3.4) and (3.6), we obtain

$$i(A, (B_R \setminus \overline{B}_\zeta) \cap P, P) = 0 - 1 = -1, \qquad i(A, (B_\zeta \setminus \overline{B}_r) \cap P, P) = 1 - 0 = 1.$$

Hence A has at least two fixed points, one on$(B_R \setminus \overline{B}_\zeta) \cap P$ and the other on $(B_\zeta \setminus \overline{B}_r) \cap P$. This proves that (1.1) has at least two positive solutions. □

4. Examples

In this section we offer two examples to illustrate our main results in Section 3.

Example 4.1. Let

$$f(t, y) := \bigg(\sum_{i=1}^n y_i\bigg)^\alpha, \quad t \in [0,1],\ y \in \mathbb{R}_+^n,$$

where $\alpha \in (0, p-1) \cup (p-1, \infty)$. If $\alpha \in (p-1, \infty)$, then (H2) and (H3) hold. If $\alpha \in (0, p-1)$, then (H4) and (H5) are satisfied.

Example 4.2. Let

$$f(t, y) = \eta\bigg(\bigg(\sum_{i=1}^n y_i\bigg)^a + \bigg(\sum_{i=1}^n y_i\bigg)^b\bigg),$$

where $0 < a < p-1 < b$, $0 < \eta < \omega^{p-1}/(n^a + n^b)$. Then (H2), (H4) and (H6) are satisfied with $\zeta = 1$.

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