Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder University Centre
Volume 38, 2011, 395–405

UNIFORMLY BOUNDED COMPOSITION OPERATORS BETWEEN GENERAL LIPSCHITZ FUNCTION NORMED SPACES

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Abstract. The notions of uniform boundedness and equidistant uniform boundedness of an operator (both weaker then usual boundedness) are introduced. The main results say that the generator of any uniformly bounded (or equidistantly uniformly bounded) composition Nemytskiĭ operator acting between general Lipschitzian normed function spaces must be affine with respect to the function variable.

1. Introduction

Given arbitrary nonempty sets X, Y, Z and a function $h: X \times Y \to Z$, the mapping $H: Y^{X} \to Z^X$ defined by

$$H(\varphi)(x) := h(x, \varphi(x)), \quad (x \in X),\ \varphi \in Y^X,$$

is called the composition (superposition or Nemytskiĭ) operator of a generator h. (Here Y^X denotes the set of all functions $\varphi: X \to Y$.)

The composition operators play important role in the theory of differential, integral and functional equations.

Let (X, d), (X, ρ) be metric spaces, $(Y, |\cdot|_Y)$, $(Z, |\cdot|_Z)$ be real normed spaces, $W \subset Y$ a convex set of a nonempty interior, and a function $h: X \times W \to Z$ be such

2010 *Mathematics Subject Classification.* Primary, 47H30.
Key words and phrases. Composition operator, uniformly bounded operator, Lipschitz function normed space.

that for any $x \in X$ the function $h(x, \cdot): W \to Z$ is continuous with respect to the second variable. We show that if H maps the set normed space $\mathrm{Lip}\,((X,d),W)$ of Lipschitzian functions into the normed space $\mathrm{Lip}\,((X,\rho),Z)$ and H is *uniformly bounded* (Definition 4.1), then

$$h(x,y) = a(x)y + b(x), \quad x \in X,\ y \in Y,$$

where $a \in \mathrm{Lip}\,((X,\rho),(L(Y,Z),\|\cdot\|))$ that is, a is a Lipschitz mapping of the metric space (X,ρ) into the normed space $L(Y,Z)$ of all linear continuous mappings of Y into Z, and $b \in \mathrm{Lip}\,((X,\rho),Z)$.

This is a special consequence of a more general Theorem 4.3. Let us add that the uniform boundedness is a much weaker assumption that the boundedness of an operator. A weaker condition of the *equidistant uniform boundedness* is also considered (Theorem 4.9). Theorem 4.3 extends and improves the earlier results in [3], [4] (cf. also [1]) where H is assumed to be Lipschitzian and in [6] where H is assumed to be uniformly continuous. As a corollary we obtain a suitable result for the Banach space of Hölder functions (Remark 3.4). Moreover, in all these papers X is assumed to be a real interval, $d = \rho$ is the euclidean metric and $Y = Z = \mathbb{R}$.

2. Preliminaries and auxiliary results

Let X be a set and $(Y, |\cdot|_Y)$ be a real normed space. By $\mathrm{B}(X,(Y,|\cdot|_Y))$, briefly $\mathrm{B}(X,Y)$, we denote the family of all bounded functions $\varphi \in Y^X$; more precisely, $\varphi \in \mathrm{B}(X,Y)$ if

$$\|\varphi\|_Y := \sup\{|\varphi(x)|_Y : x \in X\} < \infty.$$

Of course, $(\mathrm{B}(X,Y), \|\cdot\|_Y)$ is a normed space.

In the sequel (X, d_X) is a metric space.

By $\mathrm{C}_0((X,d_X),(Y,|\cdot|_Y))$, briefly $\mathrm{C}_0(X,Y)$, we denote the family of all continuous functions $\varphi \in Y^X$ and we put

$$\mathrm{C}(X,Y) := \mathrm{C}_0(X,Y) \cap \mathrm{B}(X,Y).$$

REMARK 2.1. The pair $(\mathrm{C}(X,Y), \|\cdot\|_Y)$ is a normed subspace of the space $(\mathrm{B}(X,Y), \|\cdot\|_Y)$, and the convergence with respect to the norm $\|\cdot\|_Y$ is uniform.

By $\mathrm{Lip}\,((X,d_X),(Y,|\cdot|_Y))$, briefly $\mathrm{Lip}\,(X,Y)$, we denote the family of all Lipschitz functions $\varphi \in Y^X$, i.e. such that

$$L(\varphi) := \sup\left\{\frac{|\varphi(x)-\varphi(\overline{x})|_Y}{d_X(x,\overline{x})} : x,\ \overline{x} \in X,\ x \neq \overline{x}\right\} < \infty.$$

Given $x_0 \in X$, define the function $\|\cdot\|_{\mathrm{Lip}\,(X,Y),x_0}: \mathrm{Lip}\,(X,Y) \to [0,\infty)$ by

$$\|\varphi\|_{\mathrm{Lip}\,(X,Y),x_0} := |\varphi(x_0)|_Y + L(\varphi), \quad \varphi \in \mathrm{Lip}\,(X,Y).$$

Remark 2.2. The pair $(\mathrm{Lip}\,(X,Y), \|\varphi\|_{\mathrm{Lip}\,(X,Y),x_0})$ is a normed space. Moreover, for any $x_0, x_1 \in X$, the norms $\|\varphi\|_{\mathrm{Lip}\,(X,Y),x_0}$ and $\|\varphi\|_{\mathrm{Lip}\,(X,Y),x_1}$ are equivalent.

Indeed,

$$\begin{aligned}
|\varphi(x_1)|_Y &= |\varphi(x_0) + \frac{\varphi(x_1)-\varphi(x_0)}{d_X(x_1,x_0)} d_X(x_1,x_0)|_Y \\
&\le |\varphi(x_0)|_Y + \frac{|\varphi(x_1)-\varphi(x_0)|_Y}{d_X(x_1,x_0)} d_X(x_1,x_0) \\
&\le |\varphi(x_0)|_Y + L(\varphi) d_X(x_1,x_0)
\end{aligned}$$

whence

$$\begin{aligned}
\|\varphi\|_{\mathrm{Lip}\,(X,Y),x_1} &= |\varphi(x_1)|_Y + L(\varphi) \\
&\le |\varphi(x_0)|_Y + L(\varphi)(d_X(x_1,x_0)+1) \\
&\le (d_X(x_1,x_0)+1)(|\varphi(x_0)|_Y + L(\varphi)) \\
&\le (d_X(x_1,x_0)+1)\|\varphi\|_{\mathrm{Lip}\,(X,Y),x_0}.
\end{aligned}$$

Remark 2.3. Taking into account the previous remark, in the sequel, to simplify the notation, we shall write $\|\varphi\|_{\mathrm{Lip}\,(X,Y)}$ instead of $\|\varphi\|_{\mathrm{Lip}\,(X,Y),x_0}$.

Put

$$\mathrm{BLip}\,(X,Y) := \mathrm{B}(X,Y) \cap \mathrm{Lip}\,(X,Y).$$

The pair $(\mathrm{BLip}\,(X,Y), \|\cdot\|_{\mathrm{Lip}\,(X,Y)})$ where $\|\cdot\|_{\mathrm{Lip}\,(X,Y)}\colon \mathrm{BLip}\,(X,Y) \to [0,\infty)$ is defined by

$$\|\varphi\|_{\mathrm{Lip}\,(X,Y)} := \|\varphi\|_Y + L(\varphi)$$

is a normed space. The space $\mathrm{BLip}\,(X,Y)$ is a linear subspace of $\mathrm{Lip}\,(X,Y)$. Moreover, in $\mathrm{BLip}\,(X,Y)$, the norms $\|\cdot\|_{\mathrm{Lip}\,(X,Y)}$ and $\|\cdot\|_{\mathrm{Lip}\,(X,Y),x_0}$ are equivalent for any $x_0 \in X$.

For a set $W \subset Y$ we put

$$\mathrm{Lip}\,((X,d),W) := \{\varphi \in \mathrm{Lip}\,((X,d),Y) : \varphi(X) \subset W\}.$$

In the proof of the main result we need the following

Lemma 2.4. *Let (X,d) be a metric space, $(Y,|\cdot|_Y)$ be a real normed space, $W \subset Y$ a nonempty convex set, and let $x, \overline{x} \in X$, $x \neq \overline{x}$ be fixed. Then, for arbitrary $y, \overline{y} \in Y$, the function $\varphi_{y,\overline{y}}\colon X \to Y$ defined by*

$$\varphi_{y,\overline{y}}(t) := \frac{d(t,\overline{x})y + d(t,x)\overline{y}}{d(t,\overline{x}) + d(t,x)}, \quad t \in X$$

has the following properties:

(a) *$\varphi_{y,\overline{y}}(x) = y$, $\varphi_{y,\overline{y}}(\overline{x}) = \overline{y}$;*

(b) $\varphi_{y,\overline{y}} \in \mathrm{Lip}\,((X,d),Y)$ *and*

$$L(\varphi_{y,\overline{y}}) = \frac{|\overline{y}-y|_Y}{d(\overline{x},x)};$$

(c) *if* $y,\overline{y} \in W$ *then* $\varphi_{y,\overline{y}} \in \mathrm{Lip}\,((X,d),W)$;

(d) *the set*

$$\mathcal{K}(x,\overline{x}) := \{\varphi_{y,\overline{y}} : y,\overline{y} \in Y\}$$

is a linear subspace of $\mathrm{Lip}\,((X,d),Y)$ *containing all constant functions.*

Proof. Property (a) is obvious. To prove (b) note that, for $s,t \in X$, making simple calculations and applying twice the triangle inequality, we obtain

$$\begin{aligned}
|\varphi_{y,\overline{y}}(s) &- \varphi_{y,\overline{y}}(t)|_Y \\
&= \left| \frac{d(s,\overline{x})y + d(s,x)\overline{y}}{d(s,\overline{x}) + d(s,x)} - \frac{d(t,\overline{x})y + d(t,x)\overline{y}}{d(t,\overline{x}) + d(t,x)} \right|_Y \\
&= \frac{|d(s,\overline{x})d(t,x) - d(s,x)d(t,\overline{x})|}{[d(s,\overline{x}) + d(s,x)][d(t,\overline{x}) + d(t,x)]} |\overline{y}-y|_Y \\
&= \frac{|d(t,x)[d(s,\overline{x}) - d(t,\overline{x})] + d(t,\overline{x})[d(t,x) - d(s,x)]|}{[d(s,\overline{x}) + d(s,x)][d(t,\overline{x}) + d(t,x)]} |\overline{y}-y|_Y \\
&\le \frac{d(t,x)|d(s,\overline{x}) - d(t,\overline{x})| + d(t,\overline{x})|d(t,x) - d(s,x)|}{[d(s,\overline{x}) + d(s,x)][d(t,\overline{x}) + d(t,x)]} |\overline{y}-y|_Y \\
&\le \frac{d(t,x)d(s,t) + d(t,\overline{x})d(t,s)}{[d(s,\overline{x}) + d(s,x)][d(t,\overline{x}) + d(t,x)]} |\overline{y}-y|_Y \\
&= \frac{d(s,t)\,[d(t,x) + d(t,\overline{x})]}{[d(s,\overline{x}) + d(s,x)][d(t,\overline{x}) + d(t,x)]} |\overline{y}-y|_Y \\
&= \frac{d(s,t)}{d(s,\overline{x}) + d(s,x)} |\overline{y}-y|_Y \le \frac{d(s,t)}{d(\overline{x},x)} |\overline{y}-y|_Y
\end{aligned}$$

whence, if $s \neq t$,

$$\frac{|\varphi_{y,\overline{y}}(s) - \varphi_{y,\overline{y}}(t)|_Y}{d(s,t)} \le \frac{|\overline{y}-y|_Y}{d(\overline{x},x)}.$$

For $s = \overline{x}$ and $t = x$, by property (a), this inequality becomes equality. It follows that $L(\varphi_{y,\overline{y}}) = |\overline{y}-y|_Y / d(\overline{x},x)$. Property (c) follows from the convexity of W and from the fact that, by the definition, $\varphi_{y,\overline{y}}$ is a convex combination of $y,\overline{y} \in W$. Property (d) is obvious. $\square$

By $\mathrm{K}(x,\overline{x};W)$ denote the set of all functions $\varphi \in \mathcal{K}(x,\overline{x})$ with values in W, that is

$$\mathrm{K}(x,\overline{x};W) := \mathcal{K}(x,\overline{x}) \cap \mathrm{Lip}\,((X,d),W).$$

3. Nonlinear-Lipschitz composition operators

For the normed spaces $(Y,|\cdot|_Y)$, $(Z,|\cdot|_Z)$, by $(L(Y,Z),\|\cdot\|)$, briefly $L(Y,Z)$, we denote the normed space of all linear and continuous mappings $a\colon Y \to Z$. Moreover $\mathrm{Lip}\,((X,\rho),L(Y,Z))$ stands for $\mathrm{Lip}\,((X,\rho),(L(Y,Z),\|\cdot\|))$.

Proposition 3.1. *Let (X,d), (X,ρ) be metric spaces, $(Y,|\cdot|_Y),(Z,|\cdot|_Z)$ be real normed spaces, $W\subset Y$ a convex set such that $\operatorname{int}W\neq\emptyset$, and a function $h\colon X\times W\to Z$ be such that, for any $x\in X$, the function $h(x,\cdot)\colon W\to Z$ is continuous with respect to the second variable. Suppose that for all $x,\overline{x}\in X$, $x\neq\overline{x}$, the composition operator H of the generator h maps the set $K(x,\overline{x};W)$ into the normed space $\operatorname{Lip}((X,\rho),Z)$. If for all $x,\overline{x}\in X$, $x\neq\overline{x}$, the operator H satisfies the inequality*

$$\|H(\varphi)-H(\psi)\|_{\operatorname{Lip}((X,\rho),Z)}\le\gamma(\|\varphi-\psi\|_{\operatorname{Lip}((X,d),Y)}),\quad \varphi,\psi\in \mathrm{K}(x,\overline{x};W),\tag{3.1}$$

for some function $\gamma\colon[0,\infty)\to[0,\infty)$, then there exist $a\in L(Y,Z)^X$ and $b\in Z^X$ such that

$$h(x,y)=a(x)y+b(x),\quad x\in X,\ y\in W.$$

If moreover, γ is bounded in a right-hand side neighbourhood of 0, then $a\in\operatorname{Lip}((X,\rho),L(Y,Z))$ and $b\in\operatorname{Lip}((X,\rho),Z)$.

Proof. Recall that

$$\|\varphi\|_{\mathrm{Lip}}:=|\varphi(x_0)|_Y+\sup_{s,t\in X,\ s\neq t}\frac{|\varphi(s)-\varphi(t)|_Y}{d(s,t)},$$

where x_0 can be arbitrarily fixed (cf. Remark 2.2).

By the lemma, for arbitrary $y\in W$ the constant function $\varphi(t)=y$, ($t\in X$), belongs to $\mathrm{K}(x,\overline{x};W)$. Since H maps $\mathrm{K}(x,\overline{x};W)$ into $\operatorname{Lip}((X,\rho),Z)$, the function $H(\varphi)=h(\cdot,y)\in\operatorname{Lip}((X,\rho),Z)$ and, consequently, h is continuous with respect to the first variable.

Let us fix $x,\overline{x}\in X$, $x\neq\overline{x}$, and take arbitrary y_1, y_2, $\overline{y}_1$, $\overline{y}_2\in W$. Since, in view of the lemma, the functions $\varphi_{y_1,\overline{y}_1}$, $\varphi_{y_2,\overline{y}_2}$ belong to $\mathrm{K}(x,\overline{x};W)$ and, $\mathcal{K}(x,\overline{x})$ is a linear space, we have

$$L(\varphi_{y_1,\overline{y}_1}-\varphi_{y_2,\overline{y}_2})=\frac{|y_1-y_2-\overline{y}_1+\overline{y}_2|_Y}{d(\overline{x},x)},$$

and

$$\|\varphi_{y_1,\overline{y}_1}-\varphi_{y_2,\overline{y}_2}\|_{\operatorname{Lip}((X,d),Y)}=|(\varphi_{y_1,\overline{y}_1}-\varphi_{y_2,\overline{y}_2})(x_0)|+\frac{|y_1-y_2-\overline{y}_1+\overline{y}_2|_Y}{d(\overline{x},x)}.$$

Note that modifying, if necessary, the function γ, the norms occurring in inequality (3.1) can be replaced by any equivalent ones. Therefore, taking into account Remark 2.2, we may assume that the point $x_0\in X$ in the definition of the norm $\|\varphi\|_{\operatorname{Lip}(X,Y),x_0}$ coincides with x. Then

$$\|\varphi_{y_1,\overline{y}_1}-\varphi_{y_2,\overline{y}_2}\|_{\operatorname{Lip}((X,d),Y)}=|y_1-y_2|+\frac{|y_1-y_2-\overline{y}_1+\overline{y}_2|_Y}{d(\overline{x},x)}.$$

Since, applying (3.1) with $\varphi = \varphi_{y_1,\overline{y}_1}$, $\psi = \varphi_{y_2,\overline{y}_2}$,

$$\frac{|[H(\varphi) - H(\psi)](x) - [H(\varphi) - H(\psi)](\overline{x})|}{\rho(\overline{x}, x)} \le \|H(\varphi_1) - H(\varphi_2)\|_{\mathrm{Lip}\,((X,\rho),Z)},$$

we hence get

$$\frac{|h(x, y_1) - h(x, y_2) - h(\overline{x}, \overline{y}_1) + h(\overline{x}, \overline{y}_2)|_Z}{\rho(\overline{x}, x)} \le \gamma\left(|y_1 - y_2| + \frac{|y_1 - y_2 - \overline{y}_1 + \overline{y}_2|_Y}{d(\overline{x}, x)}\right)$$

for all $x, \overline{x} \in X$, $x \ne \overline{x}$; y_1, y_2, $\overline{y}_1$, $\overline{y}_2 \in W$.

Taking arbitrary $u, v \in W$ and setting here

$$y_1 = \tau u + (1-\tau)v, \quad \overline{y}_2 = (1-\tau)u + \tau v, \quad y_2 = u, \quad \overline{y}_1 = v, \quad (\tau \in [0,1])$$

(which can be done by the convexity of W), we obtain

$$|h(x, \tau u + (1-\tau)v) - h(x, u) - h(\overline{x}, v) + h(\overline{x}, (1-\tau)u + \tau v)|_Z \le \rho(x, \overline{x})\gamma((1-\tau)|u - v|_Y)$$

for all $x, \overline{x} \in X$, $x \ne \overline{x}$, $u, v \in W$.

Letting here $\overline{x}$ tend to x and making use of the continuity of h with respect to the first variable, we hence get

$$h(x, \tau u + (1-\tau)v) + h(x, (1-\tau)u + \tau v) = h(x, u) + h(x, v)$$

for all $x \in X$, $u, v \in W$ and $\tau \in [0,1]$.

For $\tau = 1/2$ we hence get that, for every $x \in X$, the function $h(x, \cdot)$ is Jensen affine in W, that is

$$2h\left(x, \frac{u+v}{2}\right) = h(x, u) + h(x, v), \quad u, v \in W. \tag{3.2}$$

In the sequel we apply the method used in M. Kuczma [2, p. 314–315]. Let us fix arbitrary $x \in X$, $y_0 \in W$. Put

$$W_0 := W - y_0$$

and

$$g(y) := h(x, y + y_0) - h(x, y_0), \quad y \in W_0. \tag{3.3}$$

From (3.2), for all $y_1, y_2 \in W_0$, we get

$$g\left(\frac{y_1 + y_2}{2}\right) = \frac{g(y_1) + g(y_2)}{2},$$

which shows that the function g is Jensen affine in W_0. Since $0 \in W_0$ and $g(0) = 0$, by induction, we hence get that

$$2^k g\left(\frac{y}{2^k}\right) = g(y), \quad y \in W_0,\ k \in \mathbb{N}.$$

Denote by Y_W the span of W_0, that is the set of all $y \in Y$ such that

$$y = \sum_{i=1}^{n} \tau_i w_i \quad \text{for some } n \in \mathbb{N},\ \tau_i \in \mathbb{R},\ w_i \in W_0,\ i = 1, \dots, n.$$

Of course, for every $y \in Y_W$ there is a positive integer $n = n(y)$ such that $2^{-n} y \in W_0$. Put

$$f(y) := 2^n g\left(\frac{y}{2^n}\right), \quad y \in Y_W,\ n = n(y).$$

Since the value $f(y)$ does not depend on the choice of $n = n(y)$, the function $f: Y_W \to Z$ is a correctly defined. As for $y \in W_0$ one can choose $n(y) = 1$, we have

$$f(y) = g(y), \quad y \in W_0.$$

Now, taking into account that g is Jensen affine, it is easy to verify that, for arbitrary arbitrary $y_1, y_2 \in Y_W$,

$$f(y_1 + y_2) = f(y_1) + f(y_2),$$

that is f is additive in Y_W. The continuity of $h(x, \cdot\,)$ implies the continuity of f. Consequently, being additive, f must be linear, i.e. $f \in L(Y, Z)$. For $y \in W$ we have $y - y_0 \in W_0$. Hence, making use of the definition (3.3) of g, we get

$$\begin{aligned} h(x, y) &= h(x, (y - y_0) + y_0) = g(y - y_0) + h(x, y_0) \\ &= f(y - y_0) + h(x, y_0) = f(y) - f(y_0) + h(x, y_0) \end{aligned}$$

whence, setting $a(x) := f$ and $b(x) := h(x, y_0) - f(y_0)$, we obtain

$$h(x, y) = a(x)y + b(x), \quad x \in X,\ y \in W. \tag{3.4}$$

Of course, $b \in \operatorname{Lip}((X, \rho), Z)$. To show the "moreover" part, assume that there are $r_0 > 0$ and $M > 0$ such that $\gamma(\tau) \le M$ for all $\tau \in [0, r_0)$. Take $\varphi, \psi \in \operatorname{Lip}((X, d), W)$ and put $\phi := \varphi - \psi$. From (3.4) and (3.1), for all $s, t \in X$, $s \ne t$,

$$\begin{aligned} \frac{|a(s)\phi(s) - a(t)\phi(t)|_Z}{\rho(s, t)} &= \frac{|(H(\varphi) - H(\psi))(s) - (H(\varphi) - H(\psi))(t)|_Z}{\rho(s, t)} \\ &\le L(H(\varphi) - H(\psi)) \le \|H(\varphi) - H(\psi)\|_{\operatorname{Lip}((X,\rho),Z)} \\ &\le \gamma(\|\varphi - \psi\|_{\operatorname{Lip}((X,d),Y)}) = \gamma(\|\phi\|_{\operatorname{Lip}((X,d),Y)}). \end{aligned}$$

As $\operatorname{int} W \ne \emptyset$, the set $W - W := \{y_1 - y_2 : y_1, y_2 \in W\}$ is a neighbourhood of $0 \in Y$. Take arbitrary $u \in W - W$, $|u|_Z \le r_0$ and the constant functions

$\varphi, \psi \in \mathrm{K}(x, \overline{x}; W)$, $\varphi = y_1$ and $\psi = y_2$ such that $\varphi - \psi = u$. Now, from the above inequalities we get

$$\frac{|(a(s) - a(t))u|_Z}{\rho(s,t)} \leq \gamma(\|u\|_{\mathrm{Lip}\,((X,d),Y)}) = \gamma(|u|_Y) \leq M,$$

for all $s, t \in X$, $s \neq t$, and $u \in W - W$. Since

$$\frac{a(s) - a(t)}{d(s,t)} \in L(Y, Z), \quad s, t \in X, \ s \neq t,$$

it follows that

$$\left\| \frac{a(s) - a(t)}{\rho(s,t)} \right\|_{L(Y,Z)} \leq M, \quad s, t \in X, \ s \neq t,$$

which shows that $a \in \mathrm{Lip}\,((X, \rho), L(Y, Z))$. □

Remark 3.2. If the function $\gamma\colon [0, \infty) \to [0, \infty)$ is right continuous at 0 and $\gamma(0) = 0$, then the assumption of the continuity of h with respect to the second variable can be omitted, as it follows from (3.1).

Note that in the first part of the Proposition 3.1 the function $\gamma\colon [0, \infty) \to [0, \infty)$ is completely arbitrary.

As an immediate corollary of the Proposition 3.1 we obtain the following

Theorem 3.3. *Let (X, d), (X, ρ) be metric spaces, $(Y, |\cdot|_Y)$, $(Z, |\cdot|_Z)$ be real normed spaces, $W \subset Y$ a convex set such that* $\mathrm{int}\, W \neq \emptyset$*, and a function $h\colon X \times W \to Z$ be such that for any $x \in X$ the function $h(x, \cdot)\colon W \to Z$ is continuous with respect to the second variable. Suppose that the composition operator H of the generator h maps* $\mathrm{Lip}\,((X, d), W)$ *into the normed space* $\mathrm{Lip}\,((X, \rho), Z)$*. If there exists a function $\gamma\colon [0, \infty) \to [0, \infty)$, bounded from above in a right neighbourhood of* 0*, such that*

$$\|H(\varphi) - H(\psi)\|_{\mathrm{Lip}\,((X,\rho),Z)} \leq \gamma(\|\varphi - \psi\|_{\mathrm{Lip}\,((X,d),Y)}), \quad \varphi, \psi \in \mathrm{Lip}\,((X, d), W),$$

then

$$h(x, y) = a(x)y + b(x), \quad x \in X, \ y \in W,$$

for some $a \in \mathrm{Lip}\,((X, \rho), L(Y, Z))$, and $b \in \mathrm{Lip}\,((X, \rho), Z)$.

Remark 3.4. If $\gamma(t) := ct$ for some $c \geq 0$, then inequality (3.1) becomes the classical Lipschitz condition. Applying Theorem 3.3 and Remark 3.2 we obtain the improvements of the relevant earlier results for Lipschitzian Nemytskiĭ operators.

For instance, taking (X, d) and (X, ρ) such that X is real interval,

$$d(x, y) := |x - y|^{\alpha}, \quad \rho(x, y) := |x - y|^{\beta} \quad \text{for some } \alpha, \beta \in (0, 1],$$

and $(Y, |\cdot|_Y) = (Z, |\cdot|_Z) = (\mathbb{R}, |\cdot|)$ with $|\cdot|_Y = |\cdot|_Z := |\cdot|$, we obtain the main results of [6]. In the case $\alpha = \beta = 1$ one gets the first result of that type proved in [3].

4. Uniformly bounded composition operators

Definition 4.1. Let $\mathcal{Y}$ and $\mathcal{Z}$ be two metric (or normed) spaces. We say that a mapping $H\colon \mathcal{Y} \to \mathcal{Z}$ is *uniformly bounded* if, for any $t > 0$, there is a real number $\gamma(t)$ such that for any nonempty set $B \subset \mathcal{Y}$ we have

$$\operatorname{diam} B \le t \;\Rightarrow\; \operatorname{diam} H(B) \le \gamma(t).$$

Remark 4.2. Obviously, every uniformly continuous operator or Lipschitzian operator is uniformly bounded. Note that, under the assumptions of this definition, every bounded operator is uniformly bounded.

The main result of this paper reads as follows

Theorem 4.3. *Let (X,d), (X,ρ) be metric spaces, $(Y, |\cdot|_Y), (Z, |\cdot|_Z)$ be real normed spaces, $W \subset Y$ a convex set such that* $\operatorname{int} W \ne \emptyset$, *and a function $h\colon X \times W \to Z$ be such that for any $x \in X$ the function $h(x,\cdot)\colon W \to Z$ is continuous with respect to the second variable. Suppose that the composition operator H of the generator h maps the set* $\operatorname{Lip}((X,d),W)$ *into the space* $\operatorname{Lip}((X,\rho),Z)$.

If H is uniformly bounded, then there exist $a \in \operatorname{Lip}((X,\rho), L(Y,Z))$, and $b \in \operatorname{Lip}((X,\rho),Z)$ such that

$$h(x,y) = a(x)y + b(x), \quad x \in X,\ y \in W,$$

and

$$H(\varphi)(x) = a(x)\varphi(x) + b(x), \quad \varphi \in \operatorname{Lip}((X,d),W)\ (x \in X).$$

Proof. Take any $t \ge 0$ and arbitrary $\varphi, \psi \in \operatorname{Lip}((X,d),W)$ such that $\|\varphi - \psi\|_{\operatorname{Lip}((X,d),Y)} \le t$. Since $\operatorname{diam}\{\varphi,\psi\} \le t$, by the uniform boundedness of H, we have $\operatorname{diam} H(\{\varphi,\psi\}) \le \gamma(t)$, i.e.

$$\|H(\varphi) - H(\psi)\|_{\operatorname{Lip}((X,\rho),Z)} = \operatorname{diam} H(\{\varphi,\psi\}) \le \gamma(\|\varphi - \psi\|_{\operatorname{Lip}((X,d),Y)}),$$

and the result follows from Theorem 3.3. □

Remark 4.4. If the function $\gamma\colon [0,\infty) \to [0,\infty)$ in the Definition 4.1 is right-continuous at 0 and $\gamma(0) = 0$ (or if only $\gamma(0+) = 0$), then, clearly, the uniform boudedness of the involved operator reduces to its uniform continuity.

It follows that Theorem 4.3 improves the result of [6] where H is assumed to be uniformly continuous.

Remark 4.5. Under the assumptions of Theorem 4.3, the generator h of the operator H is an affine function with respect to the second variable, i.e. it has the following property: for any $x \in X$, the function $y \ni Y \to h(x,y)$ is the sum of the linear operator $a(x): Y \to Z$ and the vector $b(x)$.

Consider the following

Example 4.6. Take $X = [0,1]$, $Y = Z = W := \mathbb{R}$ and let $H: Y^X \to Z^X$ be the composition Nemytskij operator generated by the function $h: X \times Y \to Z$ given by
$$h(x,y) := \sin y, \quad x \in X, \ y \in \mathbb{R}.$$
Then, obviously, H maps $\mathrm{C}(X,Y)$ into $\mathrm{C}(X,Z)$,
$$\|H(\varphi) - H(\psi)\|_\infty \le \|\varphi - \psi\|_\infty, \quad \varphi, \psi \in C(X,Y),$$
that is H satisfies the uniform boundedness condition with the function $\gamma(t) = t$, $t \ge 0$. However, the generator h of H is not affine in the second variable.

It is also easy to see that, for any $p \in [1,\infty]$, the operator H maps $\mathrm{L}^p([0,1],\mathbb{R})$ into itself and
$$\|H(\varphi) - H(\psi)\|_p \le \|\varphi - \psi\|_p, \quad \varphi, \psi \in \mathbf{L}^p([0,1],\mathbb{R}).$$

Thus for the Banach function spaces $\mathbf{C}([0,1])$ and $\mathrm{L}^p([0,1],\mathbb{R})$ the counterparts of Theorem 4.3 are not true.

Remark 4.7. Let $(\mathrm{F}([0,1],\mathbb{R}), \|\cdot\|)$ with $\mathrm{F}([0,1],\mathbb{R}) \subset \mathbb{R}^{[0,1]}$ be a real normed function space. This example shows that if a counterpart of Theorem 4.3 holds true for the space $(\mathrm{F}([0,1],\mathbb{R}), \|\cdot\|)$, then the norm $\|\cdot\|$ must be essentially stronger than $\|\cdot\|_\infty$.

Our results show that it is a sufficient condition if the $\|\varphi\|$ depends on the slope between any different points $(s,\varphi(s))$, $(t,\varphi(t))$ of the graph of φ.

Consider the following

Definition 4.8. Let $\mathcal{Y}$ and $\mathcal{Z}$ be two metric (or normed) spaces. We say that a mapping $H: \mathcal{Y} \to \mathcal{Z}$ is *equidistantly uniformly bounded* if, for every $t > 0$ there is a nonnegative real number $\gamma(t) \ge 0$ such that for all $u, v \in B \subset \mathcal{Y}$,
$$\operatorname{diam}\{u,v\} = t \ \Rightarrow \ \operatorname{diam}\{H(u),H(v)\} \le \gamma(t).$$

Clearly, the equidistant uniform boundedness is essentially weaker condition than the uniform boundedness. The following result is a simple consequence of the Proposition 3.1.

Theorem 4.9. *Let (X,d), (X,ρ) be metric spaces, $(Y,|\cdot|_Y)$, $(Z,|\cdot|_Z)$ be real normed spaces, $W \subset Y$ a convex set such that* $\operatorname{int} W \neq \emptyset$*, and a function $h: X \times W \to Z$ be such that for any $x \in X$ the function $h(x,\cdot): W \to Z$ is continuous with respect to the second variable. Suppose that the composition operator H of the generator h maps the set* $\mathrm{Lip}\,((X,d),W)$ *into the space* $\mathrm{Lip}\,((X,\rho),Z)$*. If H*

is equidistantly uniformly bounded, then there exist $a \in L(Y,Z)^X$*, and* $b \in Z^X$ *such that*

$$h(x,y) = a(x)y + b(x), \quad x \in X, \ y \in W.$$

The assumption of the uniform boundedness and the equidistant uniform boundedness can be used to improve and extend some results Lipschitzian or uniformly continuous Nemytskiĭ operators for other normed function spaces as well as their multivalued counterparts.

Let us mention that Nemytskij composition operators are locally defined (cf. for instance [7]). The applicability of the fixed point methods in solving some functional equations involving these operators is discussed in [5].

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Manuscript received August 10, 2010

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