**T**opological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder University Centre
Volume 38, 2011, 373–393

# CONLEY INDEX OF ISOLATED EQUILIBRIA

Martin Kell

Abstract. In this paper we study stable isolated invariant sets and show that the zeroth singular homology of the Conley index characterizes stability completely. Furthermore, we investigate isolated mountain pass points of gradient-like semiflows introduced by Hofer in [4] and show that the first singular homology characterizes them completely.

The result of the last section shows that for reaction-diffusion equations

$$\begin{aligned} u_t - \Delta u &= f(u), \\ u_{|\partial\Omega} &= 0, \end{aligned}$$

the Conley index of isolated mountain pass points is equal to $\Sigma^1$ – the pointed 1-sphere. Finally we generalize the result of [1, Proposition 3.3] about mountain pass points to Alexander–Spanier cohomology.

## 1. Introduction

Some chemical reactions are described by so called reaction diffusion equations, i.e. (non-linear) parabolic partial differential equations like

$$\begin{aligned} u_t - \Delta u &= f(u), \\ u_{|\partial\Omega} &= 0, \end{aligned}$$

2010 *Mathematics Subject Classification.* 37B30, 58E05.

*Key words and phrases.* Conley index, attractor, moumtain pass point.

This paper is developed from the author's master thesis (Diplomarbeit). The author would like to thank his adviser Prof. Krzysztof Rybakowski for encouraging him to use Conley index methods and "to focus on the given semiflow".

for some $\Omega \subset \mathbb{R}^n$ with "nice" boundary. This equation induces a semiflow whose equilibria are solutions of the non-linear partial differential equation

$$\begin{aligned} -\Delta u &= f(u), \\ u_{|\partial\Omega} &= 0. \end{aligned}$$

Frequently these equilibria are isolated, that is there is no other equilibrium in a small neighbourhood of a solution. A mathematical tool to describe equilibria is the Conley homotopy index. This index has the advantage that it stays constant under perturbations of the equation.

We are going to investigate two classes of isolated equilibria: stable equilibria and "mountain pass" equilibria. If an isolated equilibrium $u_0$ attracts a small neighbourhood we call it stable. In particular this means that there is no $u$ close to $u_0$ which leaves that neighbourhood. In connection with critical point theory we will show that the frequently used statement "because the exit set is non-empty, $H_0$ is trivial" is true.

Mountain pass points were defined the first time by Hofer in [4] and the topological degree was calculated in [5]. A complete characterization via critical groups was given in [1]. All these calculations use the energy functional and its Palais–Smale condition.

Here we use a given flow and its energy function directly. Sometimes this will lead to simpler and/or more general result. We show that isolated stable equilibria are completely characterized by the zeroth homology index and isolated mountain pass points by the first homology index. Finally we will compute the homotopy index of mountain pass points for reaction diffusion equations.

## 2. Preliminaries

**2.1. Connectedness of topological spaces.** In this section we will state some simple result about path-connectedness and deformations of topological spaces. For precise definitions see [8] or [2].

Let $X$ be a topological space. For each $x \in X$ we will denote by $\mathrm{PC}_x$ the path component containing $x$, i.e. $y \in \mathrm{PC}_x$ if there is some continuous map $f\colon [0,1] \to X$ such that $f(0) = x$ and $f(1) = y$. In case of several spaces we will use $\mathrm{PC}_x^X$.

If $\emptyset \neq A \subset X$ then the quotient space $X/A$ is defined by identifying all points of $A$ to some point $[A]$. If $A = \emptyset$ then $X/A$ is the topological space $X \amalg \{p\}$ with product topology for some $p \notin X$. Both quotient spaces are treated as pointed space with base-point $[A]$ (resp. $p$). In both cases the quotient map $q\colon X \to X/A$ is continuous.

Lemma 2.1. *If $A \cap \mathrm{PC}_x \neq \emptyset$ for all $x \in X$ then $X/A$ is path-connected.*

PROOF. We will show that each $[x] \in X/A$ is path-connected to $[A]$. Let $x \in [x] \subset X$ be some representative in $[x]$. Because $A \cap \mathrm{PC}_x^X$ is not empty there is a continuous map $f_x\colon [0,1] \to X$ with $f(0) = x$ and $f(1) \in A$.

Define $\widetilde{f}\colon [0,1] \to X/A$ as $\widetilde{f} := q \circ f$. This map is continuous because $f$ and $q$ are. And

$$\mathrm{PC}_{[x]}^{X/A} = \mathrm{PC}_{[A]}^{X/A},$$

because $\widetilde{f}(0) = [x]$ and $\widetilde{f}(1) = [A]$. Since $[x]$ was arbitrary

$$\mathrm{PC}_{[x]}^{X/A} = \mathrm{PC}_{[A]}^{X/A} = \mathrm{PC}_{[y]}^{X/A}$$

for all $x, y \in X$, so that $X/A = \mathrm{PC}_{[A]}^{X/A}$ is path-connected. □

Let $X$ and $Y$ be homotopy equivalent spaces. It can be easily shown that if $X$ is path-connected, so is $Y$. Thus we say that the homotopy class $[X]$ is path-connected if some representative $X$ is path-connected.

A strong deformation retract of $X$ onto a subset $A$ is a continuous map $D\colon X \times [0,1] \to X$ with $D(x,0) = x$ for all $x \in X$ and $D(t,a) = a$ for all $(t,a) \in [0,1] \times A$. In particular we see that this does defines a homotopy equivalence and path-connectedness of $X$ implies that of $A$. Furthermore, we can show the following.

LEMMA 2.2. *Write $x \sim_{\mathrm{pc}} y$ if $x$ and $y$ are in the same path-component of $X$. Let*

$$X = \bigcup_{[x] \in X/\sim_{\mathrm{pc}}} \mathrm{PC}_x,$$

*$A \subset X$ be a subset of $X$ and $D\colon X \times [0,1] \to X$ be a strong deformation retract of $X$ onto $A$. Then*

$$A = \bigcup_{[x] \in X/\sim_{\mathrm{pc}}} D(\mathrm{PC}_x, 1)$$

*is the decomposition of $A$ into path-components.*

PROOF. According to the previous statement $D(\mathrm{PC}_x, 1) \subset A$ is path-connected and the union is equal to $A$. It remains to show that $D(\mathrm{PC}_x, 1)$ is disjoint from $D(\mathrm{PC}_y, 1)$ if $[x] \neq [y] \in X/\sim_{\mathrm{pc}}$.

Suppose this is not the case for some $x, y \in X$ with $[x] \neq [y]$. Then there is a path $f\colon [0,1] \to A$ with $f(0) = x$ and $f(1) = y$. Since $f([0,1]) \subset A \subset X$, this is also a path in $X$ and therefore $[x] = [y]$ in $X$ which is a contradiction. □

**2.2. Homology.** Finally we want to state a theorem about the first homology of the pair $(X, X \setminus \{x_0\})$ for $x_0 \in X$. This result will play a crucial role in the proof of the mountain pass characterization via the homotopy index. We assume that the reader is familiar with concepts of relative singular homology as given in [2].

All maps in the following are assumed to be continuous. A map $\sigma\colon \Delta^n \to X$ denotes a singular $n$-simplex, i.e. a continuous map from an $n$-simplex $\Delta^n$ into $X$. The sets $C_i(X,A)$ and $H_i(X,A)$ ($C_i(X)$ and $H_i(X)$ in case of $A=\emptyset$) will denote the set of singular $i$-chains and respectively the $i$-th singular homology of the pair $A \subset X$ with coefficient in the (non-trivial) group $G$.

Theorem 2.3. *Let $X$ be Hausdorff and $x_0 \in X$. Suppose there is an open neighbourhood $U \subset X$ of $x_0$ such that for every $\alpha\colon [0,1] \to U$ with $\alpha(0), \alpha(1) \neq x_0$ there is a singular 2-chain $\beta \in C_2(X)$ with*

(a) $\partial\beta = \beta_0 - \beta_1 + \beta_2$,
(b) $\beta_1, \beta_2 \in C_1(X \setminus \{x_0\})$,
(c) $\beta_0 = \sigma_\alpha$ *with* $\sigma_\alpha\colon \Delta^1 \to X$ *the induced singular 1-chains of* $\alpha$.

*Then* $H_1(X, X\setminus\{x_0\}) = 0$.

Proof. Let $\sigma$ be a relative 1-cycle, its homology class is denoted by $[\sigma]$. Using the lemma below we can assume that $\sigma$ is generated by finitely many 1-simplices with endpoints unequal to $x_0$. Because the represented homology class in such a case is the sum of the homology classes of the generating 1-simplices times a coefficient it suffices to show that each 1-simplex $\alpha$ with $\partial\alpha \in C_0(X\setminus\{x_0\})$ represents the trivial relative homology class in $H_1(X, X\setminus\{x_0\})$. This is the case if there is a $\beta \in C_2(X)$ and a $\gamma \in C_1(X\setminus\{x_0\})$ such that $\alpha = \partial\beta + \gamma$.

Let $g_\alpha\colon [0,1] \to X$ be the associated path of $\alpha$, i.e. $g_\sigma(t) = \alpha(t, 1-t)$.

*Case 1.* $g_\alpha([0,1]) \subset X\setminus\{x_0\}$. Then $\alpha \in C_1(X\setminus\{x_0\})$ so that we can choose $\gamma = \alpha$ and $\beta = 0$ and therefore $[\alpha]$ is trivial in $H_1(X, X\setminus\{x_0\})$.

*Case 2.* $g_\alpha([0,1]) \subset U$, where $U$ is the neighbourhood from the assumption of the theorem. Then there is a singular 2-chain $\beta \in C_2(X)$ with

$$\partial\beta = \beta_0 - \beta_1 + \beta_2$$

such that $\beta_0 = \alpha$ and $\beta_i \in C_1(X\setminus\{x_0\})$ for $i = 1,2$. Now choose $\gamma = \beta_1 - \beta_2 \in C_1(X\setminus\{x_0\})$ then

$$\partial\beta + \gamma = (\beta_0 - \beta_1 + \beta_2) + \beta_1 - \beta_2 = \alpha.$$

Therefore $[\alpha]$ is trivial in $H_1(X, X\setminus\{x_0\})$.

*Case 3.* $g_\alpha$ arbitrary. We claim that there is a decomposition $\{0 = t_0 < \ldots < t_{2k+1} = 1\}$ such that for $g_i = g_{\alpha|[t_{i-1},t_i]}$

$$g_{2i+1}\colon [t_{2i}, t_{2i+1}] \to X\setminus\{x_0\}, \qquad g_{2i}\colon [t_{2i-1}, t_{2i}] \to U.$$

Assuming this, the cases 1 and 2 apply to the associated 1-simplices $\sigma_{g_{2i+1}}$ and, resp. $\sigma_{g_{2i}}$, so that $[\sigma_{g_i}] = 0$ in $H_1(X, X\setminus\{x_0\})$. Therefore

$$[\alpha] = [\sigma_{g_0}] + \ldots + [\sigma_{g_{2k+1}}] = 0 + \ldots + 0 = 0,$$

i.e. $[\alpha]$ is trivial in $H_1(X, X \setminus \{x_0\})$.

The decomposition of the path can be achieved by noticing that both $X \setminus \{x_0\}$ and $U$ are open. Because $X$ is Hausdorff we can find an open neighbourhood $V \subset U$ of $x_0$ such that $g(0), g(1) \notin V$. Continuity of $g$ implies $A = g^{-1}(X \setminus \{x_0\})$ and $B = g^{-1}(V)$ are disjoint unions of open intervals in $[0,1]$. Because $[0,1]$ is compact finitely many of these intervals suffice to cover $[0,1]$ and we can assume no interval is completely contained in another one. This implies that if two intervals intersect then one belongs to $A$ and the other to $B$. Because $g(0), g(1) \notin V$ the first and last interval must be in $A$. Splitting $g$ at some arbitrary $t$ in the intersection of two neighbouring intervals we get the required decomposition. □

Lemma 2.4. *Let $\sigma \in C_1(X)$ be a relative 1-cycle, i.e. $\partial\sigma \in C_1(X \setminus \{x_0\})$. Then the relative homology class of $\sigma$ can be represented by a relative 1-cycle $\widetilde{\sigma}$ such that*

$$\widetilde{\sigma} = \sum_{i=1}^{k} d_i \widetilde{\sigma}_i$$

*where $d_i \in G$ and $\widetilde{\sigma}_i \colon \Delta^1 \to X$ is a 1-simplex with $\widetilde{\sigma}_i(0)$, $\widetilde{\sigma}_i(1) \neq x_0$.*

Proof. For distinct singular 1-simplices $\sigma_i$ and coefficients $c_i \in G \setminus \{0\}$ we have

$$\sigma = \sum_{i=1}^{l} c_i \sigma_i + \widetilde{\sigma}$$

where $\widetilde{\sigma}$ is a 1-cycle with the required property and $x_0 \in |\partial\sigma_i|$ for each $\sigma_i$. The constant 1-simplex $\sigma_i \equiv x_0$ is an $n$-boundary hence $\sigma$ and $\sigma - c_i\sigma_i$ represent the same relative homology class. So we assume that all $\sigma_i \not\equiv x_0$.

If $\partial\sigma_i = x_0 - x_0 = 0$ then we can split the simplex into $\sigma_i^1 + \sigma_i^2$ such that $\sigma_i^1(0,1) = \sigma_i^2(1,0) \neq x_0$. Then $\sigma$ and $\sigma - c_i\sigma_i + c_i\sigma_i^1 + c_i\sigma_i^2$ represent the same homology class, i.e. we can replace the 1-cycle by two 1-simplices which are not 1-cycles.

If $\rho$ is a 1-simplex denote by $\rho^-$ the reversed 1-simplex, i.e. $\rho^-(t, 1-t) = \rho(1-t, t)$. Because $\rho + \rho^-$ represents the trivial homology class in $H_1(X)$ and $H_1(X, X \setminus \{x_0\})$ for every 1-simplex $\rho$, we can replace each $c_i\sigma_i$ by $-c_i\sigma_i^-$. Thus we can assume that $\partial\sigma_i = x_0 - x_i$ for all $\sigma_i$, i.e.

$$\partial\sigma = \sum_{i=1}^{l} c_i x_0 - \sum_{i=1}^{l} c_i x_i + \partial\widetilde{\sigma}.$$

Since $\sigma$ is a relative 1-cycle, i.e. $x_0 \notin |\partial\sigma|$, we must have $\sum_{i=1}^{l} c_i = 0$. And because $c_i \neq 0$ it is obvious that $l \neq 1$. If $l = 0$ then we are done. So assume $l > 1$.

Then $\sigma_1^- + \sigma_2$ represent a path from $x_1$ to $x_2$ whose 1-simplex will be denoted by $\widetilde{\sigma}_1$ such that $\partial\widetilde{\sigma}_1 = x_1 - x_2$. Because $\sigma_1 + \sigma_1^-$ represent the trivial homology, $\sigma$ and $\sigma' = \sigma + c_1(\sigma_1^- + \sigma_2 - \widetilde{\sigma}_1) - c_1(\sigma_1 + \sigma_1^-)$ represent the same relative homology class. $\sigma'$ has the following form

$$\sigma' = (c_1 + c_2)\sigma_2 + \sum_{i=3}^{l} c_i\sigma_i + \widetilde{\sigma}'$$

where $\widetilde{\sigma}' = \widetilde{\sigma} - c_1\widetilde{\sigma}_1$ has the required form. After renaming we see that $l$ was reduced by at least 1. So after at most $l - 2$ further steps we will be done. □

## 3. Conley index

In this section we are going to show that path-connectedness of the Conley homotopy index of an isolated invariant set characterizes its the stability. We assume the reader is familiar with the Conley index for semiflows on (not necessarily locally compact) metric spaces as given in [6], [7].

A semiflow is a continuous map $\pi\colon D \to X$ where $X$ is a metric space and $D \subset \mathbb{R}^+ \times X$ the domain with $x\pi 0 = x$ and $x\pi(t + s) = (x\pi t)\pi s$ whenever this is defined. Here we are going to use $x\pi t$ for $\pi(t, x)$ if $(t, x) \in D$.

An isolated invariant set (for $\pi$) is a compact set $S \subset X$ which admits a closed isolating neighbourhood $N$ such that the maximal invariant set $A(N)$ is $K$. The sets $A^+(N)$ and $A^-(N)$ denote the positive and negative invariant sets in $N$, in particular we have $A(N) = A^+(N) \cap A^-(N)$. The set $\omega(x)$ is defined as usual and for a full left solution $\sigma\colon \mathbb{R}^- \to X$, i.e. $\sigma(s)\pi t = \sigma(s + t)$ for $t \geq 0$ and $s + t \leq 0$, $\alpha(\sigma)$ is defined as the set of limit points for sequence $(\sigma(t_n))_{n\in\mathbb{N}}$ with $t_n \to -\infty$. If $\pi$ is a flow then $\alpha(\sigma) = \alpha(\sigma(0))$ for the usual definition of $\alpha(x)$.

We will use Rybakowski's strongly $\pi$-admissibility (see [6, (H1), (H2)]) for certain bounded and closed $N \subset X$, this roughly says for longer and longer orbits inside of $N$ the sequence of endpoints has a convergent subsequence. Let $\mathcal{S}$ be the set of isolated invariant sets admitting a strongly $\pi$-admissible isolating neighbourhood and let $\mathcal{N}(K)$ be the set the strongly $\pi$-admissible isolating neighbourhoods of $K$.

For each $K$ in $\mathcal{S}$ there is a well-defined homotopy index $h(K)$. This index is the homotopy type quotient space $(B/B^-, [B^-])$ of a special pair $(B, B^-)$ where $B$ is a strongly $\pi$-admissible isolating block and $B^-$ the exit set, i.e. all points on the boundary of $B$ either leave $B$ or enter its interior immediately and there are no solutions $\sigma\colon(-\varepsilon, \varepsilon) \to B$ with $\sigma(0) \in \partial B$. Such a set always exists if $N \in \mathcal{N}(K)$ (see [6, Section 2] for definition and existence).

Now we are able to prove several results concerning stability of isolated invariant sets.

THEOREM 3.1. *Let $X$ be locally path-connected and $K \in \mathcal{S}$ be a path-connected invariant set. Then, for $N \in \mathcal{N}(K)$,*

$$h(K) \text{ is path-connected } \Leftrightarrow \ K \neq A^-(N) \ \Leftrightarrow \ H_0(h(K)) = 0.$$

*Furthermore, if $A^-(N) = K$, i.e. $K$ is stable, then $H_0(h(K)) = G$.*

PROOF. $h(K) = [Y, y_0]$ is path-connected if and only if $H_0(Y, y_0) = 0$, so we only need to show the first equivalence.

Let $N_1, N_2 \in \mathcal{N}(K)$ be any admissible neighbourhoods. If $y_1 \in A^-(N_1) \setminus K$ then there is a full left solution $\sigma$ in $N$ through $y_1$, i.e. $\sigma: \mathbb{R}^- \to N$ with $\sigma(0) = y_1$. Since $\alpha(\sigma) \subset K$ and $K \subset \operatorname{int} N_2$ there is a neighbourhood $U \subset N_2$ of some $y_{-\infty} \in \alpha(\sigma)$ such that $\sigma(-T) \in U$. Because $\sigma(\mathbb{R}^-) \subset A^-(N_1) \setminus K$ and $\sigma(-T) \notin K$, $\sigma(-T) \in A^-(N_2) \setminus K$, i.e.

$$A^-(N_1) \setminus K \neq \emptyset \ \Leftrightarrow \ A^-(N_2) \setminus K \neq \emptyset.$$

This means we only need to show the result for one isolating neighbourhood.

So let $N \in \mathcal{N}(K)$ be arbitrary. We use the fact that the semiflow $\pi$ gives us a natural path in $N$ from $x \in N$ to $x\pi t \in N$ if $(t, x) \in D$ and $x\pi[0, t] \subset N$.

If $x \in A^+(N)$ then $x\pi[0, t] \subset N$ for all $t \geq 0$. Hence $x$ and $x\pi t$ are path-connected for all $t \geq 0$.

Because $\omega(x) \subset K \subset \operatorname{int} N$ and $X$ is locally path-connected there is a path-connected neighbourhood $U \subset N$ for some $x_\infty \in \omega(x)$ ($\omega(x) \neq \emptyset$ because $N$ is strongly $\pi$-admissible). Since $x\pi t_n \to x_\infty$ for some $t_n \to \infty$ there is a $t_N > 0$ such that $x\pi t_N \in U$. Therefore there is a path in $U$ from $x_\infty$ to $x\pi t_N$, i.e. $x$ is path-connected to $x_\infty \in K$.

If $y \in A^-(N)$ then there is a full left solution $\sigma: \mathbb{R}^- \to N$ through $y$ and $\alpha(\sigma) \subset K$ holds, which is non-empty by strong $\pi$-admissibility. So again, there is a path-connected neighbourhood $V \subset N$ for some $y_{-\infty} \in \alpha(\sigma)$ and a $T > 0$ such that $\sigma(-T) \in V$. Hence $y$ is path-connected to $y_{-\infty} \in K$ via the paths from $y = \sigma(0)$ to $\sigma(-T)$ and from $\sigma(-T)$ to $y_{-\infty}$.

Now let $B \subset N$ be a strongly $\pi$-admissible, isolating block. Then for all $x \in B \setminus A^+(B)$ there is a $t(x) \in \mathbb{R}^+$ such that $x\pi[0, t(x)] \subset B$ and $x\pi t(x) \in B^-$. In particular $x$ is path-connected to $x\pi t(x) \in B^-$.

If $A^-(N) = K$ then there is an isolating block $B$ such that $B^- = \emptyset$ (see [7, I-5.5]). Therefore $B = A^+(B)$. Since $K$ is path-connected and any $x \in A^+(B)$ is path-connected to some $k \in K$, $B$ itself is path-connected, i.e.

$$H_0(h(K)) = H_0(B, \emptyset) = G.$$

If $A^-(N) \neq K$ then $A^-(B) \neq K$ and every $x \in B \setminus A^+(B)$ is connected to some $\widetilde{x} \in B^-$. Furthermore there is a $y \in A^-(B) \setminus K$ such that $y$ is path-connected to a $k \in K$ and $\widetilde{y} \in B^-$. Hence for all $x \in B$

$$\mathrm{PC}_x^B \cap B^- \neq \emptyset.$$

Using Lemma 2.1 we conclude that $B/B^-$ is path-connected and because $B$ is an isolating block

$$H_0(h(K)) = H_0(B/B^-, [B^-]) = 0. \qquad \square$$

If $\omega(x)$ and $\alpha(\sigma)$ consist of a single point we can construct a path from $x$ to $x_\infty \in \omega(x)$ resp. from $\sigma(0)$ to $y_{-\infty} \in \alpha(\sigma)$. This way we can drop the assumption that $X$ is locally path-connected.

Lemma 3.2. *Let $K \in \mathcal{S}$ and $N \in \mathcal{N}(K)$. Suppose for every $x \in A^+(N) \setminus K$ $|\omega(x)| = 1$. If $A^-(N) = K$ and $K$ is path-connected then $h(K)$ is not path-connected and*

$$H_0(h(K)) = G.$$

*If $A^-(N) \neq K$ and for every path-component $C$ of $K$ there is a full left solution $\sigma$ through some $y \in A^-(N) \setminus K$ with $|\alpha(\sigma)| = 1$ and $\alpha(\sigma) \subset C$. Then $h(K)$ is path-connected and*

$$H_0(h(K)) = 0.$$

Proof. Similarly to the previous lemma this statement does not depend on the chosen isolating neighbourhood.

We will only construct the paths from $x \in A^+(N)$ to $K$ and from $\sigma(0)$ with $|\alpha(\sigma)| = 1$ to $K$. The rest of the proof is the same as the previous one. We define the functions $f, g\colon [0,1] \to N$ as follows

$$f(t) = \begin{cases} x\pi \dfrac{\tau}{1-\tau} & \tau \in [0,1), \\ x_\infty & \tau = 1,\ x_\infty \in \omega(x), \end{cases}$$

$$g(t) = \begin{cases} \sigma\left(\dfrac{-\tau}{1-\tau}\right) & \tau \in [0,1), \\ y_{-\infty} & \tau = 1,\ y_{-\infty} \in \alpha(y). \end{cases}$$

Since $\tau/(1-\tau)$ is continuous for $\tau \in [0,1)$, $f$ and $g$ are continuous in $[0,1)$.

Let $1 \neq \tau_n \to 1$, then $t_n := \tau_n/(1-\tau_n) \to \infty$. The sets $\{x\pi t_n\}_{n\in\mathbb{N}}$ and $\{\sigma(-t_n) = \sigma(-2t_n)\pi t_n\}_{n\in\mathbb{N}}$ are precompact and every cluster point is in $\omega(x) = \{x_\infty\} \subset K$ resp. $\alpha(\sigma) = \{y_{-\infty}\} \subset K$. Therefore $f(\tau_n) \to x_\infty$ and $g(\tau_n) \to y_{-\infty}$, i.e. $f$ and $g$ are continuous in $[0,1]$ with $f(1), g(1) \in K$. $\square$

Theorem 3.3. *Let* $\{x_0\} = K \in \mathcal{S}$ *and* $N \in \mathcal{N}(K)$. *Then*

$$A^-(N) = K \;\Leftrightarrow\; h(K) = \Sigma^0 \Leftrightarrow\; H_0(h(K)) \neq 0 \Leftrightarrow\; H_0(h(K)) = G$$

*and, if* $A^-(N) = K$, *then there is a contractible isolating block* $B \in \mathcal{N}(K)$ *with* $B^- = \emptyset$.

Remark 3.4. Because path-connected spaces are also (quasi-)connected, the result holds for Alexander–Spanier cohomology $\{\overline{H}^n\}$ as well.

Proof. We only need to show the first equivalence. The rest follows from the previous lemma.

Suppose $A^-(N) = K$, then we can choose an isolating block $B \subset N$ with $B^- = \emptyset$ and $B = A^+(B)$. We define the following homotopy $H\colon B \times [0,1] \to B$

$$H(y,\tau) = \begin{cases} y\pi\dfrac{\tau}{1-\tau} & \tau \in [0,1), \\ x_0 & \tau = 1. \end{cases}$$

Since $\pi\colon B \times [0,\infty) \to B$ is defined and therefore continuous, $H$ is continuous for all $(y,\tau) \in B \times [0,1)$. So let $(y_n,\tau_n) \to (y,1)$ with $\tau_n \neq 1$. Then $t_n = \tau_n/(1-\tau_n) \to \infty$, $\{H(y_n,\tau_n) = y_n\pi t_n\}_{n\in\mathbb{N}}$ is precompact and every cluster point is in $A^-(N) = \{x_0\}$, i.e. $y_n\pi t_n \to x_0$. Hence $H$ is continuous.

Because $H(x,0) = x$ and $H(x,1) = x_0$ for all $x \in B$, $H$ is a contraction of $B$ to $x_0 \in B$ and

$$h(K) = [B \amalg \{p\}, p] = [\{x_0\} \amalg \{p\}, p] = \Sigma^0.$$

Conversely, let $h(K) = \Sigma^0$. Then $h(K)$ is not path-connected. Because $K = \{x_0\}$, $\omega(z) = \{x_0\}$ for all $z \in A^+(B)$ and $\alpha(\sigma) = \{x_0\}$ for all full left solutions $\sigma$ in $B$. This means that $A^-(N) = K$ according to the previous lemma. □

Finally we calculate the index for stable invariant sets which are local neighbourhood retracts. To show that we need the following lemma which is a variant for stable invariant set of a lemma in [6] used to prove the existence of isolating blocks.

Lemma 3.5 ([6, Lemma 2.1]). *Let* $\emptyset \neq K \in \mathcal{S}$ *be stable and* $B \in \mathcal{N}(K)$ *be an isolating block with* $B^- = \emptyset$. *Define* $F\colon B \to [0,1]$ *by*

$$F(x) := \min\{1, \operatorname{dist}(x,K)\}$$

*and for some strictly increasing* $C^\infty$*-diffeomorphism* $\alpha\colon [0,\infty) \to [1,2)$ *define* $g^-\colon B \to [0,2]$ *by*

$$g^-(x) := \sup\{\alpha(t)F(x\pi t) \mid 0 \leq t < \infty\}.$$

*Then $g^-$ is continuous and $g^-(x) = 0$ if and only if $x \in K$. Furthermore, $t \mapsto g(x\pi t)$ is strictly decreasing for $t \in \mathbb{R}^+$ and $x \in B \setminus K$.*

Using the function $g^-$ we can define a strong deformation retract of $B$ which ensures that the distance between the boundary of $B$ and $K$ is at most $\delta$.

Lemma 3.6. *Under the assumptions above let $0 < \delta < 1$ and define $t_\delta \colon B \to [0, \infty)$ by*

$$t_\delta(x) := \inf_{t \in \mathbb{R}^+} \{t \mid g^-(x\pi t) \leq \delta\}$$

*and define a deformation $H_\delta \colon B \times [0,1] \to B$ by*

$$H_\delta(x, \tau) = x\pi(\tau \cdot t_\delta(x)).$$

*Then $t_\delta$ and $H$ are well-defined and continuous. In particular*

$$\operatorname{dist}(H_\delta(x,1), K) \leq \delta$$

*and $H_\delta(x,\tau) = x$ for all $x \in B$ with* $\operatorname{dist}(x, K) \leq \delta$.

Proof. Suppose $t_\delta$ is well-defined and continuous. Then $H$ is a composition of continuous functions and therefore continuous. Furthermore, if $g^-(x) < 1$ then $\operatorname{dist}(x, K) \leq g^-(x)$. Because $g^-(x\pi t_\delta(x)) \leq \delta < 1$ by definition of $t_\delta(x)$,

$$\operatorname{dist}(H_\delta(x,1), K) \leq \delta.$$

Now we show that $t_\delta$ is well-defined and continuous: Let $x \in B$. Since $B^- = \emptyset$, $B = A^+(B)$ and thus

$$\operatorname{dist}(x\pi t, K) \to 0 \quad \text{as } t \to \infty.$$

Because $K$ is compact, there is a sequence $t_n \to \infty$ such that $x\pi t_n \to x^* \in K$ and thus $g^-(x\pi t_n) \to g(x^*) = 0$, i.e. there is an $N > 0$ such that $g^-(x\pi t_N) \leq \delta$. Because $g^-$ is decreasing along orbits

$$g^-(x\pi t) \leq g^-(x\pi t_N) \leq \delta$$

for $t \geq t_N$, i.e. $t_\delta(x) \leq t_N < \infty$.

Now, let $x_n \to x$ with $x_n, x \in B$. If $t_\delta(x) = 0$, then $g^-(x) \leq \delta$ and thus $g^-(x_n) \to g^-(x) \leq \delta$, i.e. $t_\delta(x_n) \to 0$.

If there is an $M > 0$ such that $t_\delta(x) > M$ then $g^-(x\pi M) > \delta$ and thus $g^-(x_n\pi M) > \delta$ for large $n$, i.e. $t_\delta(x_n) > M$. Similarly, if $t_\delta(x) < M$, then $g^-(x\pi M) < g^-(x\pi t_\delta(x)) \leq \delta$ and $g^-(x_n\pi M) < \delta$ for large $n$, i.e. $t_\delta(x_n) < M$. This shows that $t_\delta(x_n) \to t_\delta(x)$, i.e. $t_\delta$ is continuous. □

Definition 3.7 ((strong) local neighbourhood deformation retract). A subset $A$ of a topological space $X$ is called a *local neighbourhood deformation retract* if for every neighbourhood $V$ of $A$ there is neighbourhood $U \subset V$ of $A$ and a continuous function $D: U \times [0,1] \to U$ such that for all $x \in U$ and $a \in A$

$$D(x,0) = x, \qquad D(a,1) = a$$

and $D(x,1) \in A$. If in addition $D(a,t) = a$ for all $t$ then $A$ is called a strong local neighbourhood deformation retract.

Remark 3.8. If $A$ is a (strong) local neighbourhood deformation retract then $A$ is homotopy equivalent to $U$ from the definition. In particular the following theorem generalizes the Theorem 3.3 because stable isolated $\{x\} \in \mathcal{S}$ are locally contractible, i.e. local neighbourhood deformations retracts.

Theorem 3.9. *If $K \in \mathcal{S}$ is stable and a local neighbourhood deformation retract then $h(K) = [K \amalg \{p\}, p]$.*

Proof. Because $K$ is stable, there is an isolating block $B \in \mathcal{N}(K)$ with $B^- = \emptyset$. Since $B$ is a neighbourhood of $K$ and $K$ a local neighbourhood deformation retract, there is a neighbourhood $U \subset B$ of $K$ and a deformation retract $D: U \times [0,1] \to U$.

Let $B_\delta(x) := \{y \in X \mid d(x,y) < \delta\}$. Because $U$ is a neighbourhood of $K$ there are $\delta_x > 0$ such that $B_{\delta_x}(x) \subset U$ for every $x \in K$. Since $K$ is compact and $\{B_{\delta_x}(x)\}_{x\in K}$ is an open cover of $K$, there are finitely many $x_1, \dots, x_n \in K$ such that $K \subset U_\delta = \bigcup_{x_i} B_{\delta_i}(x_i)$ for $\delta := \max\{\delta_i\}$. In particular, for every $x \in U_\delta$, $\operatorname{dist}(x,K) < \delta$.

Let $H_\delta$ be the deformation of the previous lemma. Then $H_\delta(x,\tau) = x$ for $x \in U_\delta$ and $H_\delta(B,1) \subset U_\delta \subset U$. We define a deformation retract $H: B \times [0,1] \to B$ as follows

$$H(x,\tau) := \begin{cases} H_\delta(x, 2\tau) & \text{for } \tau \in [0,1/2], \\ D(H_\delta(x,1), 2\tau - 1) & \text{for } \tau \in [1/2,1]. \end{cases}$$

Because $D$ and $H_\delta$ are deformation retracts with image in $B$ and $H_\delta(B,1) \subset U$, this is well-defined and therefore continuous. In particular $H(x,1) \in K$ and $H(a,1) = a$ for $x \in B$ and $a \in K$ because $H_\delta(a,\tau) = a$ and $D(a,1) = a$. Therefore $B$ and $K$ are homotopy equivalent, i.e.

$$h(K) = [B \amalg \{p\}, p] = [K \amalg \{p\}, p]. \qquad \square$$

## 4. Mountain pass points

In this section we show that the first singular homology characterizes mountain pass points completely. Under further assumptions we will compute the homotopy index in the next section.

Definition 4.1 (gradient-like semiflow). Let $X$ be a metric space, $\pi$ be a (local) semiflow on $X$ and $E = \{x \in X \mid \sigma(t) \equiv x \text{ is a solution of } \pi\}$ be the set of equilibria. $\pi$ is called a *gradient-like semiflow* if there is a continuous function $f\colon X \to \mathbb{R}$ such that, for all $x \in X \setminus E$, $t \mapsto f(x\pi t)$ is strictly decreasing in $0 \leq t < \omega_x := \sup\{t > 0 \mid (t,x) \in D\}$.

Notation. We will use the following notation

$$\begin{aligned} f^c &= \{x \in X \mid f(x) \leq c\}, & \dot{f}^c &= \{x \in X \mid f(x) < c\},\\ f_c &= \{x \in X \mid f(x) \geq c\}, & \dot{f}_c &= \{x \in X \mid f(x) > c\} \end{aligned}$$

and $f_c^b = f^b \cap f_c$.

Definition 4.2 (mountain pass point [4]). Let $\pi$ be a gradient-like semiflow (w.r.t. $f$). An isolated equilibrium $x_0$ with $f(x_0) = c$ is called *mountain pass point* if there is a neighbourhood $B'$ of $x_0$ such that for any neighbourhood $B \subset B'$ of $x_0$, $f^c \cap B \setminus \{x_0\}$ is non-empty and not path-connected.

The main result of this section is the following theorem:

Theorem 4.3 (homology index of mountain pass points). *Let $X$ be a Banach space, i.e. a complete normed vector space, and let $\pi$ be a gradient-like semiflow (w.r.t. $f$) in an open set $U \subset X$. If $\{x_0\} \in \mathcal{S}$ is an isolated equilibrium then*

$$x_0 \textit{ is a mountain pass point } \Leftrightarrow \ H_1(h(\{x_0\}); G) \neq 0.$$

Remark 4.4. The if-part can be proven with the relaxed condition "$X$ is locally path-connected in $x_0$" and the "only if" part with "$X$ is locally contractible in $x_0$" (we just need $H_1(U) = 0$ for a small neighbourhood $U$ of $x_0$).

**4.1. "If" part.** We recall the following lemma from [7] which shows that for gradient-like flows the exist set can be chosen "below" the energy level of $x_0$ and that the critical groups are essentially the homology groups of the homotopy index. Usually critical groups are used without a specific (semi)flow and results are proven using Morse theory and the Palais–Smale condition on $f$. Here we only assume continuity of $f$ and strong $\pi$-admissibility of a neighbourhood of $x_0$.

Lemma 4.5 [7, III-4.8,4.9]). *Let $\{x_0\} \in \mathcal{S}$ with $f(x_0) = c$. Then there exists an isolating block $B \in \mathcal{N}(\{x_0\})$ with $B^- \subset f^{c-\varepsilon}$ for some $\varepsilon > 0$ and a strong deformation retract $\rho\colon B \times [0,1] \to B$ of $B$ onto $f^c \cap B$ and*

$$\mathsf{H}_n(h(\{x_0\})) = \mathsf{H}_n(f^c \cap B, f^c \cap B \setminus \{x_0\}) = \mathsf{H}_n(f^c \cap U, f^c \cap U \setminus \{x_0\})$$

*for any neighbourhood $U$ of $x_0$ and homology $\{\mathsf{H}_n\}$.*

The following theorem is similar to the well-known result that for mountain pass points the first critical group is non-zero. Instead of using an "artificial" pseudo-gradient flow defined via $f$ we use the deformation of the previous lemma, i.e. we will use the semiflow $\pi$ directly.

Theorem 4.6. *Let $X$ be locally path-connected in $x_0$. If $\{x_0\} \in \mathcal{S}$ is a mountain pass point for a gradient-like semiflow $\pi$ (w.r.t. $f$) with $f(x_0) = c$ then*

$$H_1(h(\{x_0\})) \neq 0$$

*for the singular homology $\{H_n\}$.*

Proof. Because $\{x_0\} \in \mathcal{S}$, every neighbourhood $U$ of $x_0$ contains some $U \supset N \in \mathcal{N}(\{x_0\})$. Let $B'$ be as in Definition 4.2. Then there exists a strongly $\pi$-admissible, isolating block $B \subset B'$ with $B^- \subset f^{c-\varepsilon}$ for some $\varepsilon > 0$.

Let $B = \bigcup_{\alpha\in I} B_\alpha$ be the decomposition of $B$ into path-components with $I = B/\sim_{\mathrm{pc}}$. Because $\rho$ from Lemma 4.5 is a strong deformation retract, $\rho(B_\alpha, 1)$ is path-connected and equals $f^c \cap B_\alpha$.

According to Lemma 2.2

$$f^c \cap B = \rho(B, 1) = \bigcup_{\alpha\in I} \rho(B_\alpha, 1) = \bigcup_{\alpha\in I} f^c \cap B_\alpha.$$

Let $B_*$ be the path-component containing $x_0$. Because $B$ is a neighbourhood of $x_0$ and $X$ is locally path-connected in $x_0$, there is a path-connected neighbourhood $U \subset B$ of $x_0$. In particular $U \subset B_*$, so $B_*$ is the maximal, path-connected neighbourhood of $x_0$ in $B \subset B'$.

Therefore $C := f^c \cap B_* \setminus \{x_0\}$ is non-empty and not path-connected. Let $J = C/\sim_{\mathrm{pc}}$. Then $|J| > 1$ and $H_0(C) = G^{|J|}$.

Because $B_*$ is path-connected, so is $f^c \cap B_* = \rho(B_*, 1)$ and $H_0(f^c \cap B_*) = G$. Obviously $H_0(C)$ and $H_0(f^c \cap B_*)$ are not isomorphic.

For the topological pair $(f^c \cap B_*, C)$ the following sequence is exact

$$\cdots \xrightarrow{j_*} H_1(f^c \cap B_*, C) \xrightarrow{\partial_1} H_0(C) \xrightarrow{i_*} H_0(f^c \cap B_*) \longrightarrow 0.$$

Suppose now $H_1(f^c \cap B_*, C) = 0$, then $\partial_1 = 0$ and therefore

$$H_0(C) \cong H_0(f^c \cap B_*),$$

which is a contradiction.

Finally $B_\alpha \setminus \{x_0\} = B_\alpha$ for all $\alpha \in I \setminus \{*\}$ and

$$\begin{aligned} H_1(h(K)) &= H_1(f^c \cap B, f^c \cap B \setminus \{x_0\}) \\ &= H_1(f^c \cap B_*, f^c \cap B_* \setminus \{x_0\}) \oplus \Bigg( \bigoplus_{\alpha\in I\setminus\{*\}} H_1(B_\alpha, B_\alpha) \Bigg) \\ &= H_1(f^c \cap B_*, f^c \cap B_* \setminus \{x_0\}) \neq 0. \end{aligned}$$

□

Corollary 4.7. *Under the assumptions of the previous theorem let $B$ and $B_*$ be chosen as in the proof. If, in addition, $X$ is locally path-connected then for every path-component $C$ of $f^c \cap B_* \setminus \{x_0\}$ there is a $y \in A^-(B) \cap C$, i.e. the path-component $C$ contains a full left solution (through $y$) leaving the isolating block.*

Proof. Let $x \in f^c \cap B_* \setminus \{x_0\}$ be arbitrary. Because $f^c \cap B_*$ is path-connected, there is a path $g\colon [0,1] \to f^c \cap B_*$ with $g(0) = x$ and $g(1) = x_0$. Since $[0,1]$ is compact, $\{x_0\}$ closed and $g$ continuous, $g^{-1}(\{x_0\})$ is compact. Let $t_0 = \min\{t \in [0,1] \mid g(t) = x_0\}$. Because $g(0) \neq x_0$, $t_0 > 0$ and $x_0 \notin g([0,t_0))$.

Therefore $g([0,t_0) \subset f^c \cap B_* \setminus \{x_0\}$. In particular $g([0,t_0))$ is in the path-component of $f^c \cap B_* \setminus \{x_0\}$ containing $x$. If this wasn't the case, then there would be a $t' \in [0,t_0)$ such that there is no path in $f^c \cap B_* \setminus \{x_0\}$ between $x$ and $g(t')$. But $h\colon [0,1] \to f^c \cap B_* \setminus \{x_0\}$ with $h(t) = g(t \cdot t_0)$ is such a path, which is a contradiction.

According to the proof of the previous theorem there is an $\varepsilon > 0$ such that $B^- \subset f^{c-\varepsilon}$. Choose $0 < \delta < \varepsilon$. Similarly to the proof of [7, III-4.8] there is unique $r(x) \in [0,\infty)$ for all $x \in B_* \cap f^c_{c-\delta} \setminus \{x_0\}$ such that $f(x\pi r(x)) = c - \delta$.

Let $0 \leq t_n < t_0$ be a sequence with $t_n \to t_0$. Because $f$ and $g$ are continuous $\lim_{t_n \to t_0} f(g(t_n)) = f(x_0) = c$, so there is an $N > 0$ such that, for all $n \geq N$,

$$f(g(t_n)) > c - \delta.$$

W.l.o.g. we can choose a sequence $t_n \to t_0$ such that this always holds.

Let $x_n = g(t_n)$ and $y_n = x_n \pi r(x_n)$. Obviously every $x_n$ and $y_n$ are in the same path-component of $f^c \cap B_* \setminus \{x_0\}$, which contains $x$.

Suppose there is an $M < \infty$ such that $r(x_n) \leq M$. Then there is a subsequence $n'$ such that $r(x_{n'}) \to r_0 < \infty$. Because $x_n \to x_0$ and [6, Lemma 1.1]

$$x_n \pi r(x_n) \to x_0 \pi r_0 = x_0 \notin f^{c-\delta}.$$

But $f^{c-\delta}$ is closed and $x_n \pi r(x_n) \in f^{c-\delta}$, i.e. the limit must be in $f^{c-\delta}$, which is a contradiction.

So we can assume $r(x_n) \nearrow \infty$. By [6, Lemma 1.1] $\{x_n \pi r(x_n)\}_{n \in \mathbb{N}}$ is precompact and every cluster point $z$ is in $A^-(B)$ and therefore $f(z) = f(x_n \pi r(x_n)) = c - \delta$.

Choose a subsequence $(x_{n'})$ such that $y_{n'} = x_{n'} \pi r(x_{n'}) \to x^* \in A^-(B) \setminus \{x_0\}$. We will finish the proof if we show that $x$ and $x^*$ are path-connected in $f^c \cap B_* \setminus \{x_0\}$.

Suppose $x^* \in \partial B$. Because $B$ is an isolating block and $A^-(B) \cap \partial B \subset B^-$, $x^* \in B^- \subset f^{c-\varepsilon}$. But this is a contradiction because $f(x^*) = c - \delta > c - \varepsilon$. Therefore $x^*$ is in the interior of $B$ and in $\dot{f}^c \cap \operatorname{int} B$, too. In particular $\dot{f}^c \cap \operatorname{int} B$ is open and disjoint from $\{x_0\}$.

Since $X$ is locally path-connected, so is $\dot{f}^c \cap \operatorname{int} B$ and there is a path-connected neighbourhood $U \subset \dot{f}^c \cap \operatorname{int} B$ of $x^*$. Because $y_{n'} \to x^*$, there is an $N > 0$ such that $y_{n'} \in U$ for all $n \geq N$.

So there is a continuous function $h: [0,1] \to U$ with $h(0) = x^*$ and $h(1) = y_N$. Because $U \subset f^c \cap B \setminus \{x_0\}$,

$$x^* \sim_{\mathrm{pc}} y_N \sim_{\mathrm{pc}} x \quad \text{in } f^c \cap B \setminus \{x_0\},$$

that is $x^*$ and $x$ are in the path-component of $f^c \cap B \setminus \{x_0\}$ and therefore they are in the same path-component of $f^c \cap B_* \setminus \{x_0\}$. □

**4.2. "Only if" part.** For the ,,only if" part we will show that the assumptions of Theorem 2.3 are satisfied if $x_0$ is not a mountain pass point. First of all let's look at the converse of the definition of a mountain pass point

Definition 4.8. An isolated equilibrium $x_0$ of a gradient-like semiflow $\pi$ (w.r.t. $f$) is not a mountain pass point if for every neighbourhood $B'$ of $x_0$ there is a neighbourhood $B \subset B'$ of $x_0$ such that $f^c \cap B \setminus \{x_0\}$ is path-connected (the empty set is path-connected).

From now on let $X$ be a Banach space. With the help of Theorem 2.3 we are now able to complete the proof:

Proof of Theorem 4.3. Since $X$ is a Banach space it is locally path-connected and therefore $U$ is locally path-connected, too. So Theorem 4.6 applies, i.e. if $x_0$ is a mountain pass point then $H_1(h(\{x_0\})) \neq 0$.

We will show the converse. Suppose $x_0$ is not a mountain pass point. Let $B$ and $B_*$ be chosen as in the proof of Theorem 4.6 such that $B_*$ is the path-component of $B$ containing $x_0$. In addition, it was shown that

$$H_1(h(\{x_0\})) = H_1(f^c \cap B_*, f^c \cap B_* \setminus \{x_0\}).$$

Suppose $f^c \cap B_* \setminus \{x_0\} = \emptyset$ then $f^c \cap B_* = \{x_0\}$ and therefore $H_1(h(\{x_0\})) = H_1(\{x_0\}) = 0$. In particular $x_0$ is stable if this happens.

So w.l.o.g. we can assume $f^c \cap B_* \setminus \{x_0\} \neq \emptyset$. Because $X$ is locally convex and $B_*$ is a neighbourhood of $x_0$, there is a convex neighbourhood $V \subset B_*$ of $x_0$ (we only need a neighbourhood with $H_1(V) = 0$). Since $x_0$ is not a mountain pass point there is a neighbourhood $U \subset V$ of $x_0$ such that $f^c \cap U \setminus \{x_0\}$ is path-connected.

If $f^c \cap U \setminus \{x_0\} = \emptyset$ then $f^c \cap U = \{x_0\}$ and according to Lemma 4.5

$$H_1(h(\{x_0\})) = H_1(f^c \cap U, f^c \cap U \setminus \{x_0\}) = H_1(\{x_0\}) = 0.$$

Otherwise $f^c \cap U \setminus \{x_0\}$ is non-empty and path-connected. Let $\alpha_0: [0,1] \to f^c \cap U$ any path with $\alpha(0), \alpha(1) \neq x_0$ and let $x^* \in f^c \cap U \setminus \{x_0\}$ be arbitrary. Because $f^c \cap U \setminus \{x_0\}$ is path-connected, there are paths $\alpha_1, \alpha_2: [0,1] \to f^c \cap U \setminus \{x_0\}$

such that $\alpha_0(1) = \alpha_1(1)$, $\alpha_1(0) = x^* = \alpha_2(0)$ and $\alpha_2(1) = \alpha_0(0)$. Denote by $\beta_i = \sigma_{\alpha_i}$ the induced 1-chains. Then the 1-chain $\gamma = \beta_0 - \beta_1 + \beta_2$ is a 1-cycle, i.e. $\partial\gamma = 0$. Since the image of $\alpha_i$ is in $f^c \cap U \subset V \cap (f^c \cap B_*)$ we regard $\beta_i$ and $\gamma$ as 1-chains in $C_1(V)$ as well as in $C_1(f^c \cap B_*)$.

Because $H_1(V) = 0$ there is a singular 2-chain $\beta \in C_2(V)$ with $\partial\beta = \gamma$.

Let $\rho$ be the strong deformation retract of Lemma 4.5. Then $\phi = i \circ \rho(\,\cdot\,, 1): V \to f^c \cap B_*$, where $i: \rho(V, 1) \to f^c \cap B_*$ is the inclusion map, induces a map $\phi_\sharp: C_n(V) \to C_n(f^c \cap B_*)$ on the singular $n$-chains. Since $\rho_{|f^c \cap B_*} = \mathrm{id}_{f^c \cap B_*}$ and $\alpha_i([0,1]) \subset f^c \cap B_*$ we have $\phi_\sharp(\beta_i) = \beta_i$. Because $\phi_\sharp$ is linear and commutes with the boundary map $\partial$

$$\partial(\phi_\sharp\beta) = \phi_\sharp(\partial\beta) = \phi_\sharp(\beta_0 - \beta_1 + \beta_2) = \beta_0 - \beta_1 + \beta_2.$$

Obviously $f^c \cap U$ is a neighbourhood of $x_0$ in $f^c \cap B_*$ and $\beta_i \in C_1(f^c \cap B_* \setminus \{x_0\})$ for $i = 1, 2$ so that Theorem 2.3 applies and

$$H_1(f^c \cap B_*, f^c \cap B_* \setminus \{x_0\}) = 0. \qquad \square$$

**4.3. Mountain pass lemma.** Let $\pi$ be a gradient-like, global semiflow w.r.t. $f$ on a locally path-connected space $X$ such that $X$ is strongly $\pi$-admissible. Let $E$ be the set of equilibria in $X$. Then every isolated equilibrium $x$ is in $\mathcal{S}(\pi)$. Because $\pi$ is gradient-like, $\omega(x) \subset E$ for all $x \in X$.

We want to show the existence of a mountain pass point in $E$. We define $\Gamma_{x,y}$ for $x, y \in X$ as follows

$$\Gamma_{x,y} = \{f \in C([0,1], X) \mid f(0) = x, f(1) = y\}.$$

Theorem 4.9. *Suppose $E$ is finite, for an $x_1 \in E$ there is an isolating block $B \in \mathcal{N}(\{x_1\})$ with $f(x) \geq \widetilde{c}$ for all $x \in \partial B$ and $\widetilde{c} > c = f(\{x_1\})$ and there is an $x_2 \in E \setminus \{x_1\}$ with $f(x_2) < \widetilde{c}$ path-connected to $x_1$, i.e. $\Gamma_{x_1,x_2} \neq \emptyset$. Then there is an equilibrium $x_0 \in E$ with*

$$f(x_0) = \inf_{g \in \Gamma_{x_1,x_2}} \sup_{t \in [0,1]} f(g(t)) \geq \widetilde{c}.$$

*If, in addition, there is a path $g \in \Gamma_{x_1,x_2}$ with $\sup_{t\in[0,1]} f(g(t)) = f(x_0)$ then there is a mountain pass point $x_m \in E$ with $f(x_m) = f(x_0)$.*

Remark 4.10. We believe that the existence of the path $g$ is necessary for the equilibrium $x_m$ with $f(x_m) = f(x_0)$ to be a mountain pass point.

Proof. Because $\Gamma_{x_1,x_2} \neq \emptyset$,

$$M := \inf_{g \in \Gamma_{x_1,x_2}} \sup_{t \in [0,1]} f(g(t))$$

is defined and $c < \widetilde{c} \leq M < \infty$. So there is a sequence $g_n \in \Gamma_{x_1,x_2}$ such that

$$\sup_{t\in[0,1]} f(g_n(t)) \leq M + \frac{1}{n}.$$

Let $E_d = \{x \in E \mid f(x) = d\}$. Suppose $E_M = \emptyset$. Because $E$ is finite, there is an $N > 0$ such that $E_{M+\varepsilon} = \emptyset$ for all $0 \leq \varepsilon \leq 1/N$.

Let $x \in f^{M+1/N}$. Because $X$ is positive $\pi$-invariant and strongly $\pi$-admissible, $\{x\pi t_n\}_{n\in\mathbb{N}}$ is precompact for some $t_n \to \infty$. In particular every cluster point is in $\omega(x) \subset E$. Choose a subsequence $n'$ such that $x\pi t_{n'} \to x^* \in E$. Because $\pi$ is gradient-like and $E_{M+\varepsilon} = \emptyset$ for all $0 \leq \varepsilon \leq 1/N$, $f(x^*) < M$. This shows that $f(x\pi t_{n'}) < M$ for large $n'$, i.e. for every $x \in f^{M+1/N}$ there is a $t(x)$ such that $f(x\pi t) < M$ for $t \geq t(x)$.

Because $\pi(\,\cdot\,, t)$ and $f$ are continuous, there is a neighbourhood $U_x$ of $x$ such that $f(y\pi t) < M$ for $y \in U_x$ and $t \geq t(x)$.

Let $g_N \in \Gamma_{x_1,x_2}$ be chosen as above. Then $f(x) \leq M + 1/N$ for every $x \in g_N^{-1}([0,1])$. Because $[0,1]$ is compact, so is $g_N([0,1])$ and there is a finite cover $U_{y_1}, \ldots, U_{y_k}$ of $g_N([0,1])$ for $y_i \in g_N([0,1])$.

Let $T := \max t(y_i)$ and define $\widetilde{g}_N(t) = g_N(t)\pi T$. Because $\pi$ is continuous and $g_N(0), g_N(1) \in E$, $\widetilde{g}_N \in \Gamma_{x_1,x_2}$ and $\sup_{t\in[0,1]} \widetilde{g}_N(t) < M$. But this is a contradiction to the definition of $M$. Therefore $E_M \neq \emptyset$.

Suppose there is a $g \in \Gamma_{x_1,x_2}$ with $g([0,1]) \subset f^M$. Let $B_x$ be an isolating block of $x \in E_M$ with $B_x^- \subset f^{M-\varepsilon}$. We can choose $B_x$ such that $B_x \cap B_y = \emptyset$ for all $x, y \in E_M$ and $x \neq y$. Because $f^M \cap B_x \setminus \{x\}$ is disjoint from $A^+(B_x)$ (see proof of [7, Theorem III-4.8]), we have $f(y\pi 1) < M$ for all $f^M \cap B_x \setminus \{x\}$. So if we define $\widetilde{g}(t) = g(t)\pi 1$ then $\widetilde{g} \in \Gamma_{x_1,x_2}$ and

$$f^{-1}(M) \cap \widetilde{g}([0,1]) \subset E_M,$$

i.e. the maximum is only achieved on the set $E_M$.

If there is no mountain pass point in $E_M$, then there are neighbourhoods $U_x \subset B_x$ such that $f^M \cap U_x \setminus \{x\}$ is path-connected. So we can construct a path $h \in \Gamma_{x_1,x_2}$ with $\sup_{t\in[0,1]} f(h(t)) \leq M$ which avoids all $x \in E_M$. (If $f^M \cap U_x \setminus \{x\}$ is empty then $x \notin g([0,1])$.) Again $\widetilde{h}(t) = h(t)\pi 1$ is in $\Gamma_{x_1,x_2}$ and

$$\sup_{t\in[0,1]} f(\widetilde{h}(t)) < M,$$

which is a contradiction to the definition of $M$. Therefore there is a mountain pass point $x_m \in E_M$. □

This can be applied to the following case. The reaction diffusion equation

$$\begin{aligned} u_t - \Delta u &= f(u), \\ u_{|\partial\Omega} &= 0, \end{aligned}$$

for some open, bounded $\Omega \subset \mathbb{R}^n$ with "nice" boundary and a "nice" function $f$ induces a semiflow $\pi$ on $U \subset X^\alpha \subset L^p(\Omega)$ (see below). Furthermore, if $u_-$ and $u_+$ are two equilibria and $u_-(x) < u_+(x)$ in $\Omega$, then the set

$$C = [u_-, u_+] := \{u \in X^\alpha \mid u_-(x) \le u(x) \le u_+(x)\}$$

is positive $\pi$-invariant and strongly $\pi$-admissible. If $u_\pm$ are stable in $C$, then there is a mountain pass point $u_0$ in $C$, which is in the interior of $C$ (see [4, Theorem 1]). The result from the next section shows that $h(\{u_0\}) = \Sigma^1$.

## 5. Semilinear parabolic equations

Now we want to compute the homotopy index of mountain pass points of parabolic equations completely. We only mention briefly the required definitions and otherwise refer to [3] and [7, Chapter II].

DEFINITION 5.1. Let $A$ be sectorial in a Banach space $X$ and $U \subset X^\alpha$ open where $X^\alpha$ for some $\alpha \in [0,1)$ is the fractional Banach space induced by $A$. Furthermore, let $f: U \to X$ be a locally Lipschitz continuous map. The following equation is called an (*autonomous*) *semilinear parabolic equation*:

$$\frac{du}{dt} + Au = f(u). \tag{5.1}$$

A solution of (5.1) through $u_0 \in U$ is a continuous map $u: [0, t_0) \to X$ with $u(0) = u_0$ such that $u$ is differentiable in $(0, t_0)$ and $u(t) \in D(A)$ for $t \in (0, t_0)$, the map $t \mapsto f(u(t))$ is locally Hölder continuous in $(0, t_0)$, $\int_0^a \|f(u(t))\|\, dt < \infty$ for some $a > 0$ and (5.1) is satisfied in $(0, t_0)$.

Assume below that the assumptions of Definition 5.1 hold and denote by $\pi$ the induced flow which exists under the condition given in the definition (see [3]).

Let $x_0$ be an isolated equilibrium. Then it admits a strongly $\pi$-admissible isolating neighbourhood. The linearization in $x_0$ is defined as $L := A - f'(x_0)$. Set $\sigma_0(L) := \sigma(L) \cap i\mathbb{R}$ and $\sigma_<(L) := \sigma(L) \cap \{\lambda \in \mathbb{C} \mid \Re\lambda < 0\}$ and denote by $X_0$ and $X_-$ the corresponding generalized eigenspaces. If $\sigma_0(L)$ is isolated in $\sigma(L)$, $X_0 \oplus X_-$ is finite dimensional then according to the index product formula [7, II-3.1]

$$h(\{x_0\}, \pi) = h(\{x_c\}, \pi_c) \wedge \Sigma^m$$

where $m = \dim X_-$, $\pi_c$ is the center flow on a $\dim X_0$-dimensional (local) center manifold (see [3] or [7, II-2.1]) and $\{x_c\}$ is isolated for $\pi_c$. Combining this formula and Theorem 4.3 we can compute the homotopy index for mountain pass points.

THEOREM 5.2. *Suppose the assumptions above hold and $\pi$ is gradient-like w.r.t. some continuous map $f$. Furthermore, suppose the following holds*

$$x_0 \text{ is isolated and } \dim X_- = 0 \;\Rightarrow\; \dim X_0 \le 1.$$

*Then*

$$x_0 \text{ is an isolated mountain pass point } \Leftrightarrow\ h(\{x_0\}) = \Sigma^1.$$

Proof. If $x_0$ is an isolated equilibrium then $\{x_0\} \in \mathcal{S}$ and thus

$$h(\{x_0\}, \pi) = h(\{x_c\}, \pi_c) \wedge \Sigma^m,$$

where $m = \dim X_-$ and $\pi_c$ is the reduced semiflow on the local center manifold.

In particular we have

$$H_{m+i}(h(\{x_0\}, \pi)) = H_{m+i}(h(\{x_c\}, \pi_c) \wedge \Sigma^m) = H_i(h(\{x_c\}, \pi_c)),$$

so that $H_j(h(\{x_0\}, \pi)) = 0$ for $j < m$.

Suppose $x_0$ is a mountain pass point then

$$H_1(h(\{x_0\}, \pi)) \neq 0.$$

Because $H_j(h(\{x_0\}, \pi)) = 0$ for $j < m$, we must have $m \leq 1$.

If $m = 1$ then

$$0 \neq H_1(h(\{x_0\}, \pi)) = H_0(h(\{x_c\}, \pi_c)).$$

Because $x_0$ is isolated and thus $x_c$, Theorem 3.3 states that $h(\{x_c\}, \pi_c) = \Sigma^0$ and therefore

$$h(\{x_0\}, \pi) = \Sigma^0 \wedge \Sigma^1 = \Sigma^1.$$

If $m = 0$ then $\dim X_0 \leq 1$. Suppose $\dim X_0 = 0$ then $h(\{x_0\}, \pi) = h(\{x_c\}, \pi_c) = \Sigma^0$. Thus $H_1(h(\{x_0\}, \pi) = 0$ which is not possible. Hence $\dim X_0 = 1$.

For one dimensional isolated equilibria we have

$$h(\{x_c\}, \pi_c) = \begin{cases} \Sigma^0 & \text{then } H_0(h(\{x_c\}, \pi_c)) \neq 0, \\ \Sigma^1 & \text{then } H_1(h(\{x_c\}, \pi_c)) \neq 0, \\ \underline{0} & \text{then } H_*(h(\{x_c\}, \pi_c)) = 0. \end{cases}$$

Because

$$H_1(h(\{x_c\}, \pi_c)) = H_1(h(\{x_0\}, \pi)) \neq 0,$$

only the second case can happen, i.e.

$$h(\{x_0\}, \pi) = \Sigma^0 \wedge h(\{x_c\}, \pi_c) = \Sigma^1.$$

It remains to show the "only if" part. Suppose $h(\{x_0\}, \pi) = \Sigma^1$. Then

$$H_1(h(\{x_0\}, \pi)) = G \neq 0.$$

Because $U \subset X^\alpha$ is open, according to Theorem 4.3 $x_0$ is a mountain pass point. □

Remark 5.3. The conclusion "$H_1 \neq 0$ then $h = \Sigma^1$" can also be made for the Alexander–Spanier cohomology $\{\overline{H}^n\}$ (also see Remark 3.4).

With the help of this theorem we are able to compute the homotopy index of isolated mountain pass points for reaction diffusion equations completely: For this type of equation $A$ is the Laplacian $-\Delta$ on $X = L^p(\Omega)$ for some open and bounded $\Omega \subset \mathbb{R}^n$ with smooth boundary. If $u \in X$ is a solution of $-\Delta u = f(u)$ and $f'(u) \in L^\infty(\Omega)$ then $L = -\Delta - f'(u)$ has simple principal eigenvalue. If we assume that $u_t - \Delta u = f(u)$ induces a semiflow on some $X^\alpha$ then the assumptions of the previous theorem are satisfied and the homotopy index of a mountain pass point is $\Sigma^1$.

Besides this we can prove a result for mountain pass points and critical groups which is proven completely in [1, Proposition 3.3]. Instead of infinite dimensional Morse theory we will use the homotopy index and generalize the statement to Alexander–Spanier cohomology.

Corollary 5.4. *Let $U \subset H$ be an open neighbourhood of $0$ in a Hilbert space $H$, $\phi\colon U \to \mathbb{R}$ a $C^2$-function satisfying the Palais–Smale condition in $0$ and $\phi(0) = 0$, $\nabla\phi(0) = 0$, $\nabla\phi(u) \neq 0$ for $u \neq 0$. Furthermore, $L = \phi''(0)\colon H \to H$ is a Fredholm operator with finite Morse index $m$ such that $\dim\ker L \leq 1$ whenever $L \geq 0$. Under these assumptions*

$$0 \textit{ is an isolated mountain pass point } \Leftrightarrow\ h(\{0\}) = \Sigma^1$$
$$\Leftrightarrow\ \mathsf{H}_1(\phi^0 \cap U, \phi^0 \cap U \setminus \{0\}) = \mathbb{Z} \ \Leftrightarrow\ \mathsf{H}_1(\phi^0 \cap U, \phi^0 \cap U \setminus \{0\}) \neq 0$$

*for $\{\mathsf{H}_n\}$ either the singular homology $\{H_n\}$ or the Alexander–Spanier cohomology $\{\overline{H}^n\}$.*

Remark 5.5. For singular homology we don't need the finite Morse index because Rybakowski proved in [7, III-4.10] that $\mathsf{H}_n(\phi^0 \cap U, \phi^0 \cap U \setminus \{0\}) = 0$ if $m = \infty$. Using [1, Proposition 3.3] shows that the linearization of a mountain pass point has finite Morse-index.

Proof. We will use the statements from the proof of Theorem 4.10 in [7], i.e. $\dot{x} = -\nabla\phi(x)$ induces a two-sided local gradient-like flow $\pi_\phi$ w.r.t. $\phi$ whose critical points ($\nabla\phi(x_0) = 0$) are exactly the equilibria of $\pi_\phi$ and the linearization in 0 is $L$.

Since $L$ is self-adjoint and Fredholm and its Morse index is finite, $L$ is sectorial and the previous theorem can be applied, i.e.

$$0 \text{ is an isolated mountain pass point } \Leftrightarrow\ h(\{0\}) = \Sigma^1.$$

Lemma 4.5 shows that

$$\mathsf{H}_n(h(\{0\})) = \mathsf{H}_n(\phi^0 \cap U, \phi^0 \cap U \setminus \{0\}).$$

According to the Remark 5.3 $\mathsf{H}_1(h(\{0\})) \neq 0$ is only possible if and only if

$$h(\{0\}) = \Sigma^1. \qquad \square$$

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*Manuscript received September 20, 2010*

MARTIN KELL
Institut für Mathematik
Universität Rostock
18051 Rostock, Germany

*E-mail address*: mkell@mis.mpg.de