Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder University Centre
Volume 38, 2011, 363–372

A ONE DIMENSIONAL PROBLEM RELATED TO THE SYMMETRY OF MINIMISERS FOR THE SOBOLEV TRACE CONSTANT IN A BALL

Olaf Torné

Abstract. The symmetry of minimisers for the best constant in the trace inequality in a ball, $S_q(\rho) = \inf_{u \in W^{1,p}(B_\rho)} \|u\|^p_{W^{1,p}(B_\rho)} / \|u\|^p_{L^q(\partial B(\rho))}$ has been studied by various authors. Partial results are known which imply radial symmetry of minimisers, or lack thereof, depending on the values of trace exponent q and the radius of the ball ρ. In this work we consider a one dimensional analogue of the trace inequality and the corresponding minimisation problem for the best constant. We describe the exact values of q and ρ for which minimisers are symmetric. We also consider the behaviour of minimisers as the symmetry breaking threshold for q and ρ is breached, and show a case in which both symmetric and nonsymmetric minimisers coexist.

1. Introduction

This note describes a one dimensional problem that is related to the study of the symmetry properties of minimisers for the best constant in the trace inequality in a ball. To motivate what follows we first review some known results.

Let B_ρ denote the ball of radius $\rho > 0$ centered at the origin in $\mathbb{R}^N$ with $N \geq 2$. Let $1 < p < \infty$ and let $1 < q < p_*$ where p_* is the critical trace

2010 *Mathematics Subject Classification.* 35J20, 26D15, 35A30, 35J60, 47J30.
Key words and phrases. Trace inequality, symmetry, symmetry breaking.

exponent. The best constant in the Sobolev trace inequality is given by

$$S_q(\rho) = \inf_{u \in W^{1,p}(B_\rho)} \frac{\int_{B_\rho} |\nabla u|^p + |u|^p \, dx}{(\int_{\partial B_\rho} |u|^q \, d\sigma)^{p/q}}. \tag{1.1}$$

If $1 < q < p_*$ this infimum is achieved by a function u which has definite sign and any nonzero multiple of u is again a minimiser. Since the minimisation problem is invariant under any rotation it is natural to ask if u is a radial function.

When $p = 2$ the results of [1], [2], [4] show that if $q \leq 2$ then any minimiser is radial. Also, if $q > 2$ is fixed and ρ is sufficiently small, or if $\rho > 0$ is fixed and $q > 2$ is sufficiently close to 2, then any minimiser is radial. On the other hand if $q > 2$ is fixed, then there is no radial minimiser provided ρ is large enough.

The case for general $1 < p < \infty$ is studied in [4]. It is shown that if $1 < q \leq p$, then any minimiser for (1.1) is radial. Moreover, under certain conditions radial symmetry is lost if either q or ρ is sufficiently large. Define the function $Q(\rho)$ by

$$Q(\rho) = \frac{1}{\lambda_1(\rho)^{p/(p-1)}} \left(1 - (N-1) \frac{\lambda_1(\rho)}{\rho} \right) + 1. \tag{1.2}$$

Here $\lambda_1(B_\rho)$ denotes the first positive eigenvalue in the Steklov problem

$$\begin{cases} \Delta_p u = |u|^{p-2} u & \text{in } B_\rho, \\ |\nabla u|^{p-2} \dfrac{\partial u}{\partial \nu} = \lambda |u|^{p-2} u & \text{on } \partial B_\rho. \end{cases} \tag{1.3}$$

It is shown that if $q > Q(\rho)$ then there is no radial minimizer for $S_q(\rho)$. Also, $Q(\rho) \downarrow p$ as $\rho \to \infty$ from which it follows that if $q > p$ is fixed, radial symmetry is lost provided ρ is sufficiently large. It is also known that if there exists a radial minimiser for some $S_{q_0}(\rho)$, then any minimiser for $S_q(\rho)$, with $q < q_0$, is radial. If a radial minimiser does exist, it is given by the unique (up to normalisation) eigenfunction associated to $\lambda_1(\rho)$.

In light of this, one may define a threshold value $Q^*(\rho)$ such that if $q < Q^*$ then any minimiser is radial, whereas if $q > Q^*$ then there does not exist a radial minimiser. Also one may show there exists a radial minimiser for $q = Q^*$ but it's not known in this case whether or not there may also exist a nonradial one.

Clearly there holds $Q^*(\rho) \leq Q(\rho)$, however thus far there do not appear to be any cases in which it is known if $Q(\rho)$ is the exact threshold for loss of radial symmetry, or merely an upper bound.

In this note we consider a one dimensional analogue of the minimisation problem (1.1) in which Q is found to be the exact threshold for symmetry breaking. Let $1 < p < \infty$, $1 < q < \infty$ and $\rho > 0$. In analogy with (1.1), we define

$$S_q(\rho) = \inf_{u \in W^{1,p}(-\rho,\rho)} \frac{\int_{-\rho}^{\rho} |u'|^p + |u|^p \, dx}{(|u(-\rho)|^q + |u(\rho)|^q)^{p/q}}. \tag{1.4}$$

As above, the infimum in (1.4) is achieved by a function of definite sign that can be assumed to be positive. In this setting the notion of radial symmetry is naturally replaced by that of an even function. As above, the case $q = p$ is referred to as the Steklov problem and the corresponding minimiser u_1 is unique up to normalisation. We have the following result.

Theorem 1.1. *Assume $p > 3/2$. Let $u \in W^{1,p}(-\rho,\rho)$ be a minimiser for $S_q(\rho)$ defined by* (1.4). *Then u is an even function if, and only if, $q \le Q(\rho)$ where Q is given by* (1.2) *with $N = 1$ and with λ_1 the first eigenvalue in the corresponding one dimensional Steklov problem.*

It is interesting to note that in the case $p = 2$, $\rho = 1$ and $N \ge 3$, Gazzini and Serra [3] find, amongst other results, that minimisers for the Rayleigh quotient (1.1) restricted to radial functions, are nondegenerate local minima over the whole of H^1 for all $q < 2_*$. In this case $Q(\rho) > 2_*$ so the condition $q < Q(\rho)$ is automatically satisfied. Furthermore, those authors observe that if one were to have $Q^* < 2_*$ then this would imply that nonradial minimisers are always far away from the radial minisers and do not branch out smoothly as the level Q^* is breached. In the case of problem (1.4) we are able to show that such counter intuitive behaviour may or may not occur depending on the value of p.

Under the assumptions of Theorem 1.1, the behaviour of minimisers is as expected in that noneven minimisers branch out from the even one when the symmetry breaking threshold is breached.

Theorem 1.2. *Assume $p > 3/2$. Any minimiser for $S_{Q(\rho)}(\rho)$ is even and, up to normalisation, is given by u_1. Also, noneven minimisers branch out from u_1 in the sense that if $q_i \downarrow Q(\rho)$, and u_{q_i} is an appropriately normalised minimiser for $S_{q_i}(\rho)$, then $u_{q_i} \to u_1$.*

When p is close to unity, we do not know if there holds $Q^*(\rho) = Q(\rho)$. However we can prove that noneven minimisers make a sudden appearance and do not branch out from even ones.

Theorem 1.3. *Let $\rho > 0$ be fixed. If $p > 1$ is sufficiently close to 1 then there exists both an even and a noneven minimiser for $S_{Q^*}(\rho)$. Moreover, if $q_i \downarrow Q^*(\rho)$, and u_{q_i} is an appropriately normalised minimiser for $S_{q_i}(\rho)$, then up to a subsequence u_{q_i} converges to a noneven minimiser.*

2. Proofs

If u is a minimiser for (1.4) then it satisfies the following equations.

$$(2.1) \qquad \begin{cases} (|u'|^{p-2}u')' = u^{p-1} \quad \text{in }]-\rho,\rho[, \\ |u'|^{p-2}u'(\pm\rho) = \pm S_q(\rho)(|u(-\rho)|^q + |u(\rho)|^q)^{p/q-1}|u(\pm\rho)|^{q-2}u(\pm\rho). \end{cases}$$

When $q = p$ this is a one dimensional version of the Steklov eigenvalue problem, the first positive eigenvalue being $\lambda_1(\rho) = S_p(\rho)$. As in the multi dimensional case, the associated eigenfunction u_1 is unique up to normalisation and it is even. If u is a solution of (2.1) such that $u(-\rho) = u(\rho)$, then up to normalisation it satisfies the same Dirichlet problem as u_1, and thus $u = u_1$. In particular, u_1 can be viewed as a function defined over all $\mathbb{R}$ and whose restriction to an interval $[-\rho, \rho]$ is the Steklov eigenfunction associated to $\lambda_1(\rho)$. To fix ideas we assume u_1 is normalised so that $u_1(0) = 1$.

If there exists an even minimiser for $S_{q_0}(\rho)$ then any minimiser for $S_q(\rho)$, with $q < q_0$ is even and is equal to u_1 up to normalisation. The proof of this fact is contained in [4] in the case $N \geq 2$ and it carries over to the current setting without difficulty. Now, since the case $q = p$ corresponds to the Steklov problem in which the minimiser is known to be even, we need only consider the case $q > p$ in what follows.

Proof of Theorem 1.1. First we prove the "if" part. Let $u > 0$ be a minimiser for (1.4) and assume that u is not even. Since any nonzero multiple of a minimiser is again a minimiser, we can assume that u has been normalised in such a way that $u(-\rho) = 1$ and $u(\rho) = a$ with $0 < a \leq 1$. Now, if $a = 1$ then, up to a further normalisation, u satisfies the same Dirichlet boundary value problem as the first Steklov eigenfunction u_1 and thus $u = u_1$. Since u_1 is even we can exclude this case and assume that $0 < a < 1$. Now u satisfies

$$(|u'|^{p-2}u')' = |u|^{p-2}u \quad \text{on }]-\rho, \rho[.$$

Multiplying both sides by u' and integrating we get

$$(p-1)(|u'|^p)' = (|u|^p)',$$

so that

$$(p-1)(|u'(\rho)|^p - |u'(-\rho)|^p) = |u(\rho)|^p - |u(-\rho)|^p = a^p - 1.$$

On the other hand, u satisfies the boundary conditions appearing in (2.1) which now take on the form

$$\begin{aligned} (-u'(-\rho))^{p-1} &= S_q(1 + a^q)^{p/q-1}, \\ u'(\rho)^{p-1} &= S_q(1 + a^q)^{p/q-1}a^{q-1}. \end{aligned}$$

Inserting this into the previous equation and rearranging gives

$$(p-1)S_q(\rho)^{p'} = f_q(a),$$

where $p' = p/(p-1)$ and

$$f_q(a) = \frac{1 - a^p}{1 - a^{p(q-1)/(p-1)}}(1 + a^q)^{p'(1-p/q)}. \tag{2.2}$$

We shall see below that f_q is decreasing on $]0,1[$ from which it follows,

$$(p-1)S_q(\rho)^{p'} = f(a) > \lim_{t\to 1} f(t) = 2^{p'(1-p/q)}\frac{p-1}{q-1}. \tag{2.3}$$

Now consider $S_q^{\text{even}}(\rho)$, the infimum (1.4) restricted to even functions. Up to normalisation an even minimiser will satisfy the same Dirichlet boundary value problem as the first Steklov eigenfunction u_1 and will thus coincide with u_1. Thus $S_q^{even}(\rho)$ is given by the Rayleigh quotient (1.4) evaluated at u_1 and we find that,

$$S_q^{\text{even}}(\rho) = 2^{1-p/q}\lambda_1(\rho). \tag{2.4}$$

Since $S_q(\rho) \le S_q^{even}(\rho)$, equations (2.3) and (2.4) now yield

$$q > 1 + \frac{1}{\lambda_1(\rho)^{p/(p-1)}} = Q(\rho),$$

which is the desired result.

It remains only to show that, under the assumptions of the theorem, f_q defined above is decreasing on the interval $]0,1[$. We have,

$$\begin{aligned}\frac{df_q}{dt}(t) = {} & \frac{t^{p-1}(1+t^q)^{(p^2-q)/(q-pq)}p}{(-1+t^{p(-1+q)/(-1+p)})^2(-1+p)} \\ & \cdot [1 + t^{-p+p(-1+q)/(-1+p)+q}(-1+p) - p \\ & + t^q(1-q) + t^{-p+p(-2+p+q)/(-1+p)}(p-q) \\ & + t^{-p+p(-1+q)/(-1+p)}(-1+q) + t^{-p+q}\,(-p+q)].\end{aligned}$$

Let the term in square brackets be denoted by $g_1(t)$. Then $g_1(0) = 1-p < 0$ and $g_1(1) = 0$ so it suffices to check that g_1 is increasing on $]0,1[$. We have

$$\begin{aligned}\frac{dg_1}{dt}(t) = t^{-(1+p-q)}\Bigg[& (p-q)^2 - \frac{p(p-q)(-1+q)}{-1+p}t^{(-p+q)/(-1+p)} \\ & + \frac{p(p-q)(-1+q)}{-1+p}t^{(-2p+p^2+q)/(-1+p)} \\ & - qt^p(-1+q) - t^{p(-1+q)/(-1+p)}(p^2+q-2pq)\Bigg].\end{aligned}$$

Denote the term in square brackets by $g_2(t)$, so $g_2(0) = (p-q)^2$ and $g_2(1) = 0$. We wish to show $g_2 \ge 0$ for all $0 \le t \le 1$.

In order to simplify notations we make the following substitutions,

$$A = \frac{(p-q)^2}{(-1+p)^2}, \quad B = \frac{(q-p)(-2p+p^2+q)}{(-1+p)^2}, \quad C = \frac{(-p^2-q+2pq)}{-1+p},$$
$$\alpha = \frac{1}{p}\frac{p^2-q}{-1+p}, \quad \beta = \frac{q}{p},$$

so that $A, B, C, \beta > 0$. We then have

$$\frac{dg_2}{dt}(t) = p(-1+q)t^{(1-2p+q)/(-1+p)}h(t^p),$$

where $h(s) = h_1(s) - h_2(s)$, $h_1(s) = A - Bs + Cs^\beta$, $h_2(s) = qs^\alpha$.

First assume that $p < q < p^2$. Then $\beta > 1$ so that h_1 is convex, and $0 \leq \alpha < 1$ so h_2 is concave. Thus $h = h_1 - h_2$ can have at most two zeros. Since $h_1(1) = h_2(1) = q$ we conclude that dg_2/dt has at most one zero on $]0,1[$. Next observe that $dg_2/dt(t) > 0$ for t close to 0, and $g_2(0) >$ and $g_2(1) = 0$ as noted above. It follows that $g_2 \geq 0$ on $]0,1[$.

Now let $q > p^2$. In this case $\alpha < 0$ so $h(0) = -\infty$, and $h(1) = 0$. We wish to show $h(s) \leq 0$ for all $0 < s < 1$. If h has a local maximum at some point $0 < s_0 < 1$, then it must hold

$$-Bs_0 = -\beta C s_0^\beta + q\alpha s_0^\alpha. \tag{2.5}$$

We define a new function $\widehat{h}$ by substituting (2.5) into the formula for $h(s)$,

$$\widehat{h}(s) = A + q(\alpha-1)s^\alpha - C(\beta-1)s^\beta,$$

so that $\widehat{h}(s_0) = h(s_0)$. Also, $\widehat{h}(0) = -\infty$ and

$$\widehat{h}(1) = A + q(\alpha-1) - C(\beta-1) = \frac{q-p}{(-1+p)^2}(p^2 - 2p + q(3-2p)).$$

If $q > p^2$ and $p > 3/2$ then $\widehat{h}(1) \leq 0$. Now if $\widehat{h}$ has a local maximum at some point $0 < s_1 < 1$, then it must hold

$$q\frac{\alpha}{\beta}(\alpha-1)s_1^\alpha = C(\beta-1)s_1^\beta,$$

and consequently

$$\widehat{h}(s_1) = A + q(\alpha-1)\left(1 - \frac{\alpha}{\beta}\right)s_1^\alpha.$$

Since $\alpha < 0$ there holds $s_1^\alpha > 1$. Also $q(\alpha-1)(1-\alpha/\beta) < 0$ so it follows

$$\widehat{h}(s_1) \leq A + q(\alpha-1)\left(1 - \frac{\alpha}{\beta}\right) = 0.$$

It follows that $h \leq 0$ and g_2 is nonnegative on $[0,1]$.

The proof of the "only if" part is similar to [4] but we sketch it here for completeness. Let u be a minimiser and assume that u is even. Then, up to normalisation, $u = u_1$ the first Steklov eigenfunction. Set $u_t(x) = u_1(x-t)$ for $t \in \mathbb{R}$ and define

$$\Phi(t) = \frac{\int_{-\rho}^{\rho} |u_t'|^p + |u_t|^p\,dx}{(|u_t(-\rho)|^q + |u_t(\rho)|^q)^{p/q}}.$$

Direct calculations show that $\Phi'(0) = 0$ and

$$\Phi''(0) = C(1 - (q-1)\lambda_1(\rho)^{p/(p-1)}),$$

where C is a positive constant. Therefore if $q > Q(\rho)$ then $\Phi''(0) < 0$ and u does not minimise (1.4) which is a contradiction. $\square$

The proof of Theorem 1.1 relies on f_q, given by (2.2), being a decreasing function on $[0,1]$. This fact is proved when either $p > 3/2$ or $1 < q < p^2$, but it does not hold for all p and q. Indeed let $q > 1$ be fixed. Then $f_q(0) = 1$ but $\lim_{p\to 1, t\to 1} f_q(t) = \infty$ so for q fixed and p close enough to 1, the function f_q is not monotonically decreasing.

Proof of Theorem 1.2. Let $q_i \to Q(\rho)$. Let u_{q_i} be a minimiser for $S_{q_i}(\rho)$, normalised so that $u_{q_i}(-\rho) = 1$ and $u_{q_i}(\rho) = a_i$ with $0 < a_i \le 1$. As in the proof of Theorem 1.1 we have

$$(p-1)S_{q_i}(\rho)^{p/(p-1)} = f_{q_i}(a_i).$$

It follows from standard arguments that $q \mapsto S_q(\rho)$ is continuous (this is true both for the usual trace constant (1.1) and in the one dimensional analogue (1.4)). Also, up to a subsequence, $a_i \to a$ for some $0 < a \le 1$. Therefore,

$$(p-1)S_{Q(\rho)}(\rho)^{p/(p-1)} = f_{Q(\rho)}(a).$$

Now $S_{Q(\rho)}(\rho)$ is realised by the first Steklov eigenfunction so that

$$(p-1)S_{Q(\rho)}(\rho)^{p/(p-1)} = (p-1)(2^{1-p/q}\lambda_1(\rho))^{p/(p-1)} = f_{Q(\rho)}(1),$$

and therefore $f_{Q(\rho)}(a) = f_{Q(\rho)}(1)$. In the proof of Theorem 1.1 it is shown that $f_{Q(\rho)}$ is a decreasing function on the interval $[0,1]$ so we must have $a = 1$. It follows that the u_{q_i} converge to the first Steklov eigenfunction u_1. $\square$

Lemma 2.1. *The first eigenvalue in the one dimensional Steklov eigenvalue problem, $\rho \mapsto \lambda_1(\rho)$, is increasing and*

$$\lambda_1' = 1 - (p-1)\lambda_1^{p/(p-1)} \quad \textit{and} \quad \lim_{\rho\to\infty} \lambda_1(\rho) = \left(\frac{1}{p-1}\right)^{(p-1)/p}.$$

Proof. A similar statement is proved in the case $N \ge 2$ in [4] and the proof therein carries to the present setting in a straightforward way. $\square$

Proof of Theorem 1.3. Firstly we note that $S_{Q^*(\rho)}(\rho)$ always has an even minimiser. Indeed, u_1 is a minimiser for all $S_q(\rho)$ with $q < Q^*(\rho)$, and $q \mapsto S_q(\rho)$ is continuous, so it follows u_1 is also a minimiser for $S_{Q^*(\rho)}(\rho)$.

Next, we observe that if $v > 0$ is a minimiser for $S_q(\rho)$, then there exist $\alpha > 0$ and $-\rho < t < \rho$ such that $v(x) = \alpha u_1(x+t)$. (Recall from the beginning of this section that u_1 is a function defined over all $\mathbb{R}$). To see this, assume first $v(-\rho) = v(\rho)$. Taking $\alpha = v(-\rho)/u_1(-\rho)$, we have $v(\pm\rho) = \alpha u_1(\pm\rho)$ so that αu_1 and v are solutions to the same Dirichlet problem on $[-\rho,\rho]$ (see equation (2.1)). Therefore $v(x) = \alpha u_1(x)$ on $[-\rho,\rho]$. Assume now, without loss of generality, $v(-\rho) < v(\rho)$. By (2.1), v is convex and $\operatorname{sign} v'(-\rho) \neq \operatorname{sign} v'(\rho)$. Therefore there exists s, with $-\rho < s < \rho$, such that $v(-\rho) = v(s)$. Now set $\alpha = v(-\rho)/u_1(-\rho + (\rho - s)/2)$ and $t = (\rho - s)/2$ and consider the function $x \mapsto \alpha u_1(x + t)$. Both this function and v satisfy the differential equation in the first line of (2.1). Also $v(-\rho) = \alpha u_1(-\rho + t)$ and $v(s) = \alpha u_1(s + t)$, so that $\alpha u_1(\,\cdot\, + t)$ and v are solutions to the same Dirichlet problem on the interval $-\rho < x < s$. It follows that $v(x) = \alpha u_1(x + t)$ on the interval $-\rho < x < s$, and therefore also on $-\rho < x < \rho$.

Let $v(x) = u_1(x + t)$, $t \neq 0$, be a minimiser for some $S_q(\rho)$. From the boundary conditions satisfied by v we have

$$\frac{|u_1'(\rho+t)|^{p-2}u_1'(\rho+t)}{u_1(\rho+t)^{q-1}} = \frac{|u_1'(-\rho+t)|^{p-2}u_1'(-\rho+t)}{u_1(-\rho+t)^{q-1}}.$$

It follows that $q = \widehat{q}(t)$ where we have defined

$$\widehat{q}(t) = 1 + \frac{\ln\left(\dfrac{u_1'(\rho+t)^{p-1}}{(-u_1'(-\rho+t))^{p-1}}\right)}{\ln\left(\dfrac{u_1(\rho+t)}{u_1(-\rho+t)}\right)}. \tag{2.6}$$

Now let $q_i \downarrow Q^*(\rho)$ and let $v_i(x) = u_1(x+t_i)$ be a minimiser for $S_{q_i}(\rho)$. Up to a subsequence, $t_i \to t \geq 0$ and it's then easy to see that $u_1(x + t)$ is a minimiser for $S_{Q^*(\rho)}(\rho)$. First assume $t = 0$. Since $\widehat{q}(t_i) = q_i \downarrow Q^*(\rho)$, there must hold $\widehat{q}(t_i) > \widehat{q}(0)$. However this is impossible if p is sufficiently close to 1 since in this case we have $\lim_{t\to 0} \widehat{q}\,'(t) = 0$ and $\lim_{t\to 0} \widehat{q}\,''(t) < 0$, as will shown below. It follows that $t_i \to t$ for some $t > 0$. Consequently, $u_1(x + t)$ is a noneven minimiser for $S_{Q^*(\rho)}(\rho)$.

It's clear that $\lim_{t\to 0} \widehat{q}\,'(t) = 0$ since $\widehat{q}$ is even, so it remains only to show $\widehat{q}\,''(0) = \lim_{t\to 0} \widehat{q}\,''(t) < 0$. Using that $u_1'(\pm\rho + t) = \pm\lambda_1^{1/(p-1)}(\rho \pm t)u_1'(\pm\rho + t)$, we can write

$$\widehat{q}(t) = p + \frac{\varphi_1(t)}{\varphi_2(t)},$$

where

$$\varphi_1(t) = \ln\frac{\lambda_1(\rho+t)}{\lambda_1(\rho-t)} \quad \text{and} \quad \varphi_2(t) = \ln\frac{u_1(\rho+t)}{u_1(-\rho+t)}.$$

Then,

$$\widehat{q}\,'' = \frac{\varphi_1''\varphi_2 - \varphi_1\varphi_2''}{\varphi_2^2} - 2\frac{\varphi_1'\varphi_2\varphi_2' - \varphi_1{\varphi_2^2}'}{\varphi_2^3}.$$

Using this expression, we apply l'Hospital's rule twice on the first fraction and thrice on the second one, to obtain

$$\widehat{q}\,''(0) = \frac{(\varphi_1''\varphi_2 - \varphi_1\varphi_2'')''(0)}{(\varphi_2^2)''(0)} - 2\frac{(\varphi_1'\varphi_2\varphi_2' - \varphi_1{\varphi_2^2}')'''(0)}{(\varphi_2^3)'''(0)}.$$

Now we formally expand the derivatives, and use that $\varphi_1(0) = \varphi_2(0) = \varphi_1''(0) = \varphi_2''(0) = 0$, to obtain,

$$\widehat{q}''(0) = \frac{1}{3}\frac{\varphi_1'''(0)\varphi_2'(0) - \varphi_2'''(0)\varphi_1'(0)}{\varphi_2'(0)^2}.$$

Using the expression for λ_1' appearing in Lemma 2.1, direct calculations yield

$$\begin{aligned}
\varphi_1'(0) &= 2\left(\frac{1}{\lambda_1(\rho)} - (p-1)\lambda_1^{p/(p-1)-1}(\rho)\right),\\
\varphi_2'(0) &= 2\lambda_1^{1/(p-1)}(\rho),\\
\varphi_1'''(0) &= 2\left(\frac{2}{\lambda_1^3(\rho)} - \left(\frac{p}{p-1} - 2\right)\lambda_1^{p/(p-1)-3}(\rho)\right)(1-(p-1)\lambda_1^{p/(p-1)}(\rho))^2\\
&\quad - 2\lambda_1^{p/(p-1)-1}(\rho)\left(\frac{-1}{\lambda_1^2(\rho)} - \lambda_1^{p/(p-1)-2}(\rho)\right)(1-(p-1)\lambda_1^{p/(p-1)}(\rho)),\\
\varphi_2'''(0) &= 2\left(\frac{2-p}{(p-1)^2}\lambda_1^{(3-2p)/(p-1)}(\rho) + \frac{-4+p}{p-1}\lambda_1^{(3-p)/(p-1)}(\rho) + 2\lambda_1^{3/(p-1)}(\rho)\right),
\end{aligned}$$

so that, after simplification,

$$\begin{aligned}
\widehat{q}''(0) &= \frac{1}{3}\left(2 - \frac{2-p}{(p-1)^2}\right)\lambda_1^{(-3p+2)/(p-1)}(\rho)\\
&\quad + \frac{-p^2+2p}{p-1}\lambda_1^{-2}(\rho) + \left(\frac{p^2}{3} - p\right)\lambda_1^{(-p+2)/(p-1)}(\rho).
\end{aligned} \tag{2.7}$$

In the following, we denote $\lambda_1(\rho) = \lambda_1(\rho;p)$ to make explicit the dependence of λ_1 on p, and also $\widehat{q}(t) = \widehat{q}(t;p)$. By Lemma 2.1 there holds

$$\lambda_1(\rho;p) < \left(\frac{1}{p-1}\right)^{(p-1)/p}, \tag{2.8}$$

and this upper bound tends to 1 as $p \to 1$. Now let $\rho > 0$ be fixed and consider a sequence $p_i \downarrow 1$. By (2.8) we can extract of a subsequence, again noted p_i, such that $\lambda_1(\rho;p_i) \to \lambda$ for some $0 \le \lambda \le 1$. The final term in (2.7) becomes negative as p approaches 1 so,

$$\begin{aligned}
\widehat{q}''(0;p_i) &\le \frac{1}{3}\left(2 - \frac{2-p_i}{(p_i-1)^2}\right)\lambda_1(\rho;p_i)^{(-3p_i+2)/(p_i-1)} + \frac{-p_i^2+2p_i}{p_i-1}\lambda_1(\rho;p_i)^{-2}\\
&= \frac{1}{(p_i-1)\lambda_1(\rho;p_i)^2}\\
&\quad \cdot \left(\frac{1}{3}\left(2(p_i-1) - \frac{2-p_i}{p_i-1}\right)\left(\frac{1}{\lambda_1(\rho;p_i)}\right)^{p_i/(p_i-1)} - p_i^2 + 2p_i\right).
\end{aligned}$$

It follows that for $\rho > 0$ fixed, $\widehat{q}''(0;p_i) \to -\infty$ as $p_i \to 1$. Since the sequence p_i was arbitrary, we conclude that $\widehat{q}''(0;p) < 0$ for p sufficiently close to 1. □

Acknowledgements. The author thanks Enrique Lami Dozo for numerous discussions relating to this work.

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Manuscript received January 31, 2011

Olaf Torné
Laboratoire MAS
Ecole Centrale Paris
Grande Voie des Vignes
92290 Châtenay-Malabry, FRANCE

E-mail address: olaf.torne@ecp.fr