TOPOLOGICAL METHODS
FOR BOUNDARY VALUE PROBLEMS
INVOLVING DISCRETE VECTOR $\phi$-LAPLACIANS

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Abstract. In this paper, using Brouwer degree arguments, we prove some existence results for nonlinear problems of the type

$$-\nabla[\phi(\Delta x_m)] = g_m(x_m, \Delta x_m) \quad (1 \leq m \leq n-1),$$

submitted to Dirichlet, Neumann or periodic boundary conditions, where $\phi(x) = |x|^{p-2}x$ ($p > 1$) or $\phi(x) = x/\sqrt{1-|x|^2}$ and $g_m: \mathbb{R}^N \to \mathbb{R}^N$ ($1 \leq m \leq n-1$) are continuous nonlinearities satisfying some additional assumptions.

1. Introduction and notation

In this paper, using Brouwer degree arguments, we prove some existence results for nonlinear problems of the type

$$-\nabla[\phi(\Delta x_m)] = g_m(x_m, \Delta x_m) \quad (1 \leq m \leq n-1),$$

submitted to Dirichlet, Neumann or periodic boundary conditions, where functions $g_m: \mathbb{R}^N \to \mathbb{R}^N$ ($1 \leq m \leq n-1$) are continuous and the discrete vector $\phi$-Laplacian operator is defined as follows.

Let $n$, $N$ be positive integers, $0 < a \leq \infty$ and $\phi: B(a) \subset \mathbb{R}^N \to \mathbb{R}^N$ be a homeomorphism such that $\phi(0) = 0$. (In what follows $B(\rho)$ denotes an open
ball with center in zero and radius $r$). For any $x = (x_0, \ldots, x_n) \in [\mathbb{R}^N]^{n+1}$ we define
\[
\Delta x_m = x_{m+1} - x_m, \quad (0 \leq m \leq n - 1)
\]
and if $|\Delta x_m| < a$ ($0 \leq m \leq n - 1$) we define
\[
\nabla \phi(\Delta x_m) = \phi(\Delta x_m) - \phi(\Delta x_m - 1), \quad (1 \leq m \leq n - 1).
\]

Discrete vector $p$-Laplacian generated by the homeomorphism
\[
h_p: \mathbb{R}^N \to \mathbb{R}^N, \quad x \mapsto |x|^{p-2}x, \quad (p > 1),
\]
and the relativistic discrete operator generated by
\[
\phi: B(1) \subset \mathbb{R}^N \to \mathbb{R}^N, \quad x \mapsto \frac{x}{\sqrt{1 - |x|^2}}.
\]
are important special cases. Here and in what follows $|\cdot|$ denotes the Euclidean norm generated by the Euclidean scalar product $(\cdot | \cdot)$.

In the $p$-Laplacian case ($\phi = h_p$), our main results (Theorems 2.4 and 2.8) are discrete versions of some interesting results from [5] (see also [4]). It is worth to point out that the main tool used in [5] is the Leray–Schauder a priori estimation method applied to some fixed point operators acting in the Sobolev space $W^{1,p}$.

In the discrete case, we use a different strategy based on the main properties of the Brouwer degree: the homotopy invariance, existence property, Borsuk’s theorem. Note also that Corollary 2.5 is a discrete version of [6, Theorem 7.1]. For interesting applications of Brouwer degree to nonlinear difference equations the reader can consult [8].

In the singular case ($a < \infty$) our main result (Theorem 3.4) is a discrete version of [1, Theorem 5] and the particular case $N = 1$ and $A = 0 = B$ is considered in [2].

If $\Omega \subset X$ is an open subset of a finite dimensional normed space $X$, $x_0 \in X$ and $S: \Omega \to X$ is a continuous function such that $x_0 \notin S(\partial \Omega)$, then $d_B[S, \Omega, x_0]$ denotes the Brouwer degree of $S$ with respect to $\Omega$ and $x_0$. For the definition and properties of the Brouwer degree see [3], [7].

2. The $p$-Laplacian case

**Dirichlet boundary value problems.** Let $f_m: \mathbb{R}^N \to \mathbb{R}^N$ be a continuous function ($1 \leq m \leq n - 1$) and consider the following nonlinear Dirichlet boundary-value problem involving the discrete vector $p$-Laplacian
\[
-\nabla [h_p(\Delta x_m)] = f_m(x_m), \quad (1 \leq m \leq n - 1), \quad x_0 = 0 = x_n.
\]
First of all notice that the solutions of (2.1) can be seen as the zeros of the continuous mapping $F: V^{n-1}_N \to [\mathbb{R}^N]^{n-1}$ defined by
\[
F_m(x) = \nabla [h_p(\Delta x_m)] + f_m(x_m), \quad (1 \leq m \leq n - 1),
\]
where the \((n - 1)N\)-dimensional vector space \(V^*_N\) is defined by
\[
V^*_N = \{ x \in [\mathbb{R}^N]^{n+1} : x_0 = 0 = x_n \}.
\]
We endow the vector space \(V^*_N\) with the norm \(\| \cdot \|_p\) defined by
\[
\| x \|_p = \left( \sum_{m=1}^{n-1} |x_m|^p \right)^{1/p}, \quad (x \in V^*_N).
\]
On the other hand, the mapping \(\| \cdot \|_{p, \Delta}\) defined by
\[
\|(x_0, \ldots, x_n)\|_{p, \Delta} = \left( \sum_{m=0}^{n-1} |\Delta x_m|^p \right)^{1/p}, \quad ((x_0, \ldots, x_n) \in V^*_N)
\]
is also a norm on \(V^*_N\). We introduce the eigenvalue-like constant
\[
(2.2) \quad \lambda_1 = \inf \left\{ \sum_{m=0}^{n-1} |\Delta x_m|^p : (x_0, \ldots, x_n) \in [\mathbb{R}^N]^{n+1} \setminus \{0\}, \ x_0 = 0 = x_n \right\}.
\]
In the next Lemma we prove that the constant \(\lambda_1\) is strictly positive and we obtain a Poincaré type inequality.

**Lemma 2.1.** The constant \(\lambda_1\) defined in (2.2) is strictly positive and
\[
(2.3) \quad \lambda_1 \| x \|^p \leq \| x \|_{p, \Delta}^p \quad \text{for all} \ x \in V^*_N.
\]

**Proof.** From the definition of \(\lambda_1\) it follows that
\[
\lambda_1 = \inf \{ \| x \|_{p, \Delta}^p : x \in V^*_N, \ |x|_p = 1 \},
\]
and using that \(\| \cdot \|_p\) and \(\| \cdot \|_{p, \Delta}\) are norms on \(V^*_N\) it follows that there exist \(x \in V^*_N\) such that \(|x|_p = 1\) and \(\lambda_1 = \| x \|_{p, \Delta}\). Hence, \(\lambda_1 > 0\) and (2.3) follows immediately from the definition of \(\lambda_1\). \(\square\)

In the next Lemma we prove a summation by parts type formula for vectors belonging to \(V^*_N\).

**Lemma 2.2.** We have that
\[
(2.4) \quad - \sum_{m=1}^{n-1} (x_m \mid \nabla [h_p(\Delta x_m)]) = \sum_{m=0}^{n-1} |\Delta x_m|^p \quad \text{for all} \ (x_0, \ldots, x_n) \in V^*_N.
\]

**Proof.** Let \((x_0, \ldots, x_n) \in V^*_N\) be fixed. For all \(1 \leq m \leq n - 1\) we have that
\[
(x_m \mid \nabla [h_p(\Delta x_m)]) = |\Delta x_m|^{p-2} (x_m \mid x_{m+1}) + |\Delta x_{m-1}|^{p-2} (x_{m-1} \mid x_m) - |\Delta x_m|^{p-2} |x_m|^2 - |\Delta x_{m-1}|^{p-2} |x_m|^2.
\]
On the other hand, for all $0 \leq m \leq n - 1$ we have that
\[
|\Delta x_m|^p = |\Delta x_m|^{p-2}|x_m|^2 - 2(x_m | x_{m+1}) + |x_{m+1}|^2.
\]
It follows that
\[
\sum_{m=1}^{n-1} (x_m | \nabla[h_p(\Delta x_m)]) = \sum_{m=1}^{n-1} |\Delta x_m|^{p-2}(x_m | x_{m+1}) + |\Delta x_{m-1}|^{p-2}(x_{m-1} | x_m) \\
- \sum_{m=1}^{n-1} |\Delta x_m|^{p-2}|x_m|^2 + |\Delta x_{m-1}|^{p-2}|x_{m-1}|^2
= \sum_{m=1}^{n-1} |\Delta x_m|^{p-2}(x_m | x_{m+1}) + \sum_{m=0}^{n-2} |\Delta x_m|^{p-2}(x_m | x_{m+1}) \\
- \sum_{m=1}^{n-1} |\Delta x_m|^{p-2}|x_m|^2 - \sum_{m=0}^{n-2} |\Delta x_m|^{p-2}|x_{m+1}|^2
= - \sum_{m=0}^{n-1} \{ |\Delta x_m|^{p-2}|x_m|^2 - 2(x_m | x_{m+1}) + |x_{m+1}|^2 \} = - \sum_{m=0}^{n-1} |\Delta x_m|^p,
\]
and the proof is completed.

Now, we consider the homotopy $\mathcal{F} : [0, 1] \times \mathbb{V}^{n-1}_{N} \to [\mathbb{R}^N]^{n-1}$ defined by
\[
\mathcal{F}_m(x) = \nabla[h_p(\Delta x_m)] + \lambda f_m(x_m), \quad (1 \leq m \leq n - 1).
\]
Notice that $\mathcal{F}(1, \cdot) = F$ and $\mathcal{F}(0, \cdot)$ is the discrete vector $p$-Laplacian operator, which is odd. On the other hand for $\lambda \in [0, 1]$ one has that $x \in \mathbb{V}^{n-1}_{N}$ is a zero of $\mathcal{F}(\lambda, \cdot)$ if and only if $x$ is a solution of the Dirichlet boundary-value problem
\[
-\nabla[h_p(\Delta x_m)] = \lambda f_m(x_m) \quad (1 \leq m \leq n - 1), \quad x_0 = 0 = x_n.
\]
In the next Lemma we obtain a priori estimations for the possible zeros of $\mathcal{F}$.

**Lemma 2.3.** If
\[
\limsup_{|x| \to \infty} \frac{\langle x | f_m(x) \rangle}{|x|^p} < \lambda_1 \quad \text{for all } 1 \leq m \leq n - 1,
\]
holds, then there exists $\rho > 0$ such that any possible zero $(\lambda, x)$ of $\mathcal{F}$ satisfies $||x||_p < \rho$.

**Proof.** Let $(\lambda, x) \in [0, 1] \times \mathbb{V}^{n-1}_{N}$ be such that $\mathcal{F}(\lambda, x) = 0$. It follows that $x = (x_0, \ldots, x_n)$ is a solution of (2.5). Multiplying (2.5) by $x_m$, summing from
1 to \( n - 1 \) and using Lemma 2.2 it follows that

\[
(2.7) \quad \sum_{m=0}^{n-1} |\Delta x_m|^p = \lambda \sum_{m=1}^{n-1} |x_m| f_m(x_m)).
\]

Using (2.6) and the continuity of \( f_m \) it follows that there exists \( \sigma \in (0, \lambda_1) \) and \( k_1 > 0 \) such that

\[
(2.8) \quad (y | f_m(y)) \leq \sigma |y|^p + k_1 \quad \text{for all} \quad y \in \mathbb{R}^N, \quad 1 \leq m \leq n - 1.
\]

From (2.3), (2.7) and (2.8) we deduce that

\[
\lambda_1 \|x\|_p^p \leq \sigma \|x\|_p^p + (n - 1)k_1.
\]

Hence, using that \( \sigma \in (0, \lambda_1) \) and \( p > 1 \) it follows that there exists \( \rho > 0 \) such that \( \|x\|_p < \rho \). \[ \square \]

**Theorem 2.4.** If (2.6) holds, then (2.1) has at least one solution.

**Proof.** Using Lemma 2.3 and the invariance of the Brouwer degree under homotopy it follows that

\[
(2.9) \quad d_B[F(1, \cdot), B(\rho), 0] = d_B[F(0, \cdot), B(\rho), 0].
\]

On the other hand \( F(0, \cdot) \) is odd, so the Borsuk theorem implies that

\[
d_B[F(0, \cdot), B(\rho), 0] \neq 0,
\]

which together with (2.9) imply that

\[
d_B[F(1, \cdot), B(\rho), 0] \neq 0.
\]

Hence, using the existence property of the Brouwer degree, we deduce that \( F(1, \cdot) \) has at least one zero which is also a solution of (2.1). \[ \square \]

An immediate consequence is the following

**Corollary 2.5.** Let \( A_m \ (1 \leq m \leq n - 1) \) be a \( N \times N \)-matrix. If there exists \( \sigma \in (0, \lambda_1) \) and \( R > 0 \) such that

\[
(x | A_m x) \leq \sigma |x|^2 \quad \text{for all} \quad |x| \geq R, \quad 1 \leq m \leq n - 1,
\]

then the Dirichlet boundary-value problem

\[
-\nabla[h_p(\Delta x_m)] = A_m[h_p(x_m)] + l_m \quad (1 \leq m \leq n - 1), \quad x_0 = 0 = x_n
\]

has at least one solution for any \( (l_1, \ldots, l_{n-1}) \in [\mathbb{R}^N]^{n-1} \).
Periodic boundary value problems. Let $f_n: \mathbb{R}^N \to \mathbb{R}^N$ be a continuous function satisfying

$$\limsup_{|x| \to \infty} \frac{|x| f_m(x)}{|x|^p} < 0 \quad (1 \leq m \leq n - 1),$$

and consider the following nonlinear periodic boundary-value problem involving the discrete vector $p$-Laplacian

$$-\nabla [h_0(\Delta x_m)] = f_m(x_m) \quad (1 \leq m \leq n - 1), \quad x_0 = x_n, \quad \Delta x_0 = \Delta x_{n-1}.$$ 

Note that, from (2.10) it follows that there exists $\sigma_1 \in (0, 1)$ and $\sigma_2 > 0$ such that

$$\langle x, f_m(x) \rangle \leq -\sigma_1 |x|^p + \sigma_2 \quad \text{for all} \quad x \in \mathbb{R}^N \quad (1 \leq m \leq n - 1),$$

and that the solutions of (2.11) can be seen as the zeros of the continuous mapping $F: U^{n-1}_N \to [\mathbb{R}^N]^{n-1}$ defined by

$$F_m(x) = \nabla [h_0(\Delta x_m)] + f_m(x_m), \quad (1 \leq m \leq n - 1),$$

where the $(n - 1)N$-dimensional vector space $U^{n-1}_N$ is defined by

$$U^{n-1}_N = \left\{ (x_0, \ldots, x_n) \in [\mathbb{R}^N]^{n+1} : x_0 = x_n, \Delta x_0 = \Delta x_{n-1} \right\}$$

and $U^{n-1}_N = \left\{ (x_0, \ldots, x_n) \in [\mathbb{R}^N]^{n+1} : x_0 = (x_1 + x_{n-1})/2 = x_n \right\}$.

We endow the vector space $U^{n-1}_N$ with the norm $|| \cdot ||_p$.

**Lemma 2.6.** If $(x_0, \ldots, x_n) \in U^{n-1}_N$, then

$$\sum_{m=1}^{n-1} (\nabla [h_0(\Delta x_m)] | x_m) \leq 0.$$ 

**Proof.** One has that

$$\sum_{m=1}^{n-1} (\nabla [h_0(\Delta x_m)] | x_m)$$

$$= (h_0(\Delta x_1) - h_0(\Delta x_0) | x_1) + \ldots + (h_0(\Delta x_{n-1}) - h_0(\Delta x_{n-2}) | x_{n-1})$$

$$= (h_0(\Delta x_1) | x_1) - (h_0(\Delta x_0) | x_1) + (h_0(\Delta x_2) | x_2) - (h_0(\Delta x_1) | x_2) + \ldots$$

$$+ (h_0(\Delta x_{n-1}) | x_{n-1}) - (h_0(\Delta x_{n-2}) | x_{n-1})$$

$$= - (h_0(\Delta x_0) | x_1) - (h_0(\Delta x_1) | x_1) - \ldots$$

$$- (h_0(\Delta x_{n-2}) | x_{n-2}) + (h_0(\Delta x_{n-1}) | x_{n-1})$$

$$= - \sum_{m=1}^{n-2} |\Delta x_m|^p \frac{1}{2} \leq 0.$$

□
Lemma 2.7. If $L: U_{N}^{n-1} \to [\mathbb{R}^N]^{n-1}$ is the odd continuous function defined by

$$L_m(x) = \nabla[h_p(\Delta x_m)] - h_p(x_m), \quad (1 \leq m \leq n-1),$$

then $d_B[L, B(\rho), 0] \neq 0$ for all $\rho > 0$.

**Proof.** Assume that $(x_0, \ldots, x_n)$ solves the problem

$$L(x_0, \ldots, x_n) = 0, \quad (x_0, \ldots, x_n) \in U_{N}^{n-1}.$$

It follows that

$$\sum_{m=1}^{n-1} \left\{ (\nabla[h_p(\Delta x_m)]|x_m) - (h_p(x_m)|x_m) \right\} = 0.$$

Using Lemma 2.6 we deduce that

$$\sum_{m=1}^{n-1} |x_m|^p = \sum_{m=1}^{n-1} (\nabla[h_p(\Delta x_m)]|x_m) \leq 0,$$

and $x_0 = \ldots = x_n = 0$. Now the result follows from Borsuk’s theorem. □

Theorem 2.8. If (2.10) holds, then (2.11) has at least one solution.

**Proof.** Let $H: [0,1] \times U_{N}^{n-1} \to [\mathbb{R}^N]^{n-1}$ be the homotopy

$$H_m(x) = \nabla[h_p(\Delta x_m)] + \lambda f_m(x_m) - (1-\lambda)h_p(x_m), \quad (1 \leq m \leq n-1).$$

It is clear that

$$H(0, \cdot) = L, \quad H(1, \cdot) = F.$$

Let also $(\lambda, x) \in [0,1] \times U_{N}^{n-1}$ be such that $H(\lambda, x) = 0$. Using (2.12) and Lemma 2.6 we deduce that

$$0 \leq \lambda \sum_{m=1}^{n-1} (f_m(x_m)) - (1-\lambda)||x||_p^p$$

$$\leq -\lambda \sigma_1 ||x||_p^p + (n-1)\sigma_2 - (1-\lambda)||x||_p^p.$$

Hence, $||x||_p < \rho$ for any $\rho > ((n-1)\sigma_2/\sigma_1)^{1/p}$. Using Lemma 2.7 and the invariance under homotopy of the Brouwer degree, it follows that

$$d_B[F, B(\rho), 0] = d_B[L, B(\rho), 0], \quad d_B[F, B(\rho), 0] \neq 0,$$

for $\rho > ((n-1)\sigma_2/\sigma_1)^{1/p}$. Then, using the existence property of the Brouwer degree it follows that $F$ has a zero which is also a solution of (2.11). □
Neumann boundary value problems. Let \( f_m: \mathbb{R}^N \to \mathbb{R}^N \) (1 \( \leq \) m \( \leq \) n − 1) be a continuous function satisfying (2.10) and consider the following nonlinear Neumann boundary-value problem involving the discrete vector p-Laplacian

\[
-\nabla [h_p(\Delta x_m)] = f_m(x_m) \quad (1 \leq m \leq n-1), \quad \Delta x_0 = 0 = \Delta x_{n-1}.
\]

Using the same strategy like in the periodic case one can prove that (2.13) has at least one solution.

3. The \( \phi \)-Laplacian case with singular \( \phi \)

Let \( n, N \) be positive integers, \( a > 0 \), \( \phi: B(a) \subset \mathbb{R}^N \to \mathbb{R}^N \) be a homeomorphism such that \( \phi(0) = 0 \); we call it singular. The space \( [\mathbb{R}^N]^{n+1} \) will be endowed we the norm

\[
||x||_\infty = \sum_{m=0}^{n} |x_m| \quad (x \in [\mathbb{R}^N]^{n+1}).
\]

Nonhomogeneous Neumann boundary value problems. Let \( A \) and \( B \) in \( \mathbb{R}^N \) be fixed.

**Lemma 3.1.** Let \( l_1, \ldots, l_{n-1} \in \mathbb{R}^N \). Forced problem

\[
\nabla [\phi(\Delta x_m)] = l_m \quad (1 \leq m \leq n-1), \quad \phi(\Delta x_0) = A, \quad \phi(\Delta x_{n-1}) = B
\]

is solvable if and only if

\[
\sum_{m=1}^{n-1} l_m = B - A.
\]

In this case the general solution \((x_0, \ldots, x_n)\) is given by

\[
x_0 \in \mathbb{R}^N, \quad x_1 = x_0 + \phi^{-1}(A),
\]

\[
x_m = x_0 + \phi^{-1}(A) + \sum_{j=1}^{m-1} \phi^{-1}(A + \sum_{k=1}^{j} l_k) \quad (2 \leq m \leq n - 1),
\]

\[
x_n = x_0 + \phi^{-1}(A) + \phi^{-1}(B) + \sum_{j=1}^{n-2} \phi^{-1}(A + \sum_{k=1}^{j} l_k).
\]

**Proof.** The proof follows by a simple computation and is left to the reader. □

Let \( g_m: \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N \) (1 \( \leq \) m \( \leq \) n − 1) be continuous and consider nonhomogeneous Neumann boundary value problem

\[
\nabla [\phi(\Delta x_m)] = g_m(x_m, \Delta x_m) \quad (1 \leq m \leq n-1), \quad \phi(\Delta x_0) = A, \quad \phi(\Delta x_{n-1}) = B.
\]
Let $Q: [\mathbb{R}^N]^{n-1} \to \mathbb{R}^N$ be defined by

$$Q(l) = \frac{1}{n-1} \sum_{m=1}^{n-1} l_m.$$  

It is clear that (3.3) can be written in the equivalent form

$$\nabla[\phi(\Delta x_m)] = g_m(x_m, \Delta x_m) - Q[s(x_m, \Delta x_m)] \quad (1 \leq m \leq n-1),$$

and

$$Q[s(x_m, \Delta x_m)] = 0,$$

where

$$s_m(x_m, \Delta x_m) = g_m(x_m, \Delta x_m) - \frac{1}{n-1}(B - A) \quad (1 \leq m \leq n-1).$$

Now, we reformulate (3.4), (3.5) as a fixed point problem. Consider the operator

$$M: [\mathbb{R}^N]^{n+1} \to [\mathbb{R}^N]^{n+1}, \quad M(x) = y,$$

where

$$y_0 = x_0 + Q(s),$$

$$y_1 = x_0 + Q(s) + \phi^{-1}(A),$$

$$y_m = x_0 + Q(s) + \phi^{-1}(A) + \frac{m-1}{n-1} \sum_{j=1}^{m} \phi^{-1} \left( A + \sum_{k=1}^{j} l_k \right) \quad (2 \leq m \leq n-1),$$

$$y_n = x_0 + Q(s) + \phi^{-1}(A) + \phi^{-1}(B) + \sum_{j=1}^{n-2} \phi^{-1} \left( A + \sum_{k=1}^{j} l_k \right),$$

where $s$ is given in (3.6) and

$$l_m = g_m(x_m, \Delta x_m) - Q(s) \quad (1 \leq m \leq n-1).$$

Then, using Lemma 3.1 one has the following

**Lemma 3.2.** The vector $x \in [\mathbb{R}^N]^{n+1}$ is a solution of (3.3) if and only if

$$M(x) = x.$$
where $s$ and $l$ are defined in (3.6), (3.7) respectively. Note that $\mathcal{M}(1, \cdot) = M$.

We introduce the following assumption:

(\(H_{g,A,B}\)) There exists $R > 0$ such that

$$
\sum_{m=1}^{n-1} g_m(x_m, \Delta x_m) \neq B - A,
$$

for all $x \in [\mathbb{R}^N]^{n+1}$ satisfying

$$
\min_{0 \leq m \leq n} |x_m| \geq R \quad \text{and} \quad \max_{0 \leq m \leq n-1} |\Delta x_m| < a.
$$

**Lemma 3.3.** Assume that (\(H_{g,A,B}\)) holds. If $(\lambda, x) \in [0, 1] \times [\mathbb{R}^N]^{n+1}$ is such that $x = \mathcal{M}(\lambda, x)$, then

$$
||x||_\infty < R + na.
$$

**Proof.** Let $(\lambda, x) \in [0, 1] \times [\mathbb{R}^N]^{n+1}$ be such that $x = \mathcal{M}(\lambda, x)$. It follows that

$$
\max_{0 \leq m \leq n-1} |\Delta x_m| < a,
$$

and

$$
\sum_{m=1}^{n-1} g_m(x_m, \Delta x_m) = B - A,
$$

implying that

$$
\min_{0 \leq m \leq n} |x_m| < R.
$$

Hence, using that

$$
|x_m| \leq \min_{0 \leq m \leq n} |x_m| + \sum_{j=0}^{n-1} |\Delta x_j| \quad (0 \leq m \leq n),
$$

we get the result. \(\square\)

Consider the continuous function

$$
\gamma: \mathbb{R}^N \to \mathbb{R}^N, \quad c \mapsto \sum_{m=1}^{n-1} g_m(c, 0).
$$

**Theorem 3.4.** If assumption (\(H_{g,A,B}\)) holds, then for all sufficiently large $\rho > 0$,

$$
d_B[I - \mathcal{M}(1, \cdot), B(\rho), 0] = (-1)^N d_B[\gamma, B(R), B - A].
$$

If furthermore

$$
d_B[\gamma, B(R), B - A] \neq 0,
$$

then (3.3) has at least one solution.

**Proof.** Note that, from assumption (\(H_{g,A,B}\)) it follows that $\gamma(c) \neq B - A$ for all $|c| \geq R$, which implies that the Brouwer degree $d_B[\gamma, B(R), B - A]$ is
well defined. On the other hand, taking $\rho > R + na$, using Lemma 3.3 and the homotopy invariance of Brouwer degree, we get that
\[ d_B[I - \mathcal{M}(1, \cdot), B(\rho), 0] = d_B[I - \mathcal{M}(0, \cdot), B(\rho), 0]. \]
But, the range of $\mathcal{M}(0, \cdot)$ is isomorphic to $\mathbb{R}^N$. Actually,
\[ \mathcal{M}_m(0, x) = x_0 + Q(s)(x) \quad (0 \leq m \leq n) \]
where $s$ is given in (3.6). Hence, using the reduction and excision properties of the Brouwer degree we have
\[
\begin{align*}
& d_B[I - \mathcal{M}(0, \cdot), B(\rho), 0] = d_B[I - \mathcal{M}(0, \cdot)|_{\mathbb{R}^N}, B(\rho), 0] \\
& \quad = (-1)^N d_B[\gamma, B(R), B - A].
\end{align*}
\]
Now, the result follows from the existence property of the Brouwer degree. □

An immediate consequence of Theorem 3.4 is the following

**Corollary 3.5.** Assume that there exists $\varepsilon \in \{-1, 1\}$ and $R > 0$ such that
\[ \varepsilon (g_m(x + y, z) - (n - 1)^{-1}(B - A)x) > 0 \]
for all $1 \leq m \leq n - 1$, $|x| \geq R$, $|y| < na$ and $|z| < a$, then (3.3) has at least one solution.

**Remark 3.6.** Similar considerations hold also for Dirichlet and periodic boundary value problems. In these cases, in order to construct the associated fixed point operators (see [2] for $N = 1$), $\phi$ must be of gradient type like in [1].

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**References**


Reduction and continuation theorems for Brouwer degree and applications to nonlinear difference equations, Opuscula Math. 28 (2008), 541–560.

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