ON NONCOERCIVE PERIODIC SYSTEMS WITH VECTOR $p$-LAPLACIAN

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Abstract. We consider nonlinear periodic systems driven by the vector $p$-Laplacian. An existence and a multiplicity theorem are proved. In the existence theorem the potential function is $p$-superlinear, but in general does not satisfy the AR-condition. In the multiplicity theorem the problem is strongly resonant with respect to the principal eigenvalue $\lambda_0 = 0$. In both of the cases the Euler–Lagrange functional is noncoercive and the method is variational.

1. Introduction

In this paper we consider the following nonlinear periodic system driven by the $p$-Laplacian operator:

\begin{equation}
\begin{cases}
-\left(|x'(t)|^{p-2}x'(t)\right)' = \nabla F(t, x(t)) \quad \text{a.e. on } T = [0, b], \\
x(0) = x(b), \quad x'(0) = x'(b)
\end{cases}
\end{equation}

Here $|\cdot|$ stands for the Euclidean norm on $\mathbb{R}^N$ and $F: T \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory mapping such that $F(t, \cdot)$ is of class $C^1$ for almost every $t \in T$.

Periodic systems were studied primarily within the context of semilinear equations (i.e. $p = 2$) and most of the works prove existence but not multiplicity results. In this direction we mention the works of M. S. Berger and M. Schecter [3] and C. L. Tang and X. P. Wu [25], who impose an anticoercivity condition.
on the potential $F(t, \cdot)$ which makes their Euler–Lagrange functional coercive. I. Ekeland and N. Ghoussoub [5] and N. Ghoussoub [10] employ the well-known Ambrosetti–Rabinowitz condition (AR-condition for short), which implies that the potential $F(t, \cdot)$ is superquadratic. On the other hand, C. L. Tang [24] considers second order systems with a subquadratic potential and uses minimax techniques, in particular the saddle point theorem. Finally, J. Mawhin and M. Willem [19] with a bounded potential function and F. Zhao and X. P. Wu [27] use the least action principle, while J. Mawhin [17] assumes that $F(t, \cdot)$ is convex and employs the dual action principle.

In contrast, the study of periodic systems driven by the vector $p$-Laplacian is in some sense lagging behind. We mention the works of R. P. Agarwal, H. Lü and D. O’Regan [1], G. Dincă and P. Jebelean [4], L. Gasinski [7], P. Jebelean [11], P. Jebelean and G. Moroşanu [13, 14], R. Manasevich and J. Mawhin [16], J. Mawhin [18], E. Papageorgiou and N. S. Papageorgiou [20–22], F. Papalini [23] and K. M. Teng and X. P. Wu [26]. In R. P. Agarwal, H. Lü and D. O’Regan [1] the authors deal with certain eigenvalue problems and prove multiplicity results valid for certain values of the parameter. L. Gasinski [7] proves multiplicity of solutions for systems with a coercive Euler–Lagrange functional. P. Jebelean [11] and P. Jebelean and G. Moroşanu [14] deal with problems with nonlinear boundary conditions and prove existence results using Szulkin’s critical point theory (see, for example [8]). Also, such type of problems, but with nonpotential right hand term, are studied in G. Dincă and P. Jebelean [4] by the a priori estimates method. R. Manasevich and J. Mawhin [16] and J. Mawhin [18] obtain existence results by degree theoretic methods. The approach in E. Papageorgiou and N. S. Papageorgiou [21] is based on the theory of nonlinear operators of monotone type and they deal with problems which may have unilateral constraints, while in [20], [22] they prove multiplicity results using minimax techniques. Part of the results from [8], [20] and [22] were extended in F. Papalini [23]. Finally, K. M. Teng and X. P. Wu [26] obtain existence and multiplicity of solutions for $p \geq 2$. In [13], [20], [22], [23] and [26] the potential function is locally Lipschitz and in general nonsmooth. So, the method of proof relies on the nonsmooth critical point theory (see L. Gasinski and N. S. Papageorgiou [8]).

Here we prove an existence result and a multiplicity result for problem (1.1). In the existence theorem we assume that the potential $F(t, \cdot)$ is $p$-superlinear, but need not to satisfy the usual in such cases AR-condition. The multiplicity theorem concerns systems which are strongly resonant with respect to the principal eigenvalue $\lambda_0 = 0$. Such problems exhibit a partial lack of compactness, in the sense that the PS-condition is not globally satisfied.
2. Preliminaries

Let \((X, \| \cdot \|)\) be a real Banach space which admits a direct sum decomposition \(X = Y \oplus V\) and let \(\varphi \in C^1(X)\). We say that \(\varphi\) has a local linking at the origin (with respect to the decomposition \((Y, V)\)) if there exists an \(r > 0\) such that

\[
\begin{align*}
\varphi(y) &\leq 0 \quad \text{for all } y \in Y \text{ with } \|y\| \leq r, \\
\varphi(v) &\geq 0 \quad \text{for all } v \in V \text{ with } \|v\| \leq r.
\end{align*}
\]

It is easy to see that if \(\varphi\) has a local linking at the origin, then \(x = 0\) is a critical point of \(\varphi\).

We say that \(\varphi\) satisfies the Palais–Smale condition at the level \(c \in \mathbb{R}\) (the PS\(_c\)-condition for short) if every sequence \(\{x_n\}_{n \geq 1} \subset X\) such that

\[
\varphi(x_n) \to c \quad \text{and} \quad \varphi'(x_n) \to 0 \quad \text{in } X^*, \quad \text{as } n \to \infty,
\]

has a strongly convergent subsequence. If \(\varphi\) satisfies the PS\(_c\)-condition at every level \(c \in \mathbb{R}\), then we say that \(\varphi\) satisfies the PS-condition. Sometimes we need to use a weaker compactness-type condition on the functional \(\varphi\). So, we say that \(\varphi\) satisfies the Cerami condition at the level \(c \in \mathbb{R}\) (the C\(_c\)-condition for short) if every sequence \(\{x_n\}_{n \geq 1} \subset X\) such that

\[
\varphi(x_n) \to c \quad \text{and} \quad (1 + \|x_n\|)\varphi'(x_n) \to 0 \quad \text{in } X^*, \quad \text{as } n \to \infty,
\]

has a strongly convergent subsequence. If \(\varphi\) satisfies the C\(_c\)-condition at every level \(c \in \mathbb{R}\), then we say that \(\varphi\) satisfies the C-condition.

The next result is essentially due to S. J. Li and M. Willem [15]. In their formulation of the result they use a gradient version of the PS-condition. Noting that the deformation theorem remains true if the functional \(\varphi\) satisfies the C-condition instead of the PS-condition (see P. Bartolo, V. Benci and D. Fortunato [2]), we can state the following version of Theorem 2 in S. J. Li and M. Willem [15].

**Theorem 2.1.** If \(X\) is a Banach space, \(X = Y \oplus V\) with \(\dim Y < \infty\) and \(\varphi \in C^1(X)\) satisfies:

(a) \(\varphi\) has a local linking at the origin;
(b) \(\varphi\) satisfies the C-condition;
(c) \(\varphi\) maps bounded sets into bounded sets;
(d) for every \(E \subset V\) finite dimensional subspace, \(\varphi|_{Y \oplus E}\) is anticoercive (i.e. \(\varphi(x) \to -\infty\) as \(\|x\| \to \infty, x \in Y \oplus E\)),

then \(\varphi\) admits at least one nontrivial critical point.

In the proof of the multiplicity result we shall use the second deformation theorem (see L. Gasinski and N. S. Papageorgiou [9, p. 628]). Let \(K\) be the
critical set of \( \varphi \), i.e. \( K = \{ x \in X \mid \varphi'(x) = 0 \} \). We introduce the following sets:
\[
\varphi^c = \{ x \in X \mid \varphi(x) \leq c \} \quad \text{(the sublevel set of } \varphi \text{ at } c \in \mathbb{R})
\]
and
\[
K_c = \{ x \in K \mid \varphi(x) = c \} \quad \text{(the critical set of } \varphi \text{ at the level } c).\]

In the next theorem we allow \( c = +\infty \), in which case \( \varphi^c \setminus K_c = X \).

**Theorem 2.2.** If \( X \) is a Banach space, \( \varphi \in C^1(X) \), \( a \in \mathbb{R} \), \( a < c \leq +\infty \), \( \varphi \) satisfies the PS\( c \)-condition for every \( r \in [a, c) \), \( \varphi^{-1}(a, c) \cap K = \emptyset \) and \( \varphi^{-1}(a) \cap K \) is finite, then there exists a homotopy \( h:[0, 1] \times (\varphi^c \setminus K_c) \to \varphi^c \) such that

(a) \( h(1, \varphi^c \setminus K_c) \subset \varphi^a \);

(b) \( h(t, x) = x \) for all \( (t, x) \in [0, 1] \times \varphi^a \);

(c) \( \varphi(h(t, x)) \leq \varphi(h(s, x)) \) for all \( t, s \in [0, 1], s \leq t \) and all \( x \in \varphi^c \setminus K_c \) (i.e. the homotopy \( h \) is \( \varphi \)-decreasing).

According to Theorem 2.2 (the second deformation theorem), the set \( \varphi^a \) is a strong deformation retract of \( \varphi^c \setminus K_c \).

Next, we present the functional framework and some basic results which are needed in the analysis of problem (1.1).

The Sobolev space
\[
W^{1,p}_{\text{per}}(T) := \{ x \in W^{1,p}(T; \mathbb{R}^N) \mid x(0) = x(b) \}
\]
is endowed with the norm
\[
\|x\| = (\|x\|_p^p + \|x\|_p^p)^{1/p},
\]
where \( \| \cdot \|_p \) stands for the usual norm on \( L^p(T; \mathbb{R}^N) \). Note that since \( W^{1,p}(T; \mathbb{R}^N) \) is embedded continuously (in fact compactly) in \( C(T; \mathbb{R}^N) \), the evaluations at \( t = 0 \) and \( t = b \) in the definition of \( W^{1,p}_{\text{per}}(T) \) make sense.

Let \( \langle \cdot, \cdot \rangle \) denote the duality brackets for the pair \( W^{1,p}_{\text{per}}(T)^*, W^{1,p}_{\text{per}}(T) \) and consider the nonlinear operator \( A: W^{1,p}_{\text{per}}(T) \to W^{1,p}_{\text{per}}(T)^* \) defined by
\[
\langle A(x), y \rangle = \int_0^b |x'(t)|^{p-2} (x'(t), y'(t)) \, dt \quad \text{for all } x, y \in W^{1,p}_{\text{per}}(T).
\]
Here \( \langle \cdot, \cdot \rangle \) stands for the usual inner product in \( \mathbb{R}^N \). It is a standard matter that \( A \) is monotone and continuous, hence it is maximal monotone. Also, the following result is known; however, for the sake of the completeness, we include a short proof.

**Proposition 2.3.** The operator \( A \) is of type \((S)_+\).

**Proof.** Let \( x_n \xrightarrow{w} x \) in \( W^{1,p}_{\text{per}}(T) \) and assume that
\[
\limsup_{n \to \infty} \langle A(x_n), x_n - x \rangle \leq 0.
\]
We need to show that \( x_n \to x \) in \( W^{1,p}_{\text{per}}(T) \). From
\[
0 \leq \langle A(x_n) - A(x), x_n - x \rangle = \langle A(x_n), x_n - x \rangle - \langle A(x), x_n - x \rangle
\]
\[
\leq \sup_{k \geq n} \langle A(x_k), x_k - x \rangle - \langle A(x), x_n - x \rangle
\]
it follows \( \langle A(x_n) - A(x), x_n - x \rangle \to 0 \) as \( n \to \infty \). Then, the inequality
\[
0 \leq (\|x'_n\|_p^{p-1} - \|x'\|^{p-1})(\|x'_n\|_p - \|x'\|_p) \leq \langle A(x_n) - A(x), x_n - x \rangle
\]
yields \( \|x'_n\|_p \to \|x'\|_p \) as \( n \to \infty \).

We know that \( x'_n \overset{w}{\rightharpoonup} x' \) in \( L^p(T;\mathbb{R}^N) \). The space \( L^p(T;\mathbb{R}^N) \) being uniformly convex, it has the Kadec–Klee property, which implies \( x'_n \to x' \) in \( L^p(T;\mathbb{R}^N) \).

We also have \( x_n \to x \) in \( C(T;\mathbb{R}^N) \) (by the compactness of the embedding of \( W^{1,p}_{\text{per}}(T) \) into \( C(T;\mathbb{R}^N) \)). Therefore, we conclude that \( x_n \to x \) in \( W^{1,p}_{\text{per}}(T) \). □

3. Existence of nontrivial solutions

In this section we prove an existence theorem for problem (1.1) under the hypothesis that the potential \( F(t, \cdot) \) exhibits \( p \)-superlinear growth near infinity, but need not to satisfy the AR-condition. The precise hypotheses on \( F \) are the following:

\((H_1)\) \( F : T \times \mathbb{R}^N \to \mathbb{R} \) is a function such that

(i) for all \( x \in \mathbb{R}^N, t \mapsto F(t, x) \) is measurable;

(ii) for almost all \( t \in T, x \mapsto F(t, x) \) is \( C^1 \) and \( F(t, 0) = 0 \);

(iii) for almost all \( t \in T \) and all \( x \in \mathbb{R}^N \)
\[
|\nabla F(t, x)| \leq a(t) + c|x|^{r-1}
\]
with \( a \in L^1(T)_+, c > 0 \) and \( p < r < \infty \);

(iv) \( \lim_{|x| \to \infty} (F(t, x)/|x|^p) = +\infty \) uniformly for almost all \( t \in T \) and there exists \( \mu > r - p \) such that
\[
(3.1) \quad \liminf_{|x| \to \infty} \frac{(\nabla F(t, x), x) - pF(t, x)}{|x|^p} > 0 \quad \text{uniformly for a.a. } t \in T;
\]

(v) \( \limsup_{x \to 0} (pF(t, x)/|x|^p) < 1/b^p \) uniformly for almost all \( t \in T \) and there exists \( \delta > 0 \) such that \( F(t, x) \geq 0 \) for almost all \( t \in T \) and all \( x \in \mathbb{R}^N \) with \( |x| \leq \delta \).

Remark 3.1. Hypothesis \((H_1)\) implies that \( F(t, \cdot) \) is \( p \)-superlinear. However, we do not assume the AR-condition, very common in such cases. We recall that the AR-condition says that there exist \( \beta > p \) and \( M > 0 \) such that
\[
(3.2) \quad 0 < \beta F(t, x) \leq (\nabla F(t, x), x) \quad \text{for a.a. } t \in T \text{ and all } |x| \geq M.
\]
Integrating (3.2) we get

\[(3.3) \quad c_1 |x|^{\beta} \leq F(t, x) \text{ for a.a. } t \in T, \text{ all } |x| \geq M, \text{ for some } c_1 > 0.\]

Clearly, (3.3) is stronger than the condition

\[\lim_{|x| \to \infty} \frac{F(t, x)}{|x|^p} = +\infty \text{ uniformly for a.a. } t \in T.\]

Here, instead of (3.2) we use the weaker condition (3.1). Note that (3.1) was used earlier in the frame of semilinear (i.e. \(p = 2\)) Hamiltonian systems by G. Fei [6]. The following example provides a function \(F\) which satisfies (3.1) but not (3.2).

**Example 3.2.** Consider the function \(F: \mathbb{R}^N \to \mathbb{R}\) (for the sake of simplicity we drop the \(t\)-dependence), defined by

\[F(x) = \frac{1}{p} |x|^p \ln(1 + |x|^\alpha)\]

with \(\alpha > 1\). Then \(F\) satisfies hypothesis \((H_1)\) (with \(r = p + \varepsilon, \varepsilon \in (0, p)\) and \(\mu = p\), but it does not satisfy the AR-condition (see (3.2)).

The Euler–Lagrange functional for problem (1.1) is defined by

\[\varphi(x) = \frac{1}{p} \|x'\|^p_p - \int_0^b F(t, x(t)) \, dt \text{ for all } x \in W^{1,p}_{\text{per}}(T).\]

It is known that \(\varphi \in C^1(W^{1,p}_{\text{per}}(T), \mathbb{R})\). Also, we shall consider the direct sum decomposition

\[W^{1,p}_{\text{per}}(T) = \mathbb{R}^N \oplus V,\]

with \(V = \{x \in W^{1,p}_{\text{per}}(T) | \int_0^b x(t) \, dt = 0\}\).

**Proposition 3.3.** If hypotheses \((H_1)\) hold, then \(\varphi\) satisfies the C-condition.

**Proof.** Let \(\{x_n\}_{n \geq 1} \subset W^{1,p}_{\text{per}}(T)\) be a sequence such that

\[(3.4) \quad |\varphi(x_n)| \leq M_1 \text{ for some } M_1 > 0 \text{ and all } n \geq 1,\]

and

\[(3.5) \quad (1 + \|x_n\|)\varphi'(x_n) \to 0 \text{ in } W^{1,p}_{\text{per}}(T)^*, \text{ as } n \to \infty.\]

We know that

\[(3.6) \quad \varphi'(x_n) = A(x_n) - N(x_n)\]

with \(N(u)(\cdot) = \nabla F(\cdot, u(\cdot))\) for all \(u \in W^{1,p}_{\text{per}}(T)\) (see, for example P. Jebelean [12]).

**Claim.** \(\{x_n\}_{n \geq 1}\) is bounded in \(W^{1,p}_{\text{per}}(T)\).
On Noncoercive Periodic Systems with Vector p-Laplacian

Suppose that the Claim is not true. By passing to a suitable subsequence if necessary, we may assume that \( \|x_n\| \to \infty \). From (3.5) and (3.6) we have

\[
\tag{3.7} \left| \langle A(x_n), u \rangle - \int_0^b (\nabla F(t, x_n), u) \, dt \right| \leq \frac{\varepsilon_n}{1 + \|x_n\|} \|u\| \quad \text{for all } u \in W^{1,p}_{\text{per}}(T),
\]

with \( \varepsilon_n \to 0^+ \). From (3.7), with \( u = x_n \), it follows

\[
\tag{3.8} -\|x_n'\|^p_p + \int_0^b (\nabla F(t, x_n), x_n) \, dt \leq \varepsilon_n \quad \text{for all } n \geq 1.
\]

Also, from (3.4) we have

\[
\tag{3.9} \|x_n'\|^p_p - \int_0^b pF(t, x_n) \, dt \leq pM_1 \quad \text{for all } n \geq 1.
\]

Adding (3.8) and (3.9), we obtain

\[
\tag{3.10} \int_0^b [(\nabla F(t, x_n), x_n) - pF(t, x_n)] \, dt \leq M_2 \quad \text{for some } M_2 > 0, \text{ all } n \geq 1.
\]

By virtue of hypothesis (H1)(iv) we can find \( \beta > 0 \) and \( M_3 = M_3(\beta) \) with

\[
\tag{3.11} 0 < \beta |x|^{\mu} \leq (\nabla F(t, x) - pF(t, x)) \quad \text{for a.a. } t \in T \text{ and all } |x| \geq M_3.
\]

Since \( F(t, 0) = 0 \), hypothesis (H1)(iii) implies that there is some \( M_4 \in L^1(T) \) such that

\[
\tag{3.12} |(\nabla F(t, x), x) - pF(t, x)| \leq M_4(t) \quad \text{for a.a. } t \in T \text{ and all } |x| < M_3.
\]

Combining (3.11) and (3.12), we infer that

\[
\tag{3.13} \beta |x|^{\mu} - c_1(t) \leq (\nabla F(t, x) - pF(t, x)) \quad \text{for a.a. } t \in T, \text{ all } x \in \mathbb{R}^N,
\]

where \( c_1(t) = M_4(t) + \beta M_3^\mu \).

We return to (3.10) and use (3.13). Then

\[
\tag{3.14} \beta \|x_n\|_\mu^{\mu} \leq M_5 \quad \text{for some } M_5 > 0, \text{ all } n \geq 1,
\]

\[
\Rightarrow \{x_n\}_{n \geq 1} \text{ is bounded in } L^\mu(T, \mathbb{R}^N).
\]

It is clear from (3.1) that we can always assume, without loss of generality, that \( \mu < r \). Then

\[
\tag{3.15} \int_0^b |x_n|^r \, dt = \int_0^b |x_n|^{r-\mu} |x_n|^{\mu} \, dt \leq \|x_n\|_\infty^{r-\mu} \int_0^b |x_n|^{\mu} \, dt \leq c_2 \|x_n\|^{r-\mu} \quad \text{for some } c_2 > 0, \text{ all } n \geq 1 \text{ (see (3.14))},
\]

\[
\tag{3.16} \Rightarrow \|x_n\|_p^p \leq c_3 \|x_n\|^{r-\mu} \quad \text{for some } c_3 > 0, \text{ all } n \geq 1 \text{ (since } p < r)
\]

\[
\Rightarrow \|x_n\|_p^p \leq c_4 \|x_n\|^{(r-\mu)p/r} \quad \text{with } c_4 = c_3^{p/r}, \text{ all } n \geq 1,
\]

\[
\Rightarrow \|x_n\|_p^p \leq c_4(1 + \|x_n\|^{r-\mu}) \quad \text{for all } n \geq 1.
\]
From (3.4), hypothesis (H1)(iii) and (3.15), we successively have
\[ \frac{1}{p} \|x'_n\|_p^p \leq M_1 + \int_0^b F(t, x_n) \, dt \]
\[ \leq M_1 + \int_0^b (a(t)|x_n(t)| + c|x_n(t)|^r) \, dt \]
\[ \leq \|x_n\|_{\infty} \|a\|_1 + c\|x_n\|^r \leq \tilde{c}\|x_n\| + c_2\|x_n\|^{r-\mu} \]
for some $\tilde{c} > 0$ and all $n \geq 1$. This together with (3.16) yield
\[ (3.17) \quad \|x_n\|_p \leq c_4 + \hat{c}\|x_n\| + c_5\|x_n\|^{r-\mu} \quad \text{for some} \quad \hat{c}, c_5 > 0 \quad \text{and all} \quad n \geq 1. \]

But recall that by hypothesis (H1)(iv) we have $p > \max\{1, r - \mu\}$. Hence, from (3.17) it follows that $\{x_n\}_{n \geq 1} \subset W^{1,p}_{\text{per}}(T)$ is bounded. This proves the Claim.

Thanks to the Claim we may assume that
\[ (3.18) \quad x_n \xrightarrow{w} x \quad \text{in} \quad W^{1,p}_{\text{per}}(T) \quad \text{and} \quad x_n \rightharpoonup x \quad \text{in} \quad C(T; \mathbb{R}^N). \]

In (3.7) we choose $u = x_n - x$. Then
\[ \left| \langle A(x_n), x_n - x \rangle - \int_0^b \langle \nabla F(t, x_n), x_n - x \rangle \, dt \right| \leq \frac{\varepsilon n}{1 + \|x_n\|} \|x_n - x\| \quad \text{for all} \quad n \geq 1. \]
Evidently
\[ \int_0^b (\nabla F(t, x_n), x_n - x) \, dt \to 0 \quad \text{as} \quad n \to \infty \]
(see (3.18) and (H1)(iii)). Hence
\[ \lim_{n \to \infty} \langle A(x_n), x_n - x \rangle \to 0 \quad \Rightarrow \quad x_n \to x \quad \text{in} \quad W^{1,p}_{\text{per}}(T) \quad \text{(see Proposition 2.3)} \quad \Rightarrow \quad \varphi \quad \text{satisfies the} \quad C\text{-condition}. \]
On account of the inequality (see J. Mawhin and M. Willem [19, p. 8]):

\[
\|x\|^p_p \leq b^p \|x'\|^p_p \quad \text{for all } x \in V,
\]

we can estimate \(\varphi(x)\) for \(x \in V\), with \(\|x\| \leq \delta_2\), as follows

\[
\varphi(x) = \frac{1}{p} \|x'\|^p_p - \int_0^b F(t, x(t)) \, dt 
\geq \frac{1}{p} \|x'\|^p_p - \frac{1}{p} \left( \frac{1}{b^p} - \varepsilon \right) \int_0^b |x(t)|^p \, dt \quad \text{(see (3.20))}
\geq \frac{\varepsilon}{p} \|x\|^p_p \geq 0.
\]

Letting \(\delta = \min\{\delta_0, \delta_2\}\), from (3.19) and (3.22) we infer that \(\varphi\) has a local linking at the origin with respect to \((\mathbb{R}^N, V)\). \(\square\)

**Proposition 3.5.** If hypotheses \((H_1)\) hold and \(E \subset V\) is a finite dimensional subspace, then \(\varphi|_{\mathbb{R}^N \oplus E}\) is anticoercive (i.e. \(\varphi(x) \to -\infty\) as \(\|x\| \to \infty\), for \(x \in \mathbb{R}^N \oplus E\)).

**Proof.** By virtue of hypothesis \((H_1)(iv)\), given \(\gamma > 0\), we can find \(M_6 = M_6(\gamma) > 0\) such that

\[
F(t, x) \geq \gamma |x|^p \quad \text{for a.a. } t \in T, \text{ all } |x| \geq M_6.
\]

On the other hand, by hypothesis \((H_1)(iii)\) we can find \(\xi_\gamma \in L^1(T)_+\) such that

\[
|F(t, x)| \leq \xi_\gamma(t) \quad \text{for a.a. } t \in T, \text{ all } |x| \leq M_6.
\]

Combining (3.23) and (3.24), we have

\[
F(t, x) \geq \gamma |x|^p - \hat{\xi}_\gamma(t) \quad \text{for a.a. } t \in T \text{ and all } x \in \mathbb{R}^N,
\]

where \(\hat{\xi}_\gamma = \xi_\gamma + \gamma M_6^p \in L^1(T)_+\). Now, let \(u \in \mathbb{R}^N \oplus E\). Then

\[
\varphi(u) = \frac{1}{p} \|u'\|^p_p - \int_0^b F(t, u(t)) \, dt \leq \frac{1}{p} \|u'\|^p_p - \gamma \|u\|^p_p + c_6
\]

with \(c_6 = \|\hat{\xi}_\gamma\|_1\) (see (3.25)). Because \(\mathbb{R}^N \oplus E\) is finite dimensional, all norms are equivalent and so from (3.26) we infer that

\[
\varphi(u) \leq \frac{1}{p} \|u\|^p_p - \gamma \|u\|^p_p + c_6 \leq \frac{1}{p} (1 - \gamma c_7) \|u\|^p_p + c_6 \quad \text{for all } u \in \mathbb{R}^N \oplus E,
\]

with \(c_7 > 0\) independent of \(\gamma\). Therefore, we can chose \(\gamma > 1/c_7\) and (3.27) shows that \(\varphi|_{\mathbb{R}^N \oplus E}\) is anticoercive. \(\square\)

Now we are ready for the existence theorem.
Theorem 3.6. If hypotheses (H1) hold, then problem (1.1) has a nontrivial solution $x_0 \in C^1(T; \mathbb{R}^N)$.

Proof. It is clear that $\varphi$ maps bounded sets into bounded sets. This together with Propositions 3.3–3.5 allow us to use Theorem 2.1, which gives the existence of some $x_0 \in W^{1,p}_0(T)$, $x_0 \neq 0$ such that $\varphi'(x_0) = 0$, which means

\begin{align}
A(x_0) = N(x_0).
\end{align}

From (3.28), a standard reasoning using integration by parts, shows that $x_0 \in C^1(T; \mathbb{R}^N)$ and solves (1.1) (see e.g. L. Gasinski and N. S. Papageorgiou [8]). □

4. Existence of multiple solutions

We prove a multiplicity theorem for problem (1.1). Our hypotheses on the potential function $F(t, x)$ incorporate systems which are strongly resonant with respect to $\lambda_0 = 0$, the principal eigenvalue of the negative vector $p$-Laplacian. The Euler–Lagrange functional $\varphi$ will be bounded below but not coercive.

The precise hypotheses on the potential function $F(t, x)$ are the following:

(H2) $F: T \times \mathbb{R}^N \to \mathbb{R}$ is a function such that:

(i) for all $x \in \mathbb{R}^N$, $t \mapsto F(t, x)$ is measurable;
(ii) for almost all $t \in T$, $x \mapsto F(t, x)$ is $C^1$ and $F(t, 0) = 0$;
(iii) for almost all $t \in T$ and all $x \in \mathbb{R}^N$

\[ |\nabla F(t, x)| \leq a_0(t)c_0(|x|) \]

with $a_0 \in L^1(T)_+$, $c_0 \in C(\mathbb{R}_+)$, $c_0 \geq 0$;
(iv) there exists a function $F_{\infty} \in L^1(T)$ such that $\int_0^b F_{\infty}(t) \, dt \leq 0$ and $F(t, x) \to F_{\infty}(t)$ for a.a. $t \in T$, as $|x| \to \infty$;
(v) there exists a function $\eta \in L^1(T)_+$, $\eta \neq 0$ such that

\[ \liminf_{x \to 0} \frac{pF(t, x)}{|x|^p} \geq \eta(t) \text{ uniformly for a.a. } t \in T; \]
(vi) $F(t, x) \leq \frac{1}{p} |x|^p$ for almost all $t \in T$ and all $x \in \mathbb{R}^N$.

Remark 4.1. Hypothesis (H2)(iv) implies that at infinity we may have strong resonance with respect to the principal eigenvalue $\lambda_0 = 0$. As it is well known, strongly resonant problems exhibit a partial lack of compactness. In our case this is reflected in Proposition 4.3 below.

Example 4.2. The function

\[ F(x) = \begin{cases} 
\frac{1}{pb^p}|x|^p & \text{if } |x| \leq 1, \\
\frac{1}{pb^p}\left(1 + (p + 1) \ln |x| \right) & \text{if } |x| > 1,
\end{cases} \]
satisfies hypotheses (H$_2$) (again, for the sake of simplicity, we dropped the $t$-dependence).

**Proposition 4.3.** If hypotheses (H$_2$) hold, then $\varphi$ satisfies the PS$_c$-condition at every level $c < -\int_0^b F_\infty(t)\,dt$.

**Proof.** Let $\{x_n\}_{n \geq 1} \subset W^{1,p}_\text{per}(T)$ be a sequence such that

\begin{equation}
\varphi(x_n) \to c, \quad \text{with } c < -\int_0^b F_\infty(t)\,dt
\end{equation}

and

\begin{equation}
\varphi'(x_n) \to 0 \quad \text{in } W^{1,p}_\text{per}(T)^*, \quad \text{as } n \to \infty.
\end{equation}

**Claim.** $\{x_n\}_{n \geq 1}$ is bounded in $W^{1,p}_\text{per}(T)$.

We proceed by contradiction. So, suppose that $\|x_n\| \to \infty$ and set $y_n = x_n/\|x_n\|$, $n \geq 1$. Then $\|y_n\| = 1$ and we may assume that

\begin{equation}
y_n \wto y \quad \text{in } W^{1,p}_\text{per}(T) \text{ and } y_n \to y \quad \text{in } C(T; \mathbb{R}^N).
\end{equation}

From (4.1) we have

\begin{equation}
|\varphi(x_n)| \leq M_7 \quad \text{for some } M_7 > 0, \quad \text{and all } n \geq 1,
\end{equation}

\[\Rightarrow \frac{1}{p} \|y_n\|_p^p \leq \frac{M_7}{\|x_n\|_p^p} + \int_0^b \frac{F(t,x_n)}{\|x_n\|_p^p} \,dt.\]

From (H$_2$)(iv), for almost all $t \in T$ there is some $M_t > 0$ such that $|F(t,x)| \leq M_t$, for all $x \in \mathbb{R}^N$. Then, by virtue of (H$_2$)(vi) and Fatou’s lemma, it follows

\begin{equation}
\limsup_{n \to \infty} \int_0^b \frac{F(t,x_n(t))}{\|x_n\|_p^p} \,dt \leq \int_0^b \limsup_{n \to \infty} \frac{F(t,x_n(t))}{\|x_n\|_p^p} \,dt = 0.
\end{equation}

So, if in (4.4) we pass to the limit as $n \to \infty$, we get $\|y\|_p = 0$ (see (4.3) and (4.5)), which means that $y = \xi \in \mathbb{R}^N$.

If $\xi = 0$, then $y_n \to 0$ in $W^{1,p}_\text{per}(T)$, a contradiction to the fact that $\|y_n\| = 1$, for all $n \geq 1$.

If $\xi \neq 0$, then $|x_n(t)| \to \infty$ for all $t \in T$, as $n \to \infty$. Then, by virtue of (H$_2$)(iv), we have

\[F(t,x_n(t)) \to F_\infty(t) \quad \text{for a.a. } t \in T, \quad \text{as } n \to \infty.
\]

Because of (4.1), given $\varepsilon > 0$, we can find $n_0 = n_0(\varepsilon) \geq 1$ such that

\[|\varphi(x_n) - c| \leq \varepsilon \quad \text{for all } n \geq n_0,
\]

\[\Rightarrow \frac{1}{p} \|x_n\|_p^p - \int_0^b F(t,x_n(t)) \,dt \leq c + \varepsilon \quad \text{for all } n \geq n_0,
\]

\[\Rightarrow - \int_0^b F_\infty(t) \,dt \leq c + \varepsilon \quad \text{(by Fatou’s lemma)}.
\]
Since $\varepsilon > 0$ was arbitrary, we let $\varepsilon \to 0^+$ and obtain
\[- \int_0^b F_\infty(t) \, dt \leq c,
\]
which contradicts the choice of $c \in \mathbb{R}$ (see (4.1)). This proves the Claim.

Due to the Claim, we may assume that
\[x_n \xrightarrow{w} x \quad \text{in} \quad W^{1,p}_\text{per}(T) \quad \text{and} \quad x_n \to x \quad \text{in} \quad C(T;\mathbb{R}^N).\]

Then using (4.2) and arguing as in the proof of Proposition 3.3, exploiting the fact that the operator $A$ is of type $(S)_+$, we conclude that $x_n \to x$ in $W^{1,p}_\text{per}(T)$.

Therefore, $\varphi$ satisfies the PS$_c$-condition at every level $c < - \int_0^b F_\infty(t) \, dt$. $\square$

Now we are ready for the multiplicity theorem.

**Theorem 4.4.** If hypotheses (H$_2$) hold, then problem (1.1) has at least two nontrivial solutions $x_0, u_0 \in C^1(T,\mathbb{R}^N)$.

**Proof.** By virtue of hypothesis (H$_2$)(v), given $\varepsilon > 0$, we can find $\delta = \delta(\varepsilon) > 0$ such that
\[(4.6) \quad \frac{1}{p}(\eta(t) - \varepsilon)|x|^p \leq F(t,x) \quad \text{for a.a. } t \in T \text{ and } x \in \mathbb{R}^N \text{ with } |x| \leq \delta.\]

Let $x = \xi \in \mathbb{R}^N$ with $|\xi| \leq \delta$. Then
\[(4.7) \quad \varphi(\xi) = - \int_0^b F(t,\xi) \, dt \leq - \frac{|\xi|^p}{p} \left[ \int_0^b \eta(t) \, dt - \varepsilon b \right] \quad (\text{see } (4.6)).\]

If we chose $\varepsilon \in (0, \|\eta\|_{1/b})$, then from (4.7) it follows that
\[(4.8) \quad \varphi(\xi) < 0.\]

We show that $\varphi$ is bounded below. Indeed, if this is not the case, then we can find a sequence $\{x_n\}_{n \geq 1} \subset W^{1,p}_\text{per}(T)$ such that
\[(4.9) \quad \varphi(x_n) \to - \infty \quad \text{as } n \to \infty.\]

Since $\varphi$ maps bounded sets into bounded sets, we may assume that $\|x_n\| \to \infty$, as $n \to \infty$. As before, let $y_n = x_n/\|x_n\|$ and assume, without any loss of generality, that
\[(4.10) \quad y_n \xrightarrow{w} y \quad \text{in} \quad W^{1,p}_\text{per}(T) \quad \text{and} \quad y_n \to y \quad \text{in} \quad C(T;\mathbb{R}^N).\]

We have
\[\frac{\varphi(x_n)}{\|x_n\|^p} = \frac{1}{p} \frac{\|y_n'\|_p^p}{\|y_n'\|_p} - \int_0^b \frac{F(t,x_n)}{\|x_n\|^p} \, dt.\]

From (4.9) and (4.10) it follows
\[0 \geq \liminf_{n \to \infty} \frac{\varphi(x_n)}{\|x_n\|^p} \geq \frac{1}{p} \frac{\|y'\|_p^p}{\|y'\|_p} - \limsup_{n \to \infty} \int_0^b \frac{F(t,x_n)}{\|x_n\|^p} \, dt \geq \frac{1}{p} \frac{\|y'\|_p^p}{\|y'\|_p}.\]
On Noncoercive Periodic Systems with Vector $p$-Laplacian

(by Fatou’s lemma; see (4.5)), meaning that $y = \xi \in \mathbb{R}^N$.

As before, if $\xi = 0$, then $y_n \to 0$ in $W^{1,p}_{\text{per}}(T)$, a contradiction to the fact that $\|y_n\| = 1$, for all $n \geq 1$. If $\xi \neq 0$, then \[ |x_n(t)| \to +\infty \] for all $t \in T$, as $n \to \infty$ and so, via (H2)(iv), (4.9) and Fatou’s lemma, we have

\[ -\infty = \lim_{n \to \infty} \varphi(x_n) \geq -\int_0^b F_\infty(t) \, dt \geq 0, \]

a contradiction. This proves that $\varphi$ is bounded below.

From (4.8) we infer

\[ -\infty < m := \inf \varphi < 0 = \varphi(0) \leq \int_0^b F_\infty(t) \, dt. \]

According to Proposition 4.3, $\varphi$ satisfies the PS$_m$-condition. Hence, we can find $x_0 \in W^{1,p}_{\text{per}}(T)$ such that

\[ m = \varphi(x_0) < 0 = \varphi(0) \]

(see, for example, L. Gasinski and N. S. Papageorgiou [9, p. 650]). From (4.11) we see that $x_0 \neq 0$ and

\[ \varphi'(x_0) = 0. \]

By virtue of (4.8), for $\rho > 0$ small enough, we have

\[ \mu := \sup \{ \varphi(x) \mid x \in \partial B_\rho \cap \mathbb{R}^N \} < 0. \]

As before, we consider the direct sum decomposition

\[ W^{1,p}_{\text{per}}(T) = \mathbb{R}^N \oplus V, \quad \text{with } V = \left\{ x \in W^{1,p}_{\text{per}}(T) \mid \int_0^b x(t) \, dt = 0 \right\}. \]

From (H2)(vi) and (3.21), for $x \in V$, we have

\[ \varphi(x) \geq \frac{1}{p} \|x'\|^p_p - \frac{1}{pb} \|x\|^p_p \geq 0 \Rightarrow \inf_V \varphi \geq 0. \]

Suppose that $x_0$ is the only nontrivial critical point of $\varphi$ (see (4.12)). Let $a := m < 0 =: c$ and apply Theorem 2.2. Then we can find a homotopy $h:[0,1] \times (\varphi^c \setminus K_c) \to \varphi^c$, such that $h(t,x) = x$ for all $(t,x) \in [0,1] \times \varphi^a$ and

\[ h(1,\varphi^c \setminus K_c) \subset \varphi^a, \]

(4.16) $\varphi(h(t,x)) \leq \varphi(h(s,x))$ for all $t, s \in [0,1]$, $s \leq t$, all $x \in \varphi^c \setminus K_c$.

Now, we consider the map $\overline{\eta}: \overline{B}_\rho \cap \mathbb{R}^N \to W^{1,p}_{\text{per}}(T)$ defined by

\[ \overline{\eta}(x) = \begin{cases} x_0 & \text{if } \|x\| \leq \rho/2, \\ h(2(\rho - \|x\|)/\rho, \rho x/\|x\|) & \text{if } \|x\| \in (\rho/2, \rho]. \end{cases} \]
If $x \in \mathbb{R}^N$, $\|x\| = \rho/2$, then $2x \in \varphi^c \setminus K_c$ (see (4.13)) and so, by (4.15)

$$h\left(\frac{2(\rho - \|x\|)}{\rho}, \frac{\rho x}{\|x\|}\right) = h(1, 2x) \in \varphi^a = \{x_0\},$$

showing that $\varphi$ is continuous (see (4.17)). If $x \in \partial B_\rho \cap \mathbb{R}^N$ then $\varphi(x) = h(0, x) = x$, because $h$ is a homotopy. Therefore

$$\varphi \in \Gamma = \{\gamma \in C(\overline{B}_\rho \cap \mathbb{R}^N, W^{1,p}_\text{per}(T)) \mid \gamma|_{\partial B_\rho \cap \mathbb{R}^N} = \text{id}|_{\partial B_\rho \cap \mathbb{R}^N}\}.$$

From L. Gasinski and N. S. Papageorgiou [9, p. 642], we know that the pair $\{\partial B_\rho \cap \mathbb{R}^N, \overline{B}_\rho \cap \mathbb{R}^N\}$ is linking with $V$ in $W^{1,p}_\text{per}(T))$. It follows that

$$\varphi(\overline{B}_\rho \cap \mathbb{R}^N) \cap V \neq \emptyset,$$

which ensures that

(4.18) \[ \sup\{\varphi(\varphi(x)) \mid x \in \overline{B}_\rho \cap \mathbb{R}^N\} \geq 0 \quad \text{(see (4.14))}. \]

On the other hand, using (4.17), (4.11), (4.13) and (4.16), we deduce

(4.19) \[ \varphi(\varphi(x)) \leq \mu < 0 \quad \text{for all } x \in \overline{B}_\rho \cap \mathbb{R}^N. \]

Comparing (4.18) and (4.19), we reach a contradiction. This proves that $\varphi$ has one more nontrivial critical point $u_0 \neq x_0$. \(\square\)

References


On Noncoercive Periodic Systems with Vector $p$-Laplacian


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