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BIFURCATIONS IN THE CASE OF TWOFOLD DEGENERATION. THE QUASI-LINEAR APPROACH

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Abstract. This paper studies typical (in a sense) bifurcations in the case of twofold degeneration linearized operator. We use an original approach of quasi-linear representation.

1 Introduction

The small eigenvectors problem of nonlinear operators dates back to the works of Euler, Poincare, A. M. Lyapunov and E. Schmidt. The important results were obtained by M. A. Krasnosel'skiĭ in late sixties. We focus on non-linear operators that vanish at zero. Therefore the nonlinear eigenvector problem is also a problem of bifurcation of zero-solution. Clearly, the various particular properties of non-linear operator require different methods of studies. Thus one can apply analytical, topological, variational, and cone methods. The main results up to the date are presented in the following publications [1], [2], [11], [14]–[18], [21], [22], [24].

Without going into details we could say that we consider the eigenvector problem $\gamma(\mathbf{A}_0 x + F(x)) = x$, where $\mathbf{A}_0$ is linear operator, and a non-linear map F approaches zero $\|F(x)\|/\|x\| \to 0$ at small $\|x\| \to 0$. The implicit function theorem suggests that the bifurcation point γ_0 is the characteristic number $\gamma_n = \gamma_0$

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of the linearized operator $\mathbf{A}_0$. There are the following famous results obtained by Krasnosel'skiĭ [16]. Firstly, if map F is variational each characteristic number γ_n is a bifurcation point. Secondly, if characteristic number γ_n has odd multiplicity, γ_n is a bifurcation point.

The case of double degeneracy $\gamma_n = \gamma_{n+1}$ is special and leads to a number of different scenarios. Let us briefly remind here main results for the double degeneracy case. This was studied by analytical methods in the case when a nonlinearity F is smooth [24]. The work [19] apply cone method assuming the map F homogeneous. The work [13] focuses on the case when the order of non-linearity smallness is two: $\|F(x)\| \sim O(\|x\|^2)$. Let us note that the above mentioned results have two common peculiarities. They all assume certain non-degeneracy conditions and they are based on the Lyapunov–Shmidt ramification equation. Therefore the main problem is to transform the given infinite-dimensional equation to the finite-dimensional ramification equation using the non-degeneracy. In the work we overcome this difficulty by using an original quasi-linear method.

Our approach is to employ the quasi-linear representation of the non-linearity F, namely $F(x) \equiv A(x)x$, where $A(x)$ is a small x-dependent operator. The quasi-linear representation was first used by A. I. Perov, P. P. Zabreĭko, and A. I. Povolotskiĭ to calculate the Leray–Schauder degree in the sixties (see § 21 in [18]). Later P. Fitzpatrick and J. Pejsachowicz developed the theory of Leray–Schauder degree for some class of quasi-linear Fredholm maps in the nineties [12]. The quasi-linear representation was first applied to the eigenvector problem in 1984 by Y. M. Dymarskiĭ [8]. Later C. Cosner used a similar method in 1988 [3]. The quasi-linear eigenvector problem is systematically studied in the monograph [9] by Y. M. Dymarskiĭ. In this work we improve the bifurcation theorem from [9] (see Chapter 7.1). This theorem is discussed in section six. We also compare our results with previous works there.

This paper is organized as follows. Next section introduces main notions and definitions. Third section formulates the main theorem. Fourth section is devoted to the a priory estimates for eigenvalues and for small eigenvectors. We continue with the new non-linear projection that compactifies quasi-linear representation in section five. Section six contains the original bifurcation theorem from [9]; it also compares the original theorem with the one from section three. Seventh section checks that the resulting quasi-linear problem satisfies the requirements of theorem from [9].

2. Main notions and definitions

Let H be a real separable Hilbert space with an inner product $\langle\cdot\,,\cdot\rangle$ and norm $\|\cdot\|$. Let also $\mathbf{A}_0\colon H \to H$ be a linear operator that satisfies the following conditions.

(i) $\mathbf{A}_0$ is a compact self-adjoint operator.

(ii) $\mathbf{A}_0$ is positive definite i.e. for any $x \neq 0$ the following inequality $\langle \mathbf{A}_0 x, x\rangle > 0$ is satisfied.

Because of the properties mentioned above the operator $\mathbf{A}_0$ has a countable collection of isolated characteristic numbers (i.e. inverse eigenvalues); the characteristic numbers are positive with finite degeneracy; the ordered characteristic numbers have limiting property $\gamma_i \to \infty$ as $i \to \infty$. In what follows, as a possible bifurcation point, we are interested in the twofold characteristic number $\gamma_n = \gamma_{n+1}$. In each eigen-space that corresponds to the eigenvalue γ_i with m_i-degeneracy one can choose m_i orthonormal in H eigenvectors. The eigenvectors that correspond to different eigenvalues are also orthogonal. The union of all eigenvectors form an orthogonal basis in H $\{e_1, e_2, \dots\}$. (These properties constitute Gilbert–Schmidt theorem [6]). For a twofold degenerate eigenvalue γ_n the corresponding eigenvectors span the plane $H^0 = \operatorname{span}\{e_n, e_{n+1}\} \subset H$. Let H^1 be an orthogonal complement to H^0 in H and π^0, π^1 the projections on H^0 and H^1 correspondingly.

Below we shall consider operators from Banach space $\mathfrak{L}^c(H)$ of linear compact self-adjoined operators. By $|\cdot|$ denote the common operator norm in $\mathfrak{L}^c(H)$.

Let nonlinear mapping F satisfies the following conditions.

(i) The mapping $F\colon H \to H$ is completely continuous (i.e. continuous and compact [16).

(ii) $F(0) = 0$.

(iii) There are such $\varepsilon, \mu, K > 0$ that under the condition $\|x_1\|, \|x_2\| < \varepsilon$, the inequality

$$\|F(x_1 + x_2) - F(x_1)\| < K(\|x_1\| + \|x_2\|)^\mu \cdot \|x_2\| \tag{2.1}$$

holds.

Lipshitz condition (iii) has the coefficient vanishing as $\|x\|^\mu$. The conditions (i)–(iii) are natural (for example, see [16]). It follows from (ii) and (iii) that

$$\|F(x)\| < K\|x\|^{1+\mu}; \tag{2.2}$$

Particularly, $\|F(x)\|/\|x\| \to 0$ as $x \to 0$.

Consider the nonlinear equation

$$\gamma(\mathbf{A}_0(x) + F(x)) = x \tag{2.3}$$

in the unknown pair $(\gamma, x) \in \mathbb{R}\times H$. If the pair (γ, x) satisfies equation and $x \neq 0$, then we say that pair is *the solution*; the number γ is called the *characteristic* one and vector x is called *the eigenvector*. We focus on small eigenvectors $\|x\| \ll 1$. We note that by condition (ii), a pair $(\gamma, 0)$ satisfies equation (2.3) for any γ. The pair $(\gamma, 0)$ is called *the trivial solution*. Now we give two main definitions.

Definition 2.1 ([17]). The number $\gamma_0 \in \mathbb{R}$ is called the bifurcation point for problem (2.3) if, for any $\delta > 0$, problem (2.3) has a solution (γ, x) satisfying the inequality

$$|\gamma - \gamma_0| + \|x\| < \delta. \tag{2.4}$$

The set of all solutions satisfying condition (2.4) is denoted by Sol_δ.

Definition 2.2. A solution branch $\mathrm{Br} \subset \mathbb{R} \times H$ of problem (2.3) is a connected component of the set of all its solutions. If the closure $\overline{\mathrm{Br}}$ of a solution branch contains a trivial solution $(\gamma_0, 0)$, the branch is said to be *emanating from the bifurcation point* $(\gamma_0, 0)$.

The concept of a continuous *eigenvector branch* was introduced by M. A. Krasnosel'skiĭ [16]. The concept of a *global continuous solution branch* was introduced by P. Rabinowitz [23]. (Paper [23] contains classical theorem about global solution branches emanating from the bifurcation point. Rabinowitz theorem was strengthened by E. Dancer [5]. Elegant statement of Rabinowitz theory is given in Nirenberg's lectures [22].) Our definition is close to Rabinowitz definition. The distinction is that Rabinowitz first takes the *closure* of set of (nontrivial) solution, and then the connected components of the obtained set are investigated.

3. The main theorem

Denote by $S_\rho = \{x \in H : \|x\| = \rho\}$ the sphere of radius ρ. We shall seek an eigenvectors $x \in S_\rho$, where radius ρ is sufficiently small.

On the plane H^0 write vector x by the polar coordinates (r, φ) $(r \geq 0$, $\varphi \in [0, 2\pi))$:

$$H^0 \ni x = r \cdot x_\varphi := r \cdot (e_n \cos\varphi + e_{n+1} \sin\varphi).$$

Then we can represent any vector $x \in H$ in the unique form

$$x = r \cdot x_\varphi + v, \quad v \in H^1,\ r^2 + ||v||^2 = \rho^2. \tag{3.1}$$

Consider the family of operators $A(x) \in (\mathfrak{L})^c(H)$ depending on variable $x \in H$; by definition, put

$$A(x)u := \frac{\langle x, u\rangle \cdot F(x) + \langle F(x), u\rangle \cdot x}{\|x\|^2} - \frac{\langle F(x), x\rangle \cdot \langle x, u\rangle \cdot x}{\|x\|^4}, \quad \text{if } x \neq 0, \tag{3.2}$$
$$A(0) := 0.$$

Using family (3.2), let us write the mapping F in the *quasilinear* form:

$$F(x) \equiv A(x)x.$$

Now nonlinear problem (2.3) is quasi-linear one:

$$\gamma(\mathbf{A}_0 + A(x))x = x. \tag{3.3}$$

Let (γ^*, x^*) be a solution of problem (3.3). Along with quasilinear eigenvector problem (3.3), let us consider associated linear self-adjoined eigenvector problem of the form

$$\gamma(\mathbf{A}_0 + A(x^*))x = x. \tag{3.4}$$

Obviously, the number γ^* is an eigenvalue of problem (3.4), and, moreover, among the eigenvectors corresponding to λ^*, there exists the vector x^*.

DEFINITION 3.1. With a solution (γ^*, x^*) of problem (3.3) we associate that *number* n and *multiplicity* m which γ^* has as an eigenvalue of associated linear problem (3.4). A solution (γ^*, x^*) of (3.3) is said to be simple or multiple if such is γ^* as an eigenvalue of (3.4).

We introduce the following nonlinear functionals

$$\begin{aligned} a(x) &= \langle A(x)e_n, e_n\rangle, & b(x) &= \langle A(x)e_n, e_{n+1}\rangle, \\ c(x) &= \langle A(x)e_{n+1}, e_{n+1}\rangle, & d(x) &= \frac{1}{2}(a(x) - c(x)). \end{aligned} \tag{3.5}$$

Considering (3.1), these functionals depend on variables φ, r, v: $a(x) = a(\varphi; r, v)$ etc. In what follows, let φ be an argument belonging to the unit circle S_1^1, the pair $(r, v) \in \mathbb{R}^+ \times H^1$ is understood as small parameter. If the inequality

$$d^2(\varphi; r, v) + b^2(\varphi; r, v) > 0,$$

holds for some r, v and for all $\varphi \in S_1^1$ then the formulas $\cos\alpha = d/\sqrt{d^2 + b^2}$, $\sin\alpha = b/\sqrt{d^2 + b^2}$ define a family of mappings $\alpha = \alpha(\varphi; r, v)$ of the circle S_1^1 to the circle S_2^1 parametrized by the angle α. The pair (r, v) is understood as small parameter of family. Denote by $\deg(\alpha)$ the oriented degree [22] of this mapping.

Let us formulate the main theorem of paper.

THEOREM 3.2. *Let there exists such $\varepsilon > 0$ for which the following conditions hold.*

(a) *There exists $\kappa > 0$ such that*

$$\sqrt{d^2(\varphi, r, 0) + b^2(\varphi, r, 0)} \geq \kappa r^\mu \tag{3.6}$$

whenever $0 < r < \varepsilon$.

(b) *Let, for a certain $0 < r_0 < \varepsilon$, the degree satisfy*

$$\deg(\alpha(\varphi, r_0, 0)) = N \neq 2. \tag{3.7}$$

Then there exists a small constant $\delta > 0$ for which the following assertions hold.

(A) *If $(\gamma, x) = (\gamma, r \cdot x_\varphi + v)$ is an arbitrary solution of problem* (2.3) *satisfying inequality* (2.4)*, then there exist constants $V, \Gamma > 0$ that*

$$\|v\| \leq V \cdot \|x\|^{1+\mu}, \quad |\gamma - \gamma_n| \leq \Gamma \cdot \|x\|^{\mu}.$$

(B) *All solutions of problem* (2.3) *for which inequality* (2.4) *hold are simple* ((2.3) *if and only if* (3.3)!)*.*

(C) *The number γ_n is a bifurcation point for problem* (2.3)*.*

(D) *For each $\rho < \delta$, problem* (2.3) *has no less than $|2 - N|$ solutions (γ, x) with number n for which $x \in S_\rho$; the same is true for the solutions with number $n + 1$.*

(I) *In addition, let the function $\alpha(\varphi; r, 0)$ be monotone in φ uniformly for all $r \in (0, \varepsilon)$. Then no less than $|2 - N|$ branches emanate from the bifurcation point γ_n; these branches contain simple solution with number n. The same is true for the branches containing solutions with number $n + 1$.*

The conditions and the statements of Theorem 3.2 are discussed in detail in Section 7.

4. A priori estimates

We first show that the desired set of solutions is compact.

Lemma 4.1. *For any fixed $\delta > 0$ the set Sol_δ of all solutions (γ, x) satisfying* (2.4) *is compact in the space $\mathbb{R} \times H$.*

Proof. First, the set of all characteristic numbers is bounded: $|\gamma - \gamma_n| < \delta$. Secondly, it follows from equation (2.3) and the compactness of mapping $\mathbf{A}_0 + F$ that the image of the ball $\mathrm{Ball}(\delta) = \{x : \|x\| < \delta\}$ is compact subset $(\mathbf{A}_0 + F)(\mathrm{Ball}(\delta)) \subset H$. Hence the solution set belongs to compact set:

$$\mathrm{Sol}_\delta \subset (\gamma_n - \delta, \gamma_n + \delta) \times (\gamma_n - \delta, \gamma_n + \delta)(\mathbf{A}_0 + F)(\mathrm{Ball}(\delta)). \qquad \square$$

Now we obtain an a priori estimates, which characterizes the variation of solution (γ, x) in plane $(\gamma_n, \mathrm{span}\{e_n, e_{n+1}\})$. At the beginning we show that the orthogonal component v (see (3.1)) is $(1 + \mu)$-order infinitesimal in comparison with the small eigenvector x.

Lemma 4.2. *There exists $\delta > 0$ such that for all solutions (γ, x) satisfying inequality* (2.4)*, the component $v \in H^1$ of eigenvector x satisfy the inequality*

$$\|v\| \leq V\|x\|^{1+\mu}, \tag{4.1}$$

where $V > 0$ is certain constant.

PROOF. The projection of equation (2.3) on subspace $H^1 \subset H$ is

$$\gamma \mathbf{A}_0 v + \pi^1 F(x) = v.$$

For sufficiently small δ, the operator

$$-\gamma \mathbf{A}_0^1 + E \colon H^1 \to H^1, \quad (-\gamma \mathbf{A}_0^1 + E)v := -\gamma \mathbf{A}_0 v + v$$

acts onto the space H^1 and is invertible on H^1. Using (2.2), we obtain

$$\begin{aligned} &\gamma \mathbf{A}_0 v + \pi^1 F(x) = v \;\Rightarrow\; v = (-\gamma \mathbf{A}_0^1 + E)^{-1}(\pi^1 F(x)) \\ \Rightarrow\; &\|v\| \le |(-\gamma \mathbf{A}_0^1 + E)^{-1}| \cdot |\pi^1| \cdot \|F(x)\| \le \sup_\gamma |(-\gamma \mathbf{A}_0^1 + E)^{-1}| \cdot 3K\|x\|^{1+\mu}, \end{aligned}$$

where $\gamma \in (\gamma_n - \delta, \gamma_n + \delta)$. (Note that we make the substitution of $3K$ for K in (2.2) deliberately; the reason for this will be explained in the proof of Lemma 6.) □

Now we estimate the variation of characteristic number γ.

LEMMA 4.3. *There exists $\delta > 0$ such that for all solutions (γ, x) satisfying inequality* (2.4)*, the variation of characteristic number satisfy the inequality*

$$|\gamma - \gamma_n| \le \Gamma \|x\|^\mu, \tag{4.2}$$

where $\Gamma > 0$ is certain constant.

PROOF. It follows from equality (2.3) and the equality $\gamma_n \mathbf{A}_0(x - v) = x - v$ that

$$\gamma_n \mathbf{A}_0 v + (\gamma - \gamma_n)\left(\frac{x - v}{\gamma_n} + \mathbf{A}_0 v\right) + F(x) = v.$$

Taking into consideration estimate (4.1), we give the parameter δ such small that

$$\left\| \frac{x - v}{\gamma_n} + \mathbf{A}_0 v \right\| > \frac{\|x\|}{2\gamma_n}.$$

Now, using estimates (2.2) and (4.1), we obtain

$$|\gamma - \gamma_n| \|x\| \le 2\gamma_n (V\|x\|^{1+\mu} + 3K\|x\|^{1+\mu} + \gamma_n |\mathbf{A}_0| V \|x\|^{1+\mu}) = \Gamma \|x\|^{1+\mu}.$$

(Above we make the substitution of $3K$ for K a second time.) □

Thus, assertion (A) of Theorem 3.2 is proved.

5. Compactification of quasi-linearity

Our next task is to define an original projection on the compact subset specified by the a priori estimates. The projection will preserve the desired solutions invariant. We will use this projection in section six to define quasi-linear representation that is a completely continuous map in the space of operators. In turn the complete continuity of the quasi-linear representation is required to construct a topological invariant (index of intersection). The latter is used to prove the existence of the solution of the non-linear problem.

The projection will be introduces in two steps. At first we introduce an auxiliary projection. We can assume that the radius ρ is small enough that $V\rho^{1+\mu} < \rho/2$ without loss of generality. Using the orthogonal decomposition of the eigenvector (3.1) we define a solid torus

$$T_\rho := \{x = rx_\varphi + v \in S_\rho : v \leq V\rho^{1+\mu}\} \tag{5.1}$$

that is homotopically equivalent to a circle: $T_\rho \sim S^1$. Let $X_{\rho,n}$ be a set of all eigenvectors of bifurcation problem (2.3), that satisfy the normalization condition $x \in S_\rho$. It follows from definition (5.1) and priori estimate (4.1) that $X_{\rho,n} \subset T_\rho$. Moreover equation (2.3) and a priori estimates (4.1) and (4.2) imply that

$$X_{\rho,n} \subset \mathrm{Comp}_{\rho,n} := \{[\gamma_n - \Gamma\rho^\mu, \gamma_n + \Gamma\rho^\mu]\overline{(\mathbf{A}_0 + F)(T_\rho)}\} \cap T_\rho. \tag{5.2}$$

Here the subset $\mathrm{Comp}_{\rho,n}$ is absolute compact because of the map $\mathbf{A}_0 + F$ is compact. Therefore the projection of $X_{\rho,n}$ on H^1 satisfies the condition

$$\pi^1(X_{\rho,n}) \subset \pi^1(\mathrm{Comp}_{\rho,n}). \tag{5.3}$$

By $\overline{\mathrm{co}}(X)$ denote the closed convex hull of $X \subset H$. It follows from embedding (5.3) that

$$\pi^1(X_{\rho,n}) \subset \overline{\mathrm{co}}(\pi^1(\mathrm{Comp}_{\rho,n})) \subset H^1,$$

where the convex set $\overline{\mathrm{co}}(\pi^1(\mathrm{Comp}_{\rho,n}))$ is absolute compact (by Mazur's theorem [7]).

Let Π be the auxiliary projection of H^1 to $\overline{\mathrm{co}}(\pi^1(\mathrm{Comp}_{\rho,n}))$ such that the set distance $\mathrm{dist}(v, \overline{\mathrm{co}}(\pi^1(\mathrm{Comp}_{\rho,n}))) := \inf \|v - x\|$, $x \in \overline{\mathrm{co}}(\pi^1(\mathrm{Comp}_{\rho,n}))$, is equal to $\|\Pi(v) - v\|$. Otherwise, the point $\Pi(v)$ actualizes the distance between v and $\overline{\mathrm{co}}(\pi^1(\mathrm{Comp}_{\rho,n}))$.

Lemma 5.1. *The following assertions hold.*

(a) *The definition of* Π *is correct: for any* $v \in H^1$ *the image* $\Pi(v)$ *is uniquely defined.*
(b) *If* $v \in \overline{\mathrm{co}}(\pi^1(\mathrm{Comp}_{\rho,n}))$, *then* $\Pi(v) = v$, $\Pi^2 = \Pi$.
(c) *The mapping* Π *is completely continuous.*
(d) $\|\Pi(v)\| \leq \|v\|$.

(e) *The norm of image satisfies the inequality:* $\|\mathrm{Im}\Pi\| \le V\rho^{1+\mu}$.

Proof. The correctness of definition follows from the fact that the subset

$$\overline{\mathrm{co}}(\pi^1(\mathrm{Comp}_{\rho,n})) \subset H^1 \subset H$$

is convex and compactness, and the space H^1 is a Hilbert one.

The second statement follows directly from the definition of Π.

The compactness of Π follows from the compactness of subset

$$\overline{\mathrm{co}}(\pi^1(\mathrm{Comp}_{\rho,n})) \subset H^1.$$

It still remains to prove that the projection Π is continuous. Consider the quadrilateral $ABDC$, where $A = v_1$, $B = v_2$, $D = \Pi(v_2)$, $C = \Pi(v_1)$. It follows from the definition of Π and the convex of subset $\overline{\mathrm{co}}(\pi^1(\mathrm{Comp}_{\rho,n})) \subset H^1$ that the angles $\angle ACD$ and $\angle BDC$ no less then $\pi/2$. Let us note that in a general case the quadrilateral $ABDC$ is three-dimensional. If we make it flat by unfolding it around one of the diagonals the resulting angles are going to be not smaller than the original once. This is before each flat angle in any trihedral angle is smaller than the sum of the their two. We keep the same notation for the plane quadrilateral $ABDC$. There are two non acute angles adjacent to the side DC of the quadrilateral $ABDC$. Therefore $\|\Pi(v_2) - \Pi(v_1)\| = \|DC\| \le \|AB\| = \|v_2 - v_1\|$.

To prove the fourth statement, consider the triangle ABO, where $A = v$, $B = \Pi(v)$, $O = 0 \in H$. It follows from the definition of Π and the convex of subset $\overline{\mathrm{co}}(\pi^1(\mathrm{Comp}_{\rho,n})) \subset H^1$ that the angles $\angle ABO$ no less then $\pi/2$. This yields that $\|\Pi(v)\| = \|OB\| \le \|OA\| = \|v\|$.

The last statement follows from the definitions of solid torus (5.1) and subset (5.2). $\square$

Let us proceed to the definition of main projection. Consider the solid torus

$$T_\Pi := \{x = rx_\varphi + v \in S_\rho : v \in \overline{\mathrm{co}}(\pi^1(\mathrm{Comp}_{\rho,n}))\} \subset T_\rho. \tag{5.4}$$

Denote by $\mathrm{Pr}\colon T_\rho \to T_\Pi$ the projection such that

$$\mathrm{Pr}(x) = \mathrm{Pr}(rx_\varphi + v) := \sqrt{\rho^2 - \|\Pi(v)\|^2}\; x_\varphi + \Pi(v).$$

Lemma 5.2. *The following assertions hold.*

(a) *If* $x \in T_\Pi$, *then* $\mathrm{Pr}(x) = x$; $\mathrm{Pr}^2 = \mathrm{Pr}$.

(b) $\|\mathrm{Pr}(x)\| = \|x\| = \rho$.

(c) *The mapping* Pr *is completely continuous.*

Proof. The first and the second assertions follow directly from the definition of Pr. The third assertion follows from the third assertion of Lemma 4 and the finite dimensionality of the space H^0. $\square$

Using the projection Pr, consider (on the solid torus T_ρ) the problem

$$\gamma(\mathbf{A}_0(x) + A(\mathrm{Pr}(x))x) = x, \quad x \in T_\rho. \tag{5.5}$$

We claim that the bifurcation problems (3.3) and (5.5) are equivalent. For this purpose, we shall prove that the a priori estimates for solutions of problem (5.5) coincide with (4.1) and (4.2).

Lemma 5.3. *There exist such $\varepsilon, \delta > 0$ that under the conditions*

$$|\gamma - \gamma_n| < \delta, \quad x \in T_\rho, \quad \rho \le \varepsilon,$$

the inequalities (4.1) *and* (4.2) *hold.*

Proof. At first, we estimate the nonlinear term in the problem (5.5) similarly to (2.2). It follows from definition (3.2), property (b) of projection Pr (see Lemma 5.2), the infinitesimality of ρ and inequality (2.2) that

$$\begin{aligned} \|A(\mathrm{Pr}(x))x\| = \bigg\| & \frac{\langle \mathrm{Pr}(x), x\rangle F(\mathrm{Pr}(x)) + \langle F(\mathrm{Pr}(x)), x\rangle \mathrm{Pr}(x)}{\|\mathrm{Pr}(x)\|^2} \\ & - \frac{\langle F(\mathrm{Pr}(x)), \mathrm{Pr}(x)\rangle \langle \mathrm{Pr}(x), x\rangle \mathrm{Pr}(x)}{\|\mathrm{Pr}(x)\|^4} \bigg\| \\ \le {} & 3\|F(\mathrm{Pr}(x))\| \le 3 \cdot K\|\mathrm{Pr}(x)\|^{1+\mu} = 3 \cdot K\|x\|^{1+\mu}. \end{aligned} \tag{5.6}$$

Now, it suffices to note that estimates (4.1) and (4.2) were obtained using the substitution of $3K$ for K in (2.2), and to apply the same arguments and inequality (5.6). □

From Lemma 5.3, the definition of projection Π and property (A) of projection Pr (see Lemma 5.2), we get the desired result.

Lemma 5.4. *There exists such $\varepsilon > 0$ that bifurcation problem* (3.3), (3.1) *and problem* (5.5) *are equivalent for each $\rho < \varepsilon$.*

Proof. Let (γ, x) be the solution of problem (3.3), (3.1). For this solution the a priori inequalities (4.1) and (4.2) hold. Hence (see definition (5.2)), $x = rx_\varphi + v \in X_{\rho,n} \subset \mathrm{Comp}_{\rho,n}$. Therefore $v \in \overline{\mathrm{co}}(\pi^1(\mathrm{Comp}_{\rho,n}))$. Now, we have that $\Pi(v) = v$ (see assertion (b) of Lemma 5.1 and $x = \mathrm{Pr}(x)$ (see (5.4) and assertion (a) of Lemma 5.2). Thus (γ, x) is the solution of problem (5.5).

Conversely, let (γ, x) be the solution of problem (5.5). Then, by Lemma 5.3, a priori estimates (4.1) and (4.2) hold. Using previous arguments, we obtain that $\mathrm{Pr}(x) = x$ in this case also. Thus (γ, x) is the solution of problem (3.3). □

6. Formulation of the theorem for quasi-linear equation

Keeping in mind the equation (5.5), by $\widetilde{A} := A \cdot \Pr$ denote the new quasi-linear representation. We shall introduce the nonlinear functionals in a way analogous to (3.5):

$$\begin{aligned} \widetilde{a}(x) &= \langle \widetilde{A}(x)e_n, e_n\rangle, & \widetilde{b}(x) &= \langle \widetilde{A}(x)e_n, e_{n+1}\rangle, \\ \widetilde{c}(x) &= \langle \widetilde{A}(x)e_{n+1}, e_{n+1}\rangle, & \widetilde{d}(x) &= \frac{1}{2}(\widetilde{a}(x) - \widetilde{c}(x)). \end{aligned}$$

Considering (3.1), $\widetilde{a}(x) = \widetilde{a}(\varphi; r, v)$ etc. $\widetilde{b}(x) = \widetilde{b}(\varphi; r, v)$, $\widetilde{c}(x) = \widetilde{c}(\varphi; r, v)$, $\widetilde{d}(x) = \widetilde{d}(\varphi; r, v)$.

The proof of Theorem 3.2 is based on Theorem 7.1. and Corollary 7.4.1 from [9]. Let us formulate this statement.

Theorem 6.1. *Consider equation* (5.5). *Let there exist positive constants* ε *and* μ *for which the following conditions hold.*

(a) *For each* $\rho < \varepsilon$, *the quasi-linear representation* $\widetilde{A}\colon T_\rho \to \mathfrak{L}^c(H)$ *is a completely continuous mapping.*

(b) *There exists a positive number* $\widetilde{K}$ *such that* $\|\widetilde{A}(x)\| \le \widetilde{K}\|x\|^\mu$, *whenever* $\|x\| = \rho < \varepsilon$.

(c) *For any* $V > 0$, *there exists a number* $\widetilde{\kappa}(V) > 0$ *such that*

$$\sqrt{\widetilde{d}^2(x) + \widetilde{b}^2(x)} > \widetilde{\kappa}(V)\|x\|^\mu, \tag{6.1}$$

whenever $\|x\| < \varepsilon$ *and* $\|v\| \le V\|x\|^{1+\mu}$.

(d) *The mapping* $\widetilde{\alpha}(\varphi; r, 0)$ *is Lipschitz in the variable* φ *uniformly for* $0 < r < \varepsilon$. *Let the degree satisfy*

$$\deg(\widetilde{\alpha}(\varphi; r_0, 0)) = N \ne 2 \quad \textit{for a certain } r_0 \in (0, \varepsilon). \tag{6.2}$$

Then there exists a small $\delta > 0$ *for which the following assertions hold.*

(A) *All solutions of problem* (5.5) *for which the inequality* $|\gamma - \gamma_n| + \|x\| < \delta$ *hold are simple.*

(B) *The degree* $\deg(\widetilde{\alpha}(\varphi; r, 0))$ *is independent of the choice of the parameter* $r \in (0, \delta)$.

(C) *The number* λ_n *is a bifurcation point for equation* (5.5).

(D) *For each* $\rho < \delta$, *problem* (5.5) *has no less than* $|2 - N|$ *solutions* (γ, x) *with number* n *for which* $x \in T_\rho$; *the same is true for the solutions with number* $n + 1$.

(E) *If the function* $\widetilde{\alpha}(\varphi; r, 0)$ *is monotone in* φ *uniformly for* $r \in (0, \varepsilon)$, *then no less than* $|2 - N|$ *branches emanate from the bifurcation point* γ_n; *these branches contain simple solution with number* n. *The same is true for the branches containing solutions with number* $n + 1$.

Now we are ready to compare Theorems 3.2 and 6.1. Firstly the Theorem 6.1 assumes the quasi-linear form of the equation (5.5) while the main Theorem 3.2 deals with the usual non-linear equation (2.3). Moreover, Theorem 6.1 requires the quasi-linear map $\widetilde{A}$ is assumed to be completely continuous (see paragraph (a)). In this work we obtain the quasi-linear form of the equation using (3.2) while the complete continuity of $\widetilde{A}$ follows from the properties of the projection Pr (see below Lemma 7.1). Secondly Theorem 6.1 includes the lower estimate (6.1) for the solid torus T_ρ. In practice it is much less useful than the lower estimate for the circle (3.6). Lastly the single condition (2.1) is much easier to check than the condition (b) and (d) from Theorem 6.1.

Note Theorem 6.1 was strengthened already in our paper [22]. But instead of natural Lipshitz condition (2.1) the following two conditions being satisfied. Firstly, it is the Lipshitz-type condition that conforms to the vector decomposition (3.1). Secondly, it is the condition (2.2), which is the corollary of (2.1) in this paper.

The proof of Theorem 6.1 is quite complex and requires bulky introduction to the subject (that can be found in [9]). At the same time condition of applicability of the Theorem 6.1 (moreover, of the Theorem 3.2) is intuitive and easy to check. It is interesting that these conditions (6.1) and (6.2) for Theorem 6.1 ((3.6) and (3.7) for Theorem 3.2) are quite unusual. Similarly the complete continuity condition $\widetilde{A}$ is also uncommon for the context. It would be interesting to give an intuitive interpretation for these results. To simply the explanation of the proof of the Theorem 6.1 we start with a brief overlook of the main steps.

In the direct product $\mathfrak{L}^c(H) \times S_\rho$ we consider the subset P of pairs (an operator, the normalized vector), i.e.:

$$P := \{(\mathbf{A}, x) \in \mathfrak{L}^c(H) \times S_\rho \mid \text{there exists } \gamma : \gamma \mathbf{A} x = x\}.$$

The subset P has a structure of analytic vector bundle over the sphere S_ρ. For definiteness, we are interested only pairs $(\mathbf{A}, x) \in P$ to which positive characteristic numbers correspond. To each point $p = (\mathbf{A}, x) \in P$ let us put in correspondence the pair (k, m) of natural numbers, the number and the multiplicity of the corresponding characteristic number γ. The manifold P is stratified with respect to the numbers and the multiplicities of its points: $P = \bigcup_{k,m} P(k, m)$. The formula of codimension $\operatorname{codim} P(k, m) = m(m-1)/2$ is valid. In particular, the submanifolds $P(k, 1)$ of "simple" points have the codimension zero; they are open subset of P.

Next, we introduce into consideration the graph

$$\operatorname{Gr} \widetilde{A} := \{(\widetilde{A}(x), x)\} \subset \mathfrak{L}^c(H) \times S_\rho$$

of mapping $\widetilde{A}$. The following assertion is laid as a base of the our method for studying the eigenvectors of quasilinear problems.

THEOREM 6.2. *A vector* $x^* \in S_\rho$ *is normed eigenvector of problem* (5.5) *if and only if* $\operatorname{Gr} \widetilde{A}(x^*) \in P$. *The number and the multiplicity of the solution* (γ^*, x^*) *are defined by the indices* (k, m) *of the stratum to which the point* $\operatorname{Gr} \widetilde{A}(x^*) \in P(k, m) \subset P$ *belongs.*

A quasi-linear representation $\widetilde{A}$ and corresponding problem (5.5) are said k-typical if $\operatorname{Gr} \widetilde{A} \cap (\bigcup_{m>1} P(k, m)) = \emptyset$. The k-typicality means that problem (5.5) has no multiple solutions with number k. We proved that the subset of k-typical quasi-linear representations is open and dense in the space of all completely continuous mapping from T_ρ to $\mathfrak{L}^c(H)$.

Then, the existence of homotopic invariant for k-typical representations is proved, namely the invariant is the intersection index $\chi_k(\widetilde{A}) = \chi(P(k, 1), \operatorname{Gr} \widetilde{A})$ of the stratum $P(k, 1)$ with graph $\operatorname{Gr} \widetilde{A}$. (Here we are in need of the complete continuity of $\widetilde{A}$.) If the intersection index $\chi_k(\widetilde{A})$ is different from zero, then problem (5.5) has at least one simple solution with number k.

The calculation of intersection index $\chi_k(\widetilde{A})$ runs into severe difficulties. But in the case of twofold degeneration, we found test condition (6.1) under which all solutions with numbers k and $k+1$ are simple. The meaning of condition (6.1) is that the image of mapping $\widetilde{A}$ is far from operators with twofold characteristic numbers $\gamma_n = \gamma_{n+1}$. Thus, under condition (6.1) indices $\chi_k(\widetilde{A})$ and $\chi_{k+1}(\widetilde{A})$ are determined and it is easy to show $\chi_k(\widetilde{A}) = -\chi_{k+1}(\widetilde{A})$. To calculate the index $\chi_k(\widetilde{A})$, the problem (5.5) is homotopic to the two-dimensional quasilinear problem

$$
\begin{aligned}
(6.3) \qquad & \gamma \widetilde{A}^{(2)}(\varphi; r) x(\varphi) = x(\varphi) \\
& \Leftrightarrow \gamma \begin{pmatrix} \widetilde{a}(\varphi; r, 0) & \widetilde{b}(\varphi; r, 0) \\ \widetilde{c}(\varphi; r, 0) & \widetilde{d}(\varphi; r, 0) \end{pmatrix} \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} = \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}.
\end{aligned}
$$

Actually, problem (6.3) is the restriction of problem (5.5) to the plane H^0 and a quasi-linear Lyapunov–Shmidt ramification equation. In addition, we use the degree of the mapping

$$\widetilde{A}^{(2)}: S_1^1 \to \mathfrak{L}(\mathbb{R}^2) \setminus \mathfrak{L}_2(\mathbb{R}^2),$$

where $\mathfrak{L}(\mathbb{R}^2)$ is the space of two-dimensional self-adjoined operators and $\mathfrak{L}_2(\mathbb{R}^2)$ is the submanifold of operators with twofold characteristic number. It turn out that $\chi_k(\widetilde{A}) = \chi_1(\widetilde{A}^{(2)}) = 2 - \deg(\widetilde{A}^{(2)})$. Finally, by factorization of the mapping $\widetilde{A}^{(2)}$, we obtain that $\deg(\widetilde{A}^{(2)}) = \deg(\widetilde{\alpha})$.

In conclusion let us note that the quasi-linear method employed in this work requires the linear component of $\mathbf{A}_0$ to be a self-adjoint operator. The main underlying idea is close to those used in theory of analytic perturbation of spectrum of the self-adjoint operator [4]. In particular the condition (6.1) is nothing but the quasi-linear form of the "non-intersection of terms" law by E. Wigner and J. von Neumann (see [20, § 79]).

7. The verification of conditions of Theorem 6.1

We show that under conditions of Theorem 3.2, conditions (a) and (b) of Theorem 6.1 hold.

LEMMA 7.1. *There exists such $\varepsilon > 0$ that the following assertions hold.*

(a) *For each $x \in T_\rho$, where $\|x\| = \rho < \varepsilon$, the inequality $|\widetilde{A}(x)| \le 3K \cdot \|x\|^\mu$ is true.*
(b) *For each $x \in T_\rho$, where $\|x\| = \rho < \varepsilon$, the operator $\widetilde{A}(x) : H \to H$ is two-dimensional.*
(c) *For each $\rho < \varepsilon$, the mapping $\widetilde{A} : T_\rho \to \mathfrak{L}^c(H)$ is completely continuous.*

PROOF. For any vector $u \in H$ ($u \neq 0$), by definition (3.2), property (b) of projection Pr (see Lemma 5.2), the infinitesimality of parameter ε and estimate (2.2), it follows that

$$\begin{aligned}
\|\widetilde{A}(x)u\| &= \|A(\Pr(x))u\| \\
&= \bigg\| \frac{\langle \Pr(x), u\rangle F(\Pr(x)) + \langle F(\Pr(x)), u\rangle \Pr(x)}{\|\Pr(x)\|^2} \\
&\qquad - \frac{\langle F(\Pr(x)), \Pr(x)\rangle \langle \Pr(x), u\rangle \Pr(x)}{\|\Pr(x)\|^4} \bigg\| \\
&\le \frac{3\|F(\Pr(x))\| \cdot \|u\|}{\|\Pr(x)\|} \le \frac{3K\|\Pr(x)\|^{1+\mu} \cdot \|u\|}{\|x\|} \\
&\le \frac{3K\|x\|^{1+\mu} \cdot \|u\|}{\|x\|} \le 3K \cdot \|x\|^\mu \|u\|.
\end{aligned}$$

Estimate (a) is proved.

The second assertion follows from definition (3.2): the image of operator $\widetilde{A}(x) = A(\Pr(x))$ belongs to the plane that spans the vectors $F(\Pr(x))$ and $\Pr(x)$.

From definition (3.2), it follows that the mapping $\widetilde{A}$ is continuous. The mapping Pr is completely continuous (see assertion (c) of Lemma 5.2). Therefore the superposition $\widetilde{A} = A(\Pr)$ is completely continuous too. □

Let us verify the condition (c) of Theorem 6.1. First, we show that estimate (6.1) is valid for the quasi-linear representation A.

LEMMA 7.2. *Suppose the conditions of Theorem* 3.2 *are fulfilled. Then there exists such small ε that under the conditions $\|x\| = \|rx_\varphi + v\| = \rho < \varepsilon$ and $\|v\| < V\rho^{1+\mu}$ ($V > 0$ is some constant), we have*

$$\sqrt{b^2(\varphi; r, v) + d^2(\varphi; r, v)} > \kappa_1(V)\rho^\mu, \tag{7.1}$$

where the coefficient $\kappa_1(V) > 0$ *depends on* V.

PROOF. First, we note that the estimate $|A(x)| \le 3K\|x\|^\mu$ is true (the proof repeats the proof of assertion (a) of Lemma 7.1). Therefore,

$$|b(\varphi;r,v)|,\ |d(\varphi;r,v)| \le 3K\rho^\mu. \tag{7.2}$$

Let us denote $C(V)$ to be some positive continuous function (not necessary the same) of a positive variable V.

Next, two preliminary similar estimates will be established:

$$|b(\varphi;r,v) - b(\varphi;r,0)| < C(V)\rho^{2\mu}, \tag{7.3}$$

$$|d(\varphi;r,v) - d(\varphi;r,0)| < C(V)\rho^{2\mu}. \tag{7.4}$$

From definitions (3.4) and (3.5), we get the following estimate

$$\begin{aligned}|b(\varphi;r,v) - b(\varphi;r,0)| &\le |b_1(\varphi;r,v) - b_1(\varphi;r,0)| \\ &\quad + |b_2(\varphi;r,v) - b_2(\varphi;r,0)| + |b_3(\varphi;r,v) - b_3(\varphi;r,0)|,\end{aligned}$$

where

$$\begin{aligned} b_1(x) &= \frac{\langle F(x), e_n\rangle\langle x, e_{n+1}\rangle}{\|x\|^2}, \qquad b_2(x) = \frac{\langle F(x), e_{n+1}\rangle\langle x, e_n\rangle}{\|x\|^2}, \\ b_3(x) &= \frac{\langle F(x), x\rangle\langle x, e_{n+1}\rangle\langle x, e_n\rangle}{\|x\|^4}. \end{aligned}$$

We shall show that each of these summands satisfies inequality of the form (7.3). Let us denote $F_1(v) := \langle F(\varphi;r,v), e_n\rangle$, $F_2(v) := \langle F(\varphi;r,v), e_{n+1}\rangle$. Now, by $\|v\| < V\rho^{1+\mu}$ and estimates (2.2), (2.1),

$$\begin{aligned}|b_1(\varphi;r,v) - b_1(\varphi;r,0)| &= \left|\frac{rF_1(v)\sin\varphi}{\rho^2} - \frac{rF_1(0)\sin\varphi}{r^2}\right| \\ &\le \frac{r|\sin\varphi|}{\rho^2}|F_1(v) - F_1(0)| + r|F_1(0)\sin\varphi|\left|\frac{1}{\rho^2} - \frac{1}{r^2}\right| \\ &< KV\rho^{2\mu} + KV^2\rho^{3\mu} < C(V)\rho^{2\mu}.\end{aligned}$$

Analogously, for the second and third summands we obtain:

$$\begin{aligned}|b_2(\varphi;r,v) - b_2(\varphi;r,0)| &= \left|\frac{rF_2(v)\cos\varphi}{\rho^2} - \frac{rF_2(0)\cos\varphi}{r^2}\right| < C(V)\rho^{2\mu}; \\ |b_3(\varphi;r,v) - b_3(\varphi;r,0)| &\le \left|\frac{F_1(v)r^3\sin\varphi\cos^2\varphi}{\rho^4} - \frac{F_1(0)r^3\sin\varphi\cos^2\varphi}{r^4}\right| \\ &\quad + \left|\frac{F_2(v)r^3\sin^2\varphi\cos\varphi}{\rho^4} - -\frac{F_2(0)r^3\sin^2\varphi\cos\varphi}{r^4}\right| \\ &\quad + \left|\frac{r^2\langle F(r,\varphi,v), v\rangle\sin\varphi\cos\varphi}{\rho^4}\right| \\ &< 2(KV\rho^{2\mu} + 2KV^2\rho^{3\mu}) + KV\rho^{2\mu} < C(V)\rho^{2\mu}.\end{aligned}$$

Inequality (7.3) is proved. The proof of (7.4) is analogous to that of (7.3).

Finally, we give the proof of Lemma 7.2. From the definition (3.5), the estimates (2.2) and (3.6), and estimates (7.2)–(7.4) we get:

$$\begin{aligned}
b^2(\varphi;r,v)+d^2(\varphi;r,v) &= (b^2(\varphi;r,v)-b^2(\varphi;r,0)) \\
&\qquad + (d^2(\varphi;r,v)-d^2(\varphi;r,0)) + (b^2(\varphi;r,0)+d^2(\varphi;r,0)) \\
&\geq (b^2(\varphi;r,0)+d^2(\varphi;r,0)) \\
&\quad - |b(\varphi;r,v)-b(\varphi;r,0)|\cdot|b(\varphi;r,v)+b(\varphi;r,0)| \\
&\quad - |d(\varphi;r,v)-d(\varphi;r,0)|\cdot|d(\varphi;r,v)+d(\varphi;r,0)| \\
&\geq \kappa^2\rho^{2\mu}-C(V)\rho^{3\mu} \geq \kappa_1^2(V)\rho^{2\mu}. \qquad \square
\end{aligned}$$

Now, we shall test that estimate of the form (7.1) holds true in the case of superposition $\widetilde{A}=A(\mathrm{Pr})$.

Lemma 7.3. *Suppose the conditions of Theorem* 3.2 *are fulfilled. Then there exists such small* ε *that under the conditions* $\|x\|=\|rx_\varphi+v\|=\rho<\varepsilon$ *and* $\|v\|<V\rho^{1+\mu}$ ($V>0$ *is some constant*), *inequality* (6.1) *holds with some coefficient* $\widetilde{\kappa}(V)>0$.

Proof. By the definition of projection Pr, property (b) of projection Pr (see Lemma 5.2) and property (d) of projection Π (see Lemma 5.1), we obtain

$$\|\pi^1(\mathrm{Pr}(x))\| = \|\pi^1(\mathrm{Pr}(rx_\varphi+v))\| = \|\Pi(v)\| \leq V\rho^{1+\mu}.$$

It remains only to use Lemma 7.2. $\square$

Let us show that under the conditions of Theorem 3.2, condition (d) of Theorem 6.1 and the additional condition in assertion (e) hold.

We first note that mentioned conditions of Theorem 6.1 are formulated under the assumption $v=0$. Hence $\mathrm{Pr}(x)=x$ (see assertion (a) of Lemma 5.2). Therefore an objects with sign "tilde" and similar objects without tilde coincide. Particularly,

$$\alpha(\varphi;r,0)\equiv\widetilde{\alpha}(\varphi;r,0), \qquad \deg(\alpha(\varphi;r,0))=\deg(\widetilde{\alpha}(\varphi;r,0))\neq 2.$$

Thus the conditions (3.7) and (6.2) coincide, and the additional conditions of both theorems from (E) also coincide.

Since a circle is compact, it is sufficient to verify that the mapping $\widetilde{\alpha}$ is local Lipschitz in the variable φ with common Lipschitz constant for all $0<r<\varepsilon$. Without loss of generality, it is sufficient to study an neighborhood of point $\varphi=0$ and to assume (see (3.6)) that the functions b and d satisfy the estimates

$$b(\varphi;r,0)>0, \quad d(\varphi;r,0)>\frac{\kappa}{2}r^\mu. \tag{7.5}$$

Thus,

$$\widetilde{\alpha}(\varphi; r, 0) = \arctan\left(\frac{b(\varphi; r, 0)}{d(\varphi; r, 0)}\right).$$

Since the function arctan is differentiable, it is sufficient to prove that the fraction b/d is Lipschitz. It follows from definitions (3.2), (3.5) and inequalities (2.1), (2.2), that

$$|b(\varphi; r, 0) - b(0; r, 0)|, \ |d(\varphi; r, 0) - d(0; r, 0)| \leq 5Kr^{\mu}\varphi. \tag{7.6}$$

By the estimates (7.6), (7.2) and (7.5), we have:

$$\begin{aligned}
&\left|\frac{b(\varphi; r, 0)}{d(\varphi; r, 0)} - \frac{b(0; r, 0)}{d(0; r, 0)}\right| \\
&\qquad \leq \frac{d(\varphi; r, 0)|b(\varphi; r, 0) - b(0; r, 0)| + b(\varphi; r, 0)|d(\varphi; r, 0) - d(0; r, 0)|}{d(\varphi; r, 0)d(0; r, 0)} \\
&\qquad \leq \left(\frac{5Kr^{\mu}}{\kappa r^{\mu}/2} + \frac{3Kr^{\mu} \cdot 5Kr^{\mu}}{(\kappa r^{\mu}/2)^2}\right)\varphi = \left(\frac{10K}{\kappa} + \frac{60K^2}{\kappa^2}\right)\varphi.
\end{aligned}$$

The verification of all conditions of Theorem 6.1 is completed. Therefore Theorem 3.2 is proved.

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