

BIFURCATION OF FREDHOLM MAPS I. THE INDEX BUNDLE AND BIFURCATION

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ABSTRACT. We associate to a parametrized family f of nonlinear Fredholm maps possessing a trivial branch of zeroes an *index of bifurcation* $\beta(f)$ which provides an algebraic measure for the number of bifurcation points from the trivial branch. The index $\beta(f)$ is derived from the index bundle of the linearization of the family along the trivial branch by means of the generalized J -homomorphism. Using the Agranovich reduction and a cohomological form of the Atiyah–Singer family index theorem, due to Fedosov, we compute the bifurcation index of a multiparameter family of nonlinear elliptic boundary value problems from the principal symbol of the linearization along the trivial branch. In this way we obtain criteria for bifurcation of solutions of nonlinear elliptic equations which cannot be achieved using the classical Lyapunov–Schmidt method.

1. Introduction and statements of the main results

1.1. Introduction. The main purpose of the article is to present a comprehensive account of the relationship between elliptic topology and bifurcation theory. More precisely, between the index bundle of a family of linear Fredholm operators and bifurcation of solutions of nonlinear elliptic equations from a trivial branch.

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Bifurcation from a trivial branch is one of the oldest notions of bifurcation in mathematics. Roughly speaking, the scheme is as follows: assuming that there is a known (trivial) branch of solutions of a parametrized family of problems, find necessary and sufficient conditions for the appearance of nontrivial solutions arbitrary close to some points (called bifurcation points) of the trivial branch. The above framework arises in several fields belonging to pure and applied mathematics, which explains the interest in the formulation of a structured theory going beyond a collection of examples.

Although the first studies of specific bifurcation phenomena can be traced back to Euler and Jacobi, bifurcation theory was born with Poincaré as a special chapter of his qualitative theory of dynamical systems. The most important tool for the analysis of bifurcation from a trivial branch is the Lyapunov–Schmidt reduction, which leads a given bifurcation problem for integral and differential equations to a locally equivalent problem for a finite number of nonlinear equations in a finite number of indeterminates.

Bifurcation can arise only at singular points of the linearization at the trivial branch, i.e. points belonging to the trivial branch at which the linearized operator in the normal direction to the branch fails to be invertible. One of the typical assumptions of the Lyapunov–Schmidt method is that singular points are isolated. Assuming this, there is a large variety of methods which, combined with the Lyapunov–Schmidt reduction, provide criteria for the appearance of nontrivial solutions close to the singular point [19], [43], [58], [24], [40].

The choice of the approach depends on the nature of the problem at hand. However, the most popular ones use either the singularity theory or topological methods. In the first case, whether the point under consideration is a bifurcation point or not, is solved investigating higher order jets of the reduced map. In the topological approach, particularly useful in the several parameter case, the presence of bifurcation is determined from topological invariants, described in Section 3.3. The books [61], [22], [35], [55] are only few of the several possible references to the first method. J. Ize’s Ph.D. thesis [38], his review [39] together with [5], [12] provide a good introduction to the second one.

In this paper we will consider bifurcation of parametrized families of Fredholm maps from a topological viewpoint which is different from the well established method mentioned above. We will not make any assumption about the nature of singular points of the linearization but we will heavily rely on the nontrivial topology of the parameter space. More precisely, we will look for homotopy invariants of the family of linearizations at points of the trivial branch whose non-vanishing entails the presence of at least one bifurcation point.

It should be noted that invariants of this type exist because the homotopy groups of the space of linear Fredholm operators between infinite dimensional

Banach spaces are nontrivial. Thus, our theory is strongly tied to homotopy theory of Fredholm operators, i.e. elliptic topology.

On the other hand, it complements the local point of view developed by J. C. Alexander and J. Ize providing criteria for bifurcation that are different from the ones that can be obtained using the Lyapunov–Schmidt reduction.

To some extent, our approach was inspired by the successful use of elliptic invariants in handling various linear PDE problems in geometry and analysis. For example, in [36] the index bundle for families was used with the purpose to find Riemannian manifolds such that the dimension of the space of harmonic spinors varies with the metric.

In [62] the same method was applied to determine spectral gaps of Dirac operators. Several generalizations of Lichnerowicz’s theorem relating the A -genus of a spin manifold to the non existence of a metric with positive scalar curvature are rooted on similar arguments. Their basic idea is to evaluate the index bundle of the relevant family of linear Fredholm operators of index 0 using family index theorems. If the index bundle is nontrivial, then $\text{Ker } L_\lambda \neq \{0\}$, for at least one value of the parameter λ . What we will show in this paper is that the above argument works for nonlinear Fredholm maps as well, but at the cost of introducing one extra tool: the generalized J -homomorphism.

Our goals are:

(1) Given a family $\{f_\lambda\}_{\lambda \in \Lambda}$ of C^1 -Fredholm maps depending continuously on a parameter belonging to a finite CW-complex Λ such that $f_\lambda(0) = 0$ for all $\lambda \in \Lambda$, we will define an *index of bifurcation points* $\beta(f)$ which, much in the same way as the Lefschetz number in fixed-point theory, provides an algebraic measure of the total number of bifurcation points of the family f . The index $\beta(f)$ takes values in a finite group $J(\Lambda)$. It only depends on the homotopy class of the family $\{L_\lambda = Df_\lambda(0) : \lambda \in \Lambda\}$ of linearizations of f at points of the trivial branch. In particular, when f is defined by a family of nonlinear elliptic differential operators, $\beta(f)$ depends only on the coefficients of leading terms of the linearization.

(2) We will introduce a local index of bifurcation $\beta(f, U)$, analogous to the local fixed-point index, which interpolates between $\beta(f)$ and the index at an isolated point derived from the Alexander–Ize bifurcation invariant. It is defined only if L_λ is invertible for λ outside of a compact subset of U and preserved by homotopies of this type. In the case of nonlinear elliptic differential operators, in general, $\beta(f, U)$ depends on lower order terms of the linearized equations as well.

(3) For particular families of nonlinear elliptic boundary value problems parametrized by \mathbb{R}^q we will compute the index of bifurcation from the principal symbol of the linearization along the trivial branch using the Agranovich

reduction, Atiyah–Singer family index theorem and known results about the generalized J -homomorphism. In this way we will obtain sufficient conditions for the existence of nontrivial solutions bifurcating from the trivial branch for nonlinear elliptic problems with general boundary conditions of Shapiro–Lopatinskiĭ type. Finally, using the local index, we will obtain conditions for the existence of multiple bifurcation points.

For families parametrized by \mathbb{R}^q the results are particularly striking. While the proofs involve some amount of algebraic topology, the complete knowledge of the J -groups of spheres and Fedosov’s formula for the Chern character of the index bundle allows to state our main bifurcation result, Theorem 1.4.1, in terms of divisibility of a number computed as an integral of a differential form constructed explicitly from the principal symbol of the linearization at the trivial branch.

Let us remark that due to the invariance of $\beta(f)$ under lower order perturbations, its nonvanishing provides stronger bifurcation results than the ones obtained using the classical approach, which always need some knowledge of the solutions of the linearized equations. On the negative side one can say that, precisely for the same reasons, $\beta(f)$ frequently vanishes. For instance, when the leading coefficients of the linearization do not depend on the parameter. In this case one has to resort to the local index in order to detect bifurcation points. Pushing the analogy with the fixed-point theory one step further, the role of the Atiyah–Singer formula in our theory is reminiscent of the role of the Lefschetz–Hopf formula there.

In the case of semilinear Fredholm maps the proof of the main abstract result, Theorem 1.2.1 is simpler, and was sketched in [49]. Simple examples of a direct calculation of the bifurcation index from the data of the problem, using elementary family index theorems, can be found in [32] and [52]. The first deals with nonlinear Sturm–Liouville problems while the second studies bifurcation of homoclinic orbits.

Here for the first time we deal with general nonlinear Fredholm maps and use the Atiyah–Singer theorem in order to compute the bifurcation index of a large family of elliptic boundary value problems with general boundary conditions. Hence, we will keep the presentation as complete and self-contained as possible. Taking into account the mixed nature of the subject, we will carefully introduce the terminology used in the paper and prove most of the assertions. Some of our results from Chapters 2 and 3 were announced without proof in [51].

The paper is structured as follows: precise statements of the results concerning item (i), $1 \leq i \leq 3$, of the above list are formulated in subsection 1.($i + 1$) of this section and proved together with some generalizations and corollaries in section $i + 1$ with the same title. Subsection 1.5 contains several comments to

related work and eventual further developments. There are three appendices. In the first we sketch out the proof of standard properties of the index bundle. The second reviews some well known results about Fredholm properties of maps induced on Hardy–Sobolev spaces by linear and nonlinear elliptic operators. The third is devoted to Fedosov’s formula for the Chern character of the index bundle of a family of elliptic pseudo-differential operators.

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1.2. Index bundle and the index of bifurcation points. Let X, Y be real Banach spaces, O be an open subset of X , and let Λ be a finite connected CW-complex. A *family of C^n -maps*, $0 \leq n \leq \infty$, *continuously parametrized by Λ* is a continuous map $f: \Lambda \times O \rightarrow Y$ such that for each $\lambda \in \Lambda$ the map $f_\lambda: O \rightarrow Y$ defined by $f_\lambda(x) = f(\lambda, x)$ is C^n and, for all $k \leq n$, the k -th derivative of f in direction x , $D_x^k f: \Lambda \times O \rightarrow L^k(X, Y)$, is continuous in norm topology of the space of $L^k(X, Y)$ of k -forms on X with values in Y .

Parametrized families of C^n -maps are a particular case of fiberwise C^n -maps, i.e. morphisms in the category of C^n -Banach manifolds over Λ (see [23], [11]). While most of our arguments have a very natural extension to this category, some problems arise related to infinite dimensional structure groups. Hence, we will consider here only the product case $\Lambda \times O$.

We will deal mainly with families of C^1 -Fredholm maps of index 0, which means that $Df_\lambda(x)$ is a Fredholm operator of index 0 for all $(\lambda, x) \in \Lambda \times O$. We will further assume everywhere in this paper that O is an open neighbourhood of the origin and that $f(\lambda, 0) = 0$ for all λ in Λ . Solutions of the equation $f(\lambda, x) = 0$ of the form $(\lambda, 0)$ are called *trivial*. The set $T = \Lambda \times \{0\}$ is called the *trivial branch*. As a rule we will identify the trivial branch with the parameter space Λ .

DEFINITION 1.2.1. A *bifurcation point from the trivial branch* of solutions of the equation $f(\lambda, x) = 0$ is a point $\lambda_* \in \Lambda$ such that every neighbourhood of $(\lambda_*, 0)$ contains nontrivial solutions of this equation.

In what follows, we will denote with $\mathcal{L}(X, Y)$ the Banach space of all bounded operators from X to Y , with $\Phi(X, Y)$ (resp. $\Phi_k(X, Y)$) the open subspace of all Fredholm operators (resp. those of index k).

The *linearization of the family f along the trivial branch* is the family of operators $L: \Lambda \rightarrow \Phi_0(X, Y)$, where $L_\lambda = Df_\lambda(0)$ is the Frechet derivative of f_λ at 0.

Bifurcation can only occur at singular points of the linearization, i.e. the points $\lambda \in \Lambda$ such that $\text{Ker } L_\lambda \neq 0$. When Λ is a smooth manifold and f is C^1 the necessity of this condition follows immediately from the implicit function

theorem. It holds in our slightly more general framework too. Indeed, in a small enough neighbourhood of a point ν such that L_ν is nonsingular, the equation $f(\lambda, x) = 0$ is equivalent to $x = L_\lambda^{-1}g(\lambda, x)$ where $g(\lambda, x) = f(\lambda, x) - L_\lambda x$. Since $g(\lambda, x) = o(\|x\|)$, by the uniqueness of the fixed point of a contraction the only solutions close to $(\nu, 0)$ are the trivial ones.

While necessary, the above condition is not sufficient for the appearance of nontrivial solutions close to the given point of the trivial branch. Hence, in general, the set $\text{Bif}(f)$ of all bifurcation points of a family f is only a proper closed subset of the set $\Sigma(L)$ of all singular points of the linearization L along the trivial branch. The purpose of the *linearized bifurcation theory* is to obtain sufficient conditions for the existence of bifurcation points of f in terms of the linearization L .

Since bifurcation arises only at points of $\Sigma(L)$, the first topological invariant that comes to mind is the obstruction to deformation of L into a family without singular points. It is well known that such an obstruction is given by an element of the reduced Grothendieck group of virtual vector bundles $\widetilde{\text{KO}}(\Lambda)$, called *family index* or *index bundle* [7], [41] and denoted with $\text{Ind } L$. However, since we are dealing with nonlinear perturbations of L , we have to take into account the generalized J -homomorphism $J: \widetilde{\text{KO}}(\Lambda) \rightarrow J(\Lambda)$ which associates to each vector bundle the stable fiberwise homotopy class of its unit sphere bundle.

Quite naturally, our bifurcation invariant is not $\text{Ind } L$ but rather its image $J(\text{Ind } L) \in J(\Lambda)$ under the generalized J -homomorphism. In fact, we have:

THEOREM 1.2.1. *Let $f: \Lambda \times O \rightarrow Y$ be a family of C^1 -Fredholm maps of index 0 parametrized by a connected finite CW-complex Λ , such that $f(\lambda, 0) = 0$. If $\Sigma(L)$ is a proper subset of Λ and $\beta(f) = J(\text{Ind } L) \neq 0$, then the family f possesses at least one bifurcation point from the trivial branch.*

The Stiefel–Whitney characteristic class $\omega(E) = 1 + \omega_1(E) + \dots, \omega_i(E) \in H^i(\Lambda; \mathbb{Z}_2)$ of a vector bundle E over Λ is invariant under addition of trivial bundles and hence it is well defined on $\widetilde{\text{KO}}(\Lambda)$. Moreover, it factorizes through $J(\Lambda)$ because, by Thom’s construction, it only depends on the stable fiberwise homotopy class of the associated sphere bundle. If p is an odd prime, the same holds for the total Wu class $q(E) = 1 + q_1(E) + \dots, q_i(E) \in H^{2(p-1)i}(\Lambda; \mathbb{Z}_p)$ [46]. In particular:

COROLLARY 1.2.2. *Let f and $\Sigma(L)$ be as in the above theorem. Then bifurcation arises if either $\omega(\text{Ind } L) \neq 1$ or $q(\text{Ind } L) \neq 1$ for some odd prime p .*

The nonvanishing of characteristic classes of the index bundle of positive degree not only entails bifurcation but also gives some information about the size of the set $\text{Bif}(f)$ of bifurcation points of f and its position in the parameter space. We will study this in a companion paper [53].

REMARK 1.2.1. The assumption $\Sigma(L) \neq \Lambda$ can be relaxed (see Section 2.4). However, it is easy to see that nonvanishing of $J(\text{Ind } L)$ only, does not imply by itself the existence of a bifurcation point.

For example, take a family L of Fredholm operators between Hilbert spaces whose kernels define a nonorientable bundle $\text{Ker } L$ over Λ and such that $\text{coker } L$ is a trivial vector bundle. Families of ordinary differential operators with this property can be found in [52] and [32]. By the above corollary, $J(\text{Ind } L) \neq 0$. Let Q and Q' be projectors on $\text{Ker } L$ and $\text{Im } L$ respectively, and let s be a nowhere vanishing section of $F = (\text{id} - Q')X \simeq \text{coker } L$. Define the family f by

$$f(\lambda, x) = L_\lambda x + \|Q_\lambda x\|^2 s(\lambda).$$

Then the linearization of f at the trivial branch is L but f has no bifurcation points.

1.3. A local index of bifurcation. Let U open subset of Λ , $f: U \times O \rightarrow Y$ be a family of C^1 -Fredholm maps parametrized by U such that $f(\lambda, 0) = 0$ and let L be the linearization of f along the trivial branch. A pair (f, U) is called *admissible* if the singular set $\Sigma(L)$ is a compact, proper subset of U . An admissible homotopy is a family of C^1 -Fredholm maps parametrized by $[0, 1] \times U$ such that the set

$$\Sigma(Dh) = \{(t, \lambda) / Dh_{(t, \lambda)}(0) \text{ is singular}\}$$

is a compact subset of $[0, 1] \times U$ and $\Sigma(Dh_i)$, $i = 0, 1$ are proper subsets of U .

Let us recall that a Kuiper space is Banach space Y such that the subspace $\text{GL}(Y)$ of all invertible operators in $\mathcal{L}(Y)$ is contractible.

The main result in Section 3 is:

THEOREM 1.3.1. *Assume that Y is a Kuiper space. There exists a local index of bifurcation which assigns to each admissible pair (f, U) an element*

$$\beta(f, U) \in J(\Lambda)$$

verifying the following properties:

- (B₁) (Existence) *If $\beta(f, U) \neq 0$, then the family f has a bifurcation point in U .*
- (B₂) (Normalization) $\beta(f, \Lambda) = \beta(f) = J(\text{Ind } L)$.
- (B₃) (Homotopy invariance) *If h is an admissible homotopy, then*

$$\beta(h_0, U) = \beta(h_1, U).$$

- (B₄) (Additivity) *Let (f, U) be admissible with $U \subset \bigcup U_i$. Put $\Sigma_i = \Sigma(f) \cap U_i$ and $f_i = f|_{U_i}$. If $\Sigma_i \cap \Sigma_j = \emptyset$ and $\bigcup \Sigma_i = \Sigma(f)$, then (f_i, U_i) are*

admissible and

$$\beta(f, U) = \sum_i \beta(f_i, U_i).$$

(B₅) (Change of parameters) *Let $\alpha: \Lambda' \rightarrow \Lambda$ be a continuous map. Let U be an open subset of Λ such that the pair (f, U) is admissible. If $U' = \alpha^{-1}(U)$ and $g: U' \times O \rightarrow Y$ is defined by $g(\lambda', x) = f(\alpha(\lambda'), x)$, then (g, U') is admissible and $\beta(g, U') = \alpha^* \beta(f, U)$, where $\alpha^*: J(\Lambda) \rightarrow J(\Lambda')$ is the homomorphism induced by α in J -groups.*

(B₆) (Isolated points) *Let λ_0 be an isolated point in $\Sigma(L)$. Assume that there exists a neighbourhood U of λ_0 homeomorphic to \mathbb{R}^n such that $\Sigma(L) \cap U = \{\lambda_0\}$. Then, identifying $J(S^n)$ with image of the stable j -homomorphism $j: \pi_{n-1} \text{GL}(\infty) \rightarrow \pi_{n-1}^s$, we have:*

$$\beta(f, U) = q^* j(\gamma_f).$$

Here γ_f is the Alexander–Ize invariant (see Section 3.3), S^n is identified with the one-point compactification U^+ of U , and $q: \Lambda \rightarrow U^+$ is the map collapsing $\Lambda - U$ to the point at infinity.

REMARK 1.3.1. A special case of (B₄) is the excision property: if (f, U) is admissible and $\Sigma \subset V \subset U$, then $\beta(f, U) = \beta(f|_V, V)$. It follows from this and (B₃) that $\beta(f, U)$ depends only on the germ of the family of linearizations $L_\lambda = Df_\lambda(0)$ at Σ .

Few words have to be said about the computation of $J(\Lambda)$ since the bifurcation index takes values in this group. $J(S^q)$ has been completely determined in the seventies [5], [37]. We will use this computation in the next subsection. In order to state the result, let $\nu_p(s)$ denote the exponent to which the prime p occurs in the prime decomposition of an integral number s . Consider the number-theoretic function m constructed as follows: the value $m(s)$ is defined through its prime decomposition by setting for $p = 2$, $\nu_2(m(s)) = 2 + \nu_2(s)$ if $s \equiv 0 \pmod{2}$ and $\nu_2(m(s)) = 1$ if the opposite is true. While, if p is an odd prime, then $\nu_p(m(s)) = 1 + \nu_p(s)$ if $s \equiv 0 \pmod{p-1}$ and 0 in the remaining cases. In particular $m(s)$ is always even. With this said, $J(S^q) = \mathbb{Z}_2$ for $q \equiv 1$ or $2 \pmod{8}$, $J(S^q) = \mathbb{Z}_{m(2s)}$ for $q = 4s$, and is trivial in the remaining cases.

The numbers $m(s)$ have a wide range of distribution (see for example [5]). However, what is important for us is that the index of bifurcation $\beta(f, U)$ is an integral \pmod{m} in the case $\Lambda = S^q$. The same holds true for $\Lambda = \mathbb{R}P^q$, the real projective space. For a finite CW-complex Λ without two-torsion in homology the order of $J(\Lambda)$ can be estimated in terms of the homology of Λ with coefficients in $J(S^q)$.

1.4. Bifurcation of solutions of nonlinear elliptic BVP. In Theorem 1.4.1 below we will state criteria for bifurcation of solutions of nonlinear elliptic boundary value problems in terms of the coefficients of the top order derivatives of linearized equations. In Theorem 1.4.2 we will consider the existence of multiple bifurcation points.

Let Ω be an open bounded subset of \mathbb{R}^n with smooth boundary $\partial\Omega$. Referring to the Appendix B for the notations, we will consider nonlinear boundary value problems of the form

$$(1.1) \quad \begin{cases} \mathcal{F}(\lambda, x, u, \dots, D^k u) = 0 & \text{for } x \in \Omega, \\ \mathcal{G}^i(\lambda, x, u, \dots, D^{k_i} u) = 0 & \text{for } x \in \partial\Omega, 1 \leq i \leq r. \end{cases}$$

Here, $u: \bar{\Omega} \rightarrow \mathbb{R}^m$ is a vector function, $\lambda \in \mathbb{R}^q$ is a parameter and, denoting with k^* the number of α 's such that $|\alpha| \leq k$,

$$\mathcal{F}: \mathbb{R}^q \times \bar{\Omega} \times \mathbb{R}^{mk^*} \rightarrow \mathbb{R}^m \quad \text{and} \quad \mathcal{G}^i: \mathbb{R}^q \times \bar{\Omega} \times \mathbb{R}^{mk_i^*} \rightarrow \mathbb{R}$$

are smooth with $\mathcal{F}(\lambda, x, 0) = 0$, $\mathcal{G}^i(\lambda, x, 0) = 0$, $1 \leq i \leq r$.

We also denote with \mathcal{F} the family of nonlinear differential operators

$$\mathcal{F}: \mathbb{R}^q \times C^\infty(\bar{\Omega}; \mathbb{R}^m) \rightarrow C^\infty(\bar{\Omega}; \mathbb{R}^m)$$

induced by the map \mathcal{F} .

The functions \mathcal{G}^i define a family of nonlinear boundary operators

$$\mathcal{G}: \mathbb{R}^q \times C^\infty(\bar{\Omega}; \mathbb{R}^m) \rightarrow C^\infty(\partial\Omega; \mathbb{R}^r),$$

$$\mathcal{G}(\lambda, x, u, \dots, D^k u) = (\tau\mathcal{G}^1(\lambda, x, u, \dots, D^{k_1} u), \dots, \tau\mathcal{G}^r(\lambda, x, u, \dots, D^{k_r} u)),$$

where τ is the restriction to the boundary.

We assume:

(H₁) For all $\lambda \in \mathbb{R}^q$, the linearization $(\mathcal{L}_\lambda(x, D), \mathcal{B}_\lambda(x, D))$ of $(\mathcal{F}_\lambda, \mathcal{G}_\lambda)$ at $u \equiv 0$, is an elliptic boundary value problem in the sense of Definition 5.2 in Appendix B.

(H₂) The coefficients $a_\alpha^{ij}, b_\alpha^{ij}$ of the linearization $(\mathcal{L}, \mathcal{B})$ extend to smooth functions defined on $S^q \times \bar{\Omega}$, where $S^q = \mathbb{R}^q \cup \{\infty\}$ is the one point compactification of \mathbb{R}^q . Moreover, the problem:

$$\begin{cases} \mathcal{L}_\infty(x, D)u(x) = \sum_{|\alpha| \leq k} a_\alpha(\infty, x)D^\alpha u(x) = f(x) & \text{for } x \in \Omega, \\ \mathcal{B}_\infty^i(x, D)u(x) = \sum_{|\alpha| \leq k_i} b_\alpha^i(\infty, x)D^\alpha u(x) = g(x) & \text{for } x \in \partial\Omega, 1 \leq i \leq r, \end{cases}$$

is elliptic and has a unique solution for every $f \in C^\infty(\bar{\Omega}; \mathbb{R}^m)$ and every $g \in C^\infty(\partial\Omega; \mathbb{R}^r)$.

(H₃) (i) The coefficients $b_\alpha^{ij}(x)$, $|\alpha| = k_i$, $1 \leq i \leq r$, of the leading terms of $\mathcal{B}(\lambda, x, D)$ are independent of λ .

- (ii) There exist a compact set $K \subset \Omega$ such that the coefficients $a_\alpha^{ij}(\lambda, x)$, $|\alpha| = k$ of the leading terms of $\mathcal{L}_\lambda(x, D)$ are independent of λ for $x \in \overline{\Omega} - K$.

Let us give a closer look to our assumptions. Since linear elliptic boundary value problems induce Fredholm operators on function spaces, (H_1) places the problem (1.1) in the framework of our abstract bifurcation theory applied to a family of nonlinear Fredholm maps f . The assumption (H_2) allows us to compute the local bifurcation index $\beta(f, \mathbb{R}^n)$ from the index bundle of the extended family. Finally, (H_3) is essential in order to carry out the Agranovich reduction showing that $\text{Ind } L$ coincides with the index bundle of a family \mathcal{S} of pseudo-differential operators whose principal symbol is the matrix function σ defined in (1.2) below.

Let $p(\lambda, x, \xi) \equiv \sum_{|\alpha|=k} a_\alpha(\lambda, x)\xi^\alpha$ be the principal symbol of \mathcal{L}_λ . Since the symbol is defined in terms of $D_j = -i\partial/\partial x_j$, $p(\lambda, x, \xi)$ is a complex matrix which verifies the *reality condition* $p(\lambda, x, -\xi) = \bar{p}(\lambda, x, \xi)$.

By ellipticity, $p(\lambda, x, \xi) \in \text{GL}(m; \mathbb{C})$ if $\xi \neq 0$. On the other hand, by (H_3) ,

$$p(\lambda, x, \xi) = p(\infty, x, \xi) \quad \text{for } x \in \overline{\Omega} - K.$$

Putting

$$\sigma(\lambda, x, \xi) = \text{id} \quad \text{for any } (\lambda, x, \xi) \text{ with } x \notin K,$$

the map $\sigma(\lambda, x, \xi) = p(\lambda, x, \xi)p(\infty, x, \xi)^{-1}$ extends to a smooth map

$$(1.2) \quad \sigma: S^q \times (\mathbb{R}^{2n} - K \times \{0\}) \rightarrow \text{GL}(m; \mathbb{C}).$$

Our bifurcation criteria will be formulated in terms of the map σ . In order to state our results we will need matrix-valued differential forms. The product of two matrices of this type is defined in the usual way, with the product of coefficients given by the wedge product of forms. The matrix of differentials $(d\sigma_{ij})$ will be denoted by $d\sigma$.

We associate to the $\text{GL}(m; \mathbb{C})$ -valued function σ of (1.2) the one form

$$\sigma^{-1}d\sigma \text{ defined on } S^q \times (\mathbb{R}^{2n} - K \times \{0\}).$$

Without loss of generality we can assume that $K \times \{0\}$ is contained in the unit ball $B^{2n} \subset \mathbb{R}^{2n}$ so that the one form $\sigma^{-1}d\sigma$ restricts (pullbacks) to the well defined one form on $S^q \times S^{2n-1}$ which will be denoted in the same way. Taking the trace of the $(q+2n-1)$ -th power of the matrix $\sigma^{-1}d\sigma$ we obtain an ordinary $(q+2n-1)$ -form $\text{tr}(\sigma^{-1}d\sigma)^{q+2n-1}$ on $S^q \times S^{2n-1}$.

For q even, we define the *degree* $d(\sigma)$ of the matrix function σ by

$$(1.3) \quad d(\sigma) = \frac{(q/2 + n - 1)!}{(2\pi i)^{(q/2+n)}(q+2n-1)!} \int_{S^q \times S^{2n-1}} \text{tr}(\sigma^{-1}d\sigma)^{q+2n-1}.$$

Proposition 5.5 in Appendix C and the integrality of the Chern character [37, Chapter 18, Theorem 9.6] imply that $d(\sigma) \in \mathbb{Z}$.

DEFINITION 1.4.1. A bifurcation point from the trivial branch for solutions of (1.1) is a point $\lambda_* \in \mathbb{R}^q$ such that there exist a sequence $(\lambda_n, u_n) \in \Lambda \times C^\infty(\overline{\Omega})$ of solutions of (1.1) with $u_n \neq 0$, $\lambda_n \rightarrow \lambda_*$ and $u_n \rightarrow 0$ uniformly with all of its derivatives.

THEOREM 1.4.1. *Let the problem*

$$(1.4) \quad \begin{cases} \mathcal{F}(\lambda, x, u, \dots, D^k u) = 0 & \text{for } x \in \Omega, \\ \mathcal{G}^i(\lambda, x, u, \dots, D^{k_i} u) = 0 & \text{for } x \in \partial\Omega, 1 \leq i \leq r, \end{cases}$$

verify assumptions (H₁), (H₂) and (H₃). If $q \equiv 0, 4 \pmod{8}$, there exists at least one bifurcation point from the trivial branch of solutions provided that $d(\sigma)$ is not divisible by $n(q)$, where

$$(1.5) \quad n(q) = \begin{cases} m(q/2) & \text{if } q \equiv 0 \pmod{8}, \\ 2m(q/2) & \text{if } q \equiv 4 \pmod{8} \end{cases}$$

and m is the number theoretic function defined at the end of Section 1.3.

Theorem 1.4.1 is stronger than the usual bifurcation results. Any lower order perturbation

$$(1.6) \quad \begin{cases} \mathcal{F}(\lambda, x, u, \dots, D^k u) + \mathcal{F}'(\lambda, x, u, \dots, D^{k-1} u) = 0 & \text{for } x \in \Omega, \\ \mathcal{G}^i(\lambda, x, u, \dots, D^{k_i} u) + \mathcal{G}'^i(\lambda, x, u, \dots, D^{k_i-1} u) = 0 & \text{for } x \in \partial\Omega, \\ & 1 \leq i \leq r, \end{cases}$$

of (1.4) with $\mathcal{F}'(\lambda, 0) = 0$, $\mathcal{G}'^i(\lambda, 0) = 0$ and such that the coefficients of the linearization of $(\mathcal{F}', \mathcal{G}')$ converge uniformly to 0 as $\lambda \rightarrow \infty$, also verifies the assumptions (H₁)–(H₃). Therefore, if $d(\sigma)$ is not divisible by $n(q)$, there must be some bifurcation point $\lambda \in \mathbb{R}^q$ for any lower order perturbation (1.6) as above.

REMARK 1.4.1. The definition of the degree of σ using differential forms explains why we have assumed that \mathcal{F} and \mathcal{G} are smooth in all of its arguments including parameters. For continuous families of linear elliptic equations with smooth coefficients, the degree of the symbol is still defined (it is called Bott's degree in [9]) but it lacks of an explicit expression like the integral formula (1.3), which is due to Fedosov. One can still formulate the above theorem in terms of Bott's degree. However, its calculation in general requires a deformation of the symbol to a simpler form.

Now, let us consider the existence of multiple bifurcation points.

Putting $H^+(\partial\Omega; \mathbb{R}^r) = \prod_{i=1}^r H^{k+s-k_i-1/2}(\partial\Omega; \mathbb{R})$, it is shown in Section 3.1 that, under the assumptions (H₁) and (H₂), the map $(\mathcal{F}, \mathcal{G})$ extends to a smooth

q -parameter family of Fredholm maps of index 0 between Hardy-Sobolev spaces:

$$(1.7) \quad h = (f, g): \mathbb{R}^q \times H^{k+s}(\Omega; \mathbb{R}^m) \rightarrow H^s(\Omega; \mathbb{C}^m) \times H^+(\partial\Omega; \mathbb{C}^r)$$

having $\mathbb{R}^q \times \{0\}$ as a trivial branch. Moreover, the Frechet derivative $Dh_\lambda(0)$ is the operator $(L_\lambda, B_\lambda): H^{k+s}(\Omega; \mathbb{R}^m) \rightarrow H^s(\Omega; \mathbb{C}^m) \times H^+(\partial\Omega; \mathbb{C}^r)$ induced by $(\mathcal{L}_\lambda, \mathcal{B}_\lambda)$.

Let $\lambda_0 \in \Sigma(L, B)$ be an isolated singular point of (L, B) . We will formulate our local bifurcation result in terms of the matrix function $R: S^{q-1} \rightarrow \text{Gl}(l; \mathbb{R})$, where $l = \dim \text{Ker } L_{\lambda_0}$, defined as follows: take a small enough closed disk D such that $D \cap \Sigma(L, B) = \{\lambda_0\}$. Then R is defined as the restriction of the linearization of the Lyapunov-Schmidt reduction of h on a neighbourhood of D to the boundary $\partial D \simeq S^{q-1}$ (see (3.17) in Section 3.3). Since R is smooth, we can consider as before the matrix differential form $R^{-1}dR$. For $q = 4s$, we define the degree $d(\lambda_0)$ of an isolated singular point $\lambda_0 \in \Sigma(L, B)$ by:

$$(1.8) \quad d(\lambda_0) = (-1)^{s+1} \frac{(2s-1)!}{(2\pi)^{2s}(4s-1)!} \int_{S^{4s-1}} \text{tr}(R^{-1}dR)^{4s-1}.$$

Much as before, by (4.19), $d(\lambda_0) \in \mathbb{Z}$.

THEOREM 1.4.2. *Let the problem (1.1) verify the assumptions (H_1) – (H_3) of Theorem 1.4.1.*

- (a) *If $\Sigma(L, B)$ consists only of isolated points, then they are finite in number, say $\{\lambda_0, \dots, \lambda_r\}$, and*

$$(1.9) \quad d(\sigma) = \sum_{i=0}^r d(\lambda_i).$$

- (b) *If λ_0 is an isolated singular point of (L, B) and $d(\lambda_0)$ is not divisible by $n(q)$, then λ_0 is a bifurcation point for solutions of (1.1). If moreover, $d(\lambda_0) \not\equiv d(\sigma) \pmod{n(q)}$, then there must be a second bifurcation point λ_* for solutions of (1.1) different from λ_0 .*

In particular there are at least two bifurcation points if $d(\lambda_0) \not\equiv 0 \pmod{n(q)}$ and either σ is independent from λ or $\sigma = \sigma^*$ or $\sigma + \sigma^*$ is a positive definite matrix.

This can be seen as follows: let S be any family of pseudo-differential operators whose principal symbol is σ . By (4.16) and (4.17), $d(\sigma)$ coincides with the evaluation of the Chern character of $\text{Ind } S$ on the fundamental class of the sphere S^q . But in all of the above cases the index bundle of S vanishes.

In the first case this is clear. In the second case, let S' be a family self-adjoint operators with principal symbol σ (it is enough to take $S' = 1/2(S + S^*)$). Then $\text{Ind } S = \text{Ind } S' = 0$, because S' is homotopic to a family of invertible operators

$S' + i \text{id}$ via the homotopy $H_t = S' + it \text{id}$. A similar homotopy leads to the same conclusion in the third case, using Garding's inequality.

REMARK 1.4.2. Let us point out that, except for the one-parameter case, the property of having isolated bifurcation points is far from being generic [53].

1.5. Comments. Our results leave many related questions open.

(a) Perhaps the most interesting one is that of *global bifurcation* which predicts the behavior of the bifurcating branch at large. Regarding this, the state of affairs is as follows: the Krasnosel'skiĭ–Rabinowitz Global Bifurcation Theorem was proved for general one-parameter families of Fredholm maps using the base-point degree in [34], [54]. Results for particular classes of Fredholm mappings of index 0 arising from nonlinear elliptic equations and systems are scattered around the literature. We mention [42], [56] among others. For a special class of bifurcation problems involving Fredholm maps a different method was developed by Zviagin in [64] (see also [65]) using a device due to J. Ize.

The extension of the Krasnosel'skiĭ–Rabinowitz theory to several-parameter families of compact perturbations of identity was carried out mainly by the work of J. C. Alexander and J. Ize. We cite here only [5], [39], [33] as a partial reference. The review paper [39] has a wide list of references for this topic. Global bifurcation for semilinear Fredholm maps was established by Bartsch in [13]. However, neither the methods of [13] nor the ones in [5] can be used for nonlinear Fredholm maps because very little is known regarding the extension properties of this class. This is particularly disappointing since the bifurcation invariant used in [34] for the proof of the global bifurcation theorem is a particular case of our bifurcation index $\beta(f, U)$. To be precise: taking $\Lambda = S^1$, viewed as one point compactification of the real line \mathbb{R} and $U = (a, b)$, under the isomorphism $J(S^1) \cong \mathbb{Z}_2$, the parity $\sigma(L, [a, b])$ used in [34] coincides with the local index of bifurcation points $\beta(f, U)$ considered here.

(b) Bifurcation from infinity also requires an improvement of our results. In the case of quasilinear Fredholm maps there is a better version of Theorem 1.2.1 which, in the presence of a priori bounds, relates the order of $J(\text{Ind } L)$ with the degree of the map f_λ [50]. This result permits to deal at the same time with bifurcation both from 0 and from infinity. However, the methods used here do not apply to the latter.

(c) As a consequence of the fact that our invariant depends only on the linearization of f at the points of the trivial branch we have to consider not only $\text{Bif}(f)$ but all of $\Sigma(f)$ in the formulation of the properties of the local bifurcation index. At a first glance this appears to be an unpleasant characteristic of our invariant since it would be preferable to deal with the set $\text{Bif}(f)$ only. Bartsch [14] defined a bifurcation index of this type for compact perturbations of identity parametrized by \mathbb{R}^n . It takes values in the stable homotopy group π_n^s . In [13]

his construction was extended to semilinear Fredholm maps. However, it is not clear how to construct an index of this type for general nonlinear maps.

On the other hand the above unpleasant characteristic is compensated by the fact that $\beta(f, U)$ lives in $J(\Lambda)$ which is computable in many cases. Indeed, π_n^s are still far from being completely understood while $J(S^n) \subset \pi_n^s$ is essentially the only known part of the stable stem.

(d) As we remarked before, one of the consequences of our theory is the relation between the nonvanishing of the Stiefel–Whitney classes of $\text{Ind } L$ and bifurcation. In [44] U. Koschorke defined characteristic classes of Fredholm morphisms between infinite-dimensional bundles. Koschorke’s classes are constructed as Poincaré duals of fundamental classes of subvarieties Σ_k whose elements are Fredholm operators (of index 0) with k -dimensional kernel. They are all computable from the Stiefel–Whitney classes of the index bundle. However, it is quite natural to ask whether Koschorke classes can be related to bifurcation in a direct way.

2. Index bundle and the index of bifurcation points

Theorem 1.2.1, is a special case of a slightly more general result which is a formula relating the order of $J(\text{Ind } L)$ in $J(\Lambda)$ with the local multiplicity of f_λ at 0. Before stating it, we must introduce three ingredients which appear in its formulation.

2.1. The index bundle. We shortly review the construction of the *index bundle* using a slightly different approach from the one in [7] which is better suited to deal with nonlinear operators. If Λ is a compact topological space, the Grothendieck group $\text{KO}(\Lambda)$ is the group completion of the abelian semigroup $\text{Vect}(\Lambda)$ of all isomorphism classes of real vector bundles over Λ . In other words, it is the quotient of the semigroup $\text{Vect}(\Lambda) \times \text{Vect}(\Lambda)$ by the diagonal sub-semigroup. The elements of $\text{KO}(\Lambda)$ are called virtual bundles. Each virtual bundle can be written as a difference $[E] - [F]$ where E, F are vector bundles over Λ and $[E]$ denotes the equivalence class of $(E, 0)$. Moreover, one can show that $[E] - [F] = 0$ in $\text{KO}(\Lambda)$ if and only if the two vector bundles become isomorphic after the addition of a trivial vector bundle to both sides. Taking complex vector bundles instead of the real ones leads to the complex Grothendieck group denoted by $K(\Lambda)$. In what follows the trivial bundle with fiber $\Lambda \times V$ will be denoted by $\Theta(V)$, $\Theta(\mathbb{R}^n)$ will be simplified to Θ^n .

Let X, Y be real Banach spaces and let $L: \Lambda \rightarrow \Phi(X, Y)$, be a continuous family of Fredholm operators. As before $L_\lambda \in \Phi(X, Y)$ will denote the value of L at the point $\lambda \in \Lambda$. Since $\text{coker } L_\lambda$ is finite dimensional, using compactness

of Λ , one can find a finite dimensional subspace V of Y such that

$$(2.1) \quad \text{Im } L_\lambda + V = Y \quad \text{for all } \lambda \in \Lambda.$$

Because of the transversality condition (2.1) the family of finite dimensional subspaces $E_\lambda = L_\lambda^{-1}(V)$ defines a vector bundle over Λ with total space

$$E = \bigcup_{\lambda \in \Lambda} \{\lambda\} \times E_\lambda.$$

Indeed, the kernels of a family of surjective Fredholm operators form a finite dimensional vector bundle [45]. Denoting with π the canonical projection of Y onto Y/V , from (2.1) it follows that operators πL_λ are surjective with

$$\text{Ker } \pi L_\lambda = E_\lambda,$$

which shows that $E \in \text{Vect}(\Lambda)$.

We define the *index bundle* $\text{Ind } L$ by:

$$(2.2) \quad \text{Ind } L = [E] - [\Theta(V)] \in \text{KO}(\Lambda).$$

If V_1 and V_2 are two subspaces verifying the transversality condition (2.1) and E, F are the corresponding vector bundles, we can suppose without loss of generality that $V_1 \subset V_2$ and hence that E is a subbundle of F . The restriction of the family L to F induces an isomorphism of F/E with the trivial bundle with fiber V_2/V_1 . Since exact sequences of vector bundles split, it follows that F is isomorphic to a direct sum of E with a trivial bundle and hence $E - \Theta(V_1)$ and $F - \Theta(V_2)$ define the same class in $\text{KO}(\Lambda)$. This shows that $\text{Ind } L$ is well defined.

The correspondence $L \mapsto \text{Ind } L$ is a natural transformation from $\pi[\cdot; \Phi(X, Y)]$ to $\text{KO}(\cdot)$ which enjoys the same homotopy invariance, additivity and logarithmic properties as the numerical index. The proofs of the above properties are sketched in Appendix A. Clearly $\text{Ind } L = 0$ if L is homotopic to a family of invertible operators.

The index bundle of a family of Fredholm operators of index 0, can be identified with the stable equivalence class of the vector bundle E arising in (2.2). Let us recall that two bundles are *stably equivalent* if they become isomorphic after addition of trivial bundles on both sides. Stable equivalence classes form a group isomorphic to the reduced Grothendieck group of Λ , i.e. the kernel $\widetilde{\text{KO}}(\Lambda)$ of the rank homomorphism $rk: \text{KO}(\cdot) \rightarrow \mathbb{Z}$. The isomorphism sends the equivalence class of F into $[F] - [\Theta^r]$ where $r = rk(F)$, [37, Theorem 3.8]. On the other hand, the index bundle of a family of Fredholm operators of index 0 belongs to $\widetilde{\text{KO}}(\Lambda)$.

2.2. J -homomorphism. Given a vector bundle E , let $S[E]$ be the associated unit sphere bundle with respect to some chosen scalar product on E . Two vector bundles E, F are said to be *stably fiberwise homotopy equivalent* if, for some n, m , (and any choice of metric) the unit sphere bundle $S(E \oplus \Theta^n)$ is fiberwise homotopy equivalent to the unit sphere bundle $S(F \oplus \Theta^m)$. Let $T(\Lambda)$ be the subgroup of $\widetilde{\text{KO}}(\Lambda)$ generated by elements $[E] - [F]$ such that E and F are stably fiberwise homotopy equivalent. Put $J(\Lambda) = \widetilde{\text{KO}}(\Lambda)/T(\Lambda)$. The projection to the quotient $J: \widetilde{\text{KO}}(\Lambda) \rightarrow J(\Lambda)$ is called *the generalized J -homomorphism*.

The group $J(\Lambda)$ was introduced by Atiyah in [8]. He proved that $J(\Lambda)$ is a finite group if Λ is a finite CW-complex by showing that $J(S^n)$ coincides with the image of the stable j -homomorphism of G. Whitehead (see Section 3.3 for details).

2.3. Parity and topological degree. The third ingredient needed in order to state our main theorem is an oriented degree theory for C^1 -Fredholm maps of index 0. The one that will be used here is the *base point degree* constructed in [54]. This construction parallels the classical approach to Brouwer degree based on regular value approximation, using an appropriate notion of orientation for Fredholm maps.

If y is a regular value of a proper differentiable map $f: \Omega \rightarrow \mathbb{R}^n$ defined on an open subset Ω of \mathbb{R}^n , Brouwer's degree of f on Ω is the integral number

$$\deg(f, \Omega, y) = \sum_{x \in f^{-1}(y)} \text{sgn det } Df(x).$$

In infinite dimensions sign of the Jacobian determinant does not exist and a useful substitute is given by the *parity* of a path of Fredholm operators of index 0 described below.

The singular set Σ of all non-invertible elements of $\Phi_0(X, Y)$ is a stratified analytic sub-variety of $\Phi_0(X, Y)$. Namely $\Sigma = \bigcup_{k \geq 1} \Sigma_k$, where each stratum

$$\Sigma_k = \{T \in \Phi_0(X, Y) / \dim \text{Ker } T = k\}$$

is an analytic submanifold of $\Phi_0(X, Y)$ of codimension k^2 . Using transversality, one can show that any continuous path γ in $\Phi_0(X, Y)$ can be arbitrarily approximated in norm by a smooth path $\tilde{\gamma}$ transversal to the strata Σ_k [31]. By dimension counting, a transversal path has no intersection with Σ_k for $k > 1$ and only a finite number of transversal intersection points with the one-codimensional stratum Σ_1 .

By definition, the *parity* of a path γ with non-singular end points is $\sigma(\gamma) = (-1)^m$ where m is the number of intersections with Σ_1 of a transversal path $\tilde{\gamma}$ close enough to γ . It is shown in [31] that the parity is well defined, it is multiplicative under concatenation of paths and invariant by homotopies which keep

end points of the path invertible. If the path is closed its parity is defined regardless of the invertibility of the end points and is invariant under free homotopies of closed paths.

Using the parity, the base point degree is defined as follows. Let O be a path connected open subset of X . A C^1 -Fredholm map $f: O \rightarrow X$ is said to be orientable if for any path γ joining two regular points of f the parity of the path $Df \circ \gamma$ depends only on the end points. A sufficient condition is that $\sigma(Df \circ \gamma) = 1$ for all closed paths in the domain. In particular, all Fredholm maps of index 0 with simply connected domain are orientable.

Let $f: O \rightarrow Y$, be an orientable Fredholm map and let Ω be any open subset of O such that the restriction of f to Ω is proper. If the set of regular points of f in O is nonempty, we choose a fixed regular point $b \in O$ (called base point) and define, for any regular value y , of the map f restricted to Ω ,

$$(2.3) \quad \deg_b(f, \Omega, y) = \sum_{x \in f^{-1}(y)} \varepsilon(x),$$

where $\varepsilon(x) = \sigma(Df \circ \gamma)$ and γ is any path joining b to x . By definition, maps without regular points have degree zero.

It was proved in [54] that that this assignment extends to an integral-valued degree theory for proper orientable C^1 -Fredholm maps of index 0. The degree is invariant under homotopies only up to sign and, as a matter of fact, there cannot be a homotopy invariant degree for general Fredholm maps extending the Leray–Schauder degree since the linear group of a Hilbert space is connected. However, what we will use here, is that the change in sign of the degree along a homotopy can be computed using the “homotopy variance property” [54, Theorem 4.1].

However the change in sign of the degree along a homotopy can be computed using the *homotopy variation property*. An admissible homotopy is a continuous family of C^1 -Fredholm maps $h: [0, 1] \times \mathcal{O} \rightarrow Y$ which is proper on closed bounded subsets of $[0, 1] \times \mathcal{O}$.

LEMMA 2.3.1. *Let $h: [0, 1] \times \mathcal{O} \rightarrow Y$ be an admissible homotopy and let Ω be an open bounded subset of X such that $0 \notin h([0, 1] \times \partial\Omega)$. If $b_i \in \mathcal{O}$ is a base point for h_i ; $i = 0, 1$, then*

$$(2.4) \quad \deg_{b_1}(h_1, \Omega, 0) = \sigma(M) \deg_{b_0}(h_0, \Omega, 0).$$

Here $M: [0, 1] \rightarrow \Phi_0(X, Y)$ is the path $L \circ \gamma$, where $L(t, x) = Dh_t(x)$ and γ is any path in $[0, 1] \times \mathcal{O}$ from $(0, b_0)$ to $(1, b_1)$.

PROOF. Assuming that h is C^1 this is the content of [34, Theorem 5.1]. In [15] P. Benevieri and M. Furi used a very simple argument which allows to extend this theorem to admissible homotopies in the above sense. We will adapt their argument to the base point degree.

First of all we show that given a point $t \in [0, 1]$ for small enough $\delta > 0$ the homotopy property (2.4) holds on the interval $[t_0 = t - \delta, t + \delta = t_1]$.

Since $\deg_b(f, \Omega, y)$ is invariant by small perturbations of y , by Sard–Smale theorem we can assume without loss of generality that 0 is a regular value of h_t . If $h_t^{-1}(0) \cap \bar{\Omega}$ is empty, being proper maps closed, there exists a $\delta > 0$ such that if $|s - t| \leq \delta$ then the same holds for h_s . Hence, in this case (2.4) is tautologically verified. If the opposite is true, being 0 a regular value of h_t , $h_t^{-1}(0) = \{x_1, \dots, x_m\}$. Applying the implicit function theorem (in the category of continuous families of C^1 -maps) on a neighbourhood of each (t, x_i) and using properness we can find a $\delta > 0$ such that, for $s \in [t_0, t_1]$, $h_s^{-1}(0) = \{\tilde{x}_1(s), \dots, \tilde{x}_m(s)\}$ where $\tilde{x}_i: [t_0, t_1] \rightarrow \Omega$ are continuous maps with $\tilde{x}_i(t) = x_i$. Taking δ small enough we will have also that each $\tilde{x}_i(s)$ is a regular point of h_s . If b_0, b_1 are base points for h_{t_0} and h_{t_1} , respectively, then

$$(2.5) \quad \deg_{b_j}(h_{t_j}, \Omega, 0) = \sum_{i=1}^n \sigma(L \circ (t_j, \gamma_i^j)),$$

where, for $j = 0, 1$ and $1 \leq i \leq m$, γ_i^j is a path in \mathcal{O} joining b_j to $\tilde{x}_i(t_j)$.

If γ is any path joining (t_0, b_0) to (t_1, b_1) , then for each i there are two ways to reach $(t_1, \tilde{x}_i(t_1))$ from (t_0, b_0) . One (say μ) is by following first the path γ and then the path (t_1, γ_i^1) , while the second (say μ') is to follow first the path (t_0, γ_i^0) and after the path $(s, \tilde{x}_i(s)); t_0 \leq s \leq t_1$. Since $[t_0, t_1] \times \mathcal{O}$ is simply connected the two paths are homotopic and by homotopy invariance of the parity $\sigma(L \circ \mu) = \sigma(L \circ \mu')$. But the path $s \rightarrow L(s, \tilde{x}_i(s))$ has parity one, being a path of isomorphisms. Now, the multiplicative property of the parity gives

$$\sigma(L \circ (t_0, \gamma_i^0)) = \sigma(L \circ \gamma) \sigma(L \circ (t_1, \gamma_i^1))$$

from which, taking in account (2.5), follows the homotopy property (2.4) on $[t_0, t_1]$.

The general case follows again from the multiplicative property of the parity by subdividing $[0, 1]$ in small enough subintervals. \square

The remaining properties of a degree theory including additivity and excision hold true without change.

2.4. The main formula. Using the base point degree we can define the multiplicity of an isolated but not necessarily regular zero of a C^1 -Fredholm map $f: O \subset X \rightarrow Y$. If x_0 is an isolated solution of $f(x_0) = 0$ its *multiplicity* is defined by $\text{mult}(f, x_0) = \deg_b(f, W, 0)$, where W is a small enough open convex neighbourhood of x_0 and b is any regular base point of f in W . Notice that the multiplicity is well defined because, being W simply connected, f is

orientable and all Fredholm maps are locally proper. Moreover, the absolute value $|\text{mult}(f, x_0)|$ is independent from the choice of the base point.

Our main formula relates the order of $J(\text{Ind } L)$ in $J(\Lambda)$ with the multiplicity of an isolated zero at a given parameter value.

THEOREM 2.4.1. *Let Λ be a finite connected CW-complex, $f: \Lambda \times O \rightarrow Y$ be a C^1 -family of Fredholm maps of index 0 and let L be the linearization of f along the trivial branch. Assume that, for small enough δ , the only solutions of the equation $f(\lambda, x) = 0$ with $\|x\| \leq \delta$ are those of the form $(\lambda, 0)$. If, for some (and hence all) $\nu \in \Lambda$, the multiplicity $k = |\text{mult}(f_\nu, 0)| \neq 0$, then*

- (a) *the first Stiefel–Whitney class $w_1(\text{Ind } L) = 0$,*
- (b) *for some $i \in \mathbb{N}$, $k^i J(\text{Ind } L) = 0$ in $J(\Lambda)$.*

In particular we have:

COROLLARY 2.4.2. *Assume that for some $\nu \in \Lambda$, $k = |\text{mult}(f_\nu, 0)|$ is defined. If $k \neq 0$, then bifurcation arises whenever either the index bundle $\text{Ind } L$ is non orientable or $J(\text{Ind } L) \neq 0$, and $k = |\text{mult}(f_\nu, 0)|$ is prime to the order of $J(\Lambda)$.*

Indeed, the first assertion is clear. In order to prove the second it is enough to observe that the order of $J(\text{Ind } L)$ in $J(\Lambda)$ divides the order of this finite group. Hence, if there is no bifurcation, by the above theorem, k cannot be prime to the order of $J(\Lambda)$.

If L_ν is invertible, then the multiplicity $\text{mult}(f_\nu, 0) = \pm 1$. Therefore, Theorem 1.2.1 is a special case of the above corollary with $k = 1$.

REMARK 2.4.1. A more precise invariant would be the order $J(\text{Ind } L)$ in $J(\Lambda)$. However, we stated the conclusion of Corollary 2.4.2 in terms of the order of the group $J(\Lambda)$ since in many important cases (e.g. spheres) the order of $J(\Lambda)$ is known. For general parameter space without 2-torsion in homology it can be estimated in terms of the homology of Λ with coefficients in $J(S^q)$. On the contrary the order of $J(\text{Ind } L)$ is a rather elusive object. There is a parallel theory in terms of codegree of the index bundle (see [13]) which gives essentially the same information as the order of $J(\text{Ind } L)$, since both numbers have the same primes on its decomposition. However, co-degree is also difficult to compute.

2.5. Proof of the main formula. First we prove (a). Chose a point $\nu \in \Lambda$. Since Λ is connected, the Hurewicz homomorphism $h: \pi_1(\Lambda, \nu) \rightarrow H_1(\Lambda; \mathbb{Z})$ is surjective. Therefore, in order to show that $w_1(\text{Ind } L) = 0$ in $H^1(\Lambda; \mathbb{Z}_2)$ it is enough to check that $\langle w_1(\text{Ind } L \circ \gamma); [S^1] \rangle = 0$ in \mathbb{Z}_2 for any closed path $\gamma: S^1 \rightarrow \Lambda$ with $\gamma(0) = \nu = \gamma(1)$. For this we will use the following proposition which relates the parity to the index bundle:

PROPOSITION 2.5.1 ([32, Proposition 2.7]). *Given a family $L: \Lambda \rightarrow \Phi_0(X, Y)$, for any closed path $\gamma: S^1 \rightarrow \Lambda$,*

$$(2.6) \quad \sigma(L \circ \gamma) = (-1)^\varepsilon,$$

where $\varepsilon = \langle w_1(\text{Ind } L); \gamma_*([S^1]) \rangle$.

By Proposition 2.5.1 we have to show that $\sigma(L \circ \gamma) = 1$ for any closed path γ in Λ based at ν . Let us choose a regular base point $b \in B(0, \delta)$ for f_ν (there must be at least one since $\text{mult}(f_\nu, 0) \neq 0$). Let $L^b(t) = D_x f(\gamma(t), b) = Df_{\gamma(t)}(b)$.

Since the parity of a closed path is invariant under free homotopies, the homotopy of closed paths $\eta(t, s) = D_x f(\gamma(t), sb)$, $0 \leq s \leq 1$, shows that

$$(2.7) \quad \sigma(L^b) = \sigma(L \circ \gamma).$$

Let $h: [0, 1] \times B(0, \delta) \rightarrow Y$ be the homotopy defined by $h(t, x) = f(\gamma(t), x)$. By assumption, there are no zeroes of h on $I \times \partial B(0, \delta)$. Hence, we can apply the homotopy property (2.5) of the base point degree to h . Since $D_x h(-, b) = L^b$, we get

$$\deg_b(f_\nu, B(0, \delta)) = \sigma(L^b) \deg_b(f_\nu, B(0, \delta)).$$

From which, being $\deg_b(f_\nu, B(0, \delta)) \neq 0$, we conclude that $\sigma(L \circ \gamma) = \sigma(L^b) = 1$. This proves the first claim.

For the second, we will incorporate parameters into a global version of the Lyapunov–Schmidt reduction (see Section 3.3) found by Renato Caccioppoli in [20] whose rigorous formulation in modern terms is due to Yu. I. Saprnov [17].

Let us choose an n -dimensional subspace V of Y such that the transversality condition $\text{Im } L + V = Y$ holds for any $\lambda \in \Lambda$. Using compactness of Λ we can find a small enough ball $B = B(0, \delta)$ such that the equation $f(\lambda, x)$ has only trivial solutions on $O = \Lambda \times B(0, \delta)$ and moreover

$$(2.8) \quad \text{Im } Df_\lambda(x) + V = Y \quad \text{for any } (\lambda, x) \in O.$$

Let π_V be a projector onto the subspace V and let $Z = \text{Im}(\text{id} - \pi_V)$. We split Y into a direct sum $Y = V \oplus Z$ and we write the map f in the form $f = (g, h)$, where $g: O \rightarrow V$ and $h: O \rightarrow Z$ are defined by $g = \pi_V f$ and $h = (\text{id} - \pi_V)f$ respectively.

Clearly (2.8) implies that for each $\lambda \in \Lambda$ and $x \in B(0, \delta)$ the differential $Dh_\lambda(x)$ is surjective. Thus $h_\lambda: B(0, \delta) \rightarrow Z$ is a submersion for all $\lambda \in \Lambda$ and $M_\lambda = h_\lambda^{-1}(0) = f_\lambda^{-1}(V)$ is a finite dimensional submanifold of $B = B(0, \delta)$.

By dimension counting, $\dim M_\lambda = n$. The tangent space to M_λ at $0 \in M_\lambda$ is $E_\lambda = \text{Ker } Dh_\lambda(0) = L_\lambda^{-1}(V)$. In particular, $\text{Ind } L$ is the stable equivalence class of the vector bundle $E = \bigcup_{\lambda \in \Lambda} \{\lambda\} \times E_\lambda$.

Since E is a finite dimensional subbundle of $\Theta(X)$, there is a family $\pi: \Lambda \rightarrow \mathcal{L}(X)$ of projectors with $\text{Im } \pi_\lambda = E_\lambda$. We will consider π as a vector bundle

morphism from $\Theta(X)$ onto E . Let $\phi: \Lambda \times B \rightarrow \Theta(Z) \oplus E$ be the (nonlinear) fiber bundle map over Λ defined by $\phi(\lambda, x) = (\lambda, h(\lambda, x), \pi_\lambda(x))$. Since $\text{Ker } Dh_\lambda(0) = \text{Im } \pi_\lambda$, $D\phi_\lambda(0)$ is an isomorphism for each $\lambda \in \Lambda$.

LEMMA 2.5.2. *The restriction of the map ϕ to a neighbourhood of the zero section $T = \Lambda \times \{0\}$ in $\Theta(X)$ is a fiberwise differentiable homeomorphism of this neighbourhood with a neighbourhood of T in $\Theta(Z) \oplus E$.*

PROOF. We first show that the restriction of ϕ to a neighbourhood of the zero section $T = \Lambda \times \{0\}$ in $\Theta(X)$ is a fiberwise differentiable local homeomorphism using the contraction mapping principle proof of the inverse mapping theorem in the category of spaces over a base [23].

Given a point $(\nu, 0) \in T$, we take a trivialization $\tau: E|_N \rightarrow N \times V$ of E on a neighbourhood N of ν . Let $\rho: \Theta(Z) \oplus E \rightarrow \Theta(Z) \oplus \Theta(V) \simeq \Theta(Y)$ be the bundle isomorphism over N defined by $\rho_\lambda(z, e) = (z, \tau_\lambda(e))$. Composing the map $\rho\phi$ on the right with $D\phi_\nu^{-1}(0)\rho_\nu^{-1}$, we obtain a map $\bar{\phi}: N \times B \rightarrow N \times X$ such that $D\bar{\phi}_\nu(0) = \text{id}$.

Since $D\phi_\nu^{-1}(0)\rho_\nu^{-1}\rho$ is an isomorphism, we have only to prove that $\bar{\phi}$ is a local homeomorphism at $(\nu, 0)$. In order to show this, eventually by taking smaller neighbourhood of $(\nu, 0)$ we can assume that $\|x - D\bar{\phi}_\lambda(x)\| \leq \|x\|/2$ for all $x \in B(0, 2\delta)$ and $\lambda \in N$. Then, for each λ in N , the map $\bar{\phi}_\lambda: B(0, \delta) \rightarrow B(0, \delta/2)$ is a homeomorphism (in fact a C^1 diffeomorphism) because $c_y(x) = x - \bar{\phi}_\lambda(x) + y$ is a contraction on $\bar{B}(0, \delta)$ for any $y \in B(0, \delta/2)$. We claim that the map

$$\bar{\phi}^{-1}: N \times B(0, \delta/2) \rightarrow N \times B(0, \delta)$$

is continuous.

Since $D\bar{\phi}_\lambda^{-1}(y)$ is continuous in both variables λ and y , taking N and δ small enough, we have $\|D\bar{\phi}_\lambda^{-1}(y)\| \leq K$ on $N \times B(0, \delta/2)$ and therefore

$$\|\bar{\phi}^{-1}(\lambda, y) - \bar{\phi}^{-1}(\lambda, z)\| \leq K\|y - z\|$$

there. On the other hand, by the continuous dependence on parameters of the fixed point of a contraction, $\bar{\phi}^{-1}(-, y)$ is continuous in the variable λ for each fixed y . The continuity of $\bar{\phi}^{-1}$ follows from this two facts. Thus $\bar{\phi}$ is a local homeomorphism and hence so is ϕ .

Finally, we observe that the restriction of ϕ to the zero section T is injective. It is easy to show that if a local homeomorphism is injective on a compact subset, then it is a homeomorphism on a neighbourhood of this set. \square

Let U and W be open neighbourhoods of T in $\Theta(X)$ and $\Theta(Z) \oplus E$ respectively such that $\phi: U \rightarrow W$ is a fiberwise differentiable fiber preserving homeomorphism between them. Then the map $\psi: E \cap W \rightarrow \Theta(X)$ defined by

$\psi(v) = \phi^{-1}(0, v)$ is a fiberwise differentiable homeomorphism of a neighbourhood of the zero section in E with its image and moreover $\psi_\lambda(W \cap E_\lambda) \subset M_\lambda$ for each $\lambda \in \Lambda$.

Now, we will use the map $g: O \rightarrow V$. Let $D(E) = D(E, r) \subset E$ be a closed disk bundle of radius r contained in $E \cap W$ and let $S(E) = \partial D(E)$ be the associated sphere bundle. Since ψ_λ sends $D_\lambda(E) - \{0\}$ into $M_\lambda - \{0\}$ and since $f_\lambda(x) = 0$ only if $x = 0$, we have that $\|g\psi(v)\| \neq 0$ for any $v \in S(E)$. Hence, if S^{n-1} is the unit sphere in V , we get a fiber bundle map $\bar{g}: S(E) \rightarrow \Lambda \times S^{n-1}$ defined by

$$(2.9) \quad \bar{g}(v) = (\lambda, \|g\psi(v)\|^{-1}g\psi(v)).$$

First, we will show that the degree of the map $\bar{g}_\nu: S(E)_\nu \rightarrow S^{n-1}$ is $\pm k$. In what follows, if M, N are oriented finite dimensional manifolds of the same dimension, $\Omega \subset M$ an open subset and $f: \Omega \rightarrow N$ is a map such that $f^{-1}(p)$ is compact, we will denote by $\deg(f, \Omega, p)$ the Brouwer degree of f in Ω with respect to p . We will use $\deg(f)$ to denote the total degree of a map between compact manifolds.

The homomorphism $\bar{g}_\nu^*: H^{n-1}(S^{n-1}; \mathbb{Z}) \rightarrow H^{n-1}(S(E)_\nu, \mathbb{Z})$ induced by \bar{g}_ν in singular cohomology coincides with the multiplication by $\pm \deg((g_\nu\psi_\nu, D(E)_\nu, 0)$ (see for example [50, Proposition 2.6]). Thus $\deg(\bar{g}_\nu) = \pm \deg((g_\nu\psi_\nu, D(E)_\nu, 0)$. But $\deg(\psi_\nu, D_\nu(E), 0) = \pm 1$ since ψ_ν is a diffeomorphism. Therefore, denoting with g'_ν the restriction of g_ν to M_ν , we have

$$\deg(\bar{g}_\nu) = \pm \deg(g_\nu\psi_\nu, D(E)_\nu, 0) = \pm \deg(g'_\nu, M_\nu, 0).$$

With the above proved, the assertion $\deg(\bar{g}_\nu) = \pm k$ is a consequence of the following reduction property of the base point degree:

PROPOSITION 2.5.3. *Let $f: \Omega \subset X \rightarrow Y$ be a proper oriented C^1 -Fredholm map of index 0. Let V be an n -dimensional subspace of Y transversal to f . Then $M = f^{-1}(V)$ is an n -dimensional oriented submanifold of Ω . The map $g: M \rightarrow V$ given by the restriction of f to M is proper. Moreover, for any base point b*

$$(2.10) \quad \deg_b(f, \Omega, 0) = \pm \deg(g, M, 0).$$

PROOF. For C^2 -maps (2.10) is a special case of [34, Theorem 5.8]. But this theorem holds for the degree of C^1 -Fredholm maps constructed in [54] with exactly the same proof. \square

From the above proposition, since the only zero of the map f_ν on $B(0, \delta)$ is 0, we get $k = \deg_0(f_\nu, B(0, \delta), 0) = \deg(g'_\nu, M_\nu, 0) = \pm \deg(\bar{g})$, which proves the assertion.

By (a), E is an orientable subbundle of the trivial bundle $\Theta(X)$. Hence, we can finish the proof of Theorem 2.4.1 using the mod- k Dold's theorem of Adams

[4, Theorem 1.1]. This theorem states that if E is an orientable vector bundle of rank n over a connected finite CW-complex Λ and if $\bar{g}: S(E) \rightarrow \Theta(S^{n-1})$ is a fiber bundle map from the sphere bundle $S(E)$ to the trivial sphere bundle of rank n such that for some (and hence any) $\lambda \in \Lambda$ the map $\bar{g}_\lambda: S(E_\lambda) \rightarrow S^{n-1}$ is of degree $\pm k$, then there exists a positive integer i such that $S(k^i E)$ is fiberwise homotopy equivalent to $S(k^i \Theta^n)$. Thus, $k^i \cdot J(\text{Ind } L) = 0$ in $J(\Lambda)$ which proves the theorem. \square

3. A local index of bifurcation

Here we will construct a local index of bifurcation points for parametrized families of Fredholm maps. We will consider only families whose range is a Kuiper space i.e. a Banach space with contractible linear group $\text{GL}(Y)$. Kuiper proved that the general linear group of a Hilbert space is contractible. Later many functional spaces were shown to be Kuiper. When nonempty, the space $\text{GL}(X, Y)$ of all isomorphisms from X to Y is homeomorphic to $\text{GL}(Y)$. Hence, if Y is Kuiper, then $\text{GL}(X, Y)$ is contractible for any Banach space X .

3.1. The local index bundle. Let $L: U \rightarrow \Phi_0(X, Y)$ be a continuous family of linear Fredholm operators defined on an open set $U \subset \Lambda$ such that $\Sigma(L)$ is a compact subset of U . We define the local index bundle as follows:

Let V be an open neighbourhood of $\Sigma(L)$ with compact closure contained in U . Since $\text{GL}(X, Y)$ is a contractible absolute neighbourhood retract, any map from a closed subset of a metric space into $\text{GL}(X, Y)$ can be extended to all of the space. In particular, the restriction of L to the boundary ∂V of V can be extended to a family $L': \Lambda - V \rightarrow \text{GL}(X, Y)$. Define $\bar{L}: \Lambda \rightarrow \Phi_0(X, Y)$ by patching L on \bar{V} with L' on $\Lambda - V$. Then \bar{L} is a family of linear Fredholm operators parametrized by Λ which coincides with L in a neighbourhood of $\Sigma(L) = \Sigma(\bar{L})$.

The *local index bundle* of the family L on U is defined by:

$$(3.1) \quad \text{Ind}(L, U) = \text{Ind}(\bar{L}) \in \widetilde{\text{KO}}(\Lambda).$$

If V_1, V_2 are two neighbourhoods of $\Sigma(L)$ with $V_2 \subset V_1$ (which we can always assume) and if \tilde{L}, \hat{L} are the corresponding extensions, then $\tilde{L}|_{\partial V_1} = \hat{L}|_{\partial V_1}$.

Let $M: \Lambda \rightarrow \text{GL}(X, Y)$ be the family defined by

$$(3.2) \quad M_\lambda = \begin{cases} \text{id} & \text{if } \lambda \in \bar{V}_1, \\ \tilde{L}_\lambda \circ \hat{L}_\lambda^{-1} & \text{if } \lambda \in \Lambda - V_1. \end{cases}$$

Then M is a family of isomorphisms verifying $M_\lambda \circ \hat{L}_\lambda = \tilde{L}_\lambda$. Since $\text{Ind } M = 0$, by the logarithmic property of the index bundle, we have $\text{Ind}(\tilde{L}) = \text{Ind}(\hat{L})$. This proves that the right hand side of (3.1) is independent of the choice of V and the extension \bar{L} .

We will need the following additivity property of the local index bundle.

PROPOSITION 3.1.1. *Let $L:U \rightarrow \Phi_0(X,Y)$ be a family such that $\Sigma = \Sigma(L)$ is compact. Let U_1, U_2 be open with $U_1 \cup U_2 \subset U$ and let $\Sigma_i = \Sigma \cap U_i$. If $\Sigma_1 \cap \Sigma_2 = \emptyset$, $\Sigma_1 \cup \Sigma_2 = \Sigma$ and if L_i , for $i = 1, 2$ are the restrictions of L to U_i , then*

$$(3.3) \quad \text{Ind}(L, U) = \text{Ind}(L_1, U_1) + \text{Ind}(L_2, U_2).$$

PROOF. Since the index bundle is invariant by composition with families of isomorphisms and since X is isomorphic to Y whenever $\Phi_0(X, Y)$ is not empty, there is no loss of generality in assuming that $X = Y$.

Let V_i be open neighbourhoods of Σ_i with $\bar{V}_i \subset U_i$, $i = 1, 2$, and such that $\bar{V}_1 \cap \bar{V}_2 = \emptyset$. Let \bar{L}_i be extensions of $L_i|_{\bar{V}_i}$ obtained as in (3.1). Using once more the fact that $\text{GL}(Y)$ is an absolute retract we can construct two families of isomorphisms parametrized by Λ , say $M_i: \Lambda \rightarrow \text{GL}(Y)$, $i = 1, 2$ such that

$$(3.4) \quad \begin{cases} M_i|_{\bar{V}_j} = \text{id} & \text{if } i = j, \\ M_i|_{\bar{V}_j} = L_i^{-1} & \text{if } i \neq j. \end{cases}$$

Put $\bar{L} = M_2 \bar{L}_2 \bar{L}_1 M_1$ and $V = V_1 \cup V_2$. It follows from (3.4) that $\bar{L}_\lambda = L_\lambda$ if $\lambda \in \bar{V}$ and that $\bar{L}_\lambda \in \text{GL}(X, Y)$ if $\lambda \notin V$. By definition of the local index bundle,

$$\begin{aligned} \text{Ind}(L, U) &= \text{Ind}(\bar{L}) = \text{Ind}(M_2 \bar{L}_2 \bar{L}_1 M_1) \\ &= \text{Ind}(\bar{L}_1) + \text{Ind}(\bar{L}_2) = \text{Ind}(L_1, U_1) + \text{Ind}(L_2, U_2). \quad \square \end{aligned}$$

In what follows we will also use $\text{Ind}_\Lambda(L, U)$ to denote the local index bundle when we want to show the dependence of this element on the parameter space.

From functoriality of the index bundle we obtain the following relation between $\text{Ind}_\Lambda(L, U)$ and the local index with respect to the one point compactification U^+ .

$$(3.5) \quad \text{Ind}_\Lambda(L, U) = q^*(\text{Ind}_{U^+}(L, U)),$$

where $q: \Lambda \rightarrow U^+$ is the map collapsing $\Lambda - U$ to the point at infinity.

The relation (3.5) suggests a different construction of the local index bundle which works for general Banach spaces. This alternative approach uses K -theory with compact support. We review shortly this theory below since we will need it in the sequel.

If Z is a locally compact space, by definition $\text{KO}_c(Z)$ is the reduced Grothendieck group $\widehat{\text{KO}}(Z^+)$ of the one-point compactification Z^+ of the space Z . However, there is a different description of this group in terms of virtual bundles with compact support [26], [10].

A *virtual bundle with compact support* is an equivalence class $[E, F, a]$ of a triple (E, F, a) , where E, F are finite dimensional real vector bundles over Z

and where $a: E \rightarrow F$ is a vector bundle morphism which is an isomorphism on the complement of a compact subset of Z . Any compact set with the above property is called *support*.

A triple having an empty support is called *trivial*. In the set of triples there is an obvious notion of direct sum and isomorphism. We define an equivalence relation by saying that two triples η_1 and η_2 are equivalent provided that there are trivial triples θ_1, θ_2 such that $\eta_1 \oplus \theta_1$ is isomorphic to $\eta_2 \oplus \theta_2$. The set of all equivalence classes is a group, isomorphic to $\text{KO}_c(Z)$. The complex K -theory with compact support $K_c(Z)$ admits an analogous description.

The above isomorphism can be constructed as follows: given a triple (E, F, a) and a relatively compact open neighbourhood V of its support, by compactness there exists a vector bundle G over \bar{V} such that $F \oplus G \cong \theta^m$. Taking $E' = E \oplus G$ and $a' = a \oplus \text{id}$ we get a triple (E', θ^m, a') over \bar{V} such that a' is an isomorphism of E' restricted to ∂V with the trivial bundle $\partial V \times \mathbb{R}^m$. We use a' in order to perform the clutching construction (see Section 3.3) of E' with the trivial bundle over $Z^+ - V$ and obtain a bundle E'' over Z^+ . It is easy to see that the map $[E, F, a] \rightarrow [E''] - [\theta^m]$ is an isomorphism. Its inverse sends $[E] - [\theta^m] \in \widetilde{\text{KO}}(Z^+)$ to the class $[E', \theta^m, a]$, where E' is the restriction of E to Z , and a is any extension to Z of a trivialization of E on an open neighbourhood U of ∞ in Z^+ restricted to $Z \cap U$.

Let Y be a general Banach space. We define $\text{Ind}(L, U)$ of a family with compact support $L: U \rightarrow \Phi_0(X, Y)$ as follows: using a finite covering of the support we can find a finite dimensional subspace F of Y such that $\text{Im } L_\lambda + F = Y$ for each $\lambda \in U$. Then the family of vector spaces $E_\lambda = L_\lambda^{-1}(F)$ is a vector bundle with a natural trivialization at infinity $a: E \rightarrow U \times F$, where $a_\lambda = L_\lambda$ restricted to E_λ . Thus $[E, U \times F, a]$ defines an element of $\widetilde{\text{KO}}_c(U)$ and it is easy to see that this element is independent of the choice of F as above. By definition,

$$(3.6) \quad \text{Ind}_\Lambda(L, U) = q^*[E, U \times F, a],$$

where q is as in (3.5). The relation (3.5) shows that the above definition coincides with the one in (3.1) when Y is a Kuiper space.

3.2. Definition and properties of $\beta(f, U)$. Let Y be a Kuiper space, let U be an open subset of a finite connected CW-complex Λ and let O be an open subset of a Banach space X .

Let $f: U \times O \subset X \rightarrow Y$ be a family of C^1 Fredholm maps such that $f(\lambda, 0) = 0$. The map f can be written in the form $f(\lambda, x) = L_\lambda x + g(\lambda, x)$ where, as before, $L_\lambda = Df_\lambda(0)$ and $g(\lambda, x) = o(\|x\|)$. In particular, $Dg_\lambda(0) = 0$ for all $\lambda \in U$.

Recall from Section 1.3 that a pair (f, U) as above is called admissible if $\Sigma(f)$ is a compact, proper subset of the open set U .

The *local bifurcation index* $\beta(f, U)$ of an admissible pair is defined by

$$(3.7) \quad \beta(f, U) = J(\text{Ind}(L, U)).$$

The rest of this subsection will be devoted to the verification of properties (B₁) to (B₅). Property (B₆) will be proved in the next subsection. Below we will use $\Sigma(f)$ to denote the singular set $\Sigma(L)$ of the linearization of f along the trivial branch.

We will recast the verification of the existence property (B₁) to our Theorem 1.2.1 by constructing a family $\bar{f}: \Lambda \times B(0, r) \rightarrow Y$ of C^1 -Fredholm maps verifying the hypothesis of this theorem and such that:

- (a) $\Sigma(\bar{f}) = \Sigma(f)$
- (b) \bar{f} coincides with f in a neighbourhood of $\Sigma(f) \times \{0\}$ in $\Lambda \times X$.

The construction of \bar{f} goes as follows: we take an open subset V of U such that $\Sigma(f) \subset V \subset \bar{V} \subset U$. Arguing as in (3.1), we extend $L|_{\bar{V}}$ to a continuous family \bar{L} defined on Λ such that $\bar{L}_\lambda \in \text{GL}(X, Y)$ for $\lambda \in \Lambda - V$. By definition,

$$(3.8) \quad \beta(f, U) = J(\text{Ind } \bar{L}).$$

Let ϕ be a continuous function on Λ with $0 \leq \phi \leq 1$, $\phi \equiv 1$ on \bar{V} and $\phi \equiv 0$ on $\Lambda - U$. For $(\lambda, x) \in \Lambda \times X$ we define

$$\bar{g}(\lambda, x) = \begin{cases} g(\lambda, \phi(\lambda)x) & \text{for } (\lambda, x) \in U \times X, \\ 0 & \text{for } (\lambda, x) \notin U \times X. \end{cases}$$

Then \bar{g} is a continuous family of C^1 -maps and clearly $D\bar{g}_\lambda(0) = 0$.

Finally, let us define \bar{f} by $\bar{f}(\lambda, x) = \bar{L}_\lambda x + \bar{g}(\lambda, x)$. Then $D\bar{f}_\lambda(0) = \bar{L}_\lambda$ and therefore, for small enough r , the restriction of \bar{f} to $\Lambda \times B(0, r)$ is a continuous family of C^1 -Fredholm maps. Clearly the map \bar{f} verifies the required conditions since it coincides with f on $V \times B(0, r)$ and has the same singular set.

It follows from (3.8) that $\beta(f, U) = J(\text{Ind } \bar{L}) = \beta(\bar{f})$. Hence, if $\beta(f, U)$ does not vanish in $J(\Lambda)$, by Theorem 1.2.1, there must be a bifurcation point of \bar{f} belonging to $\Sigma(\bar{L}) = \Sigma(L)$. Since f coincides with \bar{f} on $V \times B(0, r)$, this point must be a bifurcation point for f as well. This completes the verification of (B₁).

That (B₂) holds is clear from the definition of the local index bundle. In order to prove the additivity property (B₄), it is enough to consider the case of two open sets. Notice that, being Λ connected, if (f, U) is admissible so are (f_i, U_i) . Then (B₄) follows from Proposition 3.1.1 applying the functor J to both sides.

In order to show (B₅) let us notice that, if $\alpha: Q \rightarrow \Lambda$ is continuous, then by functoriality of the index bundle

$$(3.9) \quad \text{Ind}(L \circ \alpha, \alpha^{-1}(U)) = \alpha^* \text{Ind}(L, U).$$

If $g = f \circ (\alpha \times id_X)$, then $Dg_\lambda(0) = L \circ \alpha(\lambda)$. Applying the functor J to (3.9) we obtain (B₅).

The homotopy invariance property (B₃) follows from (B₅), since an admissible homotopy h is nothing but an admissible family of C^1 -Fredholm maps parametrized by the open subspace $V = [0, 1] \times U$ of the space $\Gamma = [0, 1] \times \Lambda$. Thus h defines an element $\beta(h, V) \in J(\Gamma)$. By (B₅), denoting with i_0 and i_1 are the top and bottom inclusion of Λ in Γ , we have $\beta(h_j, U) = i_j^* \beta(h, V)$, $j = 0, 1$. But $i_1^* = i_0^*$ and hence $\beta(h_0, U) = \beta(h_1, U)$.

If Y is a general Banach space, and we put $\beta(f, U) = J(\text{Ind}(L, U))$, where $\text{Ind}(L, U)$ is defined by (3.6), then we can show that $\beta(f, U)$ verifies properties (B₂) through (B₆) using standard properties of $\widetilde{\text{KO}}$ as generalized cohomology theory. However, the crucial property (B₁) is missed in this setting because our proof of (B₁) relies on an extension property which does not hold for general Banach spaces.

3.3. Comparison with the Alexander–Ize invariant. In order to complete the proof of Theorem 1.3.1 we have to verify the property (B₆).

We begin by introducing the Alexander–Ize invariant. Only the stable version of this invariant constructed in [5] will be considered here.

Let $f: \mathbb{R}^k \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a C^1 -family of maps from \mathbb{R}^n to itself, parametrized by \mathbb{R}^k . Assume that $f_\lambda(0) = 0$ and let L_λ be the derivative of f_λ at 0. Let λ_0 be an isolated point in the set $\Sigma(L)$. The homotopy class of the restriction of the map $\lambda \mapsto L_\lambda$ to the boundary of a small closed disk $D = D(\lambda_0, \varepsilon)$ centered at λ_0 defines an element γ_f^n in the homotopy group $\pi_{k-1} \text{GL}(n; \mathbb{R})$ (here and below we are using the fact that our target spaces are H-spaces and hence the free and pointed homotopy classes are the same). Stabilizing γ_f^n through the natural inclusion of $\text{GL}(n) = \text{Gl}(n; \mathbb{R})$ into $\text{GL}(n+1)$ one gets an element γ_f belonging to the homotopy group $\pi_{k-1} \text{GL}(\infty)$, where the space $\text{GL}(\infty) = \bigcup_{n \geq 1} \text{GL}(n)$ is endowed with the inductive topology. The element γ_f is the Alexander–Ize invariant.

Let $\pi_{k-1}^s = \text{limdir } \pi_{m+k-1} S^m$ be the $(k-1)$ -stable homotopy stem. In [5] J. C. Alexander proved that λ_0 is a bifurcation point of f provided the image of γ_f by the stable j -homomorphism $j: \pi_{k-1} \text{GL}(\infty) \rightarrow \pi_{k-1}^s$ does not vanish.

The above definition can be easily extended to continuous families of C^1 -Fredholm maps $f: \mathbb{R}^k \times X \rightarrow Y$.

Indeed, assume that λ_0 is isolated in $\Sigma(L)$ and let D be as before. A *regular parametrix* [31] for the family L is a family of isomorphisms $A: D \rightarrow \text{GL}(Y, X)$ such that $L_\lambda A_\lambda = id_Y + K_\lambda$, with $\text{Im } K_\lambda$ contained in a fixed finite dimensional subspace F of Y . Since D is contractible, any family L as above possesses a regular parametrix (see the proof of Lemma 3.3.1 below).

Putting $N_\lambda = (\text{id} + K_\lambda)|_F$, the map N sends ∂D into $\text{GL}(F)$. Choosing a basis of F we get a family of matrices in $\text{GL}(m)$ parametrized by $\partial D \simeq S^{k-1}$. By the preceding discussion, the stable homotopy class of $N|_{\partial D}$ defines an element $\gamma_f \in \pi_{n-1}\text{GL}(\infty)$ which can be shown to be independent from the choice of D and the parametrix A . By definition, the element γ_f constructed above is the *Alexander-Ize invariant* of f at λ_0 .

Let us discuss the identification of $J(S^k)$ with the image of the stable j -homomorphism of G. Whitehead.

A spherical fibration is a fibration locally fibre homotopy equivalent to a product of the base with an n -sphere. Recall that the reduced group $\widetilde{\text{KO}}(\Lambda)$ can be identified with the group of stable equivalence classes of vector bundles over Λ . In a similar way one can introduce the group $\widetilde{\text{KF}}(\Lambda)$ of stable fibre homotopy classes of spherical fibrations [25]. $\widetilde{\text{KF}}(\Lambda)$ becomes a group under the operation of fiberwise smash product.

As in the case of $\widetilde{\text{KO}}$, the group $\widetilde{\text{KF}}$ is a homotopy functor represented by the classifying space $BH(\infty)$ of the monoid $H(\infty) = \bigcup_{n \geq 1} H(n)$, where $H(n)$ is the space of all homotopy equivalences of S^n .

Since working directly with $\text{GL}(n)$ instead of the homotopy equivalent group $O(n)$ simplifies many arguments in this section, we deviate slightly from the usual convention. The later defines $H(n)$ to be the set of homotopy equivalences from S^{n-1} into itself and identifies the previously defined J homomorphism with the natural transformation which assigns to each vector bundle E its unit sphere bundle $S(E)$.

Here instead, we will consider $J: \widetilde{\text{KO}} \rightarrow \widetilde{\text{KF}}$ to be defined by the inclusion of the total space of a vector bundle in its fiberwise one-point compactification. Since the fiberwise one point compactification of a vector bundle is a suspension of its unit sphere bundle, we obtain a factorization:

$$(3.10) \quad \begin{array}{ccc} \widetilde{\text{KO}}(\Lambda) & \xrightarrow{J} & \widetilde{\text{KF}}(\Lambda) \\ & \searrow J & \nearrow \text{inclusion} \\ & & J(\Lambda) \end{array}$$

which leads to the identification of $J(\Lambda)$ with the image of the horizontal arrow.

Taking $\Lambda = S^k$ we obtain a commutative diagram

$$(3.11) \quad \begin{array}{ccc} \widetilde{\text{KO}}(S^k) & \xrightarrow{J} & \widetilde{\text{KF}}(S^k) \\ \partial_0 \downarrow & & \downarrow \partial_1 \\ \pi_{k-1}\text{GL}(\infty) & \xrightarrow{j} & \pi_{k-1}H(\infty) \end{array}$$

In the above diagram j is the homomorphism induced in homotopy by the map which assigns to each element of $\text{GL}(n)$ the obvious extension to a map from the one point compactification S^n of \mathbb{R}^n into itself. The vertical arrow ∂_0 takes the stable equivalence class of a vector bundle E over S^k to the stable homotopy class of

$$\psi_T = T_- T_+^{-1}: S^{k-1} \rightarrow \text{GL}(n),$$

where T_{\pm} are trivializations for the restrictions of E to the upper and lower hemisphere of S^k respectively. The vertical arrow ∂_1 is defined in a similar way. The homomorphisms ∂_i , $i = 0, 1$ are isomorphisms whose inverses are given by the clutching construction.

Under the identification of $\pi_{k-1}H(\infty)$ with π_{k-1}^s via the isomorphism established in Lemma 1.3 of [8], the homomorphism j in (3.11) coincides with the stable j -homomorphism of G. Whitehead and the vertical arrow ∂_1 sends $J(S^n)$ isomorphically onto $\text{Im } j$. In what follows we will identify the group $J(S^n)$ with $\text{Im } j$ by means of the restriction of ∂_1 to $J(S^n)$.

Before going to the verification of (B₆) we will need the analog of ∂_0 at the operators level. Let $\partial: \pi_k \Phi_0(X, Y) \rightarrow \pi_{k-1} \text{GL}(\infty)$, be defined as follows: let $L: S^k \rightarrow \Phi_0(X, Y)$ be a family representing the homotopy class $\alpha \in \pi_k \Phi_0(X, Y)$. We can take parametrices A_{\pm} of $L_{\pm} = L|_{D_{\pm}}$ such that, for any $\lambda \in D_{\pm}$, the operators $K_{\pm\lambda} = L_{\pm\lambda} A_{\pm\lambda} - \text{id}$ take values in the same r -dimensional subspace F of Y . Then, for $\lambda \in S^{k-1}$, the operator $A_{-\lambda}^{-1} A_{+\lambda}$ sends F into itself. By definition, $\partial(\alpha)$ is the stable homotopy class of $\phi_A: S^{k-1} \rightarrow \text{GL}(F) \cong \text{GL}(r)$ defined by $\phi_A(\lambda) = A_{-\lambda}^{-1} A_{+\lambda}|_F$.

LEMMA 3.3.1. *The diagram*

$$(3.12) \quad \begin{array}{ccc} & & \widetilde{\text{KO}}(S^k) \\ & \nearrow \text{Ind} & \downarrow \partial_0 \\ \pi_k \Phi_0(X, Y) & & \pi_{k-1} \text{GL}(\infty) \\ & \searrow \partial & \end{array}$$

is commutative.

PROOF. Let F be any subspace of Y verifying the transversality condition (2.1). Then the index bundle of L is the stable class of $E = L^{-1}(F)$. Given trivializations $T_{\pm}: E|_{D_{\pm}} \rightarrow D_{\pm} \times F$ we construct the parametrices A_{\pm} of L_{\pm} as follows: for $\lambda \in D_{\pm}$ we put

$$(3.13) \quad A_{\pm\lambda} = (Q' L_{\lambda} + T_{\pm\lambda} Q_{\lambda})^{-1}$$

where Q_λ is a continuous family of projectors of X with $\text{Im } Q_\lambda = E_\lambda$ and Q' is a projector with $\text{Ker } Q' = F$.

$Q'L_\lambda + T_{\pm\lambda}Q_\lambda$ are injective Fredholm operators of index 0 and hence are invertible for any $\lambda \in D_\pm$. Thus $A_{\pm\lambda}$ are well defined. Moreover, the image of $L_{\pm\lambda}A_{\pm\lambda} - \text{id}_Y$ is contained in F . Using parametrices A_\pm in the definition of the homomorphism ∂_0 one easily checks that on S^{k-1} , $A_{-\lambda}^{-1}A_{+\lambda} = \text{id}_Y - K_\lambda$, where K is such that $\text{Im } K_\lambda$ is contained in F . Since $Q'L_\lambda$ vanishes on E_λ , it follows that on F the operator $(Q'L_\lambda + T_{+\lambda})^{-1}$ coincides with $T_{+\lambda}^{-1}$ and hence $A_{-\lambda}^{-1}A_{+\lambda}$ restricted to F is nothing but $T_{-\lambda}T_{+\lambda}^{-1}$. Thus, with the above choice of parametrix, we have $\psi_T = \phi_A$ and therefore $\partial_0 \circ \text{Ind} = \partial$. \square

PROPOSITION 3.3.2. *Let λ_0 be the only singular point of $f:U \times O \rightarrow Y$. Assume that $U \cong \mathbb{R}^k$. Then, on $U^+ \cong S^k$, the identification $\partial_1: J(S^k) \simeq \text{Im } j$ sends $\beta_{S^k}(f,U)$ into $j(\gamma_f)$.*

PROOF. We can assume without loss of generality that λ_0 is the north pole of S^k and take in the definition of γ_f the upper hemisphere D_+ as the disk D . Let L be the linearization of the family f along the trivial branch and let \bar{L} be any extension of $L|_{D_+}$ to all of S^k such that \bar{L}_λ is an isomorphism for $\lambda \in D_-$.

Putting together the commutative diagrams (3.12) and (3.11) we obtain going up and right $J(\text{Ind } \bar{L})$ which by definition is $\beta(f,U)$. On the other hand, going down and right we get $j(\gamma_f)$. Indeed, if $A_+: D_+ \rightarrow \text{GL}(Y, X)$ is a parametrix for \bar{L}_+ , we can take $A_- = \bar{L}_-^{-1}$. Then

$$A_{-\lambda}^{-1}A_{+\lambda}|_F = \bar{L}_{-\lambda}A_{+\lambda}|_F = N_\lambda,$$

where N is the family defining the class γ_f . \square

The above proposition shows that (B_6) holds true in the case $\Lambda = S^k$. The general case now follows from this and (3.5). This completes the proof of Theorem 1.3.1.

When the family L behaves in a regular way close to λ_0 , from the above proposition, using results of J. C. Alexander and J. York in [6], we can obtain sufficient conditions for the nonvanishing of $\beta_{S^k}(f,U) \neq 0$ in a small enough neighbourhood U of λ_0 in terms of the dimension of $\text{Ker } L_{\lambda_0}$.

COROLLARY 3.3.3. *Let $f: \mathbb{R}^k \times X \rightarrow Y$ be a continuous family of C^1 -Fredholm maps and let λ_0 be such that for $L_\lambda = Df_\lambda(0)$ the following condition holds: there exists a positive number r such that for small enough δ*

$$(3.14) \quad \|L_\lambda x\| \geq r\|\lambda - \lambda_0\|\|x\| \quad \text{for } 0 \leq \|\lambda - \lambda_0\| \leq \delta.$$

Let c_k be defined by

$$(3.15) \quad \begin{array}{l} k = \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \\ c_k = \quad 1 \quad 2 \quad 4 \quad 4 \quad 8 \quad 8 \quad 8 \quad 8 \end{array} \quad c_{k+8} = c_k,$$

then, for $k \equiv 1, 2, 4, 8 \pmod{-8}$, $\dim \text{Ker } L_{\lambda_0} = m$ is a multiple of c_k . Moreover, if $m = dc_k$ with d an odd integer, then $\beta_{S^k}(f, D(\lambda_0, \delta)) \neq 0$ in $J(S^k)$.

This follows from Proposition 3.3.2 and computation of $j(\gamma_f)$ in [6].

REMARK 3.3.1. The *intrinsic derivative* of a smooth family $L: \mathbb{R}^k \rightarrow \Phi_0(X, Y)$ at $\lambda \in \mathbb{R}^k$ is the map

$$\dot{I}L(\lambda): \mathbb{R}^k \rightarrow \mathcal{L}(\text{Ker } L_\lambda; \text{coker } L_\lambda)$$

defined as follows: $\dot{I}L(\lambda)v$ is the restriction to $\text{Ker } L_\lambda$ of the ordinary Frechet derivative $DL(\lambda)v$ followed by the projection to the $\text{coker } L_\lambda$. When the family f is smooth in all variables, condition (3.14) can be checked from the intrinsic derivative of L at λ_0 . It was shown in [29] that for smooth families the regularity condition (3.14) holds if and only if for any $v \in \mathbb{R}^k$ $\dot{I}L(\lambda)v$ is an isomorphism.

In the final part of the section we will point out the relation of our construction of γ_f with the Lyapunov–Schmidt reduction. We will use this relation in the proof of Theorem 1.4.2. Moreover, we will be able to compare the approach we have chosen here with that of J. Ize in [38], [39] which uses as the unstable version of γ_f the homotopy class of the linearization at 0 of the reduced map.

For simplicity, let us assume that the isolated singular point is $\lambda_0 = 0$.

Let Q' and Q be projectors on $Y_1 = \text{Im } L_0$ and $E_0 = \text{Ker } L_0$, respectively. Then $F_0 = \text{Ker } Q' \simeq \text{coker } L_0$. Under the splitting of both Y and X into a direct sum $Y_1 \oplus F_0$ and $X_1 \oplus E_0$ the Frechet derivative $D_{x_1}Q'f(0, 0)$ in the direction of X_1 is an isomorphism.

By the implicit function theorem, there exist a map ρ defined on a neighbourhood of $(0, 0)$ in $\mathbb{R}^k \times E_0$ with values in X_1 such that, close enough to $(0, 0) \in \mathbb{R}^k \times X$, we have $Q'f(\lambda, x_1 + v) = 0$ if and only if $x_1 = \rho(\lambda, v)$. It follows that, for small (λ, x) , the solutions of $f(\lambda, x) = 0$ are in one to one correspondence with the solutions of the finite dimensional reduced system $r(\lambda, v) = 0$ (called bifurcation equation), where the map r is defined on a product neighbourhood of $(0, 0)$ in $\mathbb{R}^k \times E_0$ by

$$(3.16) \quad r(\lambda, v) = (\text{id} - Q')f(\lambda, \rho(\lambda, v) + v).$$

Clearly $r(\lambda, 0) = 0$. Let $R_\lambda = D r_\lambda(0)$ be the linearization of r at the trivial branch. Taking a small enough closed disk $D = D(0, \delta)$ centered at 0, the restriction of R to ∂D defines a map $R: S^{k-1} \rightarrow \text{GL}(E_0, F_0)$ and hence (after a choice of basis of both spaces) a family of nonsingular matrices

$$(3.17) \quad R: S^{k-1} \rightarrow \text{Gl}(m), \quad m = \dim E_0,$$

whose homotopy class depends only on the choice of orientations of E_0 and F_0 . Let us remark that the bifurcation invariant defined by J. Ize in [38] is the image of the homotopy class of R by the unstable J -homomorphism.

PROPOSITION 3.3.4. *With an appropriate choice of orientations the stable homotopy class of R in $\pi_{k-1}\mathrm{GL}(\infty)$ coincides with the Alexander–Ize invariant γ_f .*

PROOF. We will show that R is homotopic to the family of matrices N used in the definition of γ_f . This will prove the proposition.

Let $S: Y_1 \rightarrow X_1$ be the inverse of the operator L_0 restricted to X_1 . An easy calculation (see [38]) gives $R_\lambda = (\mathrm{id} - Q')L_\lambda M_\lambda$, where $M_\lambda \in \mathrm{GL}(X)$ is defined by $M_\lambda = [\mathrm{id} + SQ'(L_\lambda - L_0)]^{-1}$. For small enough D the transversality condition (2.1) is verified with $F = F_0$. Thus the family of subspaces $E_\lambda = L_\lambda^{-1}(F_0)$ form a trivial vector bundle over D .

Given a trivialization $T: E \rightarrow D \times F_0$, denoting with Q_λ the family of projectors on E_λ , the family of isomorphisms $A_\lambda = (Q'L_\lambda + T_\lambda Q_\lambda)^{-1}$ is a parametrix A of $L|_D$. Thus, each A_λ is an isomorphism and we have $L_\lambda A_\lambda = \mathrm{id} + K_\lambda$ with $\mathrm{Im} K_\lambda \subset F_0$ for all $\lambda \in D$. Arguing as in the proof of Lemma 3.3.1 we obtain $A_\lambda|_{F_0} = T_\lambda^{-1}: F_0 \rightarrow E_\lambda$. Using this in the definition of K_λ we get

$$(3.18) \quad N_\lambda = (\mathrm{id} + K_\lambda)|_{F_0} = (\mathrm{id} - Q')L_\lambda T_\lambda^{-1}.$$

We write N_λ in the form

$$(3.19) \quad N_\lambda = (\mathrm{id} - Q')L_\lambda M_\lambda (M_\lambda^{-1} T_\lambda^{-1}).$$

Observing that $M_\lambda^{-1} = \mathrm{id} + SQ'(L_\lambda - L_0)$ sends isomorphically E_λ into E_0 we have that $H_\lambda = M_\lambda^{-1} T_\lambda^{-1}$ sends F_0 isomorphically into E_0 for all $\lambda \in D$. Restricting our families to ∂D we obtain $N_\lambda = R_\lambda H_\lambda$ and hence N is homotopic to RH_0 via the homotopy $h(t, \lambda) = R_\lambda H_{t\lambda}$. Choosing basis in E_0 and F_0 such that the determinant of the matrix of H_0 is 1 we obtain a homotopy between the matrix families $R|_{\partial D}$ and $N|_{\partial D}$. \square

4. Bifurcation of solutions of nonlinear elliptic BVP

Using results from the previous chapters we will prove the criteria for bifurcation of nontrivial solutions of elliptic boundary value problems stated in Section 1.

Our strategy will be as follows: extending the Agranovich reduction [2] to parametrized families of elliptic boundary value problems we will show that $\mathrm{Ind} L$ coincides with the index bundle of a parametrized family \mathcal{S} of pseudo-differential operators of order zero on \mathbb{R}^n belonging to a class of operators introduced by R. T. Seeley in [59]. Then we will use the Atiyah–Singer theorem for operators in this class which states that $\mathrm{Ind} L$ (i.e. analytical index of the family) can be obtained from the principal class by a homomorphism called *topological index*. In our special case the topological index is an isomorphism which coincides up to sign with the inverse of the Bott isomorphism. This makes all calculations

simpler. Using Fedosov’s formula for Chern character of the index bundle and applying well known results about the kernel of J -homomorphism, due to Adams and others, we will obtain criteria for nonvanishing of $J(\text{Ind } L)$ and hence for the appearance of nontrivial solutions of the problem.

4.1. The Agranovich reduction. We will consider particular families of boundary value problems for which the reduction in the title can be carried out. We will work out the reduction for families continuously parametrized by general compact spaces since we will need this generality in [49]. Let

$$\begin{cases} \mathcal{L}_\lambda(x, D) = \sum_{|\alpha| \leq k} a_\alpha(\lambda, x) D^\alpha, \\ \mathcal{B}_\lambda^i(x, D) = \sum_{|\alpha| \leq k_i} b_\alpha^i(\lambda, x) D^\alpha, \quad 1 \leq i \leq r, \end{cases}$$

be a family of linear boundary value problems where the matrix functions $a_\alpha(\lambda, x) \in \mathbb{C}^{m \times m}$, $b_\alpha^i(\lambda, x) \in \mathbb{C}^{1 \times m}$ are smooth in x and continuously depending on a parameter λ belonging to a compact topological space Λ .

The class of problems under consideration is described by axioms (A₁) to (A₃) below.

- (A₁) For all $\lambda \in \Lambda$, the boundary value problem $(\mathcal{L}_\lambda(x, D), \mathcal{B}_\lambda(x, D))$ is elliptic. Namely, $\mathcal{L}_\lambda(x, D)$ is elliptic, properly elliptic at the boundary, and the rows of the boundary operator

$$\mathcal{B}_\lambda(x, D) = (\mathcal{B}_\lambda^1(x, D), \dots, \mathcal{B}_\lambda^r(x, D))^t$$

verify the Shapiro–Lopatinskiĭ condition with respect to $\mathcal{L}_\lambda(x, D)$ (Appendix B).

- (A₂) There exists a $\nu \in \Lambda$ such that, for every $f \in C^\infty(\bar{\Omega}; \mathbb{C}^m)$ and $g \in C^\infty(\partial\Omega; \mathbb{C}^r)$, the problem

$$\begin{cases} \mathcal{L}_\nu(x, D)u(x) = f(x) & \text{for } x \in \Omega, \\ \mathcal{B}_\nu(x, D)u(x) = g(x) & \text{for } x \in \partial\Omega, \end{cases}$$

has a unique smooth solution.

- (A₃) (i) The coefficients $b_\alpha^i(\lambda, x)$, $|\alpha| = k_i$, $1 \leq i \leq r$, of leading terms of boundary operators $\mathcal{B}_\lambda^1(x, D), \dots, \mathcal{B}_\lambda^r(x, D)$ are independent of λ .
 (ii) There exist a compact set $K \subset \Omega$ such that the coefficients $a_\alpha(\lambda, x)$, $|\alpha| = k$ of leading terms of \mathcal{L}_λ , are independent of λ for $x \in \bar{\Omega} - K$.

Under assumption (A₁) the differential operators $(\mathcal{L}_\lambda, \mathcal{B}_\lambda)$ define a continuous family of bounded semi-Fredholm operators (Appendix B)

$$(4.1) \quad (L, B): \Lambda \rightarrow \mathcal{L}(H^{2k+s}(\Omega; \mathbb{C}^m); H^s(\Omega; \mathbb{C}^m) \times H^+(\partial\Omega; \mathbb{C}^r)).$$

By (A₂) and the regularity of solutions of elliptic equations, the kernel of the operator (L_ν, B_ν) reduces to $u \equiv 0$ and its image contains a dense subspace.

Therefore, (L_ν, B_ν) is an isomorphism which on its turn, by the invariance property of the index, shows that the family (L, B) is a continuous family of Fredholm operators of index 0.

We will show that the index bundle of the family (L, B) coincides with the index bundle of a family of a particular class of pseudo-differential operators on \mathbb{R}^n introduced by R. T. Seeley in [59].

A *symbol of class* $S^k(\mathcal{O})$ is a function $\rho \in C^\infty(\mathcal{O} \times \mathbb{R}^n; \mathbb{C}^{m \times m})$ defined on $\mathcal{O} \times \mathbb{R}^n$, where \mathcal{O} is an open subset of \mathbb{R}^n , and verifying following property: for every compact subset K of \mathcal{O} there is a constant C such that, for $x \in K$,

$$(4.2) \quad |D_x^\alpha D_\xi^\beta \rho(x, \xi)| \leq C(1 + |\xi|^{k-\beta}).$$

The set $S^k(\mathcal{O})$ is naturally a Frechet space with the topology induced by the family of seminorms

$$(4.3) \quad \pi_{k,K}^{\alpha\beta}(\rho) = \sup_{x \in K, \xi \in \mathbb{R}^n} (1 + |\xi|)^{\beta-k} |D_x^\alpha D_\xi^\beta \rho(x, \xi)|.$$

A *pseudo-differential operator of order k* acting on the space $\mathcal{D}(\mathcal{O})^m$ of all smooth \mathbb{C}^m -valued functions u with compact support in \mathcal{O} is defined by an integral

$$(4.4) \quad \mathcal{Q}u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\xi} \rho(x, \xi) \hat{u}(\xi) d\xi,$$

where $\rho \in S^k(\mathcal{O})$ and \hat{u} denotes the Fourier transform of u . Every pseudo-differential operator \mathcal{Q} of order k extends to a linear continuous map

$$\mathcal{Q}: H_{\text{comp}}^{k+s}(\mathcal{O}; \mathbb{C}^m) \rightarrow H_{\text{loc}}^s(\mathcal{O}; \mathbb{C}^m).$$

Here $H_{\text{loc}}^s(\mathcal{O}; \mathbb{C}^m)$, is the space of \mathbb{C}^m -valued distributions u on \mathcal{O} such that, for all $\varphi \in \mathcal{D}(\mathcal{O})$, $\varphi u \in H^s(\mathbb{R}^n; \mathbb{C}^m)$, with the topology induced by the family of semi-norms $\|\varphi u\|_s$. The space $H_{\text{comp}}^{k+s}(\mathcal{O}; \mathbb{C}^m)$ is the union over all compact subsets K of \mathcal{O} of

$$H_K^{k+s}(\mathcal{O}; \mathbb{C}^m) = \{u \in H_{\text{loc}}^{k+s}(\mathcal{O}; \mathbb{C}^m) \mid \text{supp } u \subset K\}$$

endowed with the direct limit topology for the family of inclusions.

A pseudo-differential operator \mathcal{L} of order k is said to be elliptic if it possesses a (rough) *parametrix* or *regularizator*. This is a proper ([21]) pseudo-differential operator \mathcal{P} of order $-k$ such that both $\mathcal{L} \circ \mathcal{P} - \text{id}$ and $\mathcal{P} \circ \mathcal{L} - \text{id}$ are of order -1 . A stronger notion of parametrix is used in regularity theory but for the purpose of computing the index bundle this one will be sufficient.

Elliptic differential operators are elliptic in the above sense. As a parametrix of \mathcal{L} one can take the pseudo-differential operator \mathcal{P} associated to the symbol

$$(4.5) \quad \rho(x, \xi) = \phi(|\xi|)p^{-1}(x, \xi), \quad \text{if } x \in \mathcal{O},$$

where $p = \sum_{|\alpha|=k} a_\alpha(x)\xi^\alpha$ is the principal symbol of \mathcal{L} and ϕ is a smooth function with $\phi(r) \equiv 1$ for $r \geq 1$ and $\phi(r) \equiv 0$ on a small neighbourhood of 0.

We will deal only with pseudo-differential operators whose symbols enjoy a further property:

Outside of a small neighbourhood of $\mathcal{O} \times \{0\}$

$$(4.6) \quad \rho(x, \xi) = \rho_s(x, \xi) + \delta(x, \xi), \quad \text{where } \rho_s = \lim_{\mu \rightarrow \infty} \rho(x, \mu\xi)\mu^{-s}$$

is a homogeneous function of degree s defined on $\mathcal{O} \times (\mathbb{R}^n - \{0\})$ and δ is a symbol of order $s - 1$.

This class of pseudo-differential operators contains all differential operators, their parametrices, and is invariant under composition (when defined) and formation of adjoints. The homogeneous function ρ_s will be called the *principal symbol* of the operator. It is uniquely defined by (4.6). Moreover, the principal symbol of a composed operator is the composition of the principal symbols. Much as in the case of differential operators, a pseudo-differential operator with symbol of the form (4.6) is elliptic if and only if its principal symbol $\rho_s(x, \xi)$ is invertible for $\xi \neq 0$. Moreover, the formula (4.5) for the parametrix extends to this class.

Let us discuss now the Agranovich reduction.

The index bundle $\text{Ind}(L, B)$ of a family of elliptic boundary value problems coincides with the index bundle of the family of operators defined by the leading terms of operators $\mathcal{L}_\lambda(x, D)$ and $\mathcal{B}_\lambda(x, D)$ respectively. Indeed, the linear deformation of lower order terms to 0 produces a homotopy between the corresponding Fredholm operators induced on Hardy–Sobolev spaces. Therefore, with regard to the computation of $\text{Ind}(L, B)$ we can safely assume that both \mathcal{L} and $\mathcal{B}^1, \dots, \mathcal{B}^r$ are homogeneous polynomials of degree k and k_i respectively, which we will do from now on. In particular by (A₃) we have that \mathcal{B}_λ is independent of λ .

If K is the compact set arising in assumption (A₃), then for any $x \in \Omega - K$ we have:

$$(4.7) \quad \mathcal{L}_\lambda(x, D) = \mathcal{L}_\nu(x, D).$$

Being ellipticity an open condition, we can extend \mathcal{L} to a parametrized family of elliptic operators (again denoted by \mathcal{L}) defined on a open neighbourhood \mathcal{O} of $\bar{\Omega}$ and such that (4.7) still holds in $\mathcal{O} - K$.

For u of compact support in Ω we have:

$$(4.8) \quad \mathcal{L}_\lambda(x, D)u = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\xi} p(\lambda, x, \xi) \widehat{u}(\xi) d\xi,$$

where p is the principal symbol of the family \mathcal{L} . Let

$$(4.9) \quad \tilde{s}(\lambda, x, \xi) = \phi(|\xi|)p(\lambda, x, \xi)p(\nu, x, \xi)^{-1} + (1 - \phi(|\xi|))\text{id},$$

where ϕ is as in (4.5).

By (A₃), for $x \notin K$, $p(\lambda, x, \xi) = p(\nu, x, \xi)$. Therefore, defining $\tilde{s}(\lambda, x, \xi) = \text{id}$ outside of \mathcal{O} we can extend (4.9) to a continuous map $\tilde{s}: \Lambda \times \mathbb{R}^{2n} \rightarrow \text{GL}(m; \mathbb{C})$.

Each \tilde{s}_λ is a symbol of order 0 on \mathbb{R}^n and, by the very definition of the topology in $S^0(\mathbb{R}^n)$, \tilde{s} is a continuous family of symbols such that $\tilde{s}_\lambda(x, \xi) = \text{id}$ for $x \notin K$.

Let $\tilde{\mathcal{S}}_\lambda$ be the operator associated by (4.4) to the symbol \tilde{s}_λ . Then $\tilde{\mathcal{S}} = \{\tilde{\mathcal{S}}_\lambda\}_{\lambda \in \Lambda}$ is a family of pseudo-differential operators on \mathbb{R}^n .

It follows from (4.9) that the principal symbol of the family is given by

$$(4.10) \quad \sigma(\lambda, x, \xi) = p(\lambda, x, \xi)p(\nu, x, \xi)^{-1}$$

for $x \in K$ and is the identity at points (λ, x, ξ) with $x \notin K$. Moreover, σ extends in an obvious way to a map defined on $\Lambda \times (\mathbb{R}^{2n} - K \times \{0\})$ with values in $\text{GL}(m; \mathbb{C})$.

We will modify the family $\tilde{\mathcal{S}}$ to a family of pseudo-differential operators with the same principal symbol but which has the property of being the “identity at infinity”. For this, let $\psi: \Omega \rightarrow [0, 1]$ be a smooth function which is identically 1 on K and with compact support $K_1 \subset \Omega$ and let

$$(4.11) \quad \mathcal{S}_\lambda = \psi \tilde{\mathcal{S}}_\lambda \psi + (1 - \psi^2)\text{id}.$$

By the composition property, the principal symbol of \mathcal{S}_λ is still the same map σ defined in (4.10) and therefore each \mathcal{S}_λ is elliptic. But now, being $\psi \equiv 0$ outside of K_1 , we have

$$(4.12) \quad [\mathcal{S}_\lambda u](x) = u(x) \quad \text{for } x \notin K_1.$$

Moreover, it is easy to see that the adjoint operator \mathcal{S}_λ^* has the same property.

The class of elliptic pseudo-differential operators such that both the operator and its adjoint verify (4.12) was introduced by R. T. Seeley in [59]. It plays a central role in the proof of the index theorem in [10]. We will denote this class of operators with $\text{Ell}(\mathbb{R}^n)$. By [57, Theorem 1, Section 1.2.3.5] each operator $\mathcal{Q} \in \text{Ell}(\mathbb{R}^n)$ extends to a bounded operator Q from $H^s(\mathbb{R}^n; \mathbb{C}^m)$ into itself. Moreover, the correspondence sending the symbol ρ of the operator to the induced operator Q on $H^s(\mathbb{R}^n; \mathbb{C}^m)$ is a continuous map from $S^0(\mathbb{R}^n)$ into $\mathcal{L}(H^s(\mathbb{R}^n; \mathbb{C}^m))$ endowed with the operator norm topology. Taking into account our previous discussion, the family \mathcal{S} defined by (4.11) induces a family of bounded linear operators $S: \Lambda \rightarrow \mathcal{L}(H^s(\mathbb{R}^n; \mathbb{C}^m))$. If $(\mathcal{L}, \mathcal{B})$ is a smooth family of boundary value problems parametrized by a smooth manifold Λ , then the partials of the symbol of \mathcal{S} with

respect to the coordinates λ_i of λ admit bounds of the form (4.2). Therefore, S is a smooth family whenever $(\mathcal{L}, \mathcal{B})$ is smooth.

The following theorem is a version of the Agranovich reduction [2, Theorem 17.4] for families of elliptic boundary value problems.

THEOREM 4.1.1. *Let $(\mathcal{L}, \mathcal{B})$ be a continuous family of boundary value problems verifying assumptions (A₁)–(A₃), then the family $S: \Lambda \rightarrow \mathcal{L}(H^s(\mathbb{R}^n; \mathbb{C}^m))$ defined above is a family of Fredholm operators of index 0 and*

$$(4.13) \quad \text{Ind}(L, B) = \text{Ind } S.$$

PROOF. We will need to compare the operators S_λ and \mathcal{L}_λ . The latter is elliptic only on \mathcal{O} and may not have an elliptic extension to all of \mathbb{R}^n . This problem can be handled by constructing a compact manifold to which \mathcal{L}_λ extends, but we prefer to avoid this construction and instead we choose a compact neighbourhood W of $\bar{\Omega}$ in \mathcal{O} and we notice that, by (4.12), S_λ sends $H_W^s(\mathcal{O}; \mathbb{C}^m)$ into itself. We will consider S_λ both as a bounded operator on $H^s(\mathbb{R}^n; \mathbb{C}^m)$ and on $H_W^s(\mathcal{O}; \mathbb{C}^m)$ and will split the proof of the Theorem 4.1.1 into a sequence of lemmas:

LEMMA 4.1.2. *$S_\lambda: H_W^s(\mathcal{O}; \mathbb{C}^m) \rightarrow H_W^s(\mathcal{O}; \mathbb{C}^m)$ is Fredholm of index 0. Moreover, $S_\lambda L_\nu - L_\lambda$ is a compact operator from $H_W^s(\mathcal{O}; \mathbb{C}^m)$ into itself.*

PROOF. Each S_λ and, as a matter of fact, any elliptic pseudo-differential operator $\mathcal{Q} \in \text{Ell}(\mathbb{R}^n)$ has a parametrix \mathcal{P} of the same form. Being $\text{id} - \mathcal{P}\mathcal{Q}$ of order -1 , the induced operator $\text{id} - \mathcal{P}\mathcal{Q}: H_W^s(\mathcal{O}; \mathbb{C}^m) \rightarrow H_W^s(\mathcal{O}; \mathbb{C}^m)$ factors through $H_W^{s+1}(\mathcal{O}; \mathbb{C}^m)$. Since $H_W^{s+1}(\mathcal{O}; \mathbb{C}^m)$ is compactly embedded in $H_W^s(\mathcal{O}; \mathbb{C}^m)$, it follows that $\mathcal{P}\mathcal{Q}$ is a compact perturbation of the identity and moreover, the same holds for $\mathcal{Q}\mathcal{P}$. Therefore, $\mathcal{Q}: H_W^s(\mathcal{O}; \mathbb{C}^m) \rightarrow H_W^s(\mathcal{O}; \mathbb{C}^m)$ is Fredholm by a classical characterization of Fredholm operators. Since $S_\nu = \text{id}$, $\text{ind } S_\lambda = 0$ for all λ . The second assertion follows again from the compact embedding of $H_W^{s+1}(\mathcal{O}; \mathbb{C}^m)$ into $H_W^s(\mathcal{O}; \mathbb{C}^m)$ and the fact that the principal symbol of \mathcal{L}_λ coincides with the principal symbol of $S_\lambda \circ \mathcal{L}_\nu$ by the composition property. \square

LEMMA 4.1.3. *Each operator $S_\lambda: H^s(\mathbb{R}^n; \mathbb{C}^m) \rightarrow H^s(\mathbb{R}^n; \mathbb{C}^m)$ is Fredholm. Moreover, the index bundles of S viewed either as a family of Fredholm operators on $H_W^s(\mathcal{O}; \mathbb{C}^m)$ or as a family on $H^s(\mathbb{R}^n; \mathbb{C}^m)$ are the same.*

PROOF. We have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_W^s(\mathcal{O}; \mathbb{C}^m) & \xrightarrow{i} & H^s(\mathbb{R}^n; \mathbb{C}^m) & \xrightarrow{\pi} & H^s(\mathbb{R}^n; \mathbb{C}^m)/iH_W^s(\mathcal{O}; \mathbb{C}^m) \longrightarrow 0 \\ & & \downarrow S_\lambda & & \downarrow S_\lambda & & \downarrow \\ 0 & \longrightarrow & H_W^s(\mathcal{O}; \mathbb{C}^m) & \xrightarrow{i} & H^s(\mathbb{R}^n; \mathbb{C}^m) & \xrightarrow{\pi} & H^s(\mathbb{R}^n; \mathbb{C}^m)/iH_W^s(\mathcal{O}; \mathbb{C}^m) \longrightarrow 0 \end{array}$$

Being the support of $(S_\lambda u - u)$ contained in W , the vertical dashed arrow induced by S_λ in the quotient spaces coincides with the identity. Since an exact sequence of Hilbert spaces splits into a direct sum and since direct sums of Fredholm operators belong to the same class, each $S_\lambda: H^s(\mathbb{R}^n; \mathbb{C}^m) \rightarrow H^s(\mathbb{R}^n; \mathbb{C}^m)$ is Fredholm. Also the second assertion follows from the above diagram and the additivity of index bundle. \square

Let us take a bounded extension operator $E: H^s(\Omega; \mathbb{C}^m) \rightarrow H_W^s(\mathcal{O}, \mathbb{C}^m)$ such that the values of Eu on $\mathcal{O} - \bar{\Omega}$ depend only on the values of u on $\bar{\Omega} - K_1$. In order to obtain such an operator E it is enough to consider the extension from $H^{k+s}(\Omega; \mathbb{C}^m)$ to $H^{k+s}(\mathbb{R}^n; \mathbb{C}^m)$ constructed in [2, Section 3.6], which verifies the above property, multiplied by a smooth function which coincides with 1 on $\bar{\Omega}$ and with support in W . Finally, let $R: H_W^s(\mathcal{O}; \mathbb{C}^m) \rightarrow H^s(\Omega; \mathbb{C}^m)$ be the restriction operator.

LEMMA 4.1.4. $RS_\lambda EL_\nu - L_\lambda: H^{k+s}(\Omega; \mathbb{C}^m) \rightarrow H^s(\Omega; \mathbb{C}^m)$ is compact for all $\lambda \in \Lambda$.

PROOF. Here we closely follow the arguments used in the proof of [2, Theorem 17.4]. Since we are working with a different class of operators, we include the proof for the sake of completeness.

We will first show that:

$$(4.14) \quad S_\lambda(ER - \text{id}) = ER - \text{id} \quad \text{and} \quad (ER - \text{id})S_\lambda = ER - \text{id}.$$

Indeed, denoting with \tilde{S}_λ the operator induced by \tilde{S}_λ on Hardy-Sobolev spaces, by definition of S_λ , we have:

$$S_\lambda ER - S_\lambda = \psi \tilde{S}_\lambda \psi (ER - \text{id}) + (1 - \psi^2)(ER - \text{id}).$$

But $\psi \tilde{S}_\lambda \psi (ER - \text{id}) = 0$ because the support of the function ψ is contained in Ω and $(1 - \psi^2)(ER - \text{id}) = ER - \text{id}$ by the same reason. This proves the first relation in (4.14). The proof of the second relation is similar.

Applying R to the first equation in (4.14) we get

$$(4.15) \quad RS_\lambda ER = RS_\lambda \quad \text{for all } \lambda.$$

Let us represent L_λ defined on $H^{k+s}(\Omega; \mathbb{C}^m)$ in the form $L_\lambda = RL_\lambda E$, where the L_λ on the right hand side is viewed as an operator on $H_W^s(\mathcal{O}; \mathbb{C}^m)$. Using (4.15)

$$(4.16) \quad RS_\lambda EL_\nu - L_\lambda = RS_\lambda ERL_\nu E - RL_\lambda E = RS_\lambda L_\nu E - RL_\lambda E = R(S_\lambda L_\nu - L_\lambda)E$$

is compact by the second assertion in Lemma 4.1.2. \square

LEMMA 4.1.5. *The family S' defined by $S'_\lambda = RS_\lambda E: H^s(\Omega; \mathbb{C}^m) \rightarrow H^s(\Omega; \mathbb{C}^m)$ is a family of Fredholm operators and $\text{Ind } S' = \text{Ind } S$.*

PROOF. We will first show that S'_λ is Fredholm. Using (4.12) if $S'_\lambda u_n \rightarrow f$, then u_n restricted to $\bar{\Omega} - K_1$ converges in $H^s(\Omega - K_1)$ to the restriction of f . By the construction of E , $E u_n \rightarrow E f$ in $H^s_W(\mathcal{O} - K_1)$. It follows that $S_\lambda E u_n \rightarrow E f$. Since S_λ has a closed image, there exist a $w \in H^s_W(\mathcal{O}; \mathbb{C}^m)$ such that $S_\lambda w = E f$. But then the restriction of w to $\mathcal{O} - K_1$ coincides with $E f$ which implies that $ERw = w$ and hence $RS_\lambda ERw = RS_\lambda w = f$. This shows that $\text{Im } S'_\lambda$ is closed.

Applying R to the left of the first equation in (4.14) and E to the right of the second we get

$$(4.17) \quad S'_\lambda R = RS_\lambda \quad \text{and} \quad ES'_\lambda = S_\lambda E.$$

The second equation shows that E sends $\text{Ker } S'_\lambda$ into $\text{Ker } S_\lambda$ and since E is injective, $\dim \text{Ker } S'_\lambda$ is finite. In order to show that $\dim \text{coker } S'_\lambda$ is finite we observe that the first equation in (4.17) shows that $R: H^s_W(\mathcal{O}; \mathbb{C}^m) \rightarrow H^s(\Omega; \mathbb{C}^m)$ sends $\text{Im } S_\lambda$ into $\text{Im } S'_\lambda$ and hence induces

$$\bar{R}: H^s_W(\mathcal{O}; \mathbb{C}^m)/\text{Im } S_\lambda \rightarrow H^s(\Omega; \mathbb{C}^m)/\text{Im } S'_\lambda.$$

Being R surjective, the same holds for \bar{R} and therefore $\dim \text{coker } S'_\lambda$ is finite.

Let us show now that $\text{Ind } S' = \text{Ind } S$. If F is a finite dimensional subspace of $H^s_W(\mathcal{O}; \mathbb{C}^m)$ such that $\text{Im } S_\lambda + F = H^s_W(\mathcal{O}; \mathbb{C}^m)$ for all $\lambda \in \Lambda$, then $H = ER(F)$ enjoys the same property because $(ER - \text{id})(F) \subset \text{Im } S$ by (4.14).

Applying R to both sides we get

$$\text{Im } RS_\lambda + R(H) = H^s(\Omega; \mathbb{C}^m) \quad \text{for all } \lambda \in \Lambda.$$

But, by the first equation in (4.17), $\text{Im } RS_\lambda \subset \text{Im } S'_\lambda$, which shows that $H' = R(H) = R(F)$ is transverse to $\text{Im } S'_\lambda$ for all λ . Notice also that E sends isomorphically H' into H with inverse R . Denoting with G_λ and G'_λ the inverse images of H and H' under S_λ and S'_λ respectively, the second equation in (4.17) implies that $E(G'_\lambda) \subset G_\lambda$. On the other hand, being E injective and since

$$\dim G_\lambda = \dim H = \dim H' = \dim G'_\lambda,$$

it follows that E induces a vector bundle isomorphism between vector bundles G' and G over Λ . Thus

$$\text{Ind } S' = [G'] - \theta(H') = [G] - \theta(H) = \text{Ind } S. \quad \square$$

Now we can complete the proof of Theorem 4.1.1. Let $(\bar{L}_\lambda, \bar{B}_\lambda)_{\lambda \in \Lambda}$ be the family of operators defined as the composition

$$H^{k+s}(\Omega; \mathbb{C}^m) \xrightarrow{(L_\nu, B)} H^s(\Omega; \mathbb{C}^m) \times H^+(\partial\Omega; \mathbb{C}^r) \xrightarrow{S'_\lambda \times \text{id}} H^s(\Omega; \mathbb{C}^m) \times H^+(\partial\Omega; \mathbb{C}^r).$$

By logarithmic property of the index bundle,

$$\text{Ind}(\overline{L}, \overline{B}) = \text{Ind}(L_\nu, B) + \text{Ind}(S' \times \text{id}) = \text{Ind}(S' \times \text{id}),$$

since both L_ν and B are independent from λ . On the other hand, $L - \overline{L}$ is a family of compact operators by Lemma 4.1.4. Hence, so is $(L, B) - (\overline{L}, \overline{B})$, and therefore

$$\text{Ind}(L, B) = \text{Ind}(\overline{L}, \overline{B}) = \text{Ind}(S' \times \text{id}) = \text{Ind } S' = \text{Ind } S,$$

by Lemmas 4.1.4 and 4.1.5. \square

4.2. Proofs of the bifurcation Theorems 1.4.1 and 1.4.2.

PROOF OF THEOREM 1.4.1. It follows from the discussion in the second part of Appendix B that, for $s > n/2$, the family of nonlinear differential operators

$$(\mathcal{F}, \mathcal{G}): \mathbb{R}^q \times C^\infty(\Omega; \mathbb{R}^m) \rightarrow C^\infty(\Omega; \mathbb{R}^m) \times C^\infty(\partial\Omega; \mathbb{R}^r)$$

induces a smooth map

$$(4.18) \quad h = (f, g): \mathbb{R}^q \times H^{k+s}(\Omega; \mathbb{R}^m) \rightarrow H^s(\Omega; \mathbb{R}^m) \times H^+(\partial\Omega; \mathbb{R}^r),$$

with $\mathbb{R}^q \times \{0\}$ as a trivial branch. The Frechet derivative of h_λ at 0 is the operator (L_λ, B_λ) induced by the linearization $(\mathcal{L}_\lambda, \mathcal{B}_\lambda)$ at $u \equiv 0$. Since, for any $\lambda \in \mathbb{R}^q$, $(\mathcal{L}_\lambda, \mathcal{B}_\lambda)$ is elliptic, using Proposition 5.3, we can find a neighbourhood O of 0 in $H^{k+s}(\Omega; \mathbb{R}^m)$ such that $h: \mathbb{R}^q \times O \rightarrow H^s(\Omega; \mathbb{R}^m) \times H^+(\partial\Omega; \mathbb{R}^r)$ is a smooth family of semi-Fredholm maps.

By (H_2) , the family of boundary value problems $(\mathcal{L}, \mathcal{B})$ extends to a smooth family parametrized by S^q which clearly verifies the assumptions (A_1) to (A_3) of Section 4.1 with $\nu = \infty \in S^q$. Hence, the induced family on Hardy–Sobolev spaces also extends to a smooth family

$$(L, B): S^q \rightarrow \mathcal{L}(H^{k+s}(\Omega; \mathbb{R}^m), H^s(\Omega; \mathbb{R}^m) \times H^+(\partial\Omega; \mathbb{R}^r)).$$

Moreover, (L_∞, B_∞) is invertible by (A_2) . Thus (L_λ, B_λ) is Fredholm of index 0, for all $\lambda \in S^q$ and, by continuity of the index of semi-Fredholm operators, the map $h: \mathbb{R}^q \times O \rightarrow H^s(\Omega; \mathbb{R}^m) \times H^+(\partial\Omega; \mathbb{R}^r)$ is a smoothly parametrized family of Fredholm maps of index 0.

In order to simplify our notations, in the rest of this section we will abbreviate (L, B) to L when no confusion arises.

Since h is defined only on the open subset \mathbb{R}^q of S^q we cannot apply directly Theorem 1.2.1 to h in order to find a bifurcation point. Instead we will use the assumption (H_2) in order to compute the local index $\beta(h, \mathbb{R}^q)$ from the family index theorem applied to

$$L = (L, B): S^q \rightarrow \mathcal{L}(H^{k+s}(\Omega; \mathbb{R}^m), H^s(\Omega; \mathbb{R}^m) \times H^+(\partial\Omega; \mathbb{R}^r)).$$

Since L_λ is invertible in a neighbourhood of $\infty \in S^q$, the pair (h, \mathbb{R}^q) is admissible and the local bifurcation index $\beta(h, \mathbb{R}^q) \equiv \beta_{S^q}(h, \mathbb{R}^q)$ is defined. Being L an extension of $Dh_-(0)$ to all of S^q , by the very definition of the local bifurcation index, $\beta(h, \mathbb{R}^q) = J(\text{Ind } L)$.

If, under the hypothesis of Theorem 1.4.1, we can show that $J(\text{Ind } L) \neq 0$ in $J(S^q)$, then the family h must have a bifurcation point $\lambda \in \mathbb{R}^q$, by (B₁). This would complete the proof of the theorem, since from Proposition 5.2.2 it follows that a bifurcation point of the map h is also a bifurcation point for smooth classical solutions of (1.1) in the sense of Definition 1.4.1.

The remaining part of the proof is devoted to show that $J(\text{Ind } L) \neq 0$ in $J(S^q)$. For this, we are going to compute $J(\text{Ind } L)$ from the degree of σ using the complexification L^c of L . Since $\text{Ker } L^c = \text{Ker } L \otimes \mathbb{C}$, from definition of the index bundle in (2.2) it follows that

$$(4.19) \quad \text{Ind } L^c = c(\text{Ind } L),$$

where $c: \widetilde{\text{KO}} \rightarrow \widetilde{K}$ is the complexification homomorphism.

By Bott periodicity, $\widetilde{K}(S^q) = 0$ for q odd, while for $q = 2k$, $\widetilde{K}(S^q)$ is an infinite cyclic group. It is generated by $\xi_q = ([P^1(\mathbb{C}) \times \mathbb{C}] - [H])^k$, where H is the tautological line bundle over the complex projective space $P^1(\mathbb{C}) \cong S^2$. On the other hand, the periodicity theorem for $\widetilde{\text{KO}}$ gives $\widetilde{\text{KO}}(S^q) \cong \mathbb{Z}$ for $q \equiv 0, 4 \pmod{8}$, \mathbb{Z}_2 for $q \equiv 1, 2 \pmod{8}$ and vanishing in all remaining cases. From the homotopy sequence of the fibration of classifying spaces for $\widetilde{\text{KO}}$ and \widetilde{K} (see [60, Section 13.94]) it follows that $c: \widetilde{\text{KO}}(S^q) \rightarrow \widetilde{K}(S^q)$ is an isomorphism for $q \equiv 0 \pmod{8}$ and a monomorphism with image generated by $2\xi_q$ for $q \equiv 4 \pmod{8}$.

For $q = 4s$, we take as generator of $\widetilde{\text{KO}}(S^q)$ an element ν_q such that

$$(4.20) \quad c(\nu_q) = \begin{cases} \xi_q & \text{if } q \equiv 0 \pmod{8}, \\ 2\xi_q & \text{if } q \equiv 4 \pmod{8}. \end{cases}$$

With this choice of generators, each element $\eta \in \widetilde{K}(S^q)$ with $q = 2k$ is uniquely determined by its *degree* $d(\eta) \in \mathbb{Z}$ verifying $\eta = d(\eta)\xi_q$ and, for $q = 4s$, each element η of $\widetilde{\text{KO}}(S^q)$ has a degree defined in the same way.

By (4.20), for any $\eta \in \widetilde{\text{KO}}(S^q)$,

$$(4.21) \quad d(c(\eta)) = \begin{cases} d(\eta) & \text{if } q \equiv 0 \pmod{8}, \\ 2d(\eta) & \text{if } q \equiv 4 \pmod{8}. \end{cases}$$

The degree of an element $\eta \in \widetilde{K}(S^q)$ can be computed as a characteristic number in several ways. We will use the Chern character $\text{ch}: \widetilde{K}(\cdot) \rightarrow \mathbb{H}^*(-; \mathbb{C})$ with values in de Rham cohomology with coefficients in \mathbb{C} which is adequate to our purposes. By Bott's integrality theorem, $\text{ch} = \text{ch}_k: \widetilde{K}(S^{2k}) \rightarrow \mathbb{H}^{2k}(S^{2k}; \mathbb{C})$ is injective with image given by $\text{Im } \text{ch}_k = \mathbb{Z}u_{2k}$, where $u_{2k} = \text{ch}_k(\xi_{2k})$ is the

class of the volume form of S^{2k} [37, Chapter 18, Theorem 9.6]. Hence, for any $\eta \in \tilde{K}(S^{2k})$, we have $\text{ch}(\eta) = d(\eta)u_{2k}$ and therefore

$$(4.22) \quad d(\eta) = \langle \text{ch}(\eta); [S^{2k}] \rangle,$$

the right hand side being the evaluation of $\text{ch}(\eta)$ on the fundamental class $[S^{2k}]$ of the sphere.

Since the complexified family $(\mathcal{L}^c, \mathcal{B}^c)$ verifies assumptions (A₁)–(A₃) of Theorem 4.1.1, $\text{Ind } L^c = \text{Ind } S$, where S is induced by the family of pseudo-differential operators \mathcal{S}_λ defined by (4.11). By Fedosov's formula (see Appendix C) with $j = n + k$, we get

$$(4.23) \quad \text{ch}(\text{Ind } S) = K_k \oint_{S^{2n-1}} \text{tr}(\sigma^{-1} d\sigma)^{2(n+k)-1},$$

where \oint denotes the integral along the fiber and

$$K_k = -\frac{(n+k-1)!}{(2\pi i)^{n+k}(2n+2k-1)!}.$$

The evaluation on the fundamental class in de Rham cohomology is given by integration over the sphere. Hence, using Fubini's theorem for integration along the fiber, from (4.22) we get

$$(4.24) \quad \begin{aligned} d(\text{Ind } L^c) &= K_k \int_{S^{2k}} \oint_{S^{2n-1}} \text{tr}(\sigma^{-1} d\sigma)^{2(n+k)-1} \\ &= K_k \int_{S^{2k} \times S^{2n-1}} \text{tr}(\sigma^{-1} d\sigma)^{2(n+k)-1}, \end{aligned}$$

where the right hand side is the ordinary integration of the $(2k + 2n - 1)$ -form $\text{tr}(\sigma^{-1} d\sigma)^{2(n+k)-1}$ over $S^{2k} \times S^{2n-1}$.

Thus $d(\text{Ind } L^c)$ coincides with $d(\sigma)$ defined in (1.3). Using (4.21) we obtain

$$(4.25) \quad d(\text{Ind } L) = \begin{cases} d(\sigma) & \text{if } q \equiv 0 \pmod{8}, \\ \frac{1}{2}d(\sigma) & \text{if } q \equiv 4 \pmod{8}. \end{cases}$$

On the other hand, for $q = 4s$, $J(S^q) \simeq Z_{m(q/2)}$ and $J(\text{Ind } L) = 0$ if and only if $d(\text{Ind } L)$ is divisible by $m(q/2)$. Now, Theorem 1.4.1 follows from (4.25) and the definition of $n(q)$ in (1.5). \square

PROOF OF THEOREM 1.4.2. Let us first recall the clutching construction. Given a continuous map $G: S^{q-1} \rightarrow \text{GL}(m; \mathbb{C})$, taking two trivial complex bundles of rank m over the upper and lower hemispheres D_\pm of S^q we obtain a bundle η_G over S^q by identifying $(\lambda, v) \in \partial D_+ \times \mathbb{C}^m$ with $(\lambda, G_\lambda v) \in \partial D_- \times \mathbb{C}^m$. The isomorphism class of η_G depends only on the homotopy class of G . Moreover, the clutching construction extends to an isomorphism between $\pi_{q-1} \text{GL}(\infty; \mathbb{C})$ and $\tilde{K}(S^q)$. For $q \geq 2$, an analogous construction establishes an isomorphism of

$\pi_{q-1}\mathrm{GL}(\infty; \mathbb{R})$ with $\widetilde{\mathrm{KO}}(S^q)$ which coincides with the inverse of the isomorphism ∂_0 in Lemma 3.3.1.

Let $q = 2k$. If the map G is smooth, choosing appropriate connection-forms on D_{\pm} , one can compute $d(\eta_G)$ as

$$(4.26) \quad d(\eta_G) = \langle ch_k(\eta_G); [S^{2k}] \rangle = \frac{\pm(k-1)!}{(2\pi i)^k (2k-1)!} \int_{S^{2k-1}} \mathrm{tr}(G^{-1}dG)^{2k-1}.$$

A proof of this can be found in Section 3.2 of [28] (see also [6] in the the real case).

For $q = 4s$, let $\lambda_0 = 0$ be an isolated singular point of L . Without loss of generality we can assume that λ_0 is the north pole of S^q and that the open neighbourhood U isolating λ_0 from the rest of $\Sigma(\bar{L})$ contains the upper hemisphere D_+ .

We extend $L|_{D_+}$ to a family \tilde{L} defined on all of S^q such that \tilde{L}_λ is an isomorphism for $\lambda \in D_-$. If A_+ is any parametrix for L_+ and if we take as $A_- = \tilde{L}_-^{-1}$, then, arguing as in the proof of Proposition 3.3.2, we can show that the homomorphism ∂ of the diagram (3.12) sends \tilde{L} to the family of matrices N whose stable homotopy class is taken as definition of γ_f in Section 3.3. By commutativity of the diagram (3.12) and since the clutching construction is the inverse of ∂_0 , we have

$$(4.27) \quad \mathrm{Ind}(L, U) = \mathrm{Ind} \tilde{L} = [\eta_N].$$

As in the proof of Theorem 1.4.1 we can compute $d(\mathrm{Ind}(L, U))$ from the complexification of $[\eta_N]$. It is easy to see that $c[\eta_N]$ is the vector bundle associated by the clutching construction to the complexification N^c of N . By (4.21),

$$(4.28) \quad d(\mathrm{Ind}(L, U)) = \begin{cases} d(\eta_{N^c}) & \text{if } q \equiv 0 \pmod{8}, \\ \frac{1}{2}d(\eta_{N^c}) & \text{if } q \equiv 4 \pmod{8}. \end{cases}$$

By Proposition 3.3.4, R is homotopic to N and hence from (4.26) we obtain

$$(4.29) \quad d(\eta_{N^c}) = d(\eta_{R^c}) = (-1)^{s+1} \frac{(2s-1)!}{(2\pi)^{2s} (4s-1)!} \int_{S^{4s-1}} \mathrm{tr}(R^{c-1}dR^c)^{4s-1}.$$

The right hand side of (4.29) coincides with the degree $d(\lambda_0)$ defined in (1.8) because $\mathrm{tr}(R^{c-1}dR^c)^{4s-1} = \mathrm{tr}(R^{-1}dR)^{4s-1}$. This gives

$$(4.30) \quad d(\mathrm{Ind}(L, U)) = \begin{cases} d(\lambda_0) & \text{if } q \equiv 0 \pmod{8}, \\ \frac{1}{2}d(\lambda_0) & \text{if } q \equiv 4 \pmod{8}. \end{cases}$$

Now, the assertion (a) follows from (4.25), (4.30) and the additivity property (3.3) of the index bundle. Under the isomorphism $J(S^q) \simeq \mathbb{Z}_{m(q/2)}$, $J(\mathrm{Ind}(L, U))$ coincides with mod $m(q/2)$ reduction of $d(\mathrm{Ind}(L, U))$. Thus the first part of (b) follows from (4.30), the definition of $n(q)$ and (B₁). For the second part

it is enough to observe that if $d(\sigma) - d(\lambda_0)$ is not a multiple of $n(q)$, then $\beta(h, \Lambda - \{\lambda_0\}) \neq 0$ in $J(S^q)$, by additivity of the bifurcation index. \square

5. Appendix

A. Properties of the index bundle. Since our construction of the index bundle differs from the one in [7], [41], we briefly describe the proofs of its properties.

PROPOSITION 5.1. *The index bundle $\text{Ind } L$ verifies:*

- (a) (Functoriality) *If $L: \Lambda \rightarrow \Phi(X, Y)$ be a family of Fredholm operators and $\alpha: \Sigma \rightarrow \Lambda$ is a continuous map between compact spaces, then*

$$\text{Ind } L \circ \alpha = \alpha^*(\text{Ind } L),$$

where $\alpha^: \text{KO}(\Lambda) \rightarrow \text{KO}(\Sigma)$ is the homomorphism induced by α .*

- (b) (Homotopy invariance) *Let $H: [0, 1] \times \Lambda \rightarrow \Phi(X, Y)$ be a homotopy, then $\text{Ind } H_0 = \text{Ind } H_1$.*
- (c) (Additivity) $\text{Ind } (L \oplus M) = \text{Ind } L + \text{Ind } M$.
- (d) (Logarithmic property) $\text{Ind } (LM) = \text{Ind } L + \text{Ind } M$.
- (e) (Normalization) $\text{Ind } L = 0$ *if L is homotopic to a family in $\text{GL}(X, Y)$. Moreover, the converse holds if Y is a Kuiper space.*

PROOF. Taking the same subspace V in the definition of the index bundle for both L and $L \circ \alpha$, property (a) follows plainly from the definition of $\alpha^*(E)$. Now, (b) follows from (a) applied to the top and bottom inclusions of Λ in $[0, 1] \times \Lambda$. The proof of (c) is straightforward. Assuming $X = Y = Z$, (d) reduces to (c) thanks to a well known homotopy between $\text{id} \oplus LM$ and $L \oplus M$ [16, Theorem 7.2]. The general case follows easily from this. Another way to prove (d) is by observing that in the construction of the index bundle one can take instead of a finite dimensional subspace V of Y any finite dimensional subbundle of $\Lambda \times Y$ transverse to L . Now, if $\Theta(V)$ is transverse to LM , then $\Theta(V)$ is transverse to L and $E = L^{-1}\Theta(V)$ is transverse to M . Then, denoting by $F = M^{-1}E$, in $\text{KO}(\Lambda)$ we have

$$(5.1) \quad \text{Ind } (LM) = [F] - [\Theta(V)] = ([F] - [E]) + ([E] - [\Theta(V)]) = \text{Ind } L + \text{Ind } M.$$

The proof of (e) can be found in [30, Theorem 1.6.3]. \square

B. Elliptic boundary value problems.

B.1. *Linear elliptic boundary value problems.* We begin with a brief summary of the relevant linear theory. We will work over the field \mathbb{C} of complex numbers considering real coefficients as a special case. For nonlinear systems it becomes natural to take the opposite viewpoint.

For $\alpha = (\alpha_1, \dots, \alpha_n)$ an n -tuple of nonnegative integers, we set

$$D_j = i^{-1} \frac{\partial}{\partial x_j}, \quad D^\alpha = \prod_{i=1}^n (D_i)^{\alpha_i}, \quad |\alpha| = \sum_{i=1}^n \alpha_i \quad \text{and for } \xi \in \mathbb{C}^n, \quad \xi^\alpha = \prod_{i=1}^n \xi_i^{\alpha_i}.$$

Let Ω be an open bounded subset of \mathbb{R}^n with smooth boundary. We will consider partial differential operators acting on smooth vector functions $u: \Omega \rightarrow \mathbb{C}^m$ of the form

$$(5.2) \quad \mathcal{L}(x, D)u = \sum_{|\alpha| \leq k} a_\alpha(x) D^\alpha u(x),$$

where $a_\alpha \in C^\infty(\bar{\Omega}; \mathbb{C}^{m \times m})$. The *principal part* of \mathcal{L} is the expression (5.2) containing only the leading terms with $|\alpha| = k$. The *principal symbol* of \mathcal{L} is the matrix function p defined on $\Omega \times \mathbb{C}^n$ by

$$(5.3) \quad p(x, \xi) \equiv \sum_{|\alpha|=k} \xi^\alpha a_\alpha(x).$$

The operator $\mathcal{L}(x, D)$ is called elliptic if its principal symbol verifies

$$(5.4) \quad \det p(x, \xi) \neq 0 \quad \text{for all } x \in \bar{\Omega}, \quad \xi \in \mathbb{R}^n - \{0\}.$$

$\mathcal{L}(x, D)$ is called *properly elliptic* if $km = 2r$ and for any $x \in \partial\Omega$ and any vector $\xi \neq 0$ tangent to the boundary at x , denoting with η be the inward normal to $\partial\Omega$ at x , we have that the polynomial $\det p(x, \xi + z\eta)$ has exactly r roots in the upper half-plane $\Im z > 0$. If we introduce coordinates (y_1, \dots, y_n) at x such that $\partial\Omega$ is defined in a neighbourhood of x by $y_n = 0$, then, in terms of the ordinary differential operator $p(y_1, \dots, y_{n-1}, 0, \xi, i^{-1}d/dt)$, the above condition means that the subspaces $M^\pm(x, \xi)$ of $L^2(\mathbb{R}_\pm; \mathbb{C}^m)$ whose elements are exponentially decaying solutions of the system $p(y_1, \dots, y_{n-1}, 0, \xi, i^{-1}d/dt)v(t) = 0$ at ∞ and $-\infty$ have dimension r .

Let $\mathcal{L}(x, D)$ be an elliptic operator of order k , properly elliptic at the boundary and let $k_i, 1 \leq i \leq r$, be integers such that $0 \leq k_i \leq k - 1$. We will consider r operators $\{\mathcal{B}^1(x, D), \dots, \mathcal{B}^r(x, D)\}$ of order k_i .

$$(5.5) \quad \mathcal{B}^i(x, D)u = \sum_{|\alpha| \leq k_i} b_\alpha^i(x) D^\alpha u(x),$$

where $b_\alpha^i \in C^\infty(\bar{\Omega}; \mathbb{C}^{1 \times m})$.

The *boundary operator* is the operator matrix $\mathcal{B}(x, D)$ whose i -th row is $\mathcal{B}^i(x, D)$. Thus $\mathcal{B}(x, D) = [\mathcal{B}^1(x, D), \dots, \mathcal{B}^r(x, D)]^t$.

The principal symbol of the boundary operator $\mathcal{B}(x, D)$ is by definition the matrix function $p_b(x, \xi)$ whose i -th row is

$$(5.6) \quad p_b^i(x, \xi) = \sum_{|\alpha|=k_i} \xi^\alpha b_\alpha^i(x).$$

The boundary operator \mathcal{B} verifies the *Shapiro–Lopatinskiĭ condition* with respect to $\mathcal{L}(x, D)$ if, for each $x \in \partial\Omega$ and $\xi \in \mathbb{R}^n \setminus \{0\}$ belonging to $T_x\partial\Omega$, the subspace $M^+(x, \xi)$ is isomorphic to \mathbb{C}^r via the map

$$u \mapsto \left[p_b \left(y_1, \dots, y_{n-1}, 0, \xi, i^{-1} \frac{d}{dt} \right) v \right] (0).$$

Since the condition involves only ordinary differential equations with constant coefficients, it can be reformulated in purely algebraic terms [1] but we will not use this formulation here.

DEFINITION 5.2. Given an open bounded subset Ω of \mathbb{R}^n with smooth boundary, an elliptic boundary value problem on Ω is a pair $(\mathcal{L}, \mathcal{B})$ where $\mathcal{L} = \mathcal{L}(x, D)$ is an elliptic operator on Ω , properly elliptic at the boundary, and the boundary operator $\mathcal{B} = \mathcal{B}(x, D)$ verifies the Shapiro–Lopatinskiĭ condition with respect to \mathcal{L} .

For any manifold M , with or without boundary, there is an associated scale of Hardy-Sobolev spaces $H^s(M, \mathbb{C}^m)$, $s \in \mathbb{R}$ [21]. Every $u \in H^s(M; \mathbb{C}^m)$ has a well defined restriction to ∂M belonging to $H^{s-1/2}(\partial M; \mathbb{C}^m)$ and continuously depending on u . When $s \in \mathbb{N}$ and $M = \Omega$ an open subset of \mathbb{R}^n with smooth boundary, denoting with $D^\alpha u$ the distributional derivative, we have

$$H^s(\Omega; \mathbb{C}^m) = \{u \in L^2(\Omega; \mathbb{C}^m) \mid D^\alpha u \in L^2(\Omega; \mathbb{C}^m) \text{ for all } |\alpha| \leq s\}$$

with the norm $\|u\|_s = \sum_{|\alpha| \leq s} |D^\alpha u|_2$.

Let $\tau: C^\infty(\bar{\Omega}) \rightarrow C^\infty(\partial\Omega)$ be the trace operator. The operator

$$(\mathcal{L}, \tau\mathcal{B}): C^\infty(\bar{\Omega}; \mathbb{C}^m) \rightarrow C^\infty(\bar{\Omega}; \mathbb{C}^m) \times C^\infty(\partial\Omega; \mathbb{C}^r)$$

extends to a bounded operator

$$(5.7) \quad (L, B): H^{k+s}(\Omega; \mathbb{C}^m) \rightarrow H^s(\Omega; \mathbb{C}^m) \times H^+(\partial\Omega; \mathbb{C}^r),$$

where $H^+(\partial\Omega; \mathbb{C}^r)$ denotes $\prod_{i=1}^r H^{k+s-k_i-1/2}(\partial\Omega; \mathbb{C})$.

For any elliptic boundary value problem $(\mathcal{L}, \mathcal{B})$ the following Schauder type estimate holds [1]: there exists a constant $c > 0$ such that for any $u \in H^{k+s}(\Omega)$

$$(5.8) \quad \|u\|_{k+s} \leq c \left(\|L(u)\|_s + \sum_{i=1}^r \|B_i(u)\|_{k+s-k_i-1/2} + \|u\|_s \right).$$

It follows easily from the above estimate that the operator (L, B) has finite-dimensional kernel and closed image. Namely, (L, B) is left semi-Fredholm.

B.2. Nonlinear elliptic boundary value problems. Denoting with k^* the number of multi-indices α with $|\alpha| \leq k$, the k -jet extension

$$j_k: C^\infty(\bar{\Omega}, \mathbb{R}^m) \rightarrow C^\infty(\bar{\Omega}, \mathbb{R}^{mk^*})$$

is defined by $(j_k u(x))_\alpha = D^\alpha u(x)$ for $|\alpha| \leq k$.

Given a continuous family of smooth maps $\mathcal{F}: \Lambda \times \bar{\Omega} \times \mathbb{R}^{mk^*} \rightarrow \mathbb{R}^m$ we will informally write $\mathcal{F}(\lambda, x, u(x), \dots, D^k u(x))$ for $\mathcal{F}(\lambda, x, j_k u(x))$. As in the case of linear differential operators we will not distinguish in the notation the map \mathcal{F} from the family of nonlinear operators $\mathcal{F}: \Lambda \times C^\infty(\bar{\Omega}; \mathbb{R}^m) \rightarrow C^\infty(\bar{\Omega}; \mathbb{R}^m)$ defined by the above expression. However, we will use roman alphabet to denote the corresponding operators induced in Hardy–Sobolev spaces.

An argument based Sobolev’s embedding theorems, shows that for $s > n/2$ (which we will always assume) the family $\mathcal{F}(\lambda, x, u(x), \dots, D^k u(x))$ extends to a continuous family of smooth maps $f: \Lambda \times H^{k+s}(\Omega; \mathbb{R}^m) \rightarrow H^s(\Omega; \mathbb{R}^m)$. Moreover, if Λ is a smooth manifold and \mathcal{F} is smooth, so is f .

Indeed, for any $\lambda \in \Lambda$, the nonlinear operator f_λ is the composition of j_k with the Nemytskij operator associated to the map \mathcal{F}_λ . The operator j_k extends to a bounded linear operator from $H^{k+s}(\Omega; \mathbb{R}^m)$ to $H^s(\Omega; \mathbb{R}^m)$ while, for $s > n/2$, the associated Nemytskiĭ operator induces a smooth map from $H^s(\Omega; \mathbb{R}^m)$ into itself (see [48, Theorem 11.3]). Moreover, the argument used in the proof of [48, Theorem 11.3] automatically gives the continuous dependence on parameters of the derivatives of f_λ , if \mathcal{F} is a continuous family of smooth maps. The same argument allows to show that f is smooth if so is \mathcal{F} .

Together with \mathcal{F} , we will consider r nonlinear boundary conditions of order k_i with $0 \leq k_i \leq k - 1$. These are defined by r continuous families of smooth maps $\mathcal{G}_i: \Lambda \times \bar{\Omega} \times \mathbb{R}^{mk_i^*} \rightarrow \mathbb{R}$, $1 \leq i \leq r$.

Composing the obvious projections from \mathbb{R}^{mk^*} into $\mathbb{R}^{mk_i^*}$ with the functions \mathcal{G}_i , $1 \leq i \leq r$, we obtain a map $\mathcal{G} = (\mathcal{G}_1 \dots \mathcal{G}_r): \Lambda \times \bar{\Omega} \times \mathbb{R}^{mk^*} \rightarrow \mathbb{R}^r$ and hence a family of nonlinear *boundary operators* $\mathcal{G}: \Lambda \times C^\infty(\bar{\Omega}; \mathbb{R}^m) \rightarrow C^\infty(\partial\Omega; \mathbb{R}^r)$ defined by

$$(5.9) \quad \mathcal{G}(\lambda, u) = (\tau\mathcal{G}_1(\lambda, x, u, \dots, D^{k_1}u), \dots, \tau\mathcal{G}_r(\lambda, x, u, \dots, D^{k_r}u)),$$

where τ is the restriction to the boundary.

The above discussion, together with the well known continuity property of the trace τ , allows to conclude that the map $(\mathcal{F}, \mathcal{G})$ extends to a continuously parametrized family of smooth maps

$$(5.10) \quad (f, g): \Lambda \times H^{k+s}(\Omega; \mathbb{R}^m) \rightarrow H^s(\Omega; \mathbb{R}^m) \times H^+(\partial\Omega; \mathbb{R}^r).$$

For each fixed λ , the *linearization* of $(\mathcal{F}_\lambda, \mathcal{G}_\lambda)$ at a smooth function w is the linear operator:

$$(5.11) \quad \begin{aligned} \mathcal{L}_\lambda(x, D)u(x) &= \sum_\alpha a_\alpha(\lambda, x)D^\alpha u(x), \\ \mathcal{B}_\lambda(x, D)u(x) &= \tau \sum_\alpha b_\alpha(\lambda, x)D^\alpha u(x), \end{aligned}$$

where, denoting by $v_{j\alpha}$ the variable corresponding to $D^\alpha u_j$, the ij -entries of the matrices $a_\alpha \in C^\infty(\Lambda \times \bar{\Omega}; \mathbb{R}^{m \times m})$ and $b_\alpha \in C^\infty(\Lambda \times \bar{\Omega}; \mathbb{R}^{r \times m})$ are

$$(5.12) \quad a_\alpha^{ij}(\lambda, x) = \frac{\partial \mathcal{F}_i}{\partial v_{j\alpha}}(\lambda, x, w(x)) \quad \text{and} \quad b_\alpha^{ij}(\lambda, x, w(x)) = \frac{\partial \mathcal{G}_i}{\partial v_{j\alpha}}(\lambda, x, w(x)).$$

By [48, Theorem 11.3], for each $\lambda \in \Lambda$ and w smooth, the Frechet derivative of the map (f_λ, g_λ) at w is the operator

$$(5.13) \quad (L_\lambda, B_\lambda): H^{k+s}(\Omega; \mathbb{R}^m) \rightarrow H^s(\Omega; \mathbb{R}^m) \times H^+(\partial\Omega; \mathbb{R}^r)$$

induced on Hardy–Sobolev spaces by the differential operator (5.11).

A differentiable map is *semi-Fredholm* if the Frechet derivative at any point is a linear semi-Fredholm operator.

PROPOSITION 5.3. *Let $(\mathcal{F}, \mathcal{G})$ be as above, with $\mathcal{F}(\lambda, x, 0) = 0$, $\mathcal{G}(\lambda, x, 0) = 0$. If, for each λ , the linearization $(\mathcal{L}_\lambda, \mathcal{B}_\lambda)$ of $(\mathcal{F}_\lambda, \mathcal{G}_\lambda)$ at $u \equiv 0$ is elliptic and $s > n/2$, then there exists an open ball $B = B(0, r) \subset H^{k+s}(\Omega; \mathbb{R}^m)$ such that the map*

$$h = (f, g): \Lambda \times B \rightarrow H^s(\Omega; \mathbb{R}^m) \times H^+(\partial\Omega; \mathbb{R}^r)$$

induced by $(\mathcal{F}, \mathcal{G})$ is a continuously parametrized family of smooth semi-Fredholm maps. Moreover, if Λ is a smooth manifold and $(\mathcal{F}, \mathcal{G})$ is a smooth, then so is h .

PROOF. Since the estimate (5.8) holds, each (L_λ, B_λ) is semi-Fredholm. On the other hand, the set of all semi-Fredholm operators is open. From this, by compactness of Λ , we can find a ball $B(0, r)$ such that $D(h_\lambda)(u)$ is semi-Fredholm for any $u \in B(0, r)$. This proves the first assertion. The second is clear. \square

As a matter of fact, under our assumptions, the map $h = (f, g)$ is a family of Fredholm maps. This follows from the existence of a rough parametrix of an elliptic boundary value problem [3], [63]. While the above proposition will be sufficient for most of our needs, we will use the parametrix in order to prove that the set of bifurcation points of the family h arising in the proof of the Theorem 1.4.1 coincides with the set of bifurcation point of the elliptic system (1.1) in the sense of Definition 1.4.1.

We will be sketchy in what follows, since the method is standard and we have only to notice that the construction of a parametrix of an elliptic boundary value problem depends smoothly on parameters (see [63, Theorem 9.32], and also [2, Theorem 16.5], where boundary value problems for pseudo-differential operators with limited degree of smoothness are considered).

PROPOSITION 5.4. *Let the system (1.1) verify the assumptions of the Theorem 1.4.1, and let $s > n/2$. Then the set B of all bifurcation points of (1.1) in*

the sense of Definition 1.4.1 coincides with the set $\text{Bif}(h)$ of bifurcation points of the family

$$h: \mathbb{R}^q \times H^{k+s}(\Omega; \mathbb{R}^m) \rightarrow H^s(\Omega; \mathbb{R}^m) \times H^+(\partial\Omega; \mathbb{R}^r)$$

defined by (4.18).

PROOF. Clearly $B \subset \text{Bif}(h)$. In order to prove the opposite inclusion we will use the standard elliptic bootstrap. Keeping our previous notation, $v_{i\alpha}$ (resp. $v'_{i\alpha}$) will denote the components of a vector $v \in \mathbb{R}^{mk^*}$ (resp. $v' \in \mathbb{R}^{mk^*}$).

Since $\mathcal{F}(\lambda, x, 0) = 0$, $\mathcal{G}(\lambda, x, 0) = 0$, applying [47, Lemma 2.1] to each component of \mathcal{F} and to each \mathcal{G}_i we can write $(\mathcal{F}, \mathcal{G})$ in the form:

$$(5.14) \quad \begin{aligned} \mathcal{F}(\lambda, x, v) &= \sum_{|\alpha| \leq k} a_\alpha(\lambda, x, v) v_\alpha, \\ \mathcal{G}_i(\lambda, x, v') &= \tau \sum_{|\alpha| \leq k_i} b_\alpha^i(\lambda, x, v') v'_{i\alpha}; \quad 1 \leq i \leq r. \end{aligned}$$

where $v_\alpha = (v_{1\alpha} \dots v_{m\alpha})^t$,

In order to simplify notations, we reparametrize each family $\mathcal{B}_{\lambda, v'}^i(x, D)$ by $v \in \mathbb{R}^{mk^*}$ using the projectors $\pi: \mathbb{R}^{mk^*} \rightarrow \mathbb{R}^{mk_i^*}$. In this way we obtain a family of boundary operators

$$\mathcal{B}_{\lambda, v}(x, D) = [\mathcal{B}_{\lambda, v}^1(x, D), \dots, \mathcal{B}_{\lambda, v}^r(x, D)]^t$$

parametrized by $\mathbb{R}^q \times \mathbb{R}^{mk^*}$.

Putting $v = j_k(u)$ we have written the map

$$(\mathcal{F}, \mathcal{G}): \Lambda \times C^\infty(\bar{\Omega}; \mathbb{R}^m) \rightarrow C^\infty(\bar{\Omega}; \mathbb{R}^m) \times C^\infty(\partial\Omega; \mathbb{R}^r)$$

in the form

$$(5.15) \quad \begin{aligned} \mathcal{F}(\lambda, x, u, \dots, D^k u) &= \mathcal{L}_{\lambda, j_k(u)}(x, D)u, \\ \mathcal{G}(\lambda, x, u, \dots, D^k u) &= \tau \mathcal{B}_{\lambda, j_k(u)}(x, D)u. \end{aligned}$$

where \mathcal{L}, \mathcal{B} linear differential operators depending on parameters $(\lambda, u) \in \Lambda \times C^\infty(\bar{\Omega}; \mathbb{R}^m)$.

Now let us take $v = 0 \in \mathbb{R}^{mk^*}$ and observe that, by [47, Lemma 2.1], the pair $(\mathcal{L}_{\lambda, 0}(x, D), \mathcal{B}_{\lambda, 0}(x, D))$ coincides with the linearization (5.11) of the map $(\mathcal{F}, \mathcal{G})$ at $u = 0$, which is elliptic by hypothesis. It follows from this that for small enough ε the restriction of the family

$$\mathcal{H}_{\lambda, v}(x, D) = (\mathcal{L}_{\lambda, v}(x, D), \mathcal{B}_{\lambda, v}(x, D))$$

to $\mathbb{R}^q \times B(0, \varepsilon) \subset \mathbb{R}^q \times \mathbb{R}^{mk^*}$ is a family of elliptic boundary value problems.

Let us denote by

$$H_{\lambda, v} = (L_{\lambda, v}, B_{\lambda, v}): H^{k+s}(\Omega; \mathbb{R}^m) \rightarrow H^s(\Omega; \mathbb{R}^m) \times H^+(\partial\Omega; \mathbb{R}^r)$$

the operator induced by $\mathcal{H}_{\lambda,v}$, on Hardy–Sobolev spaces.

The construction of a rough parametrix of an elliptic boundary value problem (see the proof of [63, Theorem 9.32]) uses the inverse of the principal symbol, the canonical basis at points of the boundary and localization via smooth partitions of unity. Since each of the above objects behave well with respect to smooth variation of parameters, it follows that any smooth family of elliptic boundary value problems possesses a smooth parametrix on a neighbourhood of a given point in the parameter space.

Now, let $\lambda_* \in \text{Bif}(h)$, and let \mathcal{P} be a left parametrix of the family \mathcal{H} restricted to a neighbourhood N of $(\lambda_*, 0)$ in $\mathbb{R}^q \times B(0, \varepsilon)$.

By definition, \mathcal{P} is a family of operators

$$P_{\lambda,v}: C^\infty(\overline{\Omega}; \mathbb{R}^m) \times C^\infty(\partial\Omega; \mathbb{R}^r) \rightarrow C^\infty(\overline{\Omega}; \mathbb{R}^m)$$

smoothly varying with $(\lambda, v) \in N$ which extends to a smooth family of operators

$$P_{\lambda,v}: H^s(\Omega; \mathbb{R}^m) \times H^+(\partial\Omega; \mathbb{R}^r) \rightarrow H^{k+s}(\Omega; \mathbb{R}^m)$$

such that

$$(5.16) \quad K_{\lambda,v} = P_{\lambda,v}H_{\lambda,v} - \text{id}_{H^{k+s}(\Omega; \mathbb{R}^m)}$$

is a smooth family of bounded operators from $H^{k+s}(\Omega; \mathbb{R}^m)$ into $H^{k+s+1}(\Omega; \mathbb{R}^m)$.

Since $s > n/2$, $H_{\lambda,j^m(u)}$, $P_{\lambda,j^m(u)}$ and $K_{\lambda,j^m(u)}$ define three continuous families of bounded operators parametrized by a neighbourhood W of $(\lambda_*, 0)$ in $\mathbb{R}^q \times H^{k+s}(\Omega; \mathbb{R}^m)$.

Using (5.15) we can rewrite the restriction of h to W in the form

$$(5.17) \quad h(\lambda, u) = H_{\lambda,u}u$$

and therefore, by (5.16),

$$(5.18) \quad P_{\lambda,u}h(\lambda, u) = u + K_{\lambda,u}u$$

If $(\lambda_n, u_n) \rightarrow (\lambda_*, 0)$ and $h(\lambda_n, u_n) = 0$, by (5.18), $u_n = -K_{\lambda_n, u_n}u_n$ belongs to $H^{k+s+1}(\Omega; \mathbb{R}^m)$ and $u_n \rightarrow 0$ in $H^{k+s+1}(\Omega; \mathbb{R}^m)$ as well. Iterating this and using Sobolev embedding theorems we obtain that $u_n \rightarrow 0$ in $C^j(\Omega; \mathbb{R}^m)$ for any j , which proves that λ_* belongs to B . \square

C. Fedosov’s formula. Given a smooth manifold M , $\mathbb{H}_c^{\text{ev}}(M; \mathbb{C})$ will denote de Rham cohomology of complex valued compactly supported forms of even degree. The Chern-character is a natural transformation $\text{ch}: K_c(\cdot) \rightarrow \mathbb{H}_c^{\text{ev}}(\cdot; \mathbb{C})$ preserving the module structure over the ring $K(\cdot)$ and $\mathbb{H}^{\text{ev}}(\cdot; \mathbb{C})$, respectively. If Λ is a compact manifold, the cohomological version of the Atiyah–Singer theorem for families $\mathcal{S}: \Lambda \rightarrow \text{Ell}(\mathbb{R}^n)$ states:

$$(5.19) \quad \text{ch Ind } S = (-1)^n p_*(\text{ch}[\sigma]) \quad \text{in } \mathbb{H}^{\text{ev}}(\Lambda; \mathbb{C}).$$

Here p_* is the push-forward homomorphism in de Rham cohomology called also *integration along the fiber*. Integration along the fiber can be defined directly on differential forms. Acting on compactly supported forms on the total space of a smooth fiber bundle $\pi: E \rightarrow \Lambda$ with fiber F , the *integration along the fiber* \oint_F is defined as follows: Let us denote with $\Omega_c^*(E) = \bigoplus_i \Omega_c^i(E)$ the smooth forms of mixed degree with compact support on E . In local coordinates $(\lambda_1, \dots, \lambda_q, x_1, \dots, x_n)$, where the λ -s are coordinates on the base and the x -s are coordinates on the fiber, we can write a form $\theta \in \Omega_c^*(E)$ as $\theta = \theta' + \theta_n$, where θ' contains all terms of degree less than n in dx_1, \dots, dx_n and

$$\theta_n = \sum_{i_1, \dots, i_r} f_{i_1 \dots i_r}(x, \lambda) dx_1 \wedge \dots \wedge dx_n \wedge d\lambda_{i_1} \wedge \dots \wedge d\lambda_{i_r}.$$

By definition,

$$\oint_F \theta = \oint_F \theta_n = \sum_{i_1 \dots i_r} \left[\int_F f_{i_1 \dots i_r}(x, \lambda) dx_1 \wedge \dots \wedge dx_n \right] d\lambda_{i_1} \wedge \dots \wedge d\lambda_{i_r},$$

where the integral inside the brackets is the ordinary integral of a compactly supported form of maximal degree (see [18]).

Using Chern–Weil theory of characteristic classes for smooth vector bundles over not necessarily compact manifolds Fedosov obtained an explicit expression for the smooth form representing the Chern character of the index bundle of a family of pseudo-differential operators in $\text{Ell}(\mathbb{R}^n)$ in terms of its principal symbol.

The following proposition is an immediate consequence of [27, Corollary 6.5].

PROPOSITION 5.5. *If \mathcal{S} is a smooth family of pseudo-differential operators in $\text{Ell}(\mathbb{R}^n)$, then $\text{ch}(\text{Ind } \mathcal{S}) = p_* \text{ch}[\sigma]$ is the cohomology class of the form*

$$-\sum_{j=n}^{\infty} \frac{(j-1)!}{(2\pi i)^j (2j-1)!} \oint_{S^{2n-1}} \text{tr}(\sigma^{-1} d\sigma)^{2j-1},$$

where $S^{2n-1} = \partial B^{2n}$ is the boundary of a ball in \mathbb{R}^{2n} such that the support of σ is contained in $\Lambda \times B^{2n}$.

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