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IMPULSIVE PROBLEMS FOR FRACTIONAL EVOLUTION EQUATIONS AND OPTIMAL CONTROLS IN INFINITE DIMENSIONAL SPACES

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Abstract. In this paper, a class of impulsive fractional evolution equations and optimal controls in infinite dimensional spaces is considered. A suitable concept of a PC-mild solution is introduced and a suitable operator mapping is also constructed. By using a PC-type Ascoli–Arzela theorem, the compactness of the operator mapping is proven. Applying a generalized Gronwall inequality and Leray–Schauder fixed point theorem, the existence and uniqueness of the PC-mild solutions is obtained. Existence of optimal pairs for system governed by impulsive fractional evolution equations is also presented. Finally, an example illustrates the applicability of our results.

1. Introduction

In recent years, there has been a growing interest in the area of fractional calculus. This is mainly because fractional derivatives and fractional integrals provide more accurate models of many engineering systems than integer order derivatives and integrals. Accordingly, the theory of fractional differential

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equations has become an active area of investigation due to their applications in the fields of physics, engineering, economics and so on. For the basic theory on fractional differential equations in finite dimensional spaces, one can see the monographs of A. A. Kilbas et al. [25], V. Lakshmikantham et al. [27], K. S. Miller and B. Ross [31], I. Podlubny [35], the survey of R. P. Agarwal et al. [3]. For the basic theory on fractional differential equations involving the Caputo derivative and optimal controls in infinite dimensional spaces, one can see the papers of N. Abada et al. [1], R. P. Agarwal [2], R. P. Agarwal et al. [4], M. M. El-Borai [16], [17], E. Hernández et al. [22], M. Li et al. [28], N. Özdemir et al. [34], J. Wang and Y. Zhou [43], [44], Y. Zhou and F. Jiao [53], [54].

In order to describe dynamics of populations subject to abrupt changes as well as other phenomena such as harvesting, diseases, and so forth, some authors have used impulsive differential systems to describe the model since the last century. For the basic theory on impulsive differential equations and impulsive controls in finite dimensional spaces, the reader can refer to the monographs of D. D. Bainov and P. S. Simeonov [10], V. Lakshmikantham et al. [26], T. Yang [49]. For the basic theory on impulsive differential equations and optimal controls in infinite dimensional spaces, the reader can refer to the monograph of M. Benchohra et al. [12] and the papers of N. U. Ahmed, Y. K. Chang, J. Henderson, E. Hernández, Z. Fan, J. H. Liu, J. Liang, J. J. Nieto, R. Sakthivel, J. Wang, W. Wei, X. Xiang, etc. (see for instance [5]–[9], [15], [18]–[21], [29], [30], [33], [36]–[48] and references therein).

Impulsive fractional differential equations serve as basic fractional models to study the dynamics of processes that are subject to sudden changes in their states. Very recently, K. Balachandran and S. Kiruthika [11] studied a class of impulsive fractional evolution equations with bounded time-varying linear operator by using fractional calculus and fixed point theorems. M. Benchohra et al. [13], [14] also applied the Banach contraction principle, Schaefer's fixed point theorem and nonlinear alternative of Leray–Schauder type, measure of noncompactness to a class of impulsive fractional differential equations without unbounded operator. Although there were some papers discussing impulsive fractional differential equations without unbounded operator in infinite dimensional spaces, to our knowledge, impulsive fractional differential equations with unbounded operator in infinite dimensional spaces have not been studied extensively since it is much difficult to introduce a suitable concept of a mild solution.

Optimal control problems require minimization of a functional over a set of admissible control functions subject to dynamic constraints on the state and control variables. When the impulsive fractional differential equations describe the performance index and system dynamics, an optimal control problem reduces to an impulsive fractional optimal control problem. There has been very little work in the area of optimal control problems for system governed by impulsive fractional differential equations in finite dimensional spaces or infinite dimensional spaces.

Motivated by the above works including us [11]–[13], [37], [39], [43], [52], [53], we pay attention to investigate the fractional impulsive evolution equations of the type

$$(1.1)\qquad \begin{cases} {}^C D_t^{\alpha} x(t) = Ax(t) + f(t, x(t)), & \alpha \in (0,1),\ t \in J = [0,b],\ t \neq t_k, \\ x(0) = x_0, & \\ \Delta x(t_k) = I_k(x(t_k)), & k = 1, \ldots, \delta, \end{cases}$$

where ${}^C D_t^{\alpha}$ is the Caputo fractional derivative of order α, $A\colon D(A) \to X$ is the generator of a C_0-semigroup $\{T(t),\ t \geq 0\}$ on a Banach space X, $f\colon J \times X \to X$ is specified latter, x_0 is an element of X, $I_k\colon X \to X$ is a nonlinear map which determine the size of the jump at t_k, $0 = t_0 < t_1 < \ldots < t_\delta < t_{\delta+1} = b$, $I_k(x(t_k)) = x(t_k^+) - x(t_k^-)$, $x(t_k^+) = \lim_{h \to 0^+} = x(t_k + h)$ and $x(t_k^-) = x(t_k)$ represent respectively the right and left limits of $x(t)$ at $t = t_k$.

G. N. Mophou [32] studied system (1.1), however, E. Hernández et al. [22] showed that the concept of mild solutions (Definition 3.2, [32]) which inspired by O. K. Jaradat et al. [24] was not suitable for system (1.1) at all. So we have to introduce a new concept of a PC-mild solution (Definition 2.8) for system (1.1) based on our early works [43], [52], [53] on fractional evolution equations without impulses.

In order to obtain the existence of PC-mild solutions for fractional differential equations, some authors are used to apply Krasnosel'skiĭ's fixed point theorem or contraction mapping principle. However, the conditions for Krasnosel'skiĭ's fixed point theorem are not easy to be verified sometimes and the conditions for contraction mapping principle are too strong. Here, we use Leray–Schauder fixed point theorem to obtain the existence of solutions for system (1.1) under some easily checked conditions. First, we construct a operator $\mathcal{Q}$ associated with semigroup operators, probability density functions and impulsive terms for system (1.1), then use a PC-type Ascoli–Arzela theorem (Theorem 2.11) and overcome some difficulties to show the compactness of operator $\mathcal{Q}$ which is very important. By a new generalized Gronwall inequality with impulses and singular (Theorem 3.1), an estimate of a fixed point set $\{x = \sigma\mathcal{Q}x,\ \sigma \in [0,1]\}$ is established. Therefore, the existence of PC-mild solutions for system (1.1) is shown. Our methods are different from the original references and we give a new way to show the existence of PC-mild solutions for impulsive fractional differential equations. In addition, the new generalized Gronwall inequality with impulses and singular, which can be used in other nonlinear problems, has played an essential role in the study of impulsive fractional nonlinear differential equations in infinite dimensional spaces. Further, we also consider a Bolza problem of system governed by impulsive fractional evolution equations and an existence result of impulsive fractional optimal controls is presented.

The paper is organized as follows. In Section 2, we introduce the PC-mild solution of system (1.1) and recall some basis results. In Section 3, a new generalized Gronwall inequality with impulses and singular is established. In Section 4, the existence and uniqueness of PC-mild solutions for system (1.1) is proved. In Section 5, we introduce a class of admissible controls and an existence result of optimal controls for Bolza problem (P) is proved. At last, an example is given to demonstrate the applicability of our result.

2. Preliminaries

Let $\pounds_b(X)$ be the Banach space of all linear and bounded operators on X. For a C_0-semigroup $\{T(t),\ t \geq 0\}$ on X, we set $M = \sup_{t\in J} \|T(t)\|_{\pounds_b(X)}$. Let $C(J,X)$ be the Banach space of all X-valued continuous functions from $J=[0,b]$ into X endowed with the norm $\|x\|_C = \sup_{t\in J}\|x(t)\|$. We also introduce the set of functions $PC(J,X) \equiv \{x\colon J \to X \mid x$ is continuous at $t \in J\backslash\{t_1,\ldots,t_\delta\}$, and x is continuous from left and has right hand limits at $t \in \{t_1,\ldots,t_\delta\}\}$. Endowed with the norm

$$\|x\|_{PC} = \max\Big\{\sup_{t\in J}\|x(t+0)\|,\ \sup_{t\in J}\|x(t-0)\|\Big\},$$

it is easy to see $(PC(J,X), \|\cdot\|_{PC})$ is a Banach space.

Let us recall the following known definitions. For more details, see [25].

Definition 2.1. The fractional integral of order γ with the lower limit zero for a function f is defined as

$$I^\gamma f(t) = \frac{1}{\Gamma(\gamma)}\int_0^t \frac{f(s)}{(t-s)^{1-\gamma}}\,ds, \quad t>0,\ \gamma>0,$$

provided the right side is point-wise defined on $[0,\infty)$, where $\Gamma(\,\cdot\,)$ is the gamma function.

Definition 2.2. The Riemann–Liouville derivative of order γ with the lower limit zero for a function $f\colon[0,\infty)\to R$ can be written as

$$^LD^\gamma f(t) = \frac{1}{\Gamma(n-\gamma)}\frac{d^n}{dt^n}\int_0^t \frac{f(s)}{(t-s)^{\gamma+1-n}}\,ds, \quad t>0,\ n-1<\gamma<n.$$

Definition 2.3. The Caputo derivative of order γ for a function $f\colon[0,\infty)\to R$ can be written as

$$^CD^\gamma f(t) = {}^LD^\gamma\bigg(f(t) - \sum_{k=0}^{n-1}\frac{t^k}{k!}f^{(k)}(0)\bigg), \quad t>0,\ n-1<\gamma<n.$$

REMARK 2.4. (a) If $f(t) \in C^n[0,\infty)$, then

$$^C D^\gamma f(t) = \frac{1}{\Gamma(n-\gamma)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\gamma+1-n}} ds = I^{n-\gamma} f^{(n)}(t),\ t > 0,\ n-1 < \gamma < n.$$

(b) The Caputo derivative of a constant is equal to zero.

(c) If f is an abstract function with values in X, then integrals which appear in Definitions 2.1 and 2.2 are taken in Bochner's sense.

Let's recall that Y. Zhou and F. Jiao (Lemma 3.1 and Definition 3.1, [52]) have used the following definition of mild solutions for the problem below.

DEFINITION 2.5. By a mild solution of the following system

$$\begin{cases} ^C D_t^\alpha x(t) = Ax(t) + f(t, x(t)), \quad t \in J, \\ x(0) = x_0, \end{cases}$$

we mean that the function $x \in C(J, X)$ which satisfies the following integral equation

$$x(t) = \mathcal{T}(t)x_0 + \int_0^t (t-s)^{\alpha-1} \mathcal{S}(t-s) f(s, x(s))\, ds, \quad t \in J,$$

where

$$\begin{aligned} \mathcal{T}(t) &= \int_0^\infty \xi_\alpha(\theta) T(t^\alpha \theta)\, d\theta, \quad \mathcal{S}(t) = \alpha \int_0^\infty \theta \xi_\alpha(\theta) T(t^\alpha \theta)\, d\theta, \\ \xi_\alpha(\theta) &= \frac{1}{\alpha} \theta^{-1-1/\alpha} \varpi_\alpha(\theta^{-1/\alpha}) \geq 0, \\ \varpi_\alpha(\theta) &= \frac{1}{\pi} \sum_{n=1}^\infty (-1)^{n-1} \theta^{-n\alpha-1} \frac{\Gamma(n\alpha+1)}{n!} \sin(n\pi\alpha), \quad \theta \in (0,\infty), \end{aligned}$$

ξ_α is a probability density function defined on $(0,\infty)$, that is

$$\xi_\alpha(\theta) \geq 0, \quad \theta \in (0,\infty) \quad \text{and} \quad \int_0^\infty \xi_\alpha(\theta)\, d\theta = 1.$$

REMARK 2.6. Note that $\mathcal{T}(\,\cdot\,)$ and $\mathcal{S}(\,\cdot\,)$ are associated with noninteger α, there are no analogue of the semigroup property, i.e.

$$\mathcal{T}(t+s) \neq \mathcal{T}(t)\mathcal{T}(s), \quad \mathcal{S}(t+s) \neq \mathcal{S}(t)\mathcal{S}(s) \quad \text{for } t, s > 0.$$

According to Definitions 2.1 and 2.3, it is suitable to rewrite system (1.1) in the equivalent integral equation

$$
(2.1)\quad \begin{cases} x(t) = x_0 + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1}[Ax(s) + f(s, x(s))]\, ds, & t \in [0, t_1], \\ x(t) = x_0 + \dfrac{1}{\Gamma(\alpha)} \displaystyle\sum_{i=1}^{k} \int_{t_i-1}^{t_i} (t_i - s)^{\alpha-1}[Ax(s) + f(s, x(s))]\, ds & \\ \qquad + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{t_k}^{t} (t-s)^{\alpha-1}[Ax(s) + f(s, x(s))]\, ds & \\ \qquad + \displaystyle\sum_{i=1}^{k} I_i(x(t_i)), & t \in (t_k, t_{k+1}], \\ & k = 1, \ldots, \delta, \end{cases}
$$

provided that the integral in (2.1) exists.

Before giving the definition of mild solution of system (1.1), we firstly prove the following lemma.

LEMMA 2.7. *If* (2.1) *holds, then we have*

$$
(2.2)\quad \begin{cases} x(t) = \mathcal{T}(t)x_0 + \displaystyle\int_0^t (t-s)^{\alpha-1} \mathcal{S}(t-s) f(s, x(s))\, ds, & t \in [0, t_1], \\ x(t) = \mathcal{T}(t - t_k) \displaystyle\prod_{0<i\le k} \mathcal{T}(t_i - t_{i-1}) x_0 & \\ \qquad + \mathcal{T}(t - t_k) \displaystyle\sum_{i=1}^{k} \bigg\{ \prod_{i<j\le k} \mathcal{T}(t_j - t_{j-1}) & \\ \qquad \times \bigg[\displaystyle\int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} \mathcal{S}(t_i - s) f(s, x(s))\, ds + I_i(x(t_i)) \bigg] \bigg\} & \\ \qquad + \displaystyle\int_{t_k}^{t} (t-s)^{\alpha-1} \mathcal{S}(t-s) f(s, x(s))\, ds, & t \in (t_k, t_{k+1}], \\ & k = 1, \ldots, \delta. \end{cases}
$$

PROOF. For $t \in [0, t_1]$, by Lemma 3.1 of [52],

$$
x(t) = \mathcal{T}(t)x_0 + \int_0^t (t-s)^{\alpha-1} \mathcal{S}(t-s) f(s, x(s))\, ds,
$$

which leads to

$$
x(t_1) = \mathcal{T}(t_1)x_0 + \int_0^{t_1} (t_1 - s)^{\alpha-1} \mathcal{S}(t_1 - s) f(s, x(s))\, ds.
$$

Moreover, for $t \in (t_1, t_2]$,

$$\begin{aligned}
x(t) &= \mathcal{T}(t-t_1)x(t_1^+) + \int_{t_1}^{t} (t-s)^{\alpha-1}\mathcal{S}(t-s)f(s,x(s))\,ds \\
&= \mathcal{T}(t-t_1)(x(t_1) + \Delta x(t_1)) + \int_{t_1}^{t} (t-s)^{\alpha-1}\mathcal{S}(t-s)f(s,x(s))\,ds \\
&= \mathcal{T}(t-t_1)\mathcal{T}(t_1)x_0 \\
&\quad + \mathcal{T}(t-t_1)\bigg[\int_0^{t_1} (t_1-s)^{\alpha-1}\mathcal{S}(t_1-s)f(s,x(s))ds + I_1(x(t_1))\bigg] \\
&\quad + \int_{t_1}^{t} (t-s)^{\alpha-1}\mathcal{S}(t-s)f(s,x(s))\,ds.
\end{aligned}$$

Repeating the same procedure, we can easily deduce that (2.2) holds for any $t \in (t_k, t_{k+1}]$, $k = 1, \dots, \delta$. □

Due to Lemma 2.7, we give the following definition of the mild solution of system (1.1).

Definition 2.8. By a PC-mild solution of the system (1.1) we mean that the function $x \in PC(J,X)$ which satisfies the following integral equation

$$x(t) = \begin{cases}
\mathcal{T}(t)x_0 + \displaystyle\int_0^t (t-s)^{\alpha-1}\mathcal{S}(t-s)f(s,x(s))\,ds, & t \in [0,t_1], \\
\mathcal{T}(t-t_k)\displaystyle\prod_{0<i\le k}\mathcal{T}(t_i - t_{i-1})x_0 & \\
\qquad +\mathcal{T}(t-t_k)\displaystyle\sum_{i=1}^{k}\bigg\{\prod_{i<j\le k}\mathcal{T}(t_j - t_{j-1}) & \\
\qquad \times\bigg[\displaystyle\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\mathcal{S}(t_i-s)f(s,x(s))\,ds + I_i(x(t_i))\bigg]\bigg\} & \\
\qquad + \displaystyle\int_{t_k}^{t}(t-s)^{\alpha-1}\mathcal{S}(t-s)f(s,x(s))\,ds, & t \in (t_k, t_{k+1}], \\
& k = 1, \dots, \delta.
\end{cases}$$

The following results will be used throughout this paper.

Lemma 2.9 ([52, Lemmas 3.2–3.4]). *The operators $\mathcal{T}$ and $\mathcal{S}$ have the following properties:*

(a) *For any fixed $t \ge 0$, $\mathcal{T}(t)$ and $\mathcal{S}(t)$ are linear and bounded operators, i.e. for any $x \in X$,*

$$\|\mathcal{T}(t)x\| \le M\|x\| \quad \text{and} \quad \|\mathcal{S}(t)x\| \le \frac{qM}{\Gamma(1+q)}\|x\|.$$

(b) *$\{\mathcal{T}(t),\ t \ge 0\}$ and $\{\mathcal{S}(t),\ t \ge 0\}$ are strongly continuous.*

(c) *For every* $t > 0$, $\mathcal{T}(t)$ *and* $\mathcal{S}(t)$ *are also compact operators if* $T(t)$ *is compact.*

Lemma 2.10. *A measurable function* $f\colon J \to X$ *is Bochner integrable if* $\|f\|$ *is Lebesuge integrable.*

Lemma 2.11 (PC-type Ascoli–Arzela theorem [45, Theorem 2.1]). *Suppose* $\mathcal{W} \subset PC(J, X)$ *be a subset. If the following conditions are satisfied:*

(a) $\mathcal{W}$ *is uniformly bounded subset of* $PC(J, X)$.

(b) $\mathcal{W}$ *is equicontinuous in* (t_k, t_{k+1}), $k = 0, 1, \ldots, \delta$, *where* $t_0 = 0$, $t_{\delta+1} = b$.

(c) $\mathcal{W}(t) \equiv \{x(t) \mid x \in \mathcal{W},\ t \in J \setminus \{t_1, \ldots, t_\delta\}\}$, $\mathcal{W}(t_k + 0) \equiv \{x(t_k + 0) \mid x \in \mathcal{W}\}$ *and* $\mathcal{W}(t_k - 0) \equiv \{x(t_k - 0) \mid x \in \mathcal{W}\}$ *are relatively compact subsets of* X.

Then $\mathcal{W}$ *is a relatively compact subset of* $PC(J, X)$.

3. Gronwall's inequality with impulses and singular

In order to use Leray–Schauder fixed point theorem to show the existence and uniqueness of solutions, we need a new generalized Gronwall's inequality with impulses and singular which can also be used in other problems. It will play an essential role in the study of impulsive fractional nonlinear differential equations in Banach spaces.

Lemma 3.1. *Let* $x \in PC([0, \infty), X)$ *and satisfy the following inequality*

$$\|x(t)\| \le c_1 + c_2 \int_0^t (t - s)^{\alpha-1} \|x(s)\|\, ds + \sum_{0<t_k<t} \theta_k \|x(t_k)\|, \tag{3.1}$$

where $1 > \alpha > 0$, c_1, c_2, $\theta_k \ge 0$ *are constants. Then*

$$\|x(t)\| \le c_1 \prod_{0<t_k<t} (1 + \overline{\theta}_k) \bigg(1 + \sum_{j=1}^{n-1} \frac{c_2^j t^{j\alpha} \Gamma^j(\alpha)}{(j\alpha)\Gamma(j\alpha)}\bigg) \exp\bigg(\frac{c_2^n t^{n\alpha} \Gamma^n(\alpha)}{n\alpha(n\alpha - 1)\Gamma(n\alpha)}\bigg),$$

where

$$n = \left[\frac{1}{\alpha}\right] + 1, \qquad \overline{\theta}_i = \bigg(1 + \sum_{0<t_k\le t_i} \theta_k\bigg) \bigg(1 + \sum_{k=1}^{n} c_2^k \frac{\Gamma^k(\alpha)}{\Gamma(k\alpha)} \frac{t_i^{k\alpha}}{k\alpha}\bigg).$$

Proof. It comes from (3.1) that

$$
\begin{aligned}
\|x(t)\| &\le c_1 + c_2 \int_0^t (t-s)^{\alpha-1} \bigg[c_1 + c_2 \int_0^s (s-\tau)^{\alpha-1} \|x(\tau)\|\, d\tau \\
&\quad + \sum_{0<t_k<s} \theta_k \|x(t_k)\| \bigg]\, ds + \sum_{0<t_k<t} \theta_k \|x(t_k)\| \\
&\le \bigg(c_1 + \sum_{0<t_k<t} \theta_k \|x(t_k)\| \bigg) \\
&\quad + c_2 \int_0^t (t-s)^{\alpha-1} \bigg[c_1 + c_2 \int_0^s (s-\tau)^{\alpha-1} \|x(\tau)\|\, d\tau \\
&\quad + \sum_{0<t_k<s} \theta_k \|x(t_k)\| \bigg]\, ds \\
&\le \bigg(c_1 + \sum_{0<t_k<t} \theta_k \|x(t_k)\| \bigg) + \bigg(c_1 + \sum_{0<t_k<t} \theta_k \|x(t_k)\| \bigg) c_2 \int_0^t (t-s)^{\alpha-1}\, ds \\
&\quad + c_2^2 \int_0^t \bigg[\int_\tau^t (t-s)^{\alpha-1} (s-\tau)^{\alpha-1}\, ds \bigg] \|x(\tau)\|\, d\tau \\
&\le \bigg(c_1 + \sum_{0<t_k<t} \theta_k \|x(t_k)\| \bigg) \bigg[1 + c_2 \frac{\Gamma(\alpha) t^\alpha}{\Gamma(\alpha+1)} \bigg] \\
&\quad + c_2^2 \int_0^t \bigg[\int_\tau^t (t-s)^{\alpha-1} (s-\tau)^{\alpha-1}\, ds \bigg] \|x(\tau)\|\, d\tau.
\end{aligned}
$$

Let $y = (s-\tau)/(t-\tau)$, then

$$
\int_\tau^t (t-s)^{\alpha-1} (s-\tau)^{\alpha-1}\, ds = \int_0^1 (t-\tau)^{2\alpha-1} y^{\alpha-1} (1-y)^{\alpha-1}\, dy = \frac{\Gamma^2(\alpha)}{\Gamma(2\alpha)} (t-\tau)^{2\alpha-1}.
$$

Thus, we have

$$
\begin{aligned}
\|x(t)\| \le \bigg(c_1 + \sum_{0<t_k<t} \theta_k \|x(t_k)\| \bigg) \bigg[1 + c_2 \frac{\Gamma(\alpha) t^\alpha}{\Gamma(\alpha+1)} \bigg] \\
+ c_2^2 \frac{\Gamma^2(\alpha)}{\Gamma(2\alpha)} \int_0^t (t-\tau)^{2\alpha-1} \|x(\tau)\|\, d\tau.
\end{aligned}
$$

Using (3.1) again, we can obtain

$$
\begin{aligned}
\|x(t)\| \le \bigg(c_1 + \sum_{0<t_k<t} \theta_k \|x(t_k)\| \bigg) \bigg[1 + c_2 \frac{\Gamma(\alpha)}{\Gamma(\alpha)} \frac{t^\alpha}{\alpha} + c_2^2 \frac{\Gamma^2(\alpha)}{\Gamma(2\alpha)} \frac{t^{2\alpha}}{2\alpha} \bigg] \\
+ c_2^3 \frac{\Gamma^3(\alpha)}{\Gamma(3\alpha)} \int_0^t (t-\tau)^{3\alpha-1} \|x(\tau)\|\, d\tau.
\end{aligned}
$$

Repeating the above steps until the $(n-1)$-th step, where $n = [1/\alpha] + 1$, then we have

$$\begin{aligned}\|x(t)\| \leq \Theta_{k,t}\bigg[1 + c_2\frac{t^\alpha}{\alpha} + c_2^2\frac{\Gamma^2(\alpha)}{\Gamma(2\alpha)}\frac{t^{2\alpha}}{2\alpha} + \ldots + c_2^{n-1}\frac{\Gamma^{n-1}(\alpha)}{\Gamma((n-1)\alpha)}\frac{t^{(n-1)\alpha}}{(n-1)\alpha}\bigg]& \\ + c_2^n\frac{\Gamma^n(\alpha)}{\Gamma(n\alpha)}\int_0^t (t-\tau)^{n\alpha-1}\|x(\tau)\|\,d\tau&\end{aligned}$$

where

$$\Theta_{k,t} = \bigg(c_1 + \sum_{0<t_k<t}\theta_k\|x(t_k)\|\bigg).$$

Define

$$\begin{aligned}v(t) = \Theta_{k,t}\bigg[1 + c_2\frac{t^\alpha}{\alpha} + c_2^2\frac{\Gamma^2(\alpha)}{\Gamma(2\alpha)}\frac{t^{2\alpha}}{2\alpha} + \ldots + c_2^{n-1}\frac{\Gamma^{n-1}(\alpha)}{\Gamma((n-1)\alpha)}\frac{t^{(n-1)\alpha}}{(n-1)\alpha}\bigg]& \\ + c_2^n\frac{\Gamma^n(\alpha)}{\Gamma(n\alpha)}\int_0^t (t-\tau)^{n\alpha-1}\|x(\tau)\|\,d\tau.&\end{aligned}$$

By means of the fact that $v(t)$ is nondecreasing and $\|x(t)\| \leq v(t)$, we can arrive at

$$\begin{aligned}\frac{dv(t)}{dt} &= \Theta_{k,t}\sum_{k=1}^{n-1} c_2^k\frac{\Gamma^k(\alpha)}{\Gamma(k\alpha)}t^{k\alpha-1} + v(t)c_2^n\frac{(n\alpha-1)\Gamma^n(\alpha)}{\Gamma(n\alpha)}\int_0^t (t-\tau)^{n\alpha-2}\,d\tau \\ &= \Theta_{k,t}\sum_{k=1}^{n-1} c_2^k\frac{\Gamma^k(\alpha)}{\Gamma(k\alpha)}t^{k\alpha-1} + v(t)c_2^n\frac{(n\alpha-1)\Gamma^n(\alpha)}{\Gamma(n\alpha)}\frac{t^{n\alpha-1}}{n\alpha-1} \\ &\equiv h(t) + m(t)v(t), \quad t \neq t_k, \qquad (3.2)\\ v(t_i+0) &\leq \bigg(1 + \sum_{0<t_k\leq t_i}\theta_k\bigg)\bigg(1 + \sum_{k=1}^{n} c_2^k\frac{\Gamma^k(\alpha)}{\Gamma(k\alpha)}\frac{t_i^{k\alpha}}{k\alpha}\bigg)v(t_i) \equiv \overline{\theta}_i v(t_i), \\ v(0) &= c_1,\end{aligned}$$

where

$$h(t) = \Theta_{k,t}\sum_{k=1}^{n-1} c_2^k\frac{\Gamma^k(\alpha)}{\Gamma(k\alpha)}t^{k\alpha-1}, \qquad m(t) = c_2^n\frac{(n\alpha-1)\Gamma^n(\alpha)}{\Gamma(n\alpha)}\frac{t^{n\alpha-1}}{n\alpha-1}.$$

For $t \in (t_k, t_{k+1}]$, by (3.2) we obtain

$$\begin{aligned}v(t) &\leq \overline{\theta}_k v(t_k)\exp\bigg(\int_{t_k}^t m(s)\,ds\bigg) + \int_{t_k}^t h(s)\exp\bigg(\int_s^t m(\tau)\,d\tau\bigg)\,ds \\ &\leq c_1\prod_{0<t_k<t}\overline{\theta}_k\exp\bigg(\int_0^t m(s)ds\bigg) \\ &\quad + \int_0^t h(s)\prod_{s<t_k<t}(1+\overline{\theta}_k)\exp\bigg(\int_s^t m(\tau)\,d\tau\bigg)\,ds\end{aligned}$$

$$\leq c_1 \prod_{0<t_k<t}(1+\overline{\theta}_k)\bigg(1+\sum_{j=1}^{n-1}\frac{c_2^j t^{j\alpha}\Gamma^j(\alpha)}{(j\alpha)\Gamma(j\alpha)}\bigg)\exp\bigg(\frac{c_2^n t^{n\alpha}\Gamma^n(\alpha)}{n\alpha(n\alpha-1)\Gamma(n\alpha)}\bigg).$$

Thus

$$\|x(t)\| \leq c_1 \prod_{0<t_k<t}(1+\overline{\theta}_k)\bigg(1+\sum_{j=1}^{n-1}\frac{c_2^j t^{j\alpha}\Gamma^j(\alpha)}{(j\alpha)\Gamma(j\alpha)}\bigg)\exp\bigg(\frac{c_2^n t^{n\alpha}\Gamma^n(\alpha)}{n\alpha(n\alpha-1)\Gamma(n\alpha)}\bigg). \quad \square$$

Remark 3.2. If $x \in PC(J,X)$ and satisfies the inequality (3.1), then there exists a constant $\widetilde{M} > 0$ such that $\|x(t)\| \leq \widetilde{M} c_1$.

4. Existence and uniqueness of PC-mild solutions

In this section, we will derive the existence and uniqueness results concerning the PC-mild solution for system (1.1) under some easily checked conditions. Our result will use the PC-type Ascoli–Arzela theorem and Leray–Schauder fixed point theorem.

Let us list the following hypotheses:

- (HA) A is the infinitesimal generator of a compact semigroup $\{T(t),\ t \geq 0\}$ in X.
- (HF1) $f\colon J \times X \to X$ is measurable for $t \in J$ and for any x, $y \in X$ satisfying $\|x\|, \|y\| \leq \rho$, there exists a positive constant $L_f(\rho) > 0$ such that

$$\|f(t,x) - f(t,y)\| \leq L_f(\rho)\|x-y\|.$$

- (HF2) There exists a positive constant $M_f > 0$ such that

$$\|f(t,x)\| \leq M_f(1+\|x\|) \quad \text{for all } t \in J,\ x \in X.$$

- (HI1) $I_k\colon X \to X$, $I_k(X)$ is a bounded subset of X, $k = 1,\dots,\delta$.
- (HI2) There exist a constants $h^* > 0$, such that

$$\|I_k(x) - I_k(y)\| \leq h^*\|x-y\|, \quad \text{for all } x,y \in X, k = 1,\dots,\delta.$$

Theorem 4.1. *Assume that the hypotheses* (HA)*,* (HF1)*,* (HF2)*,* (HI1) *and* (HI2) *holds. Then system* (1.1) *has a unique PC-mild solution on* J.

Proof. Let $x_0 \in X$ be fixed. Define an operator $\mathcal{Q}$ on $PC(J,X)$ by

$$(\mathcal{Q}x)(t) = \begin{cases} \mathcal{T}(t)x_0 + \displaystyle\int_0^t (t-s)^{\alpha-1}\mathcal{S}(t-s)f\,(s,x(s))\,ds, & t \in [0,t_1], \\ \mathcal{T}(t-t_k)\displaystyle\prod_{0<i\le k}\mathcal{T}(t_i-t_{i-1})x_0 & \\ \quad + \mathcal{T}(t-t_k)\displaystyle\sum_{i=1}^{k}\bigg\{\prod_{i<j\le k}\mathcal{T}(t_j-t_{j-1}) & \\ \quad \times\bigg[\displaystyle\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\mathcal{S}(t_i-s)f(s,x(s))\,ds + I_i(x(t_i))\bigg]\bigg\} & \\ \quad + \displaystyle\int_{t_k}^{t}(t-s)^{\alpha-1}\mathcal{S}(t-s)f(s,x(s))\,ds, & t \in (t_k,t_{k+1}], \\ & k = 1,\dots,\delta. \end{cases}$$

It is obvious that $\mathcal{Q}$ is well defined mapping from $PC(J,X)$ to $PC(J,X)$ for $x \in PC(J,X)$ due to Lemma 2.10. In fact, for $0 \le \tau < t \le t_1$, by our assumptions and Lemma 2.9,

$$\begin{aligned} &\|(\mathcal{Q}x)(t) - (\mathcal{Q}x)(\tau)\| \\ &\quad\le \|\mathcal{T}(t)-\mathcal{T}(\tau)\|\|x_0\| + \frac{\alpha M}{\Gamma(1+\alpha)}\int_\tau^t (t-s)^{\alpha-1}\|f(s,x(s))\|\,ds \\ &\qquad + \sup_{s\in[0,\tau]}\|\mathcal{S}(t-s)-\mathcal{S}(\tau-s)\|\int_0^\tau (t-s)^{\alpha-1}\|f(s,x(s))\|\,ds \\ &\qquad + \frac{\alpha M\|f\|_{C([0,t_1],X)}}{\Gamma(1+\alpha)}\bigg|\int_0^\tau (\tau-s)^{\alpha-1}ds - \int_0^\tau (t-s)^{\alpha-1}\,ds\bigg| \\ &\quad\le \|\mathcal{T}(t)-\mathcal{T}(\tau)\|\|x_0\| + \frac{(t-\tau)^\alpha M\|f\|_{C([0,t_1],X)}}{\Gamma(1+\alpha)} \\ &\qquad + \frac{t_1^\alpha\|f\|_{C([0,t_1],X)}}{\alpha}\sup_{s\in[0,\tau]}\|\mathcal{S}(t-s)-\mathcal{S}(\tau-s)\| \\ &\qquad + \frac{M\|f\|_{C([0,t_1],X)}}{\Gamma(1+\alpha)}|\tau^\alpha + (t-\tau)^\alpha - t^\alpha|, \end{aligned}$$

which implies that $\mathcal{Q}x \in C([0,t_1],X)$. With analogous arguments we can obtain $\mathcal{Q}x \in C((t_1,t_2],X)$, $\mathcal{Q}x \in C((t_2,t_3],X)$, $\dots$, $\mathcal{Q}x \in C((t_\delta,b],X)$. That is, $\mathcal{Q}x \in PC(J,X)$.

For the sake of convenience, we subdivide the proof into several steps.

Step 1. $\mathcal{Q}$ is a continuous operator on $PC(J,X)$.

Let $x,y \in PC(J,X)$ and $\|x-y\|_{PC} \le 1$, then $\|y\|_{PC} \le 1 + \|x\|_{PC} = \rho$.

By our assumptions and Lemma 2.9 again, we obtain

$$\begin{aligned}
\|(\mathcal{Q}x)(t)-(\mathcal{Q}y)(t)\| &\leq \Bigg\| \mathcal{T}(t-t_k)\sum_{i=1}^{k} \times \Bigg\{ \prod_{i<j\leq k} \mathcal{T}(t_j-t_{j-1}) \\
&\quad \times \Bigg[\int_{t_{i-1}}^{t_i} (t_i-s)^{\alpha-1}\mathcal{S}(t_i-s)[f(s,x(s))-f(s,y(s))]\,ds \\
&\quad + [I_i(x(t_i))-I_i(y(t_i))] \Bigg] \Bigg\} \Bigg\| \\
&\quad + \Bigg\| \int_{t_k}^{t} (t-s)^{\alpha-1}\mathcal{S}(t-s)[f(s,x(s))-f(s,y(s))]\,ds \Bigg\| \\
&\leq \frac{\alpha M^2}{\Gamma(1+\alpha)} \sum_{i=1}^{k} M^{k-i} \Bigg[\int_{t_{i-1}}^{t_i} (t_i-s)^{\alpha-1}\|f(s,x(s))-f(s,y(s))\|\,ds \\
&\quad + \|I_i(x(t_i))-I_i(y(t_i))\| \Bigg] \\
&\quad + \frac{\alpha M}{\Gamma(1+\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1}\|f(s,x(s))-f(s,y(s))\|\,ds \\
&\leq \frac{\alpha M^2}{\Gamma(1+\alpha)} \sum_{i=1}^{k} M^{k-i} \Bigg[\int_{t_{i-1}}^{t_i} (t_i-s)^{\alpha-1} L_f(\rho)\|x-y\|_{PC}\,ds + h_i\|x-y\|_{PC} \Bigg] \\
&\quad + \frac{\alpha M}{\Gamma(1+\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} L_f(\rho)\|x-y\|_{PC}\,ds \\
&\leq \frac{\alpha M^2}{\Gamma(1+\alpha)} \sum_{i=1}^{k} M^{k-i} \left[\frac{1}{\alpha}(t_i-t_{i-1})^{\alpha} L_f(\rho) + h_i \right] \|x-y\|_{PC} \\
&\quad + \frac{\alpha M}{\Gamma(1+\alpha)} \frac{1}{\alpha}(t-t_k)^{\alpha} L_f(\rho)\|x-y\|_{PC} \\
&\leq \frac{\alpha M}{\Gamma(1+\alpha)} \sum_{i=1}^{k} M^{k+1-i} \left(\frac{b^{\alpha}}{\alpha} L_f(\rho) + h^* \right) \|x-y\|_{PC} \\
&\quad + \frac{\alpha M}{\Gamma(1+\alpha)} \frac{b^{\alpha}}{\alpha} L_f(\rho)\|x-y\|_{PC},
\end{aligned}$$

which implies that

$$\|\mathcal{Q}x-\mathcal{Q}y\|_{PC} \leq \frac{\alpha M}{\Gamma(1+\alpha)} \left[\sum_{i=1}^{\delta} M^{\delta+1-i} \left(\frac{b^{\alpha}}{\alpha} L_f(\rho) + h^* \right) + \frac{b^{\alpha}}{\alpha} L_f(\rho) \right] \|x-y\|_{PC}.$$

Thus, $\mathcal{Q}$ is a continuous operator on $PC(J,X)$.

Step 2. $\mathcal{Q}$ is a compact operator on $PC(J,X)$.

Let $\mathfrak{B}$ be a bounded subset of $PC(J,X)$, there exists a constant $\mu>0$ such that $\|x\|_{PC}\leq\mu$ for all $x\in\mathfrak{B}$. Using (HI2), there exists a constant N such that $\|I_k(x(t))\|\leq N$ for all $x\in\mathfrak{B}$, $t\in J$, $k=1,\dots,\delta$. Also using (HF2), there exists a constant ω such that $\|f(t,x(t)\| \leq M_f(1+\|x\|_{PC}) \leq M_f(1+\mu) \equiv \omega$ for all $x \in \mathfrak{B}$, $t \in J$. Further, $\mathcal{Q}\mathfrak{B}$ is a bounded subset of $PC(J,X)$. In fact, let $x \in \mathfrak{B}$, after some standard calculation,

$$\|\mathcal{Q}x\|_{PC} \leq M^{\delta}\|x_0\| + \frac{\delta M^{2+\delta}\omega b^{\alpha}}{\Gamma(1+\alpha)} + \frac{\omega b^{\alpha} M}{\Gamma(1+\alpha)} + \delta M^{\delta+1} N \equiv \rho.$$

Hence $\mathcal{Q}\mathfrak{B}$ is bounded.

Let

$$(\mathcal{Q}v)(t) = (\mathcal{Q}_1 v)(t) + (\mathcal{Q}_2 v)(t)$$

where

$$(\mathcal{Q}_1 v)(t) = \begin{cases} \mathcal{T}(t)x_0 + \displaystyle\int_0^t (t-s)^{\alpha-1}\mathcal{S}(t-s)f(s,v(s))\,ds, & t \in [0,t_1], \\ \mathcal{T}(t-t_k)\displaystyle\prod_{0<i\leq k}\mathcal{T}(t_i-t_{i-1})x_0 + \mathcal{T}(t-t_k)\sum_{i=1}^{k}\bigg\{\prod_{i<j\leq k}\mathcal{T}(t_j-t_{j-1}) & \\ \quad \times \displaystyle\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\mathcal{S}(t_i-s)f(s,v(s))\,ds\bigg\} & \\ \quad + \displaystyle\int_{t_k}^{t}(t-s)^{\alpha-1}\mathcal{S}(t-s)f(s,v(s))ds, & t \in (t_k,t_{k+1}], \\ & k=1,\dots,\delta, \end{cases}$$

and

$$(4.1)\quad (\mathcal{Q}_2 v)(t) = \begin{cases} 0, & t \in [0,t_1], \\ \mathcal{T}(t-t_k)\displaystyle\sum_{i=1}^{k}\bigg\{\prod_{i<j\leq k}\mathcal{T}(t_j-t_{j-1})I_i(v(t_i))\bigg\}, & t \in (t_k,t_{k+1}], \\ & k=1,\dots,\delta. \end{cases}$$

We first check that $\mathcal{Q}_2$ is a compact operator. Note that (4.1) and our assumptions, we know that

$$\begin{cases} W_{\mathcal{Q}_2}(t)|_{(0,t_1]} = \{0\}, & t \in (0,t_1], \\ W_{\mathcal{Q}_2}(t)|_{(t_k,t_{k+1}]} = \{(\mathcal{Q}_2 v)(t) \mid v \in \mathfrak{B}\}, & t \in (t_k,t_{k+1}],\ k=1,\dots,\delta. \end{cases}$$

are uniformly bounded and relatively compact in X. Moreover, $W_{\mathcal{Q}_2}(t_k+0)$ are also relatively compact from (HI1).

For $t_k \leq t < t+\varepsilon \leq t_{k+1}$, $\varepsilon > 0$,

$$\|(\mathcal{Q}_2 v)(t+\varepsilon) - (\mathcal{Q}_2 v)(t)\| \leq \|\mathcal{T}(t+\varepsilon-t_k) - \mathcal{T}(t-t_k)\| \times \sum_{i=1}^{\delta} M^{\delta+1-i}N.$$

Thus, the functions in $W_{\mathcal{Q}_2}$ are equicontinuous due to $\|\mathcal{T}(t+\varepsilon-t_k)-\mathcal{T}(t-t_k)\| \to 0$ as $\varepsilon \to 0$ for each fixed t_k. Now an application of the PC-type Arzela–Ascoli theorem justifies the relatively compactness of $W_{\mathcal{Q}_2}$. Therefore, $\mathcal{Q}_2$ is a compact operator.

Next, the same idea can be used to prove the compactness of $\mathcal{Q}_1$.

Since $\mathcal{T}(t)$, $t > 0$ is compact, it is easy to see

$$\begin{cases} W^1_{\mathcal{Q}_1}(t)|_{(0,t_1]} = \{\mathcal{T}(t)x_0\}, & t \in (0,t_1], \\ W^1_{\mathcal{Q}_1}(t)|_{(t_k,t_{k+1}]} = \left\{\mathcal{T}(t-t_k) \prod\limits_{0<i\le k} \mathcal{T}(t_i - t_{i-1})x_0\right\}, & t \in (t_k, t_{k+1}], \\ & k = 1, \dots, \delta, \end{cases}$$

are relatively compact in X.

Also, for each $t \in [0,t_1]$, arbitrary $t_1 > h > 0$, $\varepsilon > 0$, the set

$$\begin{aligned} &\bigg\{T(h^\alpha\varepsilon)\int_0^{t-h}(t-s)^{q-1} \\ &\qquad \times \left(\alpha\int_\varepsilon^\infty \theta\xi_\alpha(\theta)T((t-s)^\alpha\theta - h^\alpha\varepsilon)d\theta\right)f(s,v(s))\,ds \;\bigg|\; v \in \mathfrak{B}\bigg\} \\ &\quad = \left\{\alpha\int_0^{t-h}\int_\varepsilon^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)T((t-s)^\alpha\theta)f(s,v(s))\,d\theta\,ds \;\middle|\; v \in \mathfrak{B}\right\} \end{aligned}$$

is relatively compact in X since $T(h^\alpha\varepsilon)$ is compact.

After some standard calculation (see our earlier work [52]), one can obtain

$$\begin{aligned} &\alpha\int_0^{t-h}\int_\varepsilon^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)T((t-s)^\alpha\theta)f(s,v(s))\,d\theta\,ds \\ &\qquad\qquad \to \alpha\int_0^t\int_0^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)T((t-s)^\alpha\theta)f(s,v(s))\,d\theta\,ds, \end{aligned}$$

as $h \to 0$, $\varepsilon \to 0$. Thus, we can conclude that

$$\begin{aligned} W^2_{\mathcal{Q}_1}(t)|_{[0,t_1]} &= \left\{\int_0^t (t-s)^{\alpha-1}\mathcal{S}(t-s)f(s,v(s))\,ds \;\middle|\; v \in \mathfrak{B}\right\} \\ &= \left\{\alpha\int_0^t\int_0^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)T((t-s)^\alpha\theta)f(s,v(s))\,d\theta\,ds \;\middle|\; v \in \mathfrak{B}\right\} \end{aligned}$$

is relatively compact in X.

It is obvious that the set

$$\begin{aligned} W^2_{\mathcal{Q}_1}(t)|_{(t_k,t_{k+1}]} = \bigg\{&\mathcal{T}(t-t_k)\sum_{i=1}^k \bigg[\prod_{i<j\le k}\mathcal{T}(t_j - t_{j-1}) \\ &\times \int_{t_{i-1}}^{t_i}(t_i - s)^{\alpha-1}\mathcal{S}(t_i - s)f(s,v(s))\,ds\bigg] \;\bigg|\; v \in \mathfrak{B}\bigg\} \end{aligned}$$

is also relatively compact in X since $\mathcal{T}(t-t_k)$ is compact for $t - t_k > 0$.

For each $t \in (t_k, t_{k+1}]$, arbitrary $h > 0$, $\varepsilon > 0$, the set

$$\begin{aligned}\bigg\{T(h^\alpha\varepsilon)&\int_{t_k}^{t-h}(t-s)^{q-1}\\ &\times\bigg(\alpha\int_\varepsilon^\infty \theta\xi_\alpha(\theta)T((t-s)^\alpha\theta-h^\alpha\varepsilon)\,d\theta\bigg)f(s,v(s))\,ds\ \bigg|\ v\in\mathfrak{B}\bigg\}\\ &=\bigg\{\alpha\int_{t_k}^{t-h}\int_\varepsilon^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)T((t-s)^\alpha\theta)f(s,v(s))\,d\theta\,ds\ \bigg|\ v\in\mathfrak{B}\bigg\}\end{aligned}$$

is relatively compact in X since $T(h^\alpha\varepsilon)$ is compact again. After some standard calculation again (see [52]),

$$W^3_{\mathcal{Q}_1}(t)|_{(t_k,t_{k+1}]}=\bigg\{\int_{t_k}^{t}(t-s)^{\alpha-1}\mathcal{S}(t-s)f(s,v(s))\,ds\ \bigg|\ v\in\mathfrak{B}\bigg\}$$

is also relatively compact in X.

Therefore, the set

$$W_{\mathcal{Q}_1}(t)|_{(t_k,t_{k+1}]}=W^1_{\mathcal{Q}_1}(t)|_{(t_k,t_{k+1}]}+W^2_{\mathcal{Q}_1}(t)|_{(t_k,t_{k+1}]}+W^3_{\mathcal{Q}_1}(t)|_{(t_k,t_{k+1}]}$$

is relatively compact in X and $W_{\mathcal{Q}_1}(t_k+0)$ are relatively compact for $t_k \in \{t_1,\ldots,t_\delta\}$. Obviously, $W_{\mathcal{Q}_1}(t)$ is a uniformly bounded subset of $PC(J,X)$.

Now, we only need to show the piecewise equicontinuity of $W_{\mathcal{Q}_1}(t)|_{(t_k,t_{k+1}]}$.

The equicontinuity of $W^1_{\mathcal{Q}_1}(t)|_{(t_k,t_{k+1}]}$ can be proven since the fact of $\mathcal{T}(\,\cdot\,)$ is compact. Next, we check the piecewise equicontinuity of the second term $W^2_{\mathcal{Q}_1}(t)$.

For $t \in [0,t_1]$, let $0 \le t' < t'' \le t_1$, we can obtain

$$\begin{aligned}\bigg\|\int_0^{t''}(t''-s)^{\alpha-1}\mathcal{S}(t''-s)f(s,v(s))\,ds-\int_0^{t'}(t'-s)^{\alpha-1}\mathcal{S}(t'-s)f(s,v(s))\,ds\bigg\|\\ \le\bigg\|\int_{t'}^{t''}(t''-s)^{q-1}\mathcal{S}(t''-s)f(s,v(s))\,ds\bigg\|\\ +\bigg\|\int_0^{t'}(t''-s)^{\alpha-1}\mathcal{S}(t''-s)f(s,v(s))\,ds\\ -\int_0^{t'}(t'-s)^{\alpha-1}\mathcal{S}(t''-s)f(s,v(s))\,ds\bigg\|\\ +\bigg\|\int_0^{t'}(t'-s)^{q-1}\mathcal{S}(t''-s)f(s,v(s))\,ds\\ -\int_0^{t'}(t'-s)^{\alpha-1}\mathcal{S}(t'-s)f(s,v(s))\,ds\bigg\|=I_1+I_2+I_3,\end{aligned}$$

where

$$I_1 = \left\| \int_{t'}^{t''} (t''-s)^{\alpha-1} \mathcal{S}(t''-s) f(s,v(s))\, ds \right\|,$$
$$I_2 = \left\| \int_0^{t'} [(t''-s)^{\alpha-1} - (t'-s)^{\alpha-1}] \mathcal{S}(t''-s) f(s,v(s))\, ds \right\|,$$
$$I_3 = \left\| \int_0^{t'} (t'-s)^{\alpha-1} [\mathcal{S}(t''-s) - \mathcal{S}(t'-s)] f(s,v(s))\, ds \right\|.$$

Based on the discussion in our previous work [52], we know that I_1, I_2, I_3 tend to zero as $t'' \to t'$. Using the same method, one can show that the piecewise equicontinuity of the third term $W^3_{\mathcal{Q}_1}(t)|_{(t_k,t_{k+1}]}$.

By the PC-type Arzela–Ascoli theorem again, for each $t \in J$, $W_{\mathcal{Q}_1}(t)$ is relatively compact in X. Therefore, $\mathcal{Q}_1$ is a compact operator.

As a result, $\mathcal{Q}$ is compact due to $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are compact operators.

Step 3. $\mathcal{Q}$ has a fixed point in $PC(J,X)$.

According to Leray–Schauder fixed point theorem, it suffices to show the following set

$$\mathcal{K} = \{x \in PC(J,X) \mid x = \sigma \mathcal{Q} x, \sigma \in [0,1]\}$$

is a bounded subset of $PC(J,X)$.

In fact, let $x \in \mathcal{K}$, we have

$$\begin{aligned}
\|x(t)\| &= \|\mathcal{Q}(\sigma x(t))\| \\
&\le \|\mathcal{T}(t-t_k)\| \prod_{0<i\le k} \|\mathcal{T}(t_i - t_{i-1})\sigma x_0\| \\
&\quad + \|\mathcal{T}(t-t_k)\| \sum_{i=1}^{k} \Bigg\{ \prod_{i<j\le k} \|\mathcal{T}(t_j - t_{j-1})\| \\
&\quad \times \left[\int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} \|\mathcal{S}(t_i - s) f(s,\sigma x(s))\|\, ds + \|I_i(x(t_i))\| \right] \Bigg\} \\
&\quad + \int_{t_k}^{t} (t-s)^{\alpha-1} \|\mathcal{S}(t-s) f(s,\sigma x(s))\|\, ds \\
&\le M^{\delta+1}\|x_0\| + \frac{\alpha M^{\delta+1}}{\Gamma(1+\alpha)} \int_0^t (t-s)^{\alpha-1} M_f (1 + \|x(s)\|)\, ds \\
&\quad + \sum_{i=1}^{k} M^{\delta+1} (\|I_i(0)\| + h^* \|x(t_i)\|) \\
&\le \left[M^{\delta+1}\|x_0\| + \frac{b^\alpha M_f M^{\delta+1}}{\Gamma(1+\alpha)} + \delta M^{\delta+1} \|I_k(0)\| \right] \\
&\quad + \frac{\alpha M^{\delta+1} M_f}{\Gamma(1+\alpha)} \int_0^t (t-s)^{\alpha-1} \|x(s)\|\, ds + h^* M^{\delta+1} \sum_{0<t_k<t} \|x(t_k)\|.
\end{aligned}$$

By Lemma 3.1 and Remark 3.2, we know that there exist a constant $M_1^* > 0$ such that

$$\|x\|_{PC} \le M_1^* \bigg[M^{\delta+1}\|x_0\| + \frac{b^\alpha M_f M^{\delta+1}}{\Gamma(1+\alpha)} + \delta M^{\delta+1}\|I_k(0)\| \bigg] \quad \text{for all } x \in \mathcal{K}.$$

Thus, $\mathcal{K}$ is a bounded subset of $PC(J,X)$.

Now, Schauder's fixed point theorem implies that $\mathcal{Q}$ has a fixed point in $PC(J,X)$. This yields that system (1.1) has at least one PC-mild solution on J.

Step 4. Uniqueness.

Let $y(\,\cdot\,)$ be another PC-mild solution of system (1.1) with the initial value y_0. It is not difficult to verify that there exists a constant $\rho > 0$ such that $\|x\|_{PC} \le \rho$ and $\|y\|_{PC} \le \rho$. Directly calculation, we can deduce that

$$\begin{aligned} \|x(t)-y(t)\| \le M^{\delta+1}\|x_0-y_0\| + \frac{\alpha\delta M^{\delta+1}L_f(\rho)}{\Gamma(1+\alpha)} \int_0^t (t-s)^{\alpha-1}\|x(s)-y(s)\|\,ds \\ + h^* M^{\delta+1} \sum_{0<t_k<t} \|x(t_k)-y(t_k)\|. \end{aligned}$$

By Lemma 3.1 and Remark 3.2 again, there exists a constant $M_2^* > 0$ such that

$$\|x(t)-y(t)\| \le M_2^* M^{\delta+1}\|x_0-y_0\|,$$

which yields the uniqueness of $x(\,\cdot\,)$. □

5. Existence of optimal controls

Let Y be another separable reflexive Banach space from which the controls u take the value. We denote a class of nonempty closed and convex subsets of Y by $W_f(Y)$. The multifunction $\omega\colon J \to W_f(Y)$ is measurable and $\omega(\,\cdot\,) \subset E$ where E is a bounded set of Y, the admissible control set $U_{\mathrm{ad}} = \mathcal{S}_\omega^p = \{u \in L^p(E) \mid u(t) \in \omega(t) \text{ almost everywhere}\}$, $1 < p < \infty$. Then $U_{\mathrm{ad}} \ne \emptyset$ (see [23, p. 142, Proposition 1.7 and p. 174, Lemma 3.2]).

Consider the following controlled system

$$(5.1) \quad \begin{cases} {}^C D_t^\alpha x(t) = Ax(t) + f(t,x(t)) + B(t)u(t), & 0<\alpha<1,\ t \in J,\ t \ne t_k, \\ x(0) = x_0, \\ \Delta x(t_k) = I_k(x(t_k)), & k = 1,\dots,\delta. \end{cases}$$

Assumption (HB): $B \in L_\infty(J, \pounds(Y,X))$.

It is easy to see that $Bu \in L^p(J,X)$ for all $u \in U_{\mathrm{ad}}$. Define $\widetilde{f}(t,x) = f(t,x(t)) + B(t)u(t)$. It is obvious that if $\widetilde{f}$ satisfies the assumptions (HF1) and (HF2). By Theorem 4.1, we have the following existence and uniqueness result.

THEOREM 5.1. *Under the assumptions* (HA), (HF1), (HF2), (HI1), (HI2) *and* (HB), *for every $u \in U_{\mathrm{ad}}$, system* (5.1) *has a unique PC-mild solution corresponding to u given by*

$$x(t) = \begin{cases} \mathcal{T}(t)x_0 + \displaystyle\int_0^t (t-s)^{\alpha-1}\mathcal{S}(t-s)[f(s,x(s)) + B(s)u(s)]\,ds, & t \in [0,t_1], \\ \mathcal{T}(t-t_k) \displaystyle\prod_{0<i\le k} \mathcal{T}(t_i - t_{i-1})x_0 & \\ \quad + \mathcal{T}(t-t_k)\displaystyle\sum_{i=1}^{k}\bigg\{ \prod_{i<j\le k} \mathcal{T}(t_j - t_{j-1}) & \\ \quad \times \bigg[\displaystyle\int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1}\mathcal{S}(t_i - s)[f(s,x(s)) + B(s)u(s)]\,ds + I_i(x(t_i))\bigg]\bigg\} & \\ \quad + \displaystyle\int_{t_k}^{t} (t-s)^{\alpha-1}\mathcal{S}(t-s)[f(s,x(s)) + B(s)u(s)]\,ds, & t \in (t_k, t_{k+1}], \\ & k = 1, \dots, \delta, \end{cases}$$

provided that $1 > \alpha > 1/p$ for some $1 < p < \infty$.

PROOF. Compared with Theorem 4.1, the key step is to check the term containing control policy for $t \in (t_{k-1}, t_k]$.

Consider

$$\Phi(t) = \mathcal{T}(t-t_k)\sum_{i=1}^{k}\bigg\{ \prod_{i<j\le k} \mathcal{T}(t_j - t_{j-1}) \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1}\mathcal{S}(t_i - s)B(s)u(s)\,ds\bigg\} + \int_{t_k}^{t} (t-s)^{\alpha-1}\mathcal{S}(t-s)B(s)u(s)\,ds.$$

By our assumptions and Lemma 2.9, and Hölder inequality,

$$\begin{aligned} &\bigg\| \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1}\mathcal{S}(t_i - s)B(s)u(s)\,ds \bigg\| \\ &\quad \le \frac{\alpha M}{\Gamma(1+\alpha)} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1}\|B(s)u(s)\|\,ds \\ &\quad \le \frac{\alpha M\|B\|_\infty}{\Gamma(1+\alpha)} \bigg(\int_{t_i}^{t_{i-1}} (t_i - s)^{(\alpha-1)p/(p-1)}\,ds \bigg)^{(p-1)/p} \bigg(\int_0^t \|u(s)\|_Y^p\,ds \bigg)^{1/p} \\ &\quad \le \frac{\alpha M\|B\|_\infty}{\Gamma(1+\alpha)} \bigg(\frac{p-1}{p+p(\alpha-1)-1} \bigg)^{(p-1)/p} b^{(p+p(\alpha-1)-1)/p}\|u\|_{L^p(J,Y)}, \end{aligned} \tag{5.2}$$

and

$$\begin{aligned} &\bigg\| \int_{t}^{t_k} (t-s)^{\alpha-1}\mathcal{S}(t-s)B(s)u(s)\,ds \bigg\| \\ &\qquad \le \frac{\alpha M\|B\|_\infty}{\Gamma(1+\alpha)} \bigg(\frac{p-1}{p+p(\alpha-1)-1} \bigg)^{(p-1)/p} b^{(p+p(\alpha-1)-1)/p}\|u\|_{L^p(J,Y)}, \end{aligned} \tag{5.3}$$

where $\|B\|_\infty$ is the norm of operator B in Banach space $L_\infty(J, \pounds(Y,X))$. Thus, $\|(t_i - s)^{\alpha-1}\mathcal{S}(t_i - s)B(s)u(s)\|$ and $\|(t-s)^{\alpha-1}\mathcal{S}(t-s)B(s)u(s)\|$ are Lebesgue integrable with respect to $s \in [0,t]$ for all $t \in (t_k, t_{k+1}]$. From Lemma 2.10, it follows that $(t_i - s)^{\alpha-1}\mathcal{S}(t_i - s)B(s)u(s)$ and $(t-s)^{\alpha-1}\mathcal{S}(t-s)B(s)u(s)$ and are Bochner integral with respect to $s \in [0,t]$ for all $t \in (t_k, t_{k+1}]$. Hence $\Phi(\,\cdot\,) \in C((t_k, t_{k+1}], X)$. Using Theorem 4.1, one can verify it immediately. □

Theorem 5.2. *Let $\Xi \subset X$ be a bounded set, $x^1(x^2)$ be a PC-mild solution of system* (5.1) *corresponding to $(x_0^1, u_1)((x_0^2, u_2)) \in \Xi \times U_{\mathrm{ad}}$. Under assumptions of Theorem* 5.1*, there exist constants $\widehat{M}_1$, $\widehat{M}_2 > 0$ such that*

$$\|x^1 - x^2\|_{PC} \le \widehat{M}_1 \|x_0^1 - x_0^2\| + \widehat{M}_2 \|u_1 - u_2\|_{L^p(J,Y)}.$$

Proof. Since x^1 and x^2 are the PC-mild solution of system (5.1), using Lemma 3.1 and Remark 3.2, Theorem 5.1 and the boundedness of U_{ad}, it is not difficult to verify that there exists a constant $\rho > 0$ such that $\|x^1\|$, $\|x^2\| < \rho$. Similar to the discussion on Step 4 in Theorem 4.1, note that (5.2) and (5.3) one can complete the rest proof. □

Now, we consider the Bolza problem:

(P) Find $(x^0, u^0) \in PC(J,X) \times U_{\mathrm{ad}}$ such that

$$J(x^0, u^0) \le J(x^u, u), \quad \text{for all } u \in U_{\mathrm{ad}},$$

where

$$J(x^u, u) = \int_0^b \mathcal{L}(t, x^u(t), u(t))\,dt + \Psi(x^u(b)),$$

x^u denotes the PC-mild solution of system (5.1) corresponding to the control $u \in U_{\mathrm{ad}}$.

We impose some assumption on $\mathcal{L}$ and Ψ.

Assumption (HL):

(HL1) The functional $\mathcal{L}: J \times X \times Y \to R \cup \{\infty\}$ is Borel measurable.

(HL2) $\mathcal{L}(t, \cdot, \cdot)$ is sequentially lower semicontinuous on $X \times Y$ for almost all $t \in J$.

(HL3) $\mathcal{L}(t, x, \cdot)$ is convex on Y for each $x \in X$ and almost all $t \in J$.

(HL4) There exist constants $d \ge 0$, $e > 0$, φ is nonnegative and $\varphi \in L^1(J,R)$ such that

$$\mathcal{L}(t,x,u) \ge \varphi(t) + d\|x\| + e\|u\|_Y^p.$$

(HL5) The functional $\Psi: X \to R$ is continuous and nonnegative.

In order to obtain the existence of impulsive fractional optimal controls we need the following important Lemmas.

Lemma 5.3. *Assumption* (HA) *holds. Then operators* $\mathcal{E}_j\colon L^p(J,Y)\to C(J,X)$, $j=1,2$ *for some* $1>\alpha>1/p$, *given by*

$$(\mathcal{E}_1 u)(\,\cdot\,)=\int_0^{\cdot}(\,\cdot\,-s)^{\alpha-1}\mathcal{T}(\,\cdot\,-s)B(s)u(s)\,ds,$$
$$(\mathcal{E}_2 u)(\,\cdot\,)=\int_0^{\cdot}(\,\cdot\,-s)^{\alpha-1}\mathcal{S}(\,\cdot\,-s)B(s)u(s)\,ds$$

are strongly continuous.

Proof. Suppose that $\{u^n\}\subseteq L^p(J,Y)$ is bounded, we define $\mathcal{A}_n^j(t)=(\mathcal{E}_j u^n)(t)$, $j=1,2$, $t\in J$. One can verify that for any fixed $t\in J$ and $1>\alpha>1/p$, $\|\mathcal{A}_n^j(t)\|$ is bounded. By Lemma 2.9, it is not difficult to verify that $\mathcal{A}_n^j(t)$ is compact in X and is also equicontinuous. Due to Ascoli–Arzela Theorem, $\{\mathcal{A}_n^j(t)\}$ is relatively compact in $C(J,X)$. Obviously, $\mathcal{E}$ is linear and continuous. Hence, $\mathcal{E}_j$ is a strongly continuous operator due to Lebesgue dominated convergence theorem and [23]. □

By Lemma 5.3, we can obtain the following results immediately.

Lemma 5.4. *Operators* $\mathcal{H}_j\colon L^p(J,Y)\to PC(J,X)$, $j=1,2$ *for some* $1>\alpha>1/p$, *given by*

$$(\mathcal{H}_1 u)(\,\cdot\,)=\begin{cases}\displaystyle\int_0^t(t-s)^{\alpha-1}\mathcal{T}(t-s)B(s)u(s)\,ds, & t\in[0,t_1],\\ \displaystyle\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\mathcal{T}(t_2-s)B(s)u(s)\,ds, & \\ \cdots\cdots\cdots\cdots\cdots\cdots & \\ \displaystyle\int_{t_{\delta-1}}^{t_\delta}(t_\delta-s)^{\alpha-1}\mathcal{T}(t_\delta-s)B(s)u(s)\,ds, & \\ \displaystyle\int_{t_\delta}^{t}(t-s)^{\alpha-1}\mathcal{T}(t-s)B(s)u(s)\,ds, & t\in(t_\delta,T],\end{cases}$$

and

$$(\mathcal{H}_2 u)(\,\cdot\,)=\begin{cases}\displaystyle\int_0^t(t-s)^{\alpha-1}\mathcal{S}(t-s)B(s)u(s)\,ds, & t\in[0,t_1],\\ \displaystyle\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\mathcal{S}(t_2-s)B(s)u(s)\,ds, & \\ \cdots\cdots\cdots\cdots\cdots\cdots & \\ \displaystyle\int_{t_{\delta-1}}^{t_\delta}(t_\delta-s)^{\alpha-1}\mathcal{S}(t_\delta-s)B(s)u(s)\,ds, & \\ \displaystyle\int_{t_\delta}^{t}(t-s)^{\alpha-1}\mathcal{S}(t-s)B(s)u(s)\,ds, & t\in(t_\delta,T],\end{cases}$$

are strongly continuous.

Now we can give the existence of optimal controls for Bolza problem (P).

Theorem 5.5. *Under the assumptions of Theorem* 5.1 *and* (HL)*, Bolza problem* (P) *admits at least one optimal pair.*

Proof. If $\inf\{J(u) \mid u \in U_{\rm ad}\} = \infty$, there is nothing to prove. So we assume that $\inf\{J(u) \mid u \in U_{\rm ad}\} = m < \infty$. By (HL) we have

$$J(u) \geq \int_0^b \varphi(t)\,dt + d\int_0^b \|x(t)\|\,dt + e\int_0^b \|u(t)\|_Y^p\,dt + \Psi(x^u(b)) \geq -\eta > -\infty,$$

where $\eta > 0$ is a constant. Hence $m \geq -\eta > -\infty$.

By definition of infimum there exists a sequence $\{u^n\} \subset U_{\rm ad}$, such that $J(u^n) \to m$.

Since $\{u_n\}$ is bounded in $L^p(J,Y)$, there exists a subsequence, relabeled as $\{u^n\}$, and $u^0 \in L^p(J,Y)$ such that $u^n \xrightarrow{w} u^0$ in $L^p(J,Y)$. Since $U_{\rm ad}$ is closed and convex, thanking Mazur Lemma, $u^0 \in U_{\rm ad}$.

Suppose x^n is the PC-mild solution of system (5.1) corresponding to u^n $(n = 0,1,\ldots)$, then x^n satisfies the following integral equation

$$x^n(t)=\begin{cases} \mathcal{T}(t)x_0 + \displaystyle\int_0^t (t-s)^{\alpha-1}\mathcal{S}(t-s)[f(s,x^n(s)) + B(s)u^n(s)]\,ds, \\ \hfill t \in [0,t_1], \\ \mathcal{T}(t-t_k)\displaystyle\prod_{0<i\leq k}\mathcal{T}(t_i-t_{i-1})x_0 + \mathcal{T}(t-t_k)\sum_{i=1}^k\bigg\{\prod_{i<j\leq k}\mathcal{T}(t_j-t_{j-1}) \\ \quad\times\bigg[\displaystyle\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\mathcal{S}(t_i-s)[f(s,x^n(s))+B(s)u^n(s)]\,ds+I_i(x(t_i))\bigg]\bigg\} \\ \quad+\displaystyle\int_{t_k}^t (t-s)^{\alpha-1}\mathcal{S}(t-s)[f(s,x^n(s)) + B(s)u^n(s)]\,ds, \quad t \in (t_k,t_{k+1}], \\ \hfill k = 1,\ldots,\delta. \end{cases}$$

It follows from the boundedness of $\{u^n\}$ and Lemma 3.1, one can verify that there exists a $\rho^* > 0$ such that $\|x^n\|_{PC} \leq \rho^*$.

Define

$$\eta_n^i = \begin{cases} \displaystyle\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\|\mathcal{S}(t_i-s)B(s)u^n(s) - \mathcal{S}(t_i-s)B(s)u^0(s)\|\,ds, \\ \hfill i = 1,\ldots,k, \\ \displaystyle\int_{t_k}^t (t-s)^{\alpha-1}\|\mathcal{S}(t-s)B(s)u^n(s) - \mathcal{S}(t-s)B(s)u^0(s)\|\,ds, \\ \hfill k = 1,\ldots,\delta. \end{cases}$$

It comes from the compactness of $\mathcal{S}(\,\cdot\,)$ and Lemma 5.4, we obtain

$$\eta_n^i \to 0 \quad \text{in } C([t_{i-1},t_i],X) \text{ as } u^n \xrightarrow{w} u^0,\ i = 1,\ldots,\delta. \tag{5.4}$$

Now, for $t \in [0, t_1]$, directly calculation implies

$$\|x^n(t) - x^0(t)\| \leq \eta_n^1 + \frac{ML_f(\rho)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|x^n(s) - x^0(s)\| \, ds.$$

By Lemma 3.1 and Remark 3.2, there exists a $\widetilde{M} > 0$ such that

$$\|x^n(t) - x^0(t)\| \leq \widetilde{M}\eta_n^1 \equiv \widehat{C}_1 \eta_n^1, \quad \text{for } t \in [0, t_1]. \tag{5.5}$$

By (HI2) and (5.5), we also have

$$\begin{aligned}\|x^n(t_1^+) - x^0(t_1^+)\| &\leq \|x^n(t_1) - x^0(t_1)\| + \|I_1(x^n(t_1)) - I_1(x^0(t_1))\| \\ &\leq (h^* + 1)\|x^n(t_1) - x^0(t_1)\| \leq (h^* + 1)\widehat{C}_1 \eta_n^1 \equiv \widehat{C}_1' \eta_n^1.\end{aligned}$$

For $t \in (t_1, t_2]$, with the same argument, we can obtain

$$\begin{aligned}\|x^n(t) - x^0(t)\| &\leq \widehat{C}_2 \eta_n^2 \quad \text{for } t \in (t_1, t_2], \\ \|x^n(t_2^+) - x^0(t_2^+)\| &\leq \widehat{C}_2' \eta_n^2.\end{aligned}$$

Thus, in general, given any t_k, $k = 1, \dots, \delta$, and the $x^n(t_k)$, $x^0(t_k)$, prior to the jump at time t_k, we immediately following the jump as $x^n(t_k^+) = x^n(t_k) + I_k(x^n(t_k))$, $x^0(t_k^+) = x^0(t_k) + I_k(x^0(t_k))$, the associated interval $(t_k, t_{k+1}]$, we also similarly obtain

$$\begin{aligned}\|x^n(t) - x^0(t)\| \leq &\widehat{C}_{k+1} \eta_n^{k+1} \quad \text{for } t \in (t_k, t_{k+1}], \\ \|x^n(t_{k+1}^+) - x^0(t_{k+1}^+)\| &\leq \widehat{C}_{k+1}' \eta_n^{k+1}.\end{aligned}$$

Step by steps, we repeat the procedures till the time interval is exhausted. Let $\widehat{C} = \max\{\widehat{C}_1, \widehat{C}_1', \widehat{C}_2, \widehat{C}_2', \dots, \widehat{C}_{\delta+1}\}$ and $\widehat{\eta}_n = \max\left\{\eta_n^1, \eta_n^2, \dots, \eta_n^{\delta+1}\right\}$ thus we obtain

$$\|x^n - x^0\|_{PC} \leq \widehat{C}\widehat{\eta}_n. \tag{5.6}$$

From (5.4) and (5.6), we immediately have

$$x^n \to x^0 \quad \text{in } PC(J, X) \text{ as } u^n \xrightarrow{w} u^0.$$

Since $PC(J, X) \hookrightarrow L^1(J, X)$, using (HL) and Balder's theorem, we can obtain

$$\begin{aligned} m &= \lim_{n \to \infty} \int_0^b \mathcal{L}(t, x^n(t), u^n(t)) \, dt + \Psi(x^n(b)) \\ &\geq \int_0^b \mathcal{L}(t, x^0(t), u^0(t)) \, dt + \Psi(x^0(b)) \geq m. \end{aligned}$$

This shows that J attains its minimum at $u^0 \in U_{\text{ad}}$. This completes the proof.□

6. An example

In this section, an example is given to illustrate our theory.

Consider the following problem

$$(6.1)\qquad \begin{cases} {}^C D_t^\alpha x(t,y) = \dfrac{\partial^2}{\partial y^2} x(t,y) + \sqrt{x^2(t,y)+1} + |\sin(t,y)| + u(t,y), \\ \qquad\qquad\qquad\qquad\qquad\qquad \alpha = \dfrac{4}{5} \in (0,1),\ y \in \Omega = (0,\pi), \\ \qquad\qquad\qquad\qquad\qquad\qquad t \in \left[0, \dfrac{1}{3}\right) \cup \left(\dfrac{1}{3}, 1\right], \\ \Delta x(t_1,y) = \dfrac{|x(t_1,y)|}{1+|x(t_1,y)|}, \quad t_1 = \dfrac{1}{3},\ y \in \Omega, \\ x(t,y)|_{y\in\partial\Omega} = 0, \qquad\qquad\quad t>0,\ x(0,y)=0,\ y\in\Omega. \end{cases}$$

Let $X = Y = L^2(0,\pi)$ and $A\colon X \to X$ be defined by $Ax = x_{yy}$, $x \in D(A)$ where $D(A) = \{x \in X : x, x_y$ are absolutely continuous, $x(0) = x(\pi) = 0\}$. Then

$$Ax = \sum_{n=1}^{\infty} n^2 (x, x_n) x_n, \quad x \in X,$$

where $x_n(s) = \sqrt{2/\pi}\sin(ns)$, $n = 1, 2, \ldots$ is the orthogonal set of eigenfunctions of A. It can be easily shown that A is the infinitesimal generator of a compact analytic semigroup $\{T(t),\ t \geq 0\}$ in X and is given by

$$T(t)x = \sum_{n=1}^{\infty} e^{-n^2 t}(x, x_n) x_n.$$

So there exists a constant $M \geq 1$ such that $\|T(t)\| \leq M$.

The admissible control set $U_{\rm ad} = \{u \in Y \mid \|u\|_{L^2(J,Y)} \leq 1\}$ is closed and convex. Find a control $u(t,y)$ that minimizes the performance index

$$J(x,u) = \int_0^1 \int_\Omega |x(t,y)|^2\,dy\,dt + \int_0^1 \int_\Omega |u(t,y)|^2\,dy\,dt + \int_\Omega |x(1,y)|^2\,dy$$

subject to the problem (6.1).

Denote

$$\begin{aligned} x(\,\cdot\,)(y) &= x(\,\cdot\,,y), & \sin(\,\cdot\,)(y) &= \sin(\,\cdot\,,y), \\ f(\,\cdot\,,x(\,\cdot\,))(y) &= \sqrt{x^2(\,\cdot\,,y)+1} + |\sin(\,\cdot\,,y)|, \\ B(\,\cdot\,)u(\,\cdot\,)(y) &= u(\,\cdot\,,y), & I_1(x(t_1))(y) &= \frac{|x(t_1,y)|}{1+|x(t_1,y)|}. \end{aligned}$$

Thus, problem (6.1) can be rewritten as

$$\begin{cases} {}^C D_t^\alpha x(t) = Ax(t) + f(t,x(t)) + B(t)u(t), & \alpha \in (0,1),\ t \in [0,1] \setminus \{t_1\}, \\ \Delta x(t_1) = I_1(x(t_1)), & t_1 = 1/3, \\ x(0) = 0, \end{cases}$$

with the cost function

$$J(u) = \int_0^1 (\|x(t)\|^2 + \|u(t)\|_Y^2)\,dt + \|x(1)\|^2.$$

Obviously, all the assumptions in Theorem 5.5 are satisfied. Our results can be used to solve problem (6.1).

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