Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder University Centre
Volume 38, 2011, 1–15

POSITIVE SOLUTIONS FOR SECOND-ORDER FOUR-POINT BOUNDARY VALUE PROBLEMS AT RESONANCE

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Abstract. Using Leggett–Williams norm-type theorem due to D. O'Regan and M. Zima, we establish the existence of positive solution for a class of second-order four point boundary value problem under different resonance conditions. An example is given to illustrate the main results.

1. Introduction

The purpose of this paper is to study the existence of positive solution for the second-order four point boundary value problem

$$\begin{cases} x''(t) + f(t, x(t)) = 0, & t \in (0,1), \\ x(0) = \alpha x(\eta), & x(1) = \beta x(\xi), \end{cases} \tag{1.1}$$

where $\eta, \xi \in (0,1)$ under the resonance condition:

$$\alpha = \beta = 1 \tag{1.2}$$

or

$$0 < \alpha < \frac{1}{1-\eta}, \quad 0 < \beta < \frac{1}{\xi}, \quad \alpha\eta(1-\beta) + (1-\alpha)(1-\beta\xi) = 0. \tag{1.3}$$

2010 *Mathematics Subject Classification.* 34B10, 34B15.

Key words and phrases. Four-point boundary value problem, resonance, positive solution, cone, fixed point.

This work was supported by the Natural Science Foundation of Anhui Department of Education (KJ2010B163) and Anhui Provincial Natural Science Foundation (10040606Q50).

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The multi-point boundary value problems for ordinary differential equations arise in different areas of applied mathematics and physics. The study of multi-point boundary value problems for linear and nonlinear ordinary differential equations was initiated by A. V. Bitsadze and A. A. Samarskiĭ [8] and continued by V. A. Il'in, E. I. Moiseev [15], [16] and C. P. Gupta [11], [12]. Since then, by using various methods, such as Leray–Schauder continuation theorem, nonlinear alternatives of Leray–Schauder, coincidence degree theory and different fixed point theorems, the more general nonlinear multi-point boundary value problems have been studied by several authors. We refer the reader to [1], [2], [4], [10], [13], [14], [20], [21], [27], [28] and references along this line.

For problem (1.1) under the case $f(t,x) = a(t)f(x)$, B. Liu [19] established the existence of positive solutions by using Krasnosel'skiĭ–Guo fixed point theorem on cone expansion and compression and fixed point index theorem. Bai, Li and Ge [6] obtained the multiple positive solutions by using a new fixed point theorem due to Avery and Peterson. All these results were established under the condition

$$0 < \alpha < \frac{1}{1-\eta}, \quad 0 < \beta < \frac{1}{\xi}, \quad \alpha\eta(1-\beta) + (1-\alpha)(1-\beta\xi) \neq 0,$$

which ensures that the problem studied is not at resonance, that is, the associated linear operator $Lx = -x''$ is invertible. In the resonace case, Rachunková [24]–[26] obtained the existence of solutions for problem (1.1) with resonant condition (1.2) by using upper and lower solutions method and topological degree theory. Z. Bai [7] obtained the existence of solutions for problem (1.1) with resonant condition (1.3) by using coincidence degree theorey and upper and lower solutions method. B. Liu [18] considered m-point boundary value problem

$$\begin{cases} x''(t) = f(t, x(t), x'(t)) + e(t), & t \in (0,1), \\ x(0) = \alpha x(\eta), \quad x(1) = \displaystyle\sum_{j=1}^{n-2} \beta_j x(\xi_j), \end{cases}$$

where $0 < \xi_1 < \ldots < \xi_{n-2} < 1$, $\sum_{j=1}^{n-2} \beta_j = 1$. By using J. Mawhin continuation principle [22], he established the existence results of solution for this resonant problem. In paper [3], the problems in the resonance were treated too, but the four-point boundary condition in the paper was more restrictive. Namely, if the boundary points are a, b, c, d, then in the paper: $b - a = d - c$, which is not the authors' case.

It is well known that the problem of existence of positive solution to boundary value problems is very difficult when the resonant case is considered. Only few papers deal with positive solutions to boundary value problems at resonance.

Z. Bai and J. Fang [5] established the existence of positive solutions of the second-order differential equation

$$\begin{cases} (p(t)x'(t))' = f(t, x(t), x'(t)), & t \in (0,1), \\ x'(0) = 0, \quad x(1) = x(\eta), \end{cases}$$

by using a fixed point index theorem for semi-linear A-proper maps due to C. T. Cremins [9]. G. Infante and M. Zima [17] obtained the existence of positive solution for problem

$$\begin{cases} x''(t) + f(t, x(t)) = 0, & t \in (0,1), \\ x'(0) = 0, \quad x(1) = \displaystyle\sum_{i=1}^{m-2} \alpha_i x(\eta_i), \end{cases}$$

with resonant condition $\sum_{i=1}^{m-2} \alpha_i = 1$. The result was based on the Leggett–Williams norm-type theorem due to D. O'Regan and M. Zima [23].

To the best of our knowledge, positive solution of problem (1.1) at resonance has not been considered before. The main purpose of this paper is to fill this gap. In this paper we will give the sufficient conditions to ensure the existence of positive solution for problem (1.1) with resonant condition (1.2) and resonant condition (1.3). Our method is based on the Leggett–Williams norm-type theorem.

The rest of the paper is organized as follows. The preliminary definitions and lemma are given in Section 2. In Section 3, we discuss the existence of at least one positive solution for problem (1.1) with resonant condition (1.2). The existence of at least one positive solution for problem (1.1) with resonant condition (1.3) is considered in Section 4. Finally, in Section 5, we give an example to illustrate the main results.

2. Preliminaries

For the convenience of the reader,we present here the necessary definitions and a new fixed point theorem due to D. O'Regan and M. Zima. Let X, Y be real Banach spaces. A nonempty convex closed set $C \subset X$ is said to be a cone provided that

(i) $ax \in C$, for all $x \in C$, $a \geq 0$,
(ii) $x, -x \in C$ implies $x = 0$.

Note that every one cone $C \subset X$ induces an ordering in X given by $x \leq y$ if $y - x \in C$.

$L: \operatorname{dom} L \subset X \to Y$ is called a Fredholm operator with index zero, that is, $\operatorname{Im} L$ is closed and $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < \infty$, which implies that there exist continuous projections $P: X \to X$ and $Q: Y \to Y$ such that $\operatorname{Im} P = \operatorname{Ker} L$ and $\operatorname{Ker} Q = \operatorname{Im} L$. Moreover, since $\dim \operatorname{Im} Q = \operatorname{codim} \operatorname{Im} L$, there exists an isomorphism $J\colon \operatorname{Im} Q \to \operatorname{Ker} L$. Denote by L_P the restriction of L to $\operatorname{Ker} P \cap \operatorname{dom} L$ to $\operatorname{Im} L$ and its inverse by K_P, so $K_P\colon \operatorname{Im} L \to \operatorname{Ker} P \cap \operatorname{dom} L$ and the coincidence equation $Lx = Nx$ is equivalent to

$$x = (P + JQN)x + K_P(I - Q)Nx.$$

Denote $\gamma\colon X \to C$ be a retraction, that is, a continuous mapping such that $\gamma x = x$ for all $x \in C$ and

$$\Psi := P + JQN + K_P(I - Q)N \quad \text{and} \quad \Psi_\gamma := \Psi \circ \gamma.$$

Lemma 2.1 ([23]). *Let C be a cone in X and Ω_1, Ω_2 be open bounded subsets of X with $\overline{\Omega}_1 \subset \Omega_2$ and $C \cap (\overline{\Omega}_2 \setminus \Omega_1) \neq \emptyset$. Assume that $L\colon domL \subset X \to Y$ is a Fredholm operator of index zero and*

- (C1) *$QN\colon X \to Y$ is continuous and bounded, $K_P(I - Q)N\colon X \to X$ is compact on every bounded subset of X,*
- (C2) *$Lx \neq \lambda Nx$ for all $x \in C \cap \partial\Omega_2 \cap domL$ and $\lambda \in (0, 1)$,*
- (C3) *γ maps subsets of $\overline{\Omega}_2$ into bounded subsets of C.*
- (C4) *$d_B([I - (P + JQN)\gamma]|_{\operatorname{Ker} L}, \operatorname{Ker} L \cap \Omega_2, 0) \neq 0$, where d_B stands for the Brouwer degree,*
- (C5) *There exists $u_0 \in C \setminus \{0\}$ such that $\|x\| \leq \sigma(u_0)\|\Psi x\|$ for $x \in C(u_0) \cap \partial\Omega_1$, where $C(u_0) = \{x \in C : \mu u_0 \leq x$ for some $\mu > 0$ and $\sigma(u_0)$ is such that $\|x + u_0\| \geq \sigma(u_0)\|x\|$ for every $x \in C$,*
- (C6) *$(P + JQN)\gamma(\partial\Omega_2) \subset C$,*
- (C7) *$\Psi_\gamma(\overline{\Omega}_2 \setminus \Omega_1) \subset C$,*

then the equation $Lx = Nx$ has a solution in the set $C \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3. Problem (1.1) with resonant condition (1.2)

Consider the Banach spaces $X = Y = C[0, 1]$ endowed with the norm $\|x\| = \max_{0 \leq t \leq 1} |x(t)|$.

Define linear operator $L\colon \operatorname{dom} L \subset X \to Y$, $(Lx)(t) = -x''(t)$, $t \in [0, 1]$, where

$$\operatorname{dom} L = \{x \in X \mid x'' \in C[0, 1],\ x(0) = x(\eta),\ x(1) = x(\xi)\}$$

and $N\colon X \to Y$ with

$$(Nx)(t) = f(t, x(t)), \quad t \in [0, 1].$$

It is obvious that

$$\operatorname{Ker} L = \{x \in \operatorname{dom} L\colon x(t) \equiv c,\ t \in [0, 1]\}.$$

Denote the function $G(s)$ as follow:

For $0 < \eta < \xi < 1$

$$G(s) = \begin{cases} s(1-\xi) & \text{if } 0 \le s < \eta, \\ \eta(1-\xi) & \text{if } \eta \le s < \xi, \\ \eta(1-s) & \text{if } \xi < s \le 1. \end{cases}$$

For $0 < \xi < \eta < 1$,

$$G(s) = \begin{cases} s(1-\xi) & \text{if } 0 \le s < \xi, \\ (1-\eta-\xi)s + \eta\xi & \text{if } \xi \le s < \eta, \\ \eta(1-s) & \text{if } \eta < s \le 1. \end{cases}$$

Note that $G(s) \ge 0$, $s \in [0,1]$.

Denote the function $U(t,s)$ as follow:

$$U(t,s) = \begin{cases} \dfrac{(1-s)^2}{2} + \dfrac{3t^2+5}{6\int_0^1 G(s)\,ds} G(s) & \text{if } 0 \le t \le s \le 1, \\ \dfrac{(1-s)^2}{2} + s - t + \dfrac{3t^2+5}{6\int_0^1 G(s)\,ds} G(s) & \text{if } 0 \le s \le t \le 1 \end{cases}$$

and

$$\kappa := \min \left\{ 1, \min_{0 \le s \le 1} \frac{\int_0^1 G(s)\,ds}{G(s)}, \min_{t,s \in [0,1]} \frac{1}{U(t,s)} \right\} > 0.$$

Theorem 3.1. *Assume that there exists $R \in (0,\infty)$ such that $f\colon [0,1] \times [0,R] \to R$ is continuous and*

(H1) *$f(t,x) > -\kappa x$, for all $(t,x) \in [0,1] \times [0,R]$,*

(H2) *$f(t,R) < 0$, for all $t \in [0,1]$,*

(H3) *there exists $r \in (0,R)$, $t_0 \in [0,1]$, $a \in (0,1]$, $M \in (0,1)$ and continuous functions $g\colon [0,1] \to [0,\infty)$, $h\colon (0,r] \to [0,\infty)$ such that $f(t,x) \ge g(t)h(x)$ for $[t,x] \in [0,1] \times (0,r]$ and $h(x)/x^a$ is non-increasing on $(0,r]$ with*

$$\frac{h(r)}{r^a} \int_0^1 U(t_0,s) g(s)\,ds \ge \frac{1-M}{M^a},$$

then problem (1.1) *with resonance condition* (1.2) *has at least one positive solution.*

PROOF. Firstly We claim that

$$\operatorname{Im} L = \left\{ y \in Y \;\middle|\; \int_0^1 G(s) y(s)\,ds = 0 \right\}.$$

In fact, for each $y \in \{y \in Y \mid \int_0^1 G(s)y(s)\,ds = 0\}$, we take

$$x(t) = -\int_0^t (t-s)y(s)\,ds + \frac{t}{\eta} \int_0^\eta (\eta - s) y(s)\,ds.$$

It is easy to check that $-x''(t) = y(t)$, $x(0) = x(\eta)$, $x(1) = x(\xi)$. This gives $x(t) \in \operatorname{dom} L$, which means

$$\bigg\{ y \in Y \ \bigg| \ \int_0^1 G(s)y(s)\,ds = 0 \bigg\} \subset \operatorname{Im} L.$$

On the other hand, for each $y(t) \in \operatorname{Im} L$, there exists $x(t) \in \operatorname{dom} L$, that is

$$x''(t) = -y(t), \quad x(0) = x(\eta), \quad x(1) = x(\xi).$$

Thus

$$x(t) = -\int_0^t (t-s)y(s)\,ds + x'(0)t + x(0).$$

Considering the boundary condition $x(0) = x(\eta)$, $x(1) = x(\xi)$, we conclude that

$$\eta \int_0^1 (1-s)y(s)\,ds - \eta \int_0^\xi (\xi - s)y(s)\,ds - (1-\xi)\int_0^\eta (\eta - s)y(s)\,ds = 0$$

which equivalents to the conclusion that $\int_0^1 G(s)y(s)\,ds = 0$. So we have

$$\operatorname{Im} L \subset \bigg\{ y \in Y \ \bigg| \ \int_0^1 G(s)y(s)\,ds = 0 \bigg\}.$$

Thus,

$$\operatorname{Im} L = \bigg\{ y \in Y \ \bigg| \ \int_0^1 G(s)y(s)\,ds = 0 \bigg\}.$$

Clearly, $\dim \operatorname{Ker} L = 1$ and $\operatorname{Im} L$ is closed. Observe that $Y = Y_1 \oplus \operatorname{Im} L$, where

$$Y_1 = \bigg\{ y_1 \ \bigg| \ y_1 = \frac{1}{\int_0^1 G(s)\,ds} \int_0^1 G(s)y(s)\,ds, \ y \in Y \bigg\}.$$

In fact, for each $y(t) \in Y$, we have

$$\int_0^1 G(s)[y(s) - y_1]\,ds = 0.$$

This shows that $y - y_1 \in \operatorname{Im} L$. Since $Y_1 \cap \operatorname{Im} L = \{0\}$, we have $Y = Y_1 \oplus \operatorname{Im} L$. Thus L is a Fredholm operator with index zero. Then define the projections $P: X \to X$, $Q: Y \to Y$ by

$$Px = \int_0^1 x(s)\,ds, \qquad Qy = \frac{1}{\int_0^1 G(s)\,ds} \int_0^1 G(s)y(s)\,ds.$$

Clearly, $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Ker} Q = \operatorname{Im} L$ and $\operatorname{Ker} P = \{x \in X \mid \int_0^1 x(s)\,ds = 0\}$. Note that for $y \in \operatorname{Im} L$, the inverse K_P of L_P is given by

$$(K_P)y(t) = \int_0^t k(t,s)y(s)\,ds$$

where

$$k(t,s)=\begin{cases}\dfrac{(1-s)^2}{2} & \text{if } 0\le t\le s\le 1,\\ \dfrac{(1-s)^2}{2}+s-t & \text{if } 0\le s\le t\le 1.\end{cases}$$

Considering that f can be extended continuously on $[0,1]\times R$, (C1) of Lemma 2.1 is fulfilled.

Define the cone of nonnegative functions

$$C=\{x\in X \mid x(t)\ge 0,\ t\in\ [0,1]\}$$

and

$$\Omega_1=\{x\in X \mid r>|x|>M\|x\|,\ t\in[0,1]\},\quad \Omega_2=\{x\in X \mid \|x\|<R\}.$$

Clearly, Ω_1 and Ω_2 are bounded and open sets, furthermore

$$\overline{\Omega}_1=\{x\in X \mid r\ge|x|\ge M\|x\|,\ t\in[0,1]\}\subset\Omega_2,\quad C\cap\overline{\Omega}_2\setminus\Omega_1\ne\vee.$$

Let $J=I$ and $(\gamma x)(t)=|x(t)|$ for $x\in X$. Then γ is a retraction and maps subsets of $\overline{\Omega}_2$ into bounded subsets of C, which means that (C3) of Lemma 2.1 holds.

Next we confirm that (C2) of Lemma 2.1 holds. For this purpose, suppose that there exists $x_0\in C\cap\partial\Omega_2\cap\operatorname{dom}L$ and $\lambda_0\in(0,1)$ such that $Lx_0=\lambda_0 Nx_0$. Then

$$x_0''(t)+\lambda_0 f(t,x_0)=0$$

for all $t\in(0,1)$. Let $t_1\in[0,1]$ be such that $x_0(t_1)=R$. This gives

$$0\ge x''(t_1)=-\lambda_0 f(t_1,R),$$

which contradicts to (H2). Thus (C2) holds.

For $x\in\operatorname{Ker}L\cap\Omega_2$, $x(t)\equiv c$ on $[0,1]$. Define

$$H(x,\lambda)=x-\lambda|x|-\frac{\lambda}{\int_0^1 G(s)\,ds}\int_0^1 G(s)f(s,|x|)\,ds,$$

where $x\in\operatorname{Ker}L\cap\Omega_2$ and $\lambda\in[0,1]$. Suppose $H(x,\lambda)=0$. In view of (H1) we obtain

$$\begin{aligned}c&=\lambda|c|+\frac{\lambda}{\int_0^1 G(s)\,ds}\int_0^1 G(s)f(s,|c|)\,ds\\ &\ge\lambda|c|-\frac{\lambda}{\int_0^1 G(s)\,ds}\int_0^1 G(s)\kappa|c|\,ds=\lambda|c|(1-\kappa)\ge 0.\end{aligned}$$

Hence $H(x,\lambda)=0$ implies $c\ge 0$. Furthermore, if $H(R,\lambda)=0$, we get

$$0\le R(1-\lambda)=\frac{\lambda}{\int_0^1 G(s)\,ds}\int_0^1 G(s)f(s,R)\,ds,$$

contradicting to (H2). Thus $H(x,\lambda) \neq 0$ for $x \in \partial\Omega_2$ and $\lambda \in [0,1]$. Therefore

$$d_B(H(x,0), \operatorname{Ker} L \cap \Omega_2, 0) = d_B(H(x,1), \operatorname{Ker} L \cap \Omega_2, 0).$$

However

$$d_B(H(x,0), \operatorname{Ker} L \cap \Omega_2, 0) = d_B(I, \operatorname{Ker} L \cap \Omega_2, 0) = 1.$$

This gives

$$d_B([I-(P+JQN)\gamma]|_{\operatorname{Ker} L}, \operatorname{Ker} L \cap \Omega_2, 0) = d_B(H(x,1), \operatorname{Ker} L \cap \Omega_2, 0) \neq 0.$$

Let $x \in \overline{\Omega}_2 \setminus \Omega_1$ and $t \in [0,1]$. Then

$$\begin{aligned}(\Psi_\gamma x)(t) &= \int_0^1 |x(t)|\,dt + \frac{1}{\int_0^1 G(s)\,ds}\int_0^1 G(s)f(s,|x(s)|)\,ds \\ &\quad + \int_0^1 k(t,s)[f(s,|x(s)|) - \frac{1}{\int_0^1 G(s)\,ds}\int_0^1 G(\tau)f(\tau,|x(\tau)|)\,d\tau]\,ds \\ &= \int_0^1 |x(t)|\,dt + \int_0^1 U(t,s)f(s,|x(s)|)\,ds \\ &\geq \int_0^1 |x(s)|\,ds - \kappa\int_0^1 U(t,s)|x(s)|\,ds \\ &= \int_0^1 (1-\kappa U(t,s))|x(s)|\,ds \geq 0.\end{aligned}$$

Hence $\Psi_\gamma(\overline{\Omega}_2) \setminus \Omega_1 \subset C$. Moreover, since for $x \in \partial\Omega_2$, we have

$$\begin{aligned}(P+JQN)\gamma x &= \int_0^1 |x(s)|\,ds + \frac{1}{\int_0^1 G(s)\,ds}\int_0^1 G(s)f(s,|x(s)|)\,ds \\ &\geq \int_0^1 (1 - \frac{\kappa}{\int_0^1 G(s)\,ds}G(s))|x(s)|\,ds \geq 0,\end{aligned}$$

which means $(P+JQN)\gamma(\partial\Omega_2) \subset C$. This ensures that (C6), (C7) of Lemma 2.1 hold.

At last, we confirm that (C5) is satisfied. Taking $u_0(t) \equiv 1$ on $[0,1]$, we see $u_0 \in C \setminus \{0\}$, $C(u_0) = \{x \in C \mid x(t) > 0 \ on[0,1]\}$ and we can take $\sigma(u_0) = 1$. Let $x \in C(u_0) \cap \partial\Omega_1$, we have $x(t) > 0$ on $[0,1]$, $0 < \|x\| \leq r$ and $x(t) \geq M\|x\|$ on $[0,1]$.

Therefore, in view of (H3), we obtain for all $x \in C(u_0) \cap \partial\Omega_1$,

$$\begin{aligned}(\Psi x)(t_0) &= \int_0^1 x(s)\,ds + \int_0^1 U(t_0,s)f(s,x(s))\,ds \\ &\geq M\|x\| + \int_0^1 U(t_0,s)g(s)h(x(s))\,ds \\ &= M\|x\| + \int_0^1 U(t_0,s)g(s)\frac{h(x(s))}{x^a(s)}x^a(s)\,ds\end{aligned}$$

$$\begin{aligned} &\geq M\|x\| + \frac{h(r)}{r^a}\int_0^1 U(t_0,s)g(s)M^a\|x\|^a\,ds \\ &\geq M\|x\| + (1-M)\|x\| = \|x\|. \end{aligned}$$

So $\|x\| \leq \sigma(u_0)\|\Psi x\|$ for all $x \in C(u_0)\cap\partial\Omega_1$, which means (C5) of Lemma 2.1 holds.

Thus by Lemma 2.1, we confirm that the equation $Lx = Nx$ has a solution x, which implies that problem (1.1) with resonance condition (1.2) has at least one positive solution. □

4. Problem (1.1) with resonant condition (1.3)

In this section, we study the positive solution for problem (1.1) with condition (1.3). For this purpose, we also consider the Banach spaces $X = Y = C[0,1]$ and the linear operator $L\colon \operatorname{dom} L \subset X \to Y$, $(Lx)(t) = -x''(t)$, $t \in [0,1]$, where

$$\operatorname{dom} L = \{x \in X \mid x'' \in C[0,1],\ x(0) = \alpha x(\eta),\ x(1) = \beta x(\xi)\}$$

and $N\colon X \to Y$ with

$$(Nx)(t) = f(t,x(t)), \quad t \in [0,1].$$

By a simple computation,

$$\operatorname{Ker} L = \{x \in \operatorname{dom} L \mid x(t) = c[\alpha\eta + (1-\alpha)t],\ c \in R,\ t \in [0,1]\}.$$

Denote function $G(s)$ as follow: For $0<\eta<\xi<1$,

$$G(s) = \begin{cases} (1-\beta\xi+\beta\eta-\eta)s + \beta\xi(1-\eta) & \text{for } 0 \leq s < \eta, \\ \eta[(\beta-1)s + (1-\beta\xi)] & \text{for } \eta \leq s < \xi, \\ \eta(1-s) & \text{for } \xi < s \leq 1. \end{cases}$$

For $0<\xi<\eta<1$

$$G(s) = \begin{cases} (1-\beta\xi+\beta\eta-\eta)s + \beta\xi(1-\eta) & \text{for } 0 \leq s < \xi, \\ \eta(1-s) - (1-\beta\xi)(\eta-s) & \text{for } \xi \leq s < \eta, \\ \eta(1-s) & \text{for } \eta < s \leq 1. \end{cases}$$

Denote

$$U(t,s) = \begin{cases} \dfrac{(1-s)^2}{2} + \dfrac{3t^2+5}{6\int_0^1 G(s)\,ds}G(s) & \text{for } 0 \leq t \leq s \leq 1, \\ \dfrac{(1-s)^2}{2} + s - t + \dfrac{3t^2+5}{6\int_0^1 G(s)\,ds}G(s) & \text{for } 0 \leq s \leq t \leq 1, \end{cases}$$

$$\rho_1 = \begin{cases} \alpha\eta & \text{for } \alpha < 1, \\ \alpha\eta + 1 - \alpha & \text{for } \alpha > 1, \end{cases} \qquad \rho_2 = \begin{cases} \alpha\eta + 1 - \alpha & \text{for } \alpha < 1, \\ \alpha\eta & \text{for } \alpha > 1, \end{cases}$$

$$\rho_3 = \begin{cases} \dfrac{\alpha\eta}{\alpha\eta + 1 - \alpha} & \text{for } \alpha < 1, \\ \dfrac{\alpha\eta + 1 - \alpha}{\alpha\eta} & \text{for } \alpha > 1, \end{cases}$$

and

$$\kappa := \min\bigg\{1, \min_{s\in[0,1]} \frac{\rho_1 \int_0^1 G(s)\,ds}{G(s)}, \min_{t,s\in[0,1]} \frac{\rho_1}{U(t,s)}\bigg\}.$$

It's easy to check that $\kappa > 0$, $G(s) > 0$, $s \in [0,1]$ and $U(t,s) > 0$, $0 \le t,s \le 1$. Furthermore,

$$\alpha\eta + (1-\alpha)t \in [\rho_1, \rho_2], \quad t \in [0,1].$$

Theorem 4.1. *Assume that there exists $R \in (0,\infty)$ such that $f\colon [0,1] \times [0,R] \to R$ is continuous and*

(H4) *$f(t,x) > -\kappa x$ for all $(t,x) \in [0,1] \times [0,R]$,*
(H5) *$f(t,x) < 0$ for $[t,x] \in [0,1] \times [\rho_3 R, R]$,*
(H6) *there exists $r \in (0,R)$, $t_0 \in [0,1]$, $a \in (0,1]$, $M \in (0,1)$ and functions $g\colon \to [0,1] \to [0,\infty)$, $h\colon (0,r] \to [0,\infty)$ such that $f(t,x) \ge g(t)h(x)$ for $[t,x] \in [0,1] \times (0,r]$, $h(x)/x^a$ is non-increasing on $(0,r]$ with*

$$\frac{h(r)}{r^a} \int_0^1 U(t_0,s)g(s)\,ds \ge \frac{1 - M\rho_1}{M^a},$$

then the problem (1.1) *with resonance condition* (1.3) *has at least one positive solution.*

Proof. By a analogous computation with Section 3, we have

$$\operatorname{Im} L = \bigg\{ y \in Y \;\bigg|\; \int_0^1 G(s)y(s)\,ds = 0 \bigg\}.$$

Clearly, $\dim \operatorname{Ker} L = 1$ and $\operatorname{Im} L$ is closed. Observe that $Y = Y_1 + \operatorname{Im} L$, where

$$Y_1 = \bigg\{ y_1 \;\bigg|\; y_1 = \frac{1}{\int_0^1 G(s)\,ds} \int_0^1 G(s)y(s)\,ds,\ y \in Y \bigg\}.$$

Thus L is a Fredholm operator with index zero. Define the projections $P\colon X \to X$, $Q\colon Y \to Y$ by

$$\begin{aligned} Px &= \int_0^1 x(s)\,ds[\alpha\eta + (1-\alpha)t], \quad x \in X,\ t \in [0,1], \\ Qy &= \frac{1}{\int_0^1 G(s)\,ds} \int_0^1 G(s)y(s)\,ds. \end{aligned}$$

Clearly, $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Ker} Q = \operatorname{Im} L$ and $\operatorname{Ker} P = \{x \in X \mid \int_0^1 x(s)\,ds = 0\}$ and the inverse K_P of L_P is given by

$$(K_P)y(t) = \int_0^t k(t,s)y(s)\,ds$$

where

$$k(t,s) = \begin{cases} \dfrac{(1-s)^2}{2} & \text{for } 0 \le t \le s \le 1, \\ \dfrac{(1-s)^2}{2} + s - t & \text{for } 0 \le s \le t \le 1. \end{cases}$$

Considering that f can be extended continuously on $[0,1]\times R$, (C1) of Lemma 2.1 is fulfilled.

Also define the cone of nonnegative functions

$$C = \{x \in X \mid x(t) \ge 0,\ t \in [0,1]\}.$$

and

$$\Omega_1 = \{x \in X \mid r > |x| > M\|x\|,\ t \in [0,1]\}, \qquad \Omega_2 = \{x \in X \mid \|x\| < R\}.$$

Clearly, Ω_1 and Ω_2 are bounded and open sets and

$$\overline{\Omega}_1 = \{x \in X \mid r \ge |x| \ge M\|x\|,\ t \in [0,1]\} \subset \Omega_2, \quad C \cap \overline{\Omega}_2 \setminus \Omega_1 \ne \vee.$$

Let $J = I$ and $(\gamma x)(t) = |x(t)|$ for $x \in X$. It's easy to check that (C3) of Lemma 2.1 holds.

Suppose that there exists $x_0 \in C \cap \partial\Omega_2 \cap \operatorname{dom} L$ and $\lambda_0 \in (0,1)$ such that $Lx_0 = \lambda_0 N x_0$. Then

$$x_0''(t) + \lambda_0 f(t, x_0) = 0$$

for all $t \in (0,1)$. Let $t_1 \in [0,1]$ be such that $x_0(t_1) = R$. This gives

$$0 \ge x''(t_1) = -\lambda_0 f(t_1, R),$$

which contradicts to (H5). Therefore (C2) is satisfied.

To prove that (C4) holds, consider $x \in \operatorname{Ker} L \cap \Omega_2$ and define

$$H(x,\lambda) = x - \lambda|x| - \frac{\lambda}{\int_0^1 G(s)\,ds} \int_0^1 G(s) f(s,|x|)\,ds,$$

Suppose $x \in \partial\Omega_2 \cap \operatorname{Ker} L$ and $H(x,\lambda) = 0$, we see $x = (R/\rho_2)[\alpha\eta + (1-\alpha)t]$ and $\|x\| = R$. Thus for the definition of ρ_3, we have $\rho_3 R \le x(t) \le R$, which means $f(t,x) < 0$. This contradicts to

$$0 \le (1-\lambda)x = \frac{\lambda}{\int_0^1 G(s)\,ds} \int_0^1 G(s) f(s,|x|)\,ds,$$

So $H(x,\lambda) \ne 0$ for $x \in \partial\Omega_2$ and $\lambda \in [0,1]$, which induces that

$$d_B([I - (P + JQN)\gamma]|_{\operatorname{Ker} L}, \operatorname{Ker} L \cap \Omega_2, 0) = d_B(H(c,1), \operatorname{Ker} L \cap \Omega_2, 0) \ne 0.$$

Let $x \in \overline{\Omega}_2 \setminus \Omega_1$ and $t \in [0,1]$. Then

$$\begin{aligned}(\Psi_\gamma x)(t) = {}& \int_0^1 |x(s)|\, ds \times [\alpha\eta + (1-\alpha)t] + \frac{1}{\int_0^1 G(s)\, ds} \int_0^1 G(s) f(s, |x(s)|)\, ds \\ &+ \int_0^1 k(t,s) \bigg[f(s,|x(s)|) - \frac{1}{\int_0^1 G(s)\, ds} \int_0^1 G(\tau) f(\tau, |x(\tau)|)\, d\tau \bigg]\, ds \\ = {}& \int_0^1 |x(s)|\, ds \times [\alpha\eta + (1-\alpha)t] + \int_0^1 U(t,s) f(s, |x(s)|)\, ds \\ \geq {}& \rho_1 \int_0^1 |x(s)|\, ds - \kappa \int_0^1 U(t,s)|x(s)|\, ds \geq 0.\end{aligned}$$

Hence $\Psi_\gamma(\overline{\Omega}_2) \setminus \Omega_1 \subset C$. Moreover, since for $x \in \partial\Omega_2$, we have

$$\begin{aligned}(P + JQN)\gamma x = {}& \int_0^1 |x(s)|\, ds \times [\alpha\eta + (1-\alpha)t] \\ &+ \frac{1}{\int_0^1 G(s)\, ds} \int_0^1 G(s) f(s, |x(s)|)\, ds \\ \geq {}& \int_0^1 \Big(\rho_1 - \frac{\kappa}{\int_0^1 G(s)\, ds} G(s)\Big)|x(s)|\, ds \geq 0.\end{aligned}$$

We conclude that $(P + JQN)\gamma(\partial\Omega_2) \subset C$.

At last we take $u_0(t) \equiv 1$ on $[0,1]$. Then $u_0 \in C \setminus \{0\}$, $C(u_0) = \{x \in C \mid x(t) > 0 \text{ on } [0,1]\}$ and we can take $\sigma(u_0) = 1$. Let $x \in C(u_0) \cap \partial\Omega_1$. Then $x(t) > 0$ on $[0,1]$, $0 < \|x\| \leq r$ and $x(t) \geq M\|x\|$ on $[0,1]$.

Therefore, in view of (H6), we obtain, for all $x \in C(u_0) \cap \partial\Omega_1$,

$$\begin{aligned}(\Psi x)(t_0) = {}& \int_0^1 x(s)\, ds [\alpha\eta + (1-\alpha)t_0] + \int_0^1 U(t_0,s) f(s, x(s))\, ds \\ \geq {}& M\rho_1\|x\| + \int_0^1 U(t_0,s) g(s) h(x(s))\, ds \\ = {}& M\rho_1\|x\| + \int_0^1 U(t_0,s) g(s) \frac{h(x(s))}{x^a(s)} x^a(s)\, ds \\ \geq {}& M\rho_1\|x\| + \frac{h(r)}{r^a} \int_0^1 U(t_0,s) g(s) M^a \|x\|^a\, ds \\ \geq {}& M\rho_1\|x\| + (1 - M\rho_1)\|x\| = \|x\|.\end{aligned}$$

Thus $\|x\| \leq \sigma(u_0)\|\Psi x\|$ for all $x \in C(u_0) \cap \partial\Omega_1$.

By Lemma 2.1, the equation $Lx = Nx$ has a solution x, which implies that problem (1.1) with resonance condition (1.3) has at least one positive solution. $\square$

5. Example

In this section we give an example to illustrate the main results of the paper. Consider the four point boundary value problem:

$$
\begin{cases} x''(t) + \left(-\dfrac{1}{2}t^2 + \dfrac{1}{2}t + \dfrac{1}{2} \right)(x^2 - 4x + 3)\sqrt{x^2 - 6x + 10} = 0 \\ \hfill \text{for } t \in (0,1), \\ x(0) = x\left(\dfrac{1}{4}\right), \quad x(1) = x\left(\dfrac{1}{2}\right), \end{cases} \tag{5.1}
$$

where $\alpha = \beta = 1$, $\eta = 1/4$, $\xi = 1/2$ and

$$
G(s) = \begin{cases} s/2 & \text{for } 0 \le s < 1/4, \\ 1/8 & \text{for } 1/4 \le s < 1/2, \\ (1-s)/4 & \text{for } 1/2 < s \le 1. \end{cases}
$$

By a simple computation, we have

$$
\int_0^1 G(s)\,ds = \frac{5}{64}, \quad \kappa = \frac{5}{8}, \quad \int_0^1 U(0,s)\,ds = 1.
$$

We take $R = 6/5$, $r = 1/2$, $t_0 = 0$, $a = 1$, $M = 1/2$ and

$$
g(t) = -\frac{1}{2}t^2 + \frac{1}{2}t + \frac{1}{2}, \ h(x) = \sqrt{x^2 - 6x + 10}.
$$

It's easy to check that

$$
\frac{1}{2} \le g(t) \le \frac{5}{8}, \quad t \in [0,1], \qquad x^2 - 4x + 3 \ge -x, \quad x \in \left[0, \frac{6}{5}\right].
$$

We see that

(1) $f(t,x) > -(5/8)x$, for all $(t,x) \in [0,1] \times [0,6/5]$,
(2) $f(t,6/5) < 0$, for all $t \in [0,1]$,
(3) $f(t,x) \ge g(t)h(x)$ for all $[t,x] \in [0,1] \times (0,1/2]$ and

$$
\frac{h(x)}{x} = \frac{\sqrt{x^2 - 6x + 10}}{x}
$$

is non-increasing on $(0,1/2]$ with

$$
\frac{h(r)}{r^a} \int_0^1 U(0,s)g(s)\,ds \ge \frac{\sqrt{29}}{2} \int_0^1 U(0,s)\,ds \ge 1 = \frac{1-M}{M}.
$$

Thus all the conditions of Theorem 3.1 are satisfied. This ensures that resonance problem (5.1) has at least one solution, positive on $[0,1]$.

REMARK 5.1. We note that the early existence results about second-order four-point or m-point boundary value problems in [6], [7], [18], [19], [24]–[26] are not applicable to this problem.

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Manuscript received December 12, 2009

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