FIXED POINTS OF HEMI-CONVEX MULTIFUNCTIONS

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Abstract. The notion of hemi-convex multifunctions is introduced. It is shown that each convex multifunction is hemi-convex, but the converse is not true. Some fixed point results for hemi-convex multifunctions are also proved.

1. Introduction

Throughout this paper we suppose that $X$ and $Y$ are Banach spaces and $M$ is a nonempty convex subset of $X$. We denote the family of all nonempty subsets of $X$ by $2^X$ and the family of all nonempty closed and bounded subsets of $X$ by $CB(X)$. Also, we denote the Hausdorff metric on $CB(X)$ by $H$, i.e.

$$H(A, B) = \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \right\}$$

for all $A, B \in CB(X)$, where $d(x, A) = \inf_{a \in A} \|x - a\|$.

Let $T: X \to 2^Y$ be a multifunction. The graph of $T$ is defined by

$$Gr(T) = \{(x, y) : x \in X, y \in T(x)\}.$$ 

The multifunction $T$ is called closed (resp. convex) whenever $GrT$ is closed (resp. convex). Also, $T$ is called upper semi-continuous (resp. lower semi-continuous) whenever $\{x \in X : T(x) \subset A\}$ (resp. $\{x \in X : T(x) \cap A \neq \emptyset\}$) is open for
all open subsets $A$ of $Y$. Some authors work on convex multifunctions (see for example; [4]–[6] and [10]), whereas some authors work on nonconvex multifunctions (see for example [2]). In 1980, Yanagi defined the notion of semi-convex multifunctions ([9]). Later on, Bae and Park reviewed some fixed point theorems for multivalued mappings in Banach spaces by using the notion of semi-convex type multifunctions ([3]). The aim of this paper is to give the notion of hemi-convexity of multifunctions which is weaker than convexity of multifunctions. We show that this notion is independent of the notion of semi-convex multifunctions. We also prove some fixed point results for hemi-convex multifunctions.

2. Main results

**Definition 2.1.** Let $M$ be a convex subset of a Banach space $X$ and $r > 0$. We say that the multifunction $T : M \to 2^M$ is $r$-hemi-convex whenever

$$d(\lambda x + (1 - \lambda)y, T(\lambda x + (1 - \lambda)y)) \leq r$$

for all $\lambda \in [0,1]$ and $x, y \in M$ with $d(x, T(x)) < r$ and $d(y, T(y)) < r$. We say that $T$ is hemi-convex whenever $T$ is $r$-hemi-convex for all $r > 0$.

It is clear that each convex multifunction on a Banach space is a hemi-convex multifunction. Now, by providing the following example we show that the converse is not true.

**Example 2.2.** Define the multifunction $T : \mathbb{R} \to 2^\mathbb{R}$ by $T(x) = [2x, 3x]$ if $x \geq 0$ and $T(x) = [3x, 2x]$ if $x < 0$. Then $T$ is not convex whereas $T$ is hemi-convex. In fact, $(1, 2), (-1, -3) \in \text{Gr}(T)$, but for $\lambda = 1/2$ we have

$$\lambda(1, 2) + (1 - \lambda)(-1, -3) \notin \text{Gr}(T).$$

Since $d(x, T(x)) = |x|$ for all $x \in \mathbb{R}$, $T$ is a hemi-convex multifunction.

Let $M$ be a convex subset of a Banach space $X$. We say that the multifunction $T : M \to \text{CB}(X)$ is semi-convex whenever for each $x, y \in M$, $z = \lambda x + (1 - \lambda)y$, where $\lambda \in [0,1]$, and any $x_1 \in T(x), y_1 \in T(y)$, there exists $z_1 \in T(z)$ such that $\|z_1\| \leq \max\{\|x_1\|, \|y_1\|\}$ (see [9]). Now, by providing next examples, we show that the notions semi-convexity and hemi-convexity are independent, although both extend the notion of convexity of multifunctions.

**Example 2.3.** Define the multifunction $T : \mathbb{R} \to 2^\mathbb{R}$ by $T(x) = \{-x + 1\}$ if $x \geq 0$ and $T(x) = [x + 1, x + 2]$ if $x < 0$. Then $T$ is hemi-convex whereas $T$ is not semi-convex.

In fact, let $x = -1, y = 1, z = x/2 + y/2 = 0, x_1 = 0, y_1 = 0 \in T(y) = \{0\}$ and $z_1 = 1 \in T(z) = \{1\}$. Then, the relation $\|z_1\| \leq \max\{\|x_1\|, \|y_1\|\}$ does not hold. Hence, $T$ is not semi-convex.
On the other hand, \( d(x, T(x)) = |2x - 1| \) if \( x \geq 0 \) and \( d(x, T(x)) = 1 \) if \( x < 0 \). Without loss of generality, suppose that \( x < y \) and \( r > 0 \).

If \( x, y \geq 0 \), \( d(x, T(x)) < r \) and \( d(y, T(y)) < r \), then \( d(\lambda x + (1 - \lambda)y, T(\lambda x + (1 - \lambda)y)) \leq r \).

If \( x, y < 0 \), then \( d(x, T(x)) = 1 \), \( d(y, T(y)) = 1 \) and \( d(\lambda x + (1 - \lambda)y, T(\lambda x + (1 - \lambda)y)) = 1 \).

If \( x < 0 \), \( y \geq 0 \) and \( \lambda x + (1 - \lambda)y < 0 \), then \( d(x, T(x)) = 1 \) and \( d(\lambda x + (1 - \lambda)y, T(\lambda x + (1 - \lambda)y)) = 1 \).

If \( d(y, T(y)) < r \) and \( r \geq 1 \), then \( d(\lambda x + (1 - \lambda)y, T(\lambda x + (1 - \lambda)y)) \leq \min\{1, r\} \).

If \( x < 0 \), \( y \geq 0 \) and \( \lambda x + (1 - \lambda)y \geq 0 \), then \( d(x, T(x)) = 1 \), \( d(y, T(y)) = |2y - 1| \) and \(-1 \leq 2(\lambda x + (1 - \lambda)y) - 1 \leq 2y - 1 \). Thus, the relation
\[
d(\lambda x + (1 - \lambda)y, T(\lambda x + (1 - \lambda)y)) = |2(\lambda x + (1 - \lambda)y) - 1| \leq \max\{1, |2y - 1|\}
\]
implies that \( T \) is semi-convex.

**Example 2.4.** Define the multifunction \( T : \mathbb{R} \to 2^\mathbb{R} \) by \( T(x) = [x, x + 1] \) if \( x > 0 \) and \( T(x) = \{\sqrt{x}\} \) if \( x \leq 0 \). Then \( T \) is semi-convex whereas \( T \) is not semi-convex.

In fact, let \( x = 1 \), \( y = -1 \) and \( z = x/4 + 3y/4 = -1/2 \). Then, \( d(x, T(x)) = d(y, T(y)) = 0 \) while \( d(z, T(z)) = d(-1/2, -\sqrt{1/2}) > 0 \). Hence, \( T \) is not semi-convex.

Now, without loss of generality suppose that \( x < y \).

If \( x, y > 0 \) or \( x, y < 0 \) and \( z = (\lambda x + (1 - \lambda)y) \), it is easy to see that for each \( x_1 \in T(x) \) and \( y_1 \in T(y) \), there exists \( z_1 \in T(z) \) such that \( \|z_1\| \leq \max\{\|x_1\|, \|y_1\|\} \).

If \( x \leq 0 \), \( y > 0 \) and \( z = \lambda x + (1 - \lambda)y \leq 0 \), then for each \( x_1 = \sqrt{x} \in T(x) \)
and \( y_1 \in T(y) \) we have \( \sqrt{x} = x_1 \leq z_1 = \{\sqrt{\lambda x + (1 - \lambda)y}\} \leq 0 < y_1 \). Hence, \( \|z_1\| \leq \max\{\|x_1\|, \|y_1\|\} \).

If \( x \leq 0 \), \( y > 0 \) and \( z = \lambda x + (1 - \lambda)y > 0 \), then for each \( x_1 = \sqrt{x} \in T(x) \)
and \( y_1 \in T(y) \), there exists \( z_1 \in T(z) \) such that \( \sqrt{x} = x_1 \leq 0 < z_1 \leq y_1 \). Hence, \( \|z_1\| \leq \max\{\|x_1\|, \|y_1\|\} \). Therefore, \( T \) is semi-convex.

**Theorem 2.5.** Let \( T, T_n : M \to CB(M) \) be given. If \( T_n \) is a hemi-convex multifunction for all \( n \geq 1 \) and \( H(T_n(x), T(x)) \to 0 \) for all \( x \in M \), then \( T \) is a hemi-convex multifunction.

**Proof.** Fix \( \varepsilon > 0 \), \( r > 0 \), \( 0 \leq \lambda \leq 1 \) and \( x, y \in M \) with \( d(x, T(x)) < r \), \( d(y, T(y)) < r \). Choose a natural number \( N \) such that
\[
H(T_n(x), T(x)) < \varepsilon, \quad H(T_n(y), T(y)) < \varepsilon, \quad H(T_n(\lambda x + (1 - \lambda)y), T(\lambda x + (1 - \lambda)y)) < \varepsilon
\]
for all \( n \geq N \). Then, for each \( n \geq N \) we have
\[
\begin{align*}
d(x, T_n(x)) &\leq d(x, T(x)) + H(T_n(x), T(x)) < r + \varepsilon \\
d(y, T_n(y)) &\leq d(y, T(y)) + H(T_n(y), T(y)) < r + \varepsilon.
\end{align*}
\]
Thus, \( d(\lambda x + (1 - \lambda)y, T_n(\lambda x + (1 - \lambda)y)) \leq r + \varepsilon \). Hence, for each \( n \geq N \) we have
\[
\begin{align*}
d(\lambda x + (1 - \lambda)y, T(\lambda x + (1 - \lambda)y)) &\leq d(\lambda x + (1 - \lambda)y, T_n(\lambda x + (1 - \lambda)y)) \\
&\quad + H(T_n(\lambda x + (1 - \lambda)y), T(\lambda x + (1 - \lambda)y)) < r + 2\varepsilon.
\end{align*}
\]
Since \( \varepsilon \) was arbitrary, we obtain \( d(\lambda x + (1 - \lambda)y, T(\lambda x + (1 - \lambda)y)) \leq r \). Therefore, \( T \) is a hemi-convex multifunction.

**Theorem 2.6.** Let \( T: M \to \text{CB}(M) \) be an upper semi-continuous hemi-convex multifunction. Then the set of fixed points of \( T \) is convex and closed.

**Proof.** Set \( F = \{ x : x \in T(x) \} \). For each \( x, y \in F \) we have \( d(x, T(x)) = 0 \) and \( d(y, T(y)) = 0 \). Thus, \( d(T(\lambda x + (1 - \lambda)y), \lambda x + (1 - \lambda)y) = 0 \) and so \( \lambda x + (1 - \lambda)y \in F \) for all \( \lambda \in [0, 1] \), because \( T \) is a closed-valued multifunction. Since \( T \) is upper semi-continuous and closed-valued, \( \text{Gr}(T) \) is closed.

Let \( \{x_n\}_{n \geq 1} \) be a sequence in \( F \) with \( x_n \to x \). Since \( x_n \in T(x_n) \), \( (x_n, x_n) \in \text{Gr}(T) \). Hence, \( (x, x) \in \text{Gr}(T) \) and so \( x \in F \).

**Definition 2.7.** Let \( M \) be a convex subset of a Banach space \( X \) and \( r > 0 \). We say that the function \( f: X \to \mathbb{R} \) is \( r \)-hemi-convex on \( M \) whenever
\[
f(\lambda x + (1 - \lambda)y) < r
\]
for all \( \lambda \in [0, 1] \) and \( x, y \in M \) with \( f(x) < r \) and \( f(y) < r \). We say that \( f \) is hemi-convex on \( M \) whenever \( f \) is \( r \)-hemi-convex on \( M \) for all \( r > 0 \).

**Lemma 2.8.** Let \( M \) be a convex subset of a Banach space \( X \), \( \delta > 0 \), \( m \geq 2 \) and \( f: X \to \mathbb{R} \) a hemi-convex function on \( M \). If \( x_1, \ldots, x_m \in M \) with \( f(x_i) < \delta \) for \( i = 1, \ldots, m \) and \( \lambda_1, \ldots, \lambda_m \in [0, \infty) \) with \( \sum_{i=1}^{m} \lambda_i = 1 \), then
\[
f\left(\sum_{i=1}^{m} \lambda_i x_i\right) < \delta.
\]

**Proof.** We prove this by induction. For \( m = 2 \) we have nothing to prove. Suppose that this lemma holds for each \( 1 \leq k \leq m - 1 \). We have to prove it for \( m \). Note that, one can assume \( \lambda_1 \neq 0 \) and so
\[
f\left(\sum_{i=1}^{m} \lambda_i x_i\right) = f\left(\lambda_1 x_1 + \sum_{i=2}^{m} \lambda_i x_i\right) = f\left(\lambda_1 x_1 + (1 - \lambda_1) \sum_{i=2}^{m} \frac{\lambda_i}{1 - \lambda_1} x_i\right).
\]
Put \( y = \sum_{i=2}^{m} (\lambda_i/(1 - \lambda_1)) x_i \). Since \( \sum_{i=2}^{m} \lambda_i/(1 - \lambda_1) = 1 \), by assumption of the induction, we have \( f(y) < \delta \).
Now, by the case of $m = 2$, we obtain
\[
 f\left( \sum_{i=1}^{m} \lambda_i x_i \right) = f\left( \lambda_1 x_1 + (1 - \lambda_1) \sum_{i=2}^{m} \frac{\lambda_i}{1 - \lambda_1} x_i \right) = f(\lambda_1 x_1 + (1 - \lambda_1)y) < \delta.
\]
This completes the proof. \qed

**Theorem 2.9.** Let $M$ be a weakly compact subset of $X$, $T: M \to \text{CB}(X)$ a multifunction and $\inf_{x \in M} d(x, T(x)) = 0$. If the function $f: M \to [0, \infty)$, defined by $f(x) = d(x, T(x))$, is lower semi-continuous and hemi-convex on $M$, then $T$ has a fixed point in $M$.

**Proof.** Choose a sequence $\{x_n\}_{n \geq 1}$ in $M$ such that $d(x_n, T(x_n)) \to 0$. Since $M$ is weakly compact, there exists a subsequence $\{z_n\}_{n \geq 1}$ of $\{x_n\}_{n \geq 1}$ such that $z_n \xrightarrow{w} x_0$ for some $x_0 \in M$. Since $f$ is a lower semi-continuous function, for each $\varepsilon > 0$ choose $\delta > 0$ such that $f(x_0) < f(y) + \varepsilon/2$ for all $y \in M$ with $\|y - x_0\| < \delta$ (\cite{7}). Since $f(z_n) \to 0$, there exists a natural number $N$ such that $f(z_n) < \varepsilon/2$ for all $n \geq N$.

We denote again the sequence $\{z_n\}_{n \geq N}$ by $\{z_n\}_{n \geq 1}$. Since $z_n \xrightarrow{w} x_0$, there exist a sequence $\{y_i\}_{i \geq 1}$ in $M$ and a sequence $\{\alpha_{in}\}_{i, n \geq 1}$ in $[0, \infty)$ such that for each $i$ we have $y_i = \sum_{n=1}^{\infty} \alpha_{in} z_n$, where $\sum_{n=1}^{\infty} \alpha_{in} = 1$ and only finitely many $\{\alpha_{in}\}$ are not zero, and $y_i \to x_0$ originally (\cite{8; Theorem 3.13}). But, by Lemma 2.8, we have $f(y_i) < \varepsilon/2$ for all $i \geq 1$. Thus, for sufficiently large $i$, we obtain
\[
f(x_0) < f(y_i) + \frac{\varepsilon}{2} < \varepsilon.
\]
Hence, $f(x_0) = 0$ and so $x_0 \in T(x_0)$. \qed

If $T: M \to \text{CB}(M)$ is an upper semi-continuous multifunction, then the function $f(x) = d(x, T(x))$ is lower semi-continuous (\cite{1, Proposition 4.2.6}). Also, note that the function $f(x) = d(x, T(x))$ is hemi-convex whenever $T$ is so. We say that the function $f(x) = d(x, T(x))$ has the property (B) whenever $f(x_n) \to \infty$ for all sequences $\{x_n\}$ with $\|x_n\| \to \infty$. The following example shows that weak compactness of $M$ is a necessary condition in Theorem 2.9.

**Example 2.10.** Consider the multifunction $T: (0, \infty) \to 2^{(0, \infty)}$ given by
\[
T(x) = \left\{ x + \frac{1}{x} \right\}.
\]
It is clear that $T$ is a hemi-convex multifunction, $\inf_{x \in (0, \infty)} d(x, T(x)) = 0$ and the function $f(x) = d(x, T(x))$ is lower semi-continuous and hemi-convex. But it is clear that $T$ has no fixed point.

The following example shows that there are many multifunctions which satisfy the condition $\inf_{x \in M} d(x, T(x)) = 0$. 


EXAMPLE 2.11. Let $M$ be a convex and bounded subset of a Banach space $X$, $u \in M$ a fixed element and $T: M \to \text{CB}(M)$ a nonexpansive multifunction. For each $n \geq 2$ define $T_n: M \to \text{CB}(M)$ by $T_n(x) = u/n + (1-1/n)T(x)$. Since $H(T_n(x), T_n(y)) \leq (1-1/n)\|x-y\|$ for all $x, y \in M$ and $n \geq 2$, $T_n$ is a contraction multifunction and so for each $n \geq 2$ there exists $x_n \in M$ such that $x_n \in T_n(x_n)$. Note that $d(x_n, T(x_n)) \to 0$ and so $\inf_{x \in M} d(x, T(x)) = 0$.

DEFINITION 2.12. Let $M$ be a convex subset of a Banach space $X$ and $T_n: M \to \text{CB}(M)$ a sequence of multifunctions. We say that $\{T_n\}$ strongly converges to $T$ whenever for each $\varepsilon > 0$ there exists a natural number $n_0$ such that $H(T_n(x), T(x)) < \varepsilon$ for all $n \geq n_0$ and $x \in M$. In this case, we write $T_n \to T$.

THEOREM 2.13. Let $M$ be a weakly compact subset of $X$, $T: M \to \text{CB}(M)$ a multifunction and $T_n: M \to \text{CB}(M)$ an upper semi-continuous hemi-convex multifunction for all $n \geq 1$. If each $T_n$ has at least one fixed point in $M$ and $T_n \to T$, then $T$ has a fixed point.

Proof. Since each $T_n$ has at least one fixed point in $M$, $\inf_{x \in M} d(x, T_n(x)) = 0$ for all $n \geq 1$. Let $\varepsilon > 0$ be given. Choose a natural number $n_0$ such that $H(T_n(x), T(x)) < \varepsilon$ for all $n \geq n_0$. Since $d(x, T(x)) \leq d(x, T_n(x)) + H(T_n(x), T(x)) \leq d(x, T_n(x)) + \varepsilon$, for all $n \geq n_0$, $\inf_{x \in M} d(x, T(x)) \leq \varepsilon$. Hence, $\inf_{x \in M} d(x, T(x)) = 0$. By Theorem 2.5, $T$ is hemi-convex and so is the function $f(x) = d(x, T(x))$. Since $T$ is upper semi-continuous, the function $f(x) = d(x, T(x))$ is lower semi-continuous. Now by using Theorem 2.9, $T$ has a fixed point.

The next example shows that strong convergence of the sequence $\{T_n\}_{n \geq 1}$ is a necessary condition in Theorem 2.13.

EXAMPLE 2.14. Let $X = \mathbb{R}$ and $M = [0, 2]$. Define $T: M \to \text{CB}(M)$ by $T(x) = \{x + 1\}$ if $x < 1$, $T(x) = \{x - 1\}$ if $x > 1$ and $T(x) = \{0, 2\}$ if $x = 1$. Moreover, for each $n \geq 2$, let $T_n: M \to \text{CB}(M)$ be defined by $T_n(x) = T(x)$ if $x \neq 1/n$ and $T_n(x) = [0, 2]$ if $x = 1/n$. It is easily seen that $T_n$ is upper semi-continuous, $d(x, T_n(x)) = 1$ if $x \neq 1/n$, $d(x, T_n(x)) = 0$ if $x = 1/n$ and $T_n$ has a fixed point for each $n \geq 2$. This implies that $T_n$ is hemi-convex. Evidently $H(T_n(x), T(x)) \to 0$ for all $x \in M$, but $T$ has no fixed point.

THEOREM 2.15. Let $X$ be an uniformly convex Banach space, $T: X \to \text{CB}(X)$ an upper semi-continuous hemi-convex multifunction, $\inf_{x \in M} d(x, T(x)) = 0$. If the function $f(x) = d(x, T(x))$ has the property (B), then $T$ has a fixed point.
Proof. Choose a sequence \( \{x_n\} \) in \( X \) such that \( f(x_{n+1}) \leq f(x_n) \) and \( f(x_n) \to 0 \). Now, for each \( n \geq 1 \) define \( F_n = \{ x \in X : f(x) \leq f(x_n) \} \). Since the function \( f(x) = d(x,T(x)) \) has the property (B), each \( F_n \) is a nonempty bounded subset of \( X \). Since \( T \) is upper semi-continuous, the function \( f(x) = d(x,T(x)) \) is lower semi-continuous and so each \( F_n \) is a closed subset of \( X \). Also, each \( F_n \) is convex because \( T \) is a hemi-convex multifunction. Now by using [1, Theorem 2.3.14], there exists \( x_0 \in X \) such that \( x_0 \in \bigcap_{n=1}^{\infty} F_n \). Thus, \( f(x_0) \leq f(x_n) \) for all \( n \geq 1 \). Hence, \( f(x_0) = 0 \) and so \( x_0 \in T(x_0) \). □

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References


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