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# MAPS ON GRAPHS CAN BE DEFORMED TO BE COINCIDENCE FREE

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Abstract. We give a construction to remove coincidence points of continuous maps on graphs (1-complexes) by changing the maps by homotopies. When the codomain is not homeomorphic to the circle, we show that any pair of maps can be changed by homotopies to be coincidence free. This means that there can be no nontrivial coincidence index, Nielsen coincidence number, or coincidence Reidemeister trace in this setting, and the results of our previous paper "A formula for the coincidence Reidemeister trace of selfmaps on bouquets of circles" are invalid.

## 1. Introduction

Let $X$ and $Y$ be graphs, which are always assumed to be nontrivial. Throughout, we will consider continuous maps $f, g\colon X \to Y$ (continuous maps of $X$ and $Y$ as dimension 1 CW-complexes) and examine the coincidence set

$$\mathrm{Coin}(f,g) = \{x \mid f(x) = g(x)\}.$$

The paper [3] attempts, in the special case of bouquets of circles, to study coincidence points of $f$ and $g$ by computing the Reidemeister trace, which would then allow the computation of the Nielsen number of the pair $(f,g)$. This Nielsen number would be a lower bound on the minimal number of coincidence points achievable by deforming $f$ and $g$.

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A serious error in [3] renders the approach fundamentally misguided. The approach makes heavy use of the coincidence index, which is not well-behaved for bouquets of circles. Our main result (Theorem 2.3) is that maps $f, g\colon X \to Y$ of graphs with $Y$ not homeomorphic to the circle can always be changed by homotopy to be coincidence free. Thus any coincidence index in this setting must always be zero, and so any Nielsen number or Reidemeister trace which were being computed in [3] must have the value zero.

In Section 2 we give our main result. We conclude in Section 3 with a note on the specific errors in [3].

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## 2. Removing coincidences by homotopy

Our strategy for removing coincidences can be intuitively described using a road traffic analogy. Consider a coincidence point which occurs on the interior of an edge of the domain space. Then we parameterize this edge (1-cell) as the time interval $[0, 1]$, and we can view the maps $f$ and $g$ as being represented by a pair of points which travel around the space $Y$.

Let us imagine that these points represent cars traveling on a network of single-lane roads (so that the cars may not pass one another), and a coincidence point of the maps will represent a collision of the cars. A removal of a coincidence point by a homotopy will consist of a strategy for letting the two cars pass one another without colliding.

Avoiding a collision is possible provided that there is a fork in the network of roads where at least three roads meet: When two cars are about to collide, one of them reverses direction until the fork is reached. At this point, the car which reversed direction moves onto the third road and allows the other to pass. The cars can now proceed back to their original meeting point, this time with their positions reversed. Repeating this process before each imminent collision allows the cars to complete their trips without colliding.

This strategy is formalized as follows:

Theorem 2.1. *Let $f, g\colon X \to Y$ be maps of connected graphs with $Y$ not a manifold, and let $x \in \mathrm{Coin}(f, g)$ be a coincidence point in the interior of some edge. Then there is an arbitrarily small neighbourhood $U$ of $x$ on which $f$ and $g$ can be changed by homotopy to be coincidence free.*

Proof. Let $\sigma$ be the edge (1-cell) containing $x$, which we identify with its attaching map $\sigma\colon [0, 1] \to X$. Let $x = \sigma(t_0)$, and let $U = \sigma([t_0 - \epsilon, t_0 + \epsilon])$ for some small $\epsilon > 0$. We may assume that $f(x) = g(x)$ is a point on the interior of some

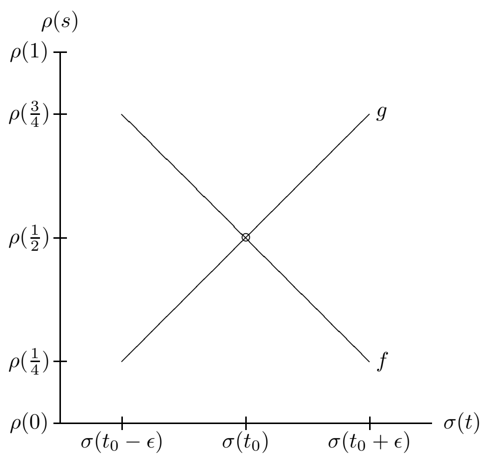


Figure 1. Behavior of $f$ and $g$ on $U$

1-cell $\rho$: $[0, 1] \to Y$. We can parameterize $\sigma$ and $\rho$ so that $f(x) = g(x) = \rho(1/2)$, and that $f$ and $g$ behave according to the graph in Figure 1. (We may perhaps have to interchange the roles of $f$ and $g$.)

The assumption that $Y$ is not a manifold means that we may choose the CW-complex structure on $Y$ so that the vertex $\sigma(0)$ meets two other 1-cells $\gamma, \lambda$: $[0, 1] \to Y$ with $\gamma(0) = \lambda(0) = \sigma(0)$. This vertex is the "fork in the road". Now we change $f$ and $g$ by homotopy on $U$ to maps $f'$ and $g'$ according to Figure 2. Informally, the two maps retreat to the fork point, use the fork to pass one another without colliding, and return to their original positions at time $t_0 + \epsilon$.

The maps $f'$ and $g'$ are free of coincidences on $U$, and the theorem is proved.□

The above theorem implies our main result, with the help of a lemma which is true for much more general spaces, though we only require it for complexes. Its proof is an exercise.

Lemma 2.2. *Let $f, g$: $X \to Y$ where $X$ and $Y$ are connected complexes, and let $x \in \mathrm{Coin}(f, g)$. Then for any neighbourhood $U$ of $x$, we may change $f$ and $g$ by homotopy on $U$ so that $x$ is no longer a coincidence point.*

The lemma above means that we may assume that every coincidence of our maps occurs on the interior of an edge, and then Theorem 2.1 can be applied repeatedly to remove them. Thus we obtain:

Theorem 2.3. *If $f, g$: $X \to Y$ are maps on connected graphs with $Y$ not homeomorphic to the circle, then $f$ and $g$ can be changed by homotopy to be coincidence free.*

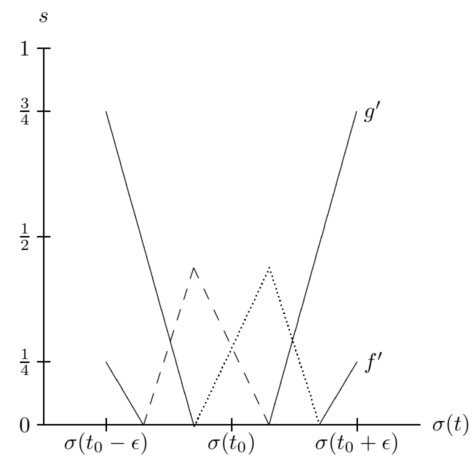


FIGURE 2. Behavior of $f'$ and $g'$ on $U$. Solid line indicates values in $\rho(s)$, dotted line indicates values in $\gamma(s)$, and dashed line indicates values in $\lambda(s)$

PROOF. If $Y$ is homeomorphic to the interval $[0, 1]$, then $f$ and $g$ are trivially nullhomotopic and thus can be made to be coincidence free by deforming them into different constant maps. Thus we may assume that $Y$ is not a manifold, and we may freely use Theorem 2.1.

First, we may change our maps by homotopy to be "linear" as in [3] so that $\mathrm{Coin}(f, g)$ is a finite set. Furthermore by the lemma we may assume that all coincidences occur at interior points of edges. Then repeated application of Theorem 2.1 will remove all coincidences. $\square$

See the end of Section 3 for a note on the case where $Y$ is the circle.

The above theorem highlights the fact that coincidence theory on graphs is not a generalization of fixed point theory. It is certainly possible for a selfmap $f$ on e.g. a bouquet of 2 circles to have fixed points which cannot be removed by homotopy, even though (by Theorem 2.3) any coincidences of $f$ with the identity map *can* be removed. This occurs because our removal construction changes the second map by homotopy as well as the first. This distinction does not occur between fixed point and coincidence theory on manifolds and some other spaces, as demonstrated by R. Brooks in [1], but Brooks' result does not apply to complexes in general.

## 3. The error of [3], and the coincidence index

The formula given for the Reidemeister trace in [3] uses essentially two ingredients: the computation of the Reidemeister class for each coincidence point, and the computation of the coincidence index for each coincidence point. The material concerning the Reidemeister class is essentially correct, and the material concerning the index is incorrect.

The error specifically arises on page 43 of [3]: "Near any point $x$ other than $x_0$, the space $X$ is an orientable differentiable manifold, and we define the coincidence index as usual for that setting." This formulation of the coincidence index is not well-behaved under homotopy. If, over the course of the homotopy, the coincidence value (the common value of $f(x)$ and $g(x)$) travels through the wedge point $y_0$, this "index" will change unpredictably.

In fact, two fundamental properties of the coincidence index are that it is invariant under homotopies of $f$ and $g$, and that the index is zero when $f$ and $g$ are coincidence-free on $U$. Since (by Theorem 2.3) the coincidence set for maps of graphs can always be made empty by homotopies, any "coincidence index" in this setting must always give the value zero.

Some nontrivial indices can be defined by restricting the structure of either the domain or the codomain spaces. D. L. Gonçalves in [2] gives a coincidence index for maps from a complex into a manifold of the same dimension, which suffices to address the exceptional case from Section 2, the case where $Y$ is the circle. In this case Gonçalves's index does provide a nontrivial coincidence index which generalizes the fixed point index.

Thus there are many examples of maps $f,g\colon X \to S^1$ for which $\mathrm{MC}(f,g)$, the minimal number of coincidence points when $f$ and $g$ are changed by homotopy, is nonzero (this will occur whenever Gonçalves's index is nonzero). In particular when $X$ is also $S^1$, it is known that $\mathrm{MC}(f,g)$ is the Nielsen number $N(f,g) = |\deg(f) - \deg(g)|$, which is easily made nonzero.

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