Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 37, 2011, 303–341

ON SECOND-ORDER BOUNDARY VALUE PROBLEMS IN BANACH SPACES: A BOUND SETS APPROACH

Jan Andres — Luisa Malaguti — Martina Pavlačková

Abstract. The existence and localization of strong (Carathéodory) solutions is obtained for a second-order Floquet problem in a Banach space. The combination of applied degree arguments and bounding (Liapunov-like) functions allows some solutions to escape from a given set. The problems concern both semilinear differential equations and inclusions. The main theorem for upper-Carathéodory inclusions is separately improved for Marchaud inclusions (i.e. for globally upper semicontinuous right-hand sides) in the form of corollary. Three illustrative examples are supplied.

1. Introduction

Let E be a Banach space (with the norm $\|\cdot\|$) satisfying the Radon–Nikodym property (e.g. reflexivity) and let us consider the Floquet boundary value problem (b.v.p.)

$$
(1.1)\qquad \begin{cases} \ddot{x}(t) + A(t)\dot{x}(t) + B(t)x(t) \in F(t, x(t), \dot{x}(t)), & \text{for a.a. } t \in [0, T], \\ x(T) = Mx(0), \quad \dot{x}(T) = N\dot{x}(0). \end{cases}
$$

2010 *Mathematics Subject Classification.* 34A60, 34B15, 47H04.

Key words and phrases. Second-order Floquet problem, bounding functions, solutions in a given set, evolution equations, condensing multivalued operators..

The first and the third named authors were supported by the Council of Czech Government (MSM 6198959214).

The second named author was supported by the national research project PRIN “Ordinary Differential Equations and Applications”

Throughout the paper, we assume (for the related definitions, see the next Section 2) that

- ($1_{\rm i}$) $A, B\colon [0,T] \to \mathcal{L}(E)$ are Bochner integrable, where $\mathcal{L}(E)$ stands for the Banach space of all linear, bounded transformations $L\colon E \to E$ endowed with the sup-norm,
- ($1_{\rm ii}$) $F\colon [0,T] \times E \times E \multimap E$ is an upper-Carathéodory multivalued mapping,
- ($1_{\rm iii}$) $M,\ N \in \mathcal{L}(E)$.

The notion of a solution will be understood in a strong (i.e. Carathéodory) sense. Namely, by a *solution* of problem (1.1), we mean a function $x\colon [0,T] \to E$ whose first derivative $\dot{x}(\,\cdot\,)$ is absolutely continuous and satisfies (1.1), for almost all $t \in [0,T]$.

Problems of this type can be related to those for *abstract nonlinear wave equations* in Hilbert spaces. For $t \in [0,T]$ and $\xi \in \Omega$, where Ω is a nonempty, bounded domain in $\mathbb{R}^n$ with a Lipschitz boundary $\partial\Omega$, consider the functional evolution equation

$$\frac{\partial^2 u}{\partial t^2} + a\frac{\partial u}{\partial t} + \widetilde{B}u(t,\,\cdot\,) + \mathcal{B}||u(t,\,\cdot\,)||^{p-2}u = \varphi(t,u), \tag{1.2}$$

where $u = u(t,\xi)$, subject to boundary conditions

$$u(T,\,\cdot\,) = Mu(0,\,\cdot\,), \qquad \frac{\partial u(T,\,\cdot\,)}{\partial t} = N\frac{\partial u(0,\,\cdot\,)}{\partial t}. \tag{1.3}$$

Assume that $a \geq 0$, $\mathcal{B} \geq 0$, $p > 1$ are constants, $\widetilde{B}\colon L^2(\Omega) \to L^2(\Omega)$ is a linear operator and that $\varphi\colon [0,T] \times \mathbb{R} \to \mathbb{R}$ is sufficiently regular. The problem under consideration can be still restricted by a constraint:

$$u(t,\,\cdot\,) \in \overline{K} := \{e \in L^2(\Omega) \mid ||e|| \leq r\}, \quad t \in [0,T].$$

Taking $x(t) := u(t,\,\cdot\,)$ with $x \in AC^1([0,T], L^2(\Omega))$, $A(t) \equiv A := a$, $B(t)\colon L^2(\Omega) \to L^2(\Omega)$ defined by $x = u(t,\,\cdot\,) \to \widetilde{B}x$, $f\colon [0,T] \times L^2(\Omega) \to L^2(\Omega)$ defined by $(t,v) \to \varphi(t, v(\,\cdot\,))$, and $F(t,x,y) \equiv F(t,x) := -\mathcal{B}||x||^{p-2}x + f(t,x)$, the above problem can be rewritten into the form (1.1), possibly together with $x(t) \in \overline{K}, t \in [0,T]$, where $K \subset L^2(\Omega)$ is a nonempty, open, convex subset of $L^2(\Omega)$ containing 0.

If $\varphi(t,\,\cdot\,)$ is e.g. bounded, but discontinuous at finitely many points, then the Filippov regularization $\widetilde{\varphi}$ of $\varphi(t,\,\cdot\,)$ (cf. e.g. [6], [9]) can lead to a multivalued problem (1.1).

An interesting case occurs when $E = L^2(\Omega)$ and $\widetilde{B}u(t,\,\cdot\,) := -\Delta u(t,\,\cdot\,)$; equation (1.2) then becomes a hyperbolic equation (see e.g. [22, Chapter 5.2]). Since such a $\widetilde{B}$ is defined only on $W^{2,2}(\Omega) \subset E = L^2(\Omega)$, it does not satisfy condition ($1_{\rm i}$) and the related model can not be attached with the techniques developed in this work. Moreover, the Laplace operator is not bounded on $W^{2,2}(\Omega)$, as required in ($1_{\rm i}$). Indeed, the main purpose of the present paper is to prove the existence of a Carathéodory solution $x \in AC^1([0,T],E)$ to problem (1.1) in a given set Q. Section 6 also contains an applications of our results to the b.v.p. (1.2), (1.3), where $B \in \mathcal{L}(E)$.

Since the application of degree arguments will tendentiously allow some solutions of (1.1) to escape from Q, the crucial condition of the related continuation principle developed in Section 3 consists in guaranteeing the fixed point free boundary of Q w.r.t. an admissible homotopical bridge starting from (1.1) (see condition (e) in Proposition 3.1 below). This requirement will be verified by means of Liapunov-like bounding functions, i.e. via a bound sets technique (whence the title).

That is also why the whole Section 4 is devoted to this technique applied to Floquet problem (1.1) and in fact, as pointed out in remarks, to Floquet problems with general upper-Carathéodory differential inclusions (i.e. for A and B possibly equal to 0 in (1.1)). We distinguish two cases, namely when

(i) A, B are Bochner integrable transformations and F is an upper-Carathéodory mapping, and
(ii) A, B are continuous transformations and F is globally upper semicontinuous (i.e. a Marchaud mapping).

Unlike in the first case, the second one allows us to apply bounding functions which can be strictly localized on the boundaries of given bound sets.

The application of bounding functions to problems in abstract spaces was so far, to our best knowledge, exclusively related to first-order problems (see e.g. [5], [23], [24]). Moreover, guiding functions can only be (globally) applied in L^2-spaces or so, but not in general Banach spaces like here, as documented in [5] (see the related references therein). In this light, the bound sets approach to second-order problems in Banach spaces brings the main novelty of our paper.

Similarly as in finite-dimensional Euclidean spaces, the geometry concerning second-order problems, reflecting the behaviour of controlled trajectories, is much more sophisticated than for first-order problems. Moreover, to express desired transversality conditions in terms of bounding functions, it requires for second-order problems in Banach spaces to employ newly dual spaces, etc. On the other hand, the sufficient existence conditions are, in principle, better than those for equivalent first-order problems.

Although the main results formulated in Theorem 5.1 and Corollary 5.2 are rather abstract, they can be suitably applied for obtaining effective criteria of solvability of (1.1), as demonstrated especially by the third illustrative example supplied in Section 6.

Since the most important particular cases of the Floquet problem are related to a periodic problem ($M = N = \mathrm{id}$) and to an anti-periodic problem ($M = N = -\mathrm{id}$), the comparison of the obtained criteria with those of the other authors should preferably concern these two cases. However, since the methods applied by other authors in this field are significantly different from ours (see e.g. [1], [9], [12], [14], [7], [17], [18], [20], [21], [26]), we resigned to make such a comparison. If the localization of solutions, as the main advantage of our results, was guaranteed somewhere else, then it was almost exclusively done, in the frame of the viability theory, by means of various Nagumo-type (cone-type) conditions. Nevertheless, in the majority of quoted papers, the localization of solutions can be detected only with difficulties.

2. Preliminaries

Let E be a Banach space having the Radon–Nikodym property (see e.g. [19, pp. 694–695]) and $[0,T] \subset \mathbb{R}$ be a closed interval. By the symbol $L^1([0,T],E)$, we shall mean the set of all Bochner integrable functions $x\colon [0,T] \to E$. For the definition and properties, see e.g. [19, pp. 693–701]. The symbol $AC^1([0,T],E)$ will denote the set of functions $x\colon [0,T] \to E$ whose first derivative $\dot{x}(\,\cdot\,)$ is absolutely continuous. Then $\ddot{x} \in L^1([0,T],E)$ and the fundamental theorem of calculus (the Newton–Leibniz formula) holds (see e.g. [3, pp. 243–244], [19, pp. 695–696]). In the sequel, we shall always consider $AC^1([0,T],E)$ as a subspace of the Banach space $C^1([0,T],E)$. Given $C \subset E$ and $\varepsilon > 0$, the symbol $B(C,\varepsilon)$ will denote, as usually, the set $C + \varepsilon B$, where B is the open unit ball in E, i.e. $B = \{x \in E \mid ||x|| < 1\}$.

For each $L \in \mathcal{L}(E \times E)$, there exist unique $L_{ij} \in \mathcal{L}(E)$, $i,j = 1,2$, such that

$$L(x,y) = (L_{11}x + L_{12}y, L_{21}x + L_{22}y),$$

where $(x,y) \in E \times E$. For the sake of simplicity, we shall use the notation

$$L = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}.$$

We shall also need the following definitions and notions from multivalued analysis. Let X, Y be two metric spaces. We say that F is a *multivalued mapping* from X to Y (written $F\colon X \multimap Y$) if, for every $x \in X$, a nonempty subset $F(x)$ of Y is given. We associate with F its graph Γ_F, the subset of $X \times Y$, defined by

$$\Gamma_F := \{(x,y) \in X \times Y \mid y \in F(x)\}.$$

A multivalued mapping $F\colon X \multimap Y$ is called *upper semicontinuous* (shortly, u.s.c.) if, for each open subset $U \subset Y$, the set $\{x \in X \mid F(x) \subset U\}$ is open in X.

A multivalued mapping $F\colon X \multimap Y$ is called *compact* if the set $F(X) = \bigcup_{x \in X} F(x)$ is contained in a compact subset of Y and it is called *quasi-compact* if it maps compact sets onto relatively compact sets.

The relationship between upper semicontinuous mappings and compact mappings with closed graphs is expressed by the following proposition (see, e.g. [15]).

Proposition 2.1. *Let X, Y be metric spaces and $F\colon X \multimap Y$ be a quasi-compact mapping with a closed graph. Then F is u.s.c.*

We say that a multivalued mapping $F\colon [0,T] \multimap Y$ with closed values is a *step multivalued mapping* if there exists a finite family of disjoint measurable subsets I_k, $k = 1, \ldots, n$ such that $[0,T] = \bigcup I_k$ and F is constant, on every I_k. A multivalued mapping $F\colon [0,T] \multimap Y$ with closed values is called *strongly measurable* if there exists a sequence of step multivalued mappings $\{F_n\}$ such that $d_H(F_n(t), F(t)) \to 0$ as $n \to \infty$, for almost all $t \in [0,T]$, where d_H stands for the Hausdorff distance.

Let us note that if Y is a Banach space, then a strongly measurable mapping $F\colon [0,T] \multimap Y$ with compact values possesses a single-valued strongly measurable selection.

Let $J = [0,T]$ be a given compact interval. A multivalued mapping $F\colon J \times X \multimap Y$ is called an *upper-Carathéodory mapping* if the map $F(\,\cdot\,, x)\colon J \multimap Y$ is strongly measurable, for all $x \in X$, the map $F(t, \,\cdot\,)\colon X \multimap Y$ is u.s.c. for almost all $t \in J$, and the set $F(t,x)$ is compact and convex, for all $(t,x) \in J \times X$.

For more details concerning multivalued analysis, see e.g. [3], [De], [13], [15].

Definition 2.2. Let N be a partially ordered set, E be a Banach space and let $P(E)$ denote the family of all subsets of E. A function $\beta\colon P(E) \to N$ is called a *measure of non-compactness* (m.n.c.) in E if $\beta(\overline{\operatorname{co}\Omega}) = \beta(\Omega)$, for all $\Omega \in P(E)$, where $\overline{\operatorname{co}\Omega}$ denotes the closed convex hull of Ω.

A m.n.c. β is called:

(a) *monotone* if $\beta(\Omega_1) \le \beta(\Omega_2)$, for all $\Omega_1 \subset \Omega_2 \subset E$,
(b) *nonsingular* if $\beta(\{x\} \cup \Omega) = \beta(\Omega)$, for all $x \in E$ and $\Omega \subset E$,
(c) *invariant with respect to the union with compact sets* if $\beta(K \cup \Omega) = \beta(\Omega)$, for every relatively compact $K \subset E$ and every $\Omega \subset E$.
(d) *regular* when $\beta(\Omega) = 0$ if and only if Ω is relatively compact.

It is obvious that the m.n.c. which is invariant with respect to the union with compact sets is also nonsingular.

The typical example of an m.n.c. is the *Hausdorff measure of noncompactness* γ defined, for all $\Omega \subset E$ by

$$\gamma(\Omega) := \inf \left\{ \varepsilon > 0 \;\middle|\; \exists x_1, \ldots, x_n \in E\colon \Omega \subset \bigcup_{i=1}^{n} B(\{x_i\}, \varepsilon) \right\}.$$

The Hausdorff measure of noncompactness is monotone and nonsingular. Moreover, if $L \in \mathcal{L}(E)$ and $\Omega \subset E$, then (see, e.g. [15])

$$\gamma(L\Omega) \le ||L||_{\mathcal{L}(E)} \gamma(\Omega). \tag{2.1}$$

Let $\{f_n\} \subset L([0,T],E)$ be such that $||f_n(t)|| \le \alpha(t)$, $\gamma(\{f_n(t)\}) \le c(t)$, for almost all $t \in [0,T]$, all $n \in \mathbb{N}$ and suitable $\alpha, c \in L([0,T],\mathbb{R})$, then (cf. [15])

$$\gamma\bigg(\bigg\{\int_0^T f_n(t)\,dt\bigg\}\bigg) \le 2\int_0^T c(t), \quad \text{for a.a. } t \in [0,T]. \tag{2.2}$$

Moreover, for all subsets Ω of E (see e.g. [5]),

$$\gamma\bigg(\bigcup_{\lambda\in[0,1]} \lambda\Omega\bigg) \le \gamma(\Omega). \tag{2.3}$$

Let us now introduce the function

$$\begin{aligned}\mu(\Omega) := \max_{\{w_n\}_n\subset\Omega} \bigg(&\sup_{t\in[0,T]} [\gamma(\{w_n(t)\}_n) + \gamma(\{\dot{w}_n(t)\}_n)], \\ &\operatorname{mod}_C(\{w_n\}_n) + \operatorname{mod}_C(\{\dot{w}_n\}_n)\bigg),\end{aligned} \tag{2.4}$$

defined on the bounded $\Omega \subset C^1([0,T],E)$, where the ordering is induced by the positive cone in $\mathbb{R}^2$ and where $\operatorname{mod}_C(\Omega)$ denotes the modulus of continuity of a subset $\Omega \subset C([0,T],E)$ ([1]). Such a μ is a m.n.c. in $C^1([0,T],E)$, as proven in the following lemma, where the properties of μ will be also discussed. We will use μ in order to solve problem (1.1) (cf. Theorem 5.1).

Lemma 2.3. *The function μ given by* (2.4) *defines an m.n.c. in $C^1([0,T],E)$; such a μ is monotone, invariant with respect to the union with compact sets and regular.*

Proof. At first, we show that μ is well-defined, i.e. that the maximum in (2.4) is reached. Indeed, let $\{x_n^{(m)}\}_n \subset \Omega$ and $\{y_n^{(m)}\}_n \subset \Omega$ be two sequences of denumerable sets such that, as $m \to \infty$,

$$\begin{gathered}\sup_{t\in[0,T]} [\gamma(\{x_n^{(m)}(t)\}_n) + \gamma(\{\dot{x}_n^{(m)}(t)\}_n)] \to \sup_{\{w_n\}_n\subset\Omega} [\gamma(\{w_n(t)\}_n) + \gamma(\{\dot{w}_n(t)\}_n)], \\ \operatorname{mod}_C[(\{y_n^{(m)}\}_n) + (\{\dot{y}_n^{(m)}\}_n)] \to \sup_{\{\dot{w}_n\}_n\subset\Omega} [\operatorname{mod}_C(\{w_n\}_n) + \operatorname{mod}_C(\{\dot{w}_n\}_n)].\end{gathered}$$

It is easy to see that the denumerable set

$$\{z_n\}_n := \bigg\{\bigg(\bigcup_{m=1}^{\infty} x_n^{(m)}, \bigcup_{m=1}^{\infty} y_n^{(m)}\bigg)\bigg\}_n$$

is such that

$$\mu(\Omega) = \bigg(\sup_{t\in[0,T]} [\gamma(\{z_n(t)\}_n) + \gamma(\{\dot{z}_n(t)\}_n)], \operatorname{mod}_C(\{z_n\}_n) + \operatorname{mod}_C(\{\dot{z}_n\}_n)\bigg).$$

([1]) The m.n.c. $\operatorname{mod}_C(\Omega)$ is a monotone, nonsingular and algebraically subadditive on $C([0,T],E)$ (cf. e.g. [15]) and it is equal to zero if and only if all the elements $x \in \Omega$ are equi-continuos.

Thus μ is well-defined. Observe that μ is also monotone, because if $\Omega_1 \subset \Omega_2 \subset C^1([0,T],E)$ are bounded, then the maximum for $\mu(\Omega_2)$ is taken on a larger set than for $\mu(\Omega_1)$, and so $\mu(\Omega_1) \le \mu(\Omega_2)$. We now prove the equality $\mu(\overline{\mathrm{co}}\,\Omega) = \mu(\Omega)$. By the monotonicity of μ, it is sufficient to prove that $\mu(\overline{\mathrm{co}}\,\Omega) \le \mu(\Omega)$. Let $\{y_n\}_n \subset (\overline{\mathrm{co}}\,\Omega)$ be such that

$$\mu(\overline{\mathrm{co}}\,\Omega) = \Big(\sup_{t\in[0,T]}[\gamma(\{y_n(t)\}_n) + \gamma(\{\dot y_n(t)\}_n)], \mathrm{mod}_C(\{y_n\}_n) + \mathrm{mod}_C(\{\dot y_n\}_n)\Big).$$

Hence, we can find $\{x_n\}_n$ such that $\{y_n\}_n \subset \overline{\mathrm{co}}\,\{x_n\}_n$. According to the monotonicity of the Hausdorff m.n.c. and of $\mathrm{mod}_C(\Omega)$, we obtain that

$$\begin{aligned}\gamma(\{y_n(t)\}_n) &\le \gamma(\overline{\mathrm{co}}\,\{x_n(t)\}_n) = \gamma(\{x_n(t)\}_n), \quad \text{for each } t\in[0,T],\\ \mathrm{mod}_C(\{y_n\}_n) &\le \mathrm{mod}_C(\mathrm{co}\,\{x_n\}_n) = \mathrm{mod}_C(\{x_n\}_n),\end{aligned}$$

implying that $\mu(\{y_n\}_n) \le \mu(\{x_n\}_n) \le \mu(\Omega)$.

Now, we prove that μ is invariant with respect to the union with compact sets. Let $K \subset C^1([0,T],E)$ be relatively compact. Then, in view of monotonicity, $\mu(\Omega) \le \mu(\Omega\cup K)$, for all bounded $\Omega \subset C^1([0,T],E)$. The reverse inequality $\mu(\Omega\cup K) \le \mu(\Omega)$ can be proven as follows. Let $\{y_n\}_n \subset \Omega\cup K$ be a sequence where the maximum in the definition of $\mu(\Omega\cup K)$ is reached. Then

$$\gamma(\{y_n(t)\}) = \gamma((\{y_n\}\cap\Omega)(t) \cup (\{y_n\}\cap K)(t)) = \gamma((\{y_n\}\cap\Omega)(t)),$$

for all $t\in[0,T]$, and

$$\mathrm{mod}_C(\{y_n\}) = \mathrm{mod}_C((\{y_n\}\cap\Omega)\cup(\{y_n\}\cap K)) = \mathrm{mod}_C(\{y_n\}\cap\Omega).$$

Put

$$\dot\Omega := \{x\in C([0,T],E) \mid \exists y\in\Omega\colon x(t) = \dot y(t),\ \text{for all } t\in[0,T]\}$$

and

$$\dot K := \{x\in C([0,T],E) \mid \exists y\in K\colon x(t) = \dot y(t),\ \text{for all } t\in[0,T]\}.$$

It is easy to see that both K and $\dot K$ are relatively compact in $C([0,T],E)$. Consequently,

$$\gamma(\{\dot y_n(t)\}) = \gamma((\{\dot y_n\}\cap\dot\Omega)(t) \cup (\{\dot y_n\}\cap\dot K)(t)) = \gamma((\{\dot y_n\}\cap\dot\Omega)(t))$$

and

$$\mathrm{mod}_C(\{\dot y_n\}) = \mathrm{mod}_C((\{\dot y_n\}\cap\dot\Omega)\cup(\{\dot y_n\}\cap\dot K)) = \mathrm{mod}_C(\{\dot y_n\}\cap\dot\Omega).$$

Therefore,

$$\begin{aligned}\mu(\Omega\cup K) = \Big(&\sup_{t\in[0,T]}[\gamma((\{y_n\}\cap\Omega)(t)) + \gamma((\{\dot y_n\}\cap\dot\Omega)(t))],\\ &\mathrm{mod}_C(\{y_n\}\cap\Omega) + \mathrm{mod}_C(\{\dot y_n\}\cap\dot\Omega)\Big) \le \mu(\Omega).\end{aligned}$$

Thus, the m.n.c. μ is invariant with respect to the union with compact sets, and so nonsingular as well.

It remains to show that μ is regular. If the set Ω is relatively compact, then each sequence $\{w_n\}_n \subset \Omega$ is also relatively compact. It implies that $\gamma(\{w_n(t)\}) = \gamma(\{\dot{w}_n(t)\}) = 0$, for every $t \in [0,T]$, and also that $\mathrm{mod}_C(\{w_n\}) = \mathrm{mod}_C(\{\dot{w}_n\}) = 0$. Hence, $\mu(\Omega) = (0,0)$.

On the other hand, if $\mu(\Omega) = (0,0)$, then $\gamma(\{w_n(t)\}) = \gamma(\{\dot{w}_n(t)\}) = \mathrm{mod}_C(\{w_n\}) = \mathrm{mod}_C(\{\dot{w}_n\}) = 0$, for each $t \in [0,T]$, and every $\{w_n\}_n \subset \Omega$. So, both $\{w_n\}_n$ and $\{\dot{w}_n\}_n$ are equi-continuous and, according to the regularity of the Hausdorff measure, the sets $\{w_n(t)\}_n$, $\{\dot{w}_n(t)\}_n$ are relatively compact, for every t. The well-known Arzelà–Ascoli lemma can be then applied to verify the relative compactness of $\{w_n\}_n$ which completes the proof. $\square$

Definition 2.4. Let E be a Banach space and $X \subset E$. A multivalued mapping $F\colon X \multimap E$ with compact values is called *condensing with respect to an m.n.c.* β (shortly, β*-condensing*) if, for every $\Omega \subset X$ such that $\beta(F(\Omega)) \geq \beta(\Omega)$, it holds that Ω is relatively compact.

A family of mappings $G\colon X \times [0,1] \multimap E$ with compact values is called β-*condensing* if, for every $\Omega \subset X$ such that $\beta(G(\Omega \times [0,1])) \geq \beta(\Omega)$, it holds that Ω is relatively compact.

The following convergence result will be also employed.

Lemma 2.5 (cf. [3, Lemma III.1.30]). *Let E be a Banach space and assume that the sequence of absolutely continuous functions $x_k\colon [0,T] \to E$ satisfies the following conditions:*

(a) *the set $\{x_k(t) \mid k \in \mathbb{N}\}$ is relatively compact, for every $t \in [0,T]$,*
(b) *there exists $\alpha \in L^1([0,T],[0,\infty))$ such that*

$$||\dot{x}_k(t)|| \leq \alpha(t), \quad \text{for a.a. } t \in [0,T] \text{ and for all } k \in \mathbb{N},$$

(c) *the set $\{\dot{x}_k(t) \mid k \in \mathbb{N}\}$ is weakly relatively compact, for almost all $t \in [0,T]$.*

Then there exists a subsequence of $\{x_k\}$ (for the sake of simplicity denoted in the same way as the sequence) converging to an absolutely continuous function $x\colon [0,T] \to E$ in the following way:

(i) *$\{x_k\}$ converges uniformly to x, in $C([0,T],E)$,*
(ii) *$\{\dot{x}_k\}$ converges weakly in $L^1([0,T],E)$ to $\dot{x}$.*

The following lemma is well-known when the Banach spaces E_1 and E_2 coincide (see, e.g. [25, p. 88]). The present slight modification, for $E_1 \neq E_2$, was proved in [4].

Lemma 2.6. *Let $[0,T] \subset \mathbb{R}$ be a compact interval, let E_1, E_2 be Banach spaces and let $F\colon [0,T] \times E_1 \multimap E_2$ be a multivalued mapping satisfying the following conditions:*

(a) *$F(\,\cdot\,,x)$ has a strongly measurable selection, for every $x \in E_1$,*
(b) *$F(t,\,\cdot\,)$ is u.s.c., for a.a. $t \in [0,T]$,*
(c) *the set $F(t,x)$ is compact and convex, for all $(t,x) \in [0,T] \times E_1$.*

Assume in addition that, for every nonempty, bounded set $\Omega \subset E_1$, there exists $\nu = \nu(\Omega) \in L^1([0,T],(0,\infty))$ such that

$$||F(t,x)|| \le \nu(t),$$

for almost all $t \in [0,T]$ and every $x \in \Omega$. Let us define the Nemytskiĭ operator $N_F\colon C([0,T],E_1) \multimap L^1([0,T],E_2)$ in the following way:

$$N_F(x) := \{f \in L^1([0,T],E_2) \mid f(t) \in F(t,x(t)),\ \text{a.e. on } [0,T]\},$$

for every $x \in C([0,T],E_1)$. Then, if sequences $\{x_k\} \subset C([0,T],E_1)$ and $\{f_k\} \subset L^1([0,T],E_2)$, $f_k \in N_F(x_k)$, $k \in \mathbb{N}$, are such that $x_k \to x$ in $C([0,T],E_1)$ and $f_k \to f$ weakly in $L^1([0,T],E_2)$, then $f \in N_F(x)$.

It will be also convenient to recall some basic facts concerning evolution equations. For a suitable introduction and more details, we refer, e.g. to [8], [16], [22].

Hence, let $C\colon [0,T] \to \mathcal{L}(E)$ be Bochner integrable and let $f \in L([0,T],E)$. Given $x_0 \in E$, consider the linear initial value problem

$$\dot{x}(t) = C(t)x(t) + f(t), \quad x(0) = x_0. \tag{2.5}$$

It is well-known (see, e.g. [8]) that, for the uniquely solvable problem (2.5), there exists the evolution operator $\{U(t,s)\}_{(t,s)\in\Delta}$, where $\Delta := \{(t,s)\colon 0 \le s \le t \le T\}$, such that

$$||U(t,s)|| \le \mathrm{e}^{\int_s^t ||C(\tau)||\,d\tau}, \quad \text{for all } (t,s) \in \Delta; \tag{2.6}$$

in addition, the unique solution $x(\,\cdot\,)$ of (2.5) is given by

$$x(t) = U(t,0)x_0 + \int_0^t U(t,s)f(s)\,ds, \qquad t \in [0,T].$$

Given $D \in \mathcal{L}(E)$, the linear Floquet b.v.p.

$$\begin{cases} \dot{x}(t) = C(t)x(t) + f(t), \\ x(T) = Dx(0) \end{cases} \tag{2.7}$$

associated with the equation in (2.5), satisfies the following property.

Lemma 2.7 (cf. [5]). *If the linear operator* $D - U(T,0)$ *is invertible, then* (2.7) *admits a unique solution given, for all* $t \in [0,T]$, *by*

$$x(t) = U(t,0)[D - U(T,0)]^{-1} \int_0^T U(T,\tau)f(\tau)\,d\tau + \int_0^t U(t,\tau)f(\tau)\,d\tau. \tag{2.8}$$

Remark 2.8. Denoting

$$\Lambda := \mathrm{e}^{\int_0^T ||C(s)||\,ds}, \quad \Gamma := ||[D - U(T,0)]^{-1}||,$$

we obtain, in view of (2.6), (2.8) and the growth estimate imposed on $C(t)$, the following inequality for the solution $x(\,\cdot\,)$ of (2.7)

$$||x(t)|| \le \Lambda(\Lambda\Gamma + 1) \int_0^T ||f(s)||\,ds.$$

Now, consider the second-order linear Floquet b.v.p.

$$\begin{cases} \ddot{x}(t) + A(t)\dot{x}(t) + B(t)x(t) = f(t), & \text{for a.a. } t \in [0,T], \\ x(T) = Mx(0), \ \dot{x}(T) = N\dot{x}(0), \end{cases} \tag{2.9}$$

where A, B are Bochner integrable and $f \in L^1([0,T],E)$, and let

$$||(x,y)||_{E\times E} := \sqrt{||x||^2 + ||y||^2}, \quad \text{for all } x, y \in E.$$

Problem (2.9) is equivalent to the following first-order linear one

$$\begin{cases} \dot{\xi}(t) + C(t)\xi(t) = h(t), & \text{for a.a. } t \in [0,T], \\ \xi(T) = \widetilde{D}\xi(0), \end{cases} \tag{2.10}$$

where

$$\xi = (x,y) = (x,\dot{x}), \tag{2.11}$$

$$h(t) = (0, f(t)), \tag{2.12}$$

$$C(t)\colon E\times E \to E\times E, \quad (x,y) \mapsto (-y,\ B(t)x + A(t)y), \tag{2.13}$$

$$\widetilde{D}\colon E\times E \to E\times E, \quad (x,y) \mapsto (Mx,\ Ny). \tag{2.14}$$

Let us denote, for all $(t,s) \in [0,T]\times[0,T]$, by

$$U(t,s) := \begin{pmatrix} U_{11}(t,s) & U_{12}(t,s) \\ U_{21}(t,s) & U_{22}(t,s) \end{pmatrix}$$

the evolution operator associated with

$$\begin{cases} \dot{\xi}(t) + C(t)\xi(t) = h(t), & \text{for a.a. } t \in [0,T], \\ \xi(0) = \xi_0, \end{cases} \tag{2.15}$$

where ξ, h and C are defined by relations (2.11), (2.12) and (2.13), respectively, and $\xi_0 \in E \times E$. It is easy to see that $||C(t)|| \leq 1 + ||A(t)|| + ||B(t)||$ and, according to (2.6), we obtain

$$||U(t,s)|| \leq \mathrm{e}^{\int_0^T (1+||A(t)||+||B(t)||)\,dt}, \quad \text{for all } (t,s) \in \Delta.$$

Consequently, for all $i, j = 1, 2$,

$$||U_{ij}(t,s)|| \leq \mathrm{e}^{\int_0^T (1+||A(t)||+||B(t)||)\,dt}, \quad \text{for all } (t,s) \in \Delta. \tag{2.16}$$

Moreover, if we denote

$$[\widetilde{D} - U(T,0)]^{-1} := \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}$$

and put

$$k := ||[\widetilde{D} - U(T,0)]^{-1}||, \tag{2.17}$$

then $||K_{ij}|| \leq k$, for $i, j = 1, 2$, and the solution $x(\,\cdot\,)$ of (2.9) and its derivative $\dot{x}(\,\cdot\,)$ take, for all $t \in [0,T]$, the forms

$$\begin{aligned} x(t) &= A_1(t) \int_0^T U_{12}(T,\tau) f(\tau)\,d\tau \\ &\quad + A_2(t) \int_0^T U_{22}(T,\tau) f(\tau)\,d\tau + \int_0^t U_{12}(t,\tau) f(\tau)\,d\tau, \end{aligned} \tag{2.18}$$

$$\begin{aligned} \dot{x}(t) &= A_3(t) \int_0^T U_{12}(T,\tau) f(\tau)\,d\tau \\ &\quad + A_4(t) \int_0^T U_{22}(T,\tau) f(\tau)\,d\tau + \int_0^t U_{22}(t,\tau) f(\tau)\,d\tau, \end{aligned} \tag{2.19}$$

where

$$\begin{aligned} A_1(t) &:= U_{11}(t,0)K_{11} + U_{12}(t,0)K_{21}, \\ A_2(t) &:= U_{11}(t,0)K_{12} + U_{12}(t,0)K_{22}, \\ A_3(t) &:= U_{21}(t,0)K_{11} + U_{22}(t,0)K_{21}, \\ A_4(t) &:= U_{21}(t,0)K_{12} + U_{22}(t,0)K_{22}, \end{aligned}$$

for all $t \in [0,T]$. It holds that

$$||A_i(t)|| \leq 2k\mathrm{e}^{\int_0^T (1+||A(t)||+||B(t)||)\,dt}, \quad \text{for } i = 1,2,3,4 \text{ and } t \in [0,T]. \tag{2.20}$$

If there exists $\alpha \in L^1([0,T],[0,\infty))$ such that $||f(t)|| \leq \alpha(t)$, for almost all $t \in [0,T]$, then it immediately follows from Remark 2.8 that the following estimates hold for each solution $x(\,\cdot\,)$ of (2.9) and its derivative $\dot{x}(\,\cdot\,)$:

$$||x(t)|| \leq Z(Zk+1) \int_0^T \alpha(s)\,ds \quad \text{and} \quad ||\dot{x}(t)|| \leq Z(Zk+1) \int_0^T \alpha(s)\,ds,$$

where

$$Z := \mathrm{e}^{\int_0^T (||A(s)||+||B(s)||+1)\,ds} \tag{2.21}$$

with k defined in (2.17).

3. Continuation principle

In this section, consider the general multivalued b.v.p.

$$\begin{cases} \ddot{x}(t) \in \varphi(t, x(t), \dot{x}(t)) & \text{for a.a. } t \in J, \\ x \in S, \end{cases} \tag{3.1}$$

where $J = [0, T]$ is a given compact interval, $\varphi: J \times E \times E \multimap E$ is an upper-Carathéodory mapping. Furthermore, let $S \subset AC^1(J, E)$.

We also introduce the set $Q \subset AC^1(J, E)$ of candidate solutions of the b.v.p. (3.1) and associate to this problem a family of problems depending on two parameters $q \in Q$ and $\lambda \in [0, 1]$. The family of associated problems will be defined in such a way that if $\mathfrak{T}: Q \times [0, 1] \multimap AC^1(J, E)$ is its corresponding solution mapping, then all fixed points of the map $\mathfrak{T}(\,\cdot\,, 1)$ are solutions of (3.1) (see condition (3.2) below). In order to study the fixed point set of $\mathfrak{T}(\,\cdot\,, 1)$, a suitable topological degree technique will be employed.

Proposition 3.1. *Let us consider the b.v.p.* (3.1) *and let* $H: [0, T] \times E \times E \times E \times E \times [0, 1] \multimap E$ *be an upper-Carathéodory mapping such that*

$$H(t, c, d, c, d, 1) \subset \varphi(t, c, d), \quad \textit{for all } (t, c, d) \in [0, T] \times E \times E. \tag{3.2}$$

Moreover, assume that the following conditions hold:

(a) *There exist a closed set* $S_1 \subset S$ *and a closed, convex set* $Q \subset C^1([0, T], E)$ *with a non-empty interior* $\operatorname{Int} Q$ *such that each associated problem*

$$P(q,\lambda) \qquad \begin{cases} \ddot{x}(t) \in H(t, x(t), \dot{x}(t), q(t), \dot{q}(t), \lambda) & \textit{for a.a. } t \in [0, T], \\ x \in S_1, \end{cases}$$

where $q \in Q$ *and* $\lambda \in [0, 1]$, *has a non-empty, convex set of solutions (denoted by* $\mathfrak{T}(q, \lambda)$*).*

(b) *For every non-empty, bounded set* $\Omega \subset E \times E$, *there exists* $\nu_\Omega \in L^1([0, T], [0, \infty))$ *such that*

$$||H(t, x, y, q(t), \dot{q}(t), \lambda)|| \le \nu_\Omega(t),$$

for almost all $t \in [0, T]$ *and all* $(x, y) \in \Omega$, $q \in Q$ *and* $\lambda \in [0, 1]$.

(c) *The solution mapping* $\mathfrak{T}$ *is quasi-compact and* μ*-condensing with respect to a monotone and nonsingular measure of noncompactness* μ *defined on* $C^1([0, T], E)$.

(d) *For each* $q \in Q$*, the set of solutions of the problem* $P(q,0)$ *is a subset of* $\operatorname{Int} Q$*, i.e.* $\mathfrak{T}(q,0) \subset \operatorname{Int} Q$*, for all* $q \in Q$*.*

(e) *For each* $\lambda \in (0,1)$*, the solution mapping* $\mathfrak{T}(\,\cdot\,,\lambda)$ *has no fixed points on the boundary* ∂Q *of* Q*.*

Then the b.v.p. (3.1) *has a solution in* Q*.*

Proof. Let us observe that, according to condition (3.2), every fixed point of the solution mapping $\mathfrak{T}(\,\cdot\,,1)$ is a solution of the original problem (3.1) lying in Q. Thus, if the intersection $\operatorname{Fix}(\mathfrak{T}(\,\cdot\,,1)) \cap \partial Q$ is nonempty, then the b.v.p. (3.1) has a solution in Q and we are done. Otherwise, condition (e) can be reformulated (according to the above consideration and assumption (d)) as follows:

$$\operatorname{Fix}(\mathfrak{T}(\,\cdot\,,\lambda)) \cap \partial Q = \emptyset, \quad \text{for all } \lambda \in [0,1]. \tag{3.3}$$

Now, we will show that the solution mapping $\mathfrak{T}\colon Q \times [0,1] \multimap AC^1([0,T],E)$ is a u.s.c. mapping with compact values. Consequently, the properties of the solution mapping together with condition (3.3) will allow us to define the topological degree of $\mathfrak{T}$ and to prove that the b.v.p. (3.1) has a solution in Q.

At first, let us prove, by means of Lemmas 2.5 and 2.6, that the solution mapping $\mathfrak{T}$ has a closed graph $\Gamma_{\mathfrak{T}}$. For this purpose, let $\{q_k,\lambda_k,x_k\} \subset \Gamma_{\mathfrak{T}}$ be a sequence such that $(q_k,\lambda_k,x_k) \to (q_0,\lambda_0,x_0)$ in $C^1([0,T],E)\times\mathbb{R}\times C^1([0,T],E)$ as $k \to \infty$, where $q_0 \in Q$, $\lambda_0 \in [0,1]$ and $x_0 \in C^1([0,T],E)$. Since $\dot{x}_k(t) \to \dot{x}_0(t)$, the sequence $\{\dot{x}_k(t)\}_{k=1}^{\infty}$ is relatively compact, for all $t \in [0,T]$. Moreover, since $\{x_k,\dot{x}_k\}$ is uniformly convergent on $[0,T]$, there exists a constant $M > 0$ such that

$$||x_k(t)|| \le M \quad \text{and} \quad ||\dot{x}_k(t)|| \le M, \quad \text{for all } t \in [0,T] \text{ and } k \in \mathbb{N}. \tag{3.4}$$

According to the estimates in (3.4) and condition (b), there exists $\nu \in L^1([0,T],[0,\infty))$ such that

$$||H(x_k(t),\dot{x}_k(t),q_k(t),\dot{q}_k(t),\lambda_k)|| \le \nu(t),$$

for almost all $t \in [0,T]$ and all $k \in \mathbb{N}$. Therefore, $||\ddot{x}_k(t)|| \le \nu(t)$, for almost all $t \in [0,T]$ and all $k \in \mathbb{N}$.

Now, let us show that, for almost all $t \in [0,T]$, $\{\ddot{x}_k(t)\}$ is relatively compact. For this purpose, let $t \in [0,T]$ be such that

$$\ddot{x}_k(t) \in H(t,x_k(t),\dot{x}_k(t),q_k(t),\dot{q}_k(t),\lambda_k), \quad \text{for all } k \in \mathbb{N}.$$

Since $H(t,\,\cdot\,)$ is u.s.c., for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$H(t,x,y,u,v,\lambda) \subset H(t,x_0(t),\dot{x}_0(t),q_0(t),\dot{q}_0(t),\lambda_0) + \varepsilon B,$$

for all $(x,y,u,v,\lambda) \in E \times E \times E \times E \times [0,1]$ satisfying

$$||(x,y,u,v,\lambda) - (x_0(t),\dot{x}_0(t),q_0(t),\dot{q}_0(t),\lambda_0)|| < \delta.$$

The fact that $(q_k, \dot{q}_k, \lambda_k, x_k, \dot{x}_k) \to (q_0, \dot{q}_0, \lambda_0, x_0, \dot{x}_0)$ ensures the existence of $k_0 \in \mathbb{N}$ such that

$$H(t, x_k(t), \dot{x}_k(t), q_k(t), \dot{q}_k(t), \lambda_k) \subset H(t, x_0(t), \dot{x}_0(t), q_0(t), \dot{q}_0(t), \lambda_0) + \varepsilon B,$$

for all $k \geq k_0$. Thus,

$$\{\ddot{x}_k(t)\}_{k=1}^{\infty} \subset \bigcup_{k=1}^{k_0} H(t, x_k(t), \dot{x}_k(t), q_k(t), \dot{q}_k(t), \lambda_k) \\ \cup H(t, x_0(t), \dot{x}_0(t), q_0(t), \dot{q}_0(t), \lambda_0) + \varepsilon B.$$

Since H has compact values, the sequence $\{\ddot{x}_k(t)\}$ is relatively compact, for almost all $t \in [0,T]$.

The above reasonings imply that the sequence $\{\dot{x}_k\}$ satisfies all assumptions of Lemma 2.5. Thus, there exists a subsequence of $\{\dot{x}_k\}$, for the sake of simplicity denoted in the same way as the sequence, such that $\{\ddot{x}_k\}$ converges weakly to $\ddot{x}_0$ in $L^1([0,T], E)$.

If we set $y_k := \dot{x}_k$ and $z_k := (x_k, y_k)$, then $\dot{z}_k = (\dot{x}_k, \dot{y}_k) = (\dot{x}_k, \ddot{x}_k) \to (\dot{x}_0, \ddot{x}_0)$ weakly in $L^1([0,T], E)$. Let us now consider the system

$$\dot{z}_k(t) \in H^*(t, z_k(t), q_k(t), \dot{q}_k(t), \lambda_k), \quad \text{for a.a. } t \in [0,T],$$

where $H^*(t, z_k(t), q_k(t), \dot{q}_k(t), \lambda_k) = (y_k(t), H(t, z_k(t), q_k(t), \dot{q}_k(t), \lambda_k))$.

Applying Lemma 2.6, for $f_k := \dot{z}_k$, $f := (\dot{x}_0, \ddot{x}_0)$, $x_k := (z_k, q_k, \dot{q}_k, \lambda_k)$, it follows that

$$(\dot{x}_0(t), \ddot{x}_0(t)) \in H^*(t, x_0(t), \dot{x}_0(t), q_0(t), \dot{q}_0(t), \lambda_0),$$

for almost all $t \in [0,T]$, i.e.

$$\ddot{x}_0(t) \in H(t, x_0(t), \dot{x}_0(t), q_0(t), \dot{q}_0(t), \lambda_0), \quad \text{for a.a. } t \in [0,T].$$

Moreover, since S_1 is closed, $x_0 \in S_1$, and so the solution mapping $\mathfrak{T}$ has a closed graph.

Thus, the set $\mathfrak{T}(q, \lambda)$ is closed, for all $(q, \lambda) \in Q \times [0,1]$, which (together with condition (c)) implies that $\mathfrak{T}$ has compact values. Furthermore, according to Proposition 2.1, $\mathfrak{T}$ is a u.s.c. mapping. Therefore, we can conclude that $\mathfrak{T}$ is a u.s.c. mapping with compact, convex values which is condensing on the closed set Q. This ensures that both the topological degree (see e.g. [15]) as well as the fixed point index (see e.g. [3]) can be defined on open sets with fixed point free boundaries. Moreover, both the degree and the index satisfy the standard properties. In particular, $\mathfrak{T}$ is an admissible homotopy according to (3.3) and the multivalued vector-fields $\phi_0(\,\cdot\,) := \mathrm{id} - \mathfrak{T}(\,\cdot\,, 0)$, $\phi_1(\,\cdot\,) := \mathrm{id} - \mathfrak{T}(\,\cdot\,, 1)$ are homotopic as well, and so $\deg_{C^1([0,T],E)}(\phi_1, Q) = \deg_{C^1([0,T],E)}(\phi_0, Q)$. Furthermore, since $\mathfrak{T}(Q \times \{0\}) \subset \operatorname{Int} Q$, the localization property of the degree ensures that $\deg_{C^1([0,T],E)}(\phi_0, Q) = \deg_Q(\phi_0, Q) = 1$. Therefore, the nonemptiness of $\mathrm{Fix}(\mathfrak{T}(\,\cdot\,,1))$ is ensured by the existence property of the degree which completes the proof. □

4. Bound sets technique

The continuation principle formulated in Proposition 3.1 requires, in particular, the existence of a suitable set $Q \subset AC^1(J,E)$ of candidate solutions. The set Q must satisfy the transversality condition (d), i.e. it must have fixed-point free boundary with respect to the solution mapping $\mathfrak{T}$. Since the direct verification of the transversality condition is usually a difficult task, we will devote this section to a bound sets technique which can be used for guaranteeing this condition. For this purpose, we will define the set Q as $Q = C^1([0,T], \overline{K})$, where K is nonempty and open in E and $\overline{K}$ denotes its closure.

Hence, let us consider the Floquet boundary value problem (1.1) and let $V: E \to \mathbb{R}$ be a C^1-function satisfying

(H1) $V|_{\partial K} = 0$,
(H2) $V(x) \leq 0$, for all $x \in \overline{K}$.

Definition 4.1. A nonempty open set $K \subset E$ is called a *bound set* for the b.v.p. (1.1) if every solution x of (1.1) such that $x(t) \in \overline{K}$, for each $t \in [0,T]$, does not satisfy $x(t^*) \in \partial K$, for any $t^* \in [0,T]$.

Let E' be the Banach space dual to E and let us denote by $\langle\,\cdot\,,\,\cdot\,\rangle$ the pairing (the duality relation) between E and E', i.e. for all $\Phi \in E'$ and $x \in E$, we put $\Phi(x) := \langle \Phi, x\rangle$.

Proposition 4.2. *Let $K \subset E$ be an open set such that $0 \in K$. Moreover, let $M\partial K = \partial K$. Assume that the function $V \in C^1(E,\mathbb{R})$ has a locally Lipschitz Fréchet derivative $\dot{V}_x$ and satisfies conditions* (H1) *and* (H2). *Suppose, moreover, that there exists $\varepsilon > 0$ such that, for all $x \in \overline{K} \cap B(\partial K, \varepsilon)$, $t \in (0,T)$ and $y \in E$, the following condition*

$$\limsup_{h \to 0^-} \frac{\langle \dot{V}_{x+hy} - \dot{V}_x, y\rangle}{h} + \langle \dot{V}_{x+hy}, w\rangle > 0 \tag{4.1}$$

holds, for all $w \in F(t,x,y) - A(t)y - B(t)x$, and that

$$\langle \dot{V}_{Mx}, Nz\rangle \cdot \langle \dot{V}_x, z\rangle > 0 \quad \text{or} \quad \langle \dot{V}_{Mx}, Nz\rangle = \langle \dot{V}_x, z\rangle = 0, \tag{4.2}$$

for all $x \in \partial K$ and $z \in E$. Then K is a bound set for the Floquet problem (1.1).

Proof. Let $x: [0,T] \to \overline{K}$ be a solution of problem (1.1). We assume, by a contradiction, that there exists $t^* \in [0,T]$ such that $x(t^*) \in \partial K$. According to the boundary condition in (1.1) and in view of $M\partial K = \partial K$, we can take, without any loss of generality, $t^* \in (0,T]$.

Since $\dot{V}_x$ is locally Lipschitz, there exist a neighbourhood U of $x(t^*)$ and a constant $L > 0$ such that $\dot{V}|_U$ is Lipschitz with constant L. Let $\delta > 0$ be such that $x(t) \in U \cap B(\partial K, \varepsilon)$, for each $t \in [t^* - \delta, t^*]$.

In order to get the desired contradiction, let us define the function $g\colon [0,T] \to \mathbb{R}$ as the composition $g(t) := (V \circ x)(t)$. According to the regularity properties of x and V, $g \in C^1([0,T],\mathbb{R})$. Since $g(t^*) = 0$ and $g(t) \leq 0$, for all $t \in [0,T]$, t^* is a local maximum point for g. Therefore, $\dot{g}(t^*) \geq 0$ and $\dot{g}(t^*) = 0$, when $t^* \in (0,T)$. Moreover, there exists a point $t^{**} \in (t^* - \delta, t^*)$ such that $\dot{g}(t^{**}) \geq 0$.

According to boundary conditions, if $t^* = T$, then also $x(0) \in \partial K$ and

$$\dot{g}(0) = \langle \dot{V}_{x(0)}, \dot{x}(0) \rangle \leq 0.$$

Moreover, since $x(T) = Mx(0)$ and $\dot{x}(T) = N\dot{x}(0)$, we have

$$\dot{g}(T) = \langle \dot{V}_{x(T)}, \dot{x}(T) \rangle = \langle \dot{V}_{Mx(0)}, N\dot{x}(0) \rangle \geq 0.$$

Condition (4.2) then implies

$$\langle \dot{V}_{x(0)}, \dot{x}(0) \rangle = \langle V_{Mx(0)}, N\dot{x}(0) \rangle = 0$$

which is equivalent to $\dot{g}(0) = \dot{g}(T) = 0$.

Since $\dot{g}(t) = \langle V_{x(t)}, \dot{x}(t) \rangle$, where $\dot{V}_{x(t)}$ is locally Lipschitz and $\dot{x}(t)$ is absolutely continuous on $[t^* - \delta, t^*]$, $\ddot{g}(t)$ exists, for almost all $t \in [t^* - \delta, t^*]$. Consequently,

$$0 \geq -\dot{g}(t^{**}) = \dot{g}(t^*) - \dot{g}(t^{**}) = \int_{t^{**}}^{t^*} \ddot{g}(s)\,ds. \tag{4.3}$$

Let $t \in (t^{**}, t^*)$ be such that $\ddot{g}(t)$ and $\ddot{x}(t)$ exist. Then,

$$\lim_{h \to 0} \frac{\dot{x}(t+h) - \dot{x}(t)}{h} = \ddot{x}(t)$$

and, therefore, there exists a function $a(h)$, $a(h) \to 0$ as $h \to 0$ such that, for each h,

$$\dot{x}(t+h) = \dot{x}(t) + h[\ddot{x}(t) + a(h)].$$

Moreover, since $x \in C^1([0,T],E)$, there exists a function $b(h)$, $b(h) \to 0$ as $h \to 0$ such that, for each h,

$$x(t+h) = x(t) + h[\dot{x}(t) + b(h)].$$

Consequently, we obtain

$$\begin{aligned}
\ddot{g}(t) = \lim_{h \to 0} \frac{\dot{g}(t+h) - \dot{g}(t)}{h} &= \limsup_{h \to 0^-} \frac{\dot{g}(t+h) - \dot{g}(t)}{h} \\
&= \limsup_{h \to 0^-} \frac{\langle \dot{V}_{x(t+h)}, \dot{x}(t+h) \rangle - \langle V_{x(t)}, \dot{x}(t) \rangle}{h} \\
&= \limsup_{h \to 0^-} \frac{\langle \dot{V}_{x(t)+h[\dot{x}(t)+b(h)]}, \dot{x}(t) + h[\ddot{x}(t) + a(h)] \rangle - \langle \dot{V}_{x(t)}, \dot{x}(t) \rangle}{h}
\end{aligned}$$

$$\begin{aligned}
&\geq \limsup_{h\to 0^-} \frac{\langle \dot V_{x(t)+h\dot x(t)}, \dot x(t)+h[\ddot x(t)+a(h)]\rangle - \langle \dot V_{x(t)}, \dot x(t)\rangle}{h} \\
&\quad - L\cdot |b(h)|\cdot |\dot x(t)+h[\ddot x(t)+a(h)]| \\
&= \limsup_{h\to 0^-} \frac{\langle \dot V_{x(t)+h\dot x(t)}, \dot x(t)+h\ddot x(t)\rangle - \langle \dot V_{x(t)}, \dot x(t)\rangle}{h} \\
&\quad - L\cdot |b(h)|\cdot |\dot x(t)+h[\ddot x(t)+a(h)]| + \langle \dot V_{x(t)+h\dot x(t)}, a(h)\rangle.
\end{aligned}$$

Since

$$\langle \dot V_{x(t)+h\dot x(t)}, a(h)\rangle - L\cdot |b(h)|\cdot |\dot x(t)+h[\ddot x(t)+a(h)]| \to 0 \quad \text{as } h\to 0,$$

$$\begin{aligned}
\ddot g(t) &\geq \limsup_{h\to 0^-} \frac{\langle \dot V_{x(t)+h\dot x(t)}, \dot x(t)+h\ddot x(t)\rangle - \langle \dot V_{x(t)}, \dot x(t)\rangle}{h} \\
&= \limsup_{h\to 0^-} \frac{\langle \dot V_{x(t)+h\dot x(t)} - \dot V_{x(t)}, \dot x(t)\rangle}{h} + \langle \dot V_{x(t)+h\dot x(t)}, \ddot x(t)\rangle > 0,
\end{aligned}$$

according to assumption (4.1), it leads to a contradiction with (4.3). □

Remark 4.3. Observe that Proposition 4.2 holds, without any loss of generality, for the general upper-Carathéodory differential inclusion in (1.1), i.e. for $A = B \equiv 0$.

If the mapping $F(t,x,y) - A(t)y - B(t)x$ is globally u.s.c. in (t,x,y), then the transversality conditions can be localized directly on the boundary of K, as will be shown in the following propositions.

Proposition 4.4. *Let $K \subset E$ be a nonempty open set, $F\colon [0,T]\times E\times E \multimap E$ be an upper semicontinuous multivalued mapping with nonempty, compact, convex values and A and B be continuous. Assume that there exists a function $V \in C^1(E,\mathbb{R})$ with a locally Lipschitz Fréchet derivative $\dot V_x$ which satisfies conditions* (H1) *and* (H2)*. Suppose moreover that, for all $x \in \partial K$, $t \in (0,T)$ and $y \in E$ with*

$$\langle \dot V_x, y\rangle = 0, \tag{4.4}$$

the following condition holds

$$\liminf_{h\to 0} \frac{\langle \dot V_{x+hy}, y+hw\rangle}{h} > 0, \tag{4.5}$$

for all $w \in F(t,x,y) - A(t)y - B(t)x$. Then all solutions $x\colon [0,T]\to \overline{K}$ of problem (1.1) *satisfy $x(t)\in K$, for every $t\in(0,T)$.*

Proof. Let $x\colon [0,T]\to\overline{K}$ be a solution of problem (1.1). We assume, by a contradiction, that there exists $t_0\in(0,T)$ such that $x(t_0)\in\partial K$.

Let us define the function $g\colon [-t_0, T-t_0] \to (-\infty, 0]$ as the composition $g(h) := (V\circ x)(t_0+h)$. Then $g(0) = 0$ and $g(h)\leq 0$, for all $h\in[-t_0, T-t_0]$, i.e.

there is a local maximum for g at the point 0, and so $\dot{g}(0) = \langle \dot{V}_{x(t_0)}, \dot{x}(t_0)\rangle = 0$. Consequently, $v := \dot{x}(t_0)$ satisfies condition (4.4).

Since $\dot{V}_x$ is locally Lipschitz, there exist a neighborhood U of $x(t_0)$ and a constant $L > 0$ such that $\dot{V}|_U$ is Lipschitz with constant L.

Let $\{h_k\}_{k=1}^{\infty}$ be an arbitrary decreasing sequence of positive numbers such that $h_k \to 0^+$ as $k \to \infty$, $x(t_0 + h) \in U$, for all $h \in (0, h_1)$.

Since $g(0) = 0$ and $g(h) \leq 0$, for all $h \in (0, h_k]$, there exists, for each $k \in \mathbb{N}$, $h_k^* \in (0, h_k)$ such that $\dot{g}(h_k^*) \leq 0$.

Since $x \in C^1([0,T], E)$, for each $k \in \mathbb{N}$,

$$x(t_0 + h_k^*) = x(t_0) + h_k^*[\dot{x}(t_0) + b_k^*], \tag{4.6}$$

where $b_k^* \to 0$ as $k \to \infty$.

If we define, for each $t \in [0,T]$,

$$P(t, x(t), \dot{x}(t)) := -A(t)\dot{x}(t) - B(t)x(t) + F(t, x(t), \dot{x}(t)), \tag{4.7}$$

then (1.1) can be written in the form

$$\ddot{x}(t) \in P(t, x(t), \dot{x}(t)), \quad \text{for a.a. } t \in [0,T].$$

Let

$$\zeta := \left\{ \frac{\dot{x}(t_0 + h_k^*) - \dot{x}(t_0)}{h_k^*},\ k \in \mathbb{N} \right\}$$

and let $\varepsilon > 0$ be given. As a consequence of the regularity assumptions on F, A and B and of the continuity of both x and $\dot{x}$, there exists $\overline{\delta} = \overline{\delta}(\varepsilon) > 0$ such that, for each $t \in (0,T)$, $|t - t_0| \leq \overline{\delta}$, it follows that

$$P(t, x(t), \dot{x}(t)) \subset P(t_0, x(t_0), \dot{x}(t_0)) + \varepsilon\overline{B}.$$

Subsequently, according to the Mean Value Theorem (see e.g. [6, Theorem 0.5.3]), there exists $k_\varepsilon \in \mathbb{N}$ such that, for each $k > k_\varepsilon$,

$$\frac{\dot{x}(t_0 + h_k^*) - \dot{x}(t_0)}{h_k^*} = \frac{1}{h_k^*}\int_{t_0}^{t_0 + h_k^*} \ddot{x}(s)\, ds \in P(t_0, x(t_0), \dot{x}(t_0)) + \varepsilon\overline{B}.$$

Therefore,

$$\zeta \subset \left\{ \frac{\dot{x}(t_0 + h_k^*) - \dot{x}(t_0)}{h_k^*},\ k = 1, \ldots, k(\varepsilon) \right\} \cup P(t_0, x(t_0), \dot{x}(t_0)) + \varepsilon\overline{B}.$$

Since P has compact values and ε is arbitrary, we obtain that ζ is a relatively compact set. Thus, there exist a subsequence, for the sake of simplicity denoted as the sequence, of $\{(\dot{x}(t_0 + h_k^*) - \dot{x}(t_0))/h_k^*\}$ and $w \in E$ such that

$$\frac{\dot{x}(t_0 + h_k^*) - \dot{x}(t_0)}{h_k^*} \to w \tag{4.8}$$

as $k \to \infty$ implying, for the arbitrariness of $\varepsilon > 0$,

$$w \in P(t_0, x(t_0), \dot{x}(t_0)).$$

As a consequence of property (4.8), there exists a sequence $\{a_k^*\}_{k=1}^\infty$, $a_k^* \to 0$ as $k \to \infty$, such that

$$\dot{x}(t_0 + h_k^*) = \dot{x}(t_0) + h_k^*[w + a_k^*], \tag{4.9}$$

for each $k \in \mathbb{N}$. Since $h_k^* > 0$ and $\dot{g}(h_k^*) \le 0$, in view of (4.6) and (4.9),

$$0 \ge \frac{\dot{g}(h_k^*)}{h_k^*} = \frac{\langle \dot{V}_{x(t_0+h_k^*)}, \dot{x}(t_0 + h_k^*)\rangle}{h_k^*} = \frac{\langle \dot{V}_{x(t_0)+h_k^*[\dot{x}(t_0)+b_k^*]}, \dot{x}(t_0) + h_k^*[w + a_k^*]\rangle}{h_k^*}.$$

Since $h_k^* \in (0, h_k) \subset (0, h_1)$, for all $k \in \mathbb{N}$, we have, according to (4.6), that $x(t_0) + h_k^*[\dot{x}(t_0) + b_k^*] \in U$, for each $k \in \mathbb{N}$. Since $b_k^* \to 0$ as $k \to \infty$, it is possible to find $k_0 \in \mathbb{N}$ such that, for all $k \ge k_0$, it holds that $x(t_0) + \dot{x}(t_0)h_k^* \in U$. By means of the local Lipschitzianity of $\dot{V}$, for all $k \ge k_0$,

$$\begin{aligned}
0 &\ge \frac{\dot{g}(h_k^*)}{h_k^*} \\
&= \frac{\langle \dot{V}_{x(t_0)+h_k^*[\dot{x}(t_0)+b_k^*]} - \dot{V}_{x(t_0)+h_k^*\dot{x}(t_0)} + \dot{V}_{x(t_0)+h_k^*\dot{x}(t_0)}, \dot{x}(t_0) + h_k^*[w + a_k^*]\rangle}{h_k^*} \\
&\ge \frac{\langle \dot{V}_{x(t_0)+h_k^*\dot{x}(t_0)}, \dot{x}(t_0) + h_k^*[w + a_k^*]\rangle}{h_k^*} - L \cdot |b_k^*| \cdot |\dot{x}(t_0) + h_k^*[w + a_k^*]| \\
&= \frac{\langle \dot{V}_{x(t_0)+h_k^*\dot{x}(t_0)}, \dot{x}(t_0) + h_k^* w\rangle}{h_k^*} \\
&\quad - L \cdot |b_k^*| \cdot |\dot{x}(t_0) + h_k^*[w + a_k^*]| + \langle \dot{V}_{x(t_0)+h_k^*\dot{x}(t_0)}, a_k^*\rangle.
\end{aligned}$$

Since $\langle \dot{V}_{x(t_0)+h_k^*\dot{x}(t_0)}, a_k^*\rangle - L \cdot |b_k^*| \cdot |\dot{x}(t_0) + h_k^*[w + a_k^*]| \to 0$ as $k \to \infty$,

$$\liminf_{h \to 0^+} \frac{\langle \dot{V}_{x(t_0)+h\dot{x}(t_0)}, \dot{x}(t_0) + hw\rangle}{h} \le 0. \tag{4.10}$$

If we consider, instead of the sequence $\{h_k\}_{k=1}^\infty$, an increasing sequence $\{\overline{h}_k\}_{k=1}^\infty$ of negative numbers such that $\overline{h}_k \to 0^-$ as $k \to \infty$, $x(t_0 + h) \in U$ for all $h \in (\overline{h}_1, 0)$, we are able to find, for each $k \in \mathbb{N}$, $\overline{h}_k^* \in (\overline{h}_k, 0)$ such that $\dot{g}(\overline{h}_k^*) \ge 0$. Therefore, using the same procedure as in the first part of the proof, we obtain, for $k \in \mathbb{N}$ sufficiently large, that

$$\begin{aligned}
0 \ge \frac{\dot{g}(\overline{h}_k^*)}{\overline{h}_k^*} \ge &\frac{\langle \dot{V}_{x(t_0)+\overline{h}_k^*\dot{x}(t_0)}, \dot{x}(t_0) + \overline{h}_k^*\overline{w}\rangle}{\overline{h}_k^*} \\
&- L \cdot |\overline{b}_k^*| \cdot |\dot{x}(t_0) + \overline{h}_k^*[\overline{w} + \overline{a}_k^*]| + \langle \dot{V}_{x(t_0)+\overline{h}_k^*\dot{x}(t_0)}, \overline{a}_k^*\rangle,
\end{aligned}$$

where $\overline{a}_k^* \to 0$, $\overline{b}_k^* \to 0$ as $k \to \infty$ and $\overline{w} \in P(t_0, x(t_0), \dot{x}(t_0))$.

This means that $\langle \dot{V}_{x(t_0)+\overline{h}_k^* \dot{x}(t_0)}, \overline{a}_k^* \rangle - L \cdot |\overline{b}_k^*| \cdot |\dot{x}(t_0) + \overline{h}_k^* [\overline{w} + \overline{a}_k^*]| \to 0$ as $k \to \infty$ which implies

$$\liminf_{h \to 0^-} \frac{\langle \dot{V}_{x(t_0)+h\dot{x}(t_0)}, \dot{x}(t_0) + h\overline{w} \rangle}{h} \leq 0. \tag{4.11}$$

Inequalities (4.10) and (4.11) are in a contradiction with condition (4.5), because $x(t_0) \in \partial K$, $\dot{x}(t_0)$ satisfies condition (4.4) and $w, \overline{w} \in P(t_0, x(t_0), \dot{x}(t_0))$.□

Remark 4.5. Observe that Proposition 4.4 holds, without any loss of generality, for the general second-order problem (3.1), i.e. for $A = B \equiv 0$.

Proposition 4.6. *Let $K \subset E$ be a nonempty open set, $F\colon [0,T] \times E \times E \multimap E$ be an upper semicontinuous mapping with nonempty, compact, convex values and A and B be continuous. Assume that there exists a function $V \in C^1(E, \mathbb{R})$ with a locally Lipschitz–Fréchet derivative $\dot{V}_x$ which satisfies conditions* (H1) *and* (H2). *Moreover, let M be invertible and such that*

$$M(\partial K) = \partial K. \tag{4.12}$$

Furthermore, assume that, for all $x \in \partial K$, $t \in (0,T)$ and $y \in E$ satisfying (4.4), *condition* (4.5) *holds, for all $w \in F(t,x,y) - A(t)y - B(t)x$. At last, assume that, for all $x \in \partial K$ and $y \in E$ with*

$$\langle \dot{V}_x, y \rangle \leq 0 \leq \langle \dot{V}_{Mx}, Ny \rangle, \tag{4.13}$$

at least one of the following conditions

$$\liminf_{h \to 0^+} \frac{\langle \dot{V}_{x+hy}, y + hw_1 \rangle}{h} > 0 \tag{4.14}$$

or

$$\liminf_{h \to 0^-} \frac{\langle \dot{V}_{Mx+hNy}, Ny + hw_2 \rangle}{h} > 0 \tag{4.15}$$

holds, for all $w_1 \in F(0,x,y) - A(0)y - B(0)x$ or, for all $w_2 \in F(T, Mx, Ny) - A(T)Ny - B(T)Mx$, respectively. Then K is a bound set for problem (1.1).

Proof. Applying Proposition 4.4, we only need to show that if $x\colon [0,T] \to \overline{K}$ is a solution of problem (1.1), then $x(0) \in K$ and $x(T) \in K$. As in the proof of Proposition 4.4, we argue by a contradiction. Since $x(0) \in \partial K$ if and only if $x(T) \in \partial K$ (according to condition (4.12) and the properties of M), we can take, without any loss of generality, a solution of (1.1) satisfying $x(0) \in \partial K$. Following the same reasoning as in the proof of Proposition 4.4, for $t_0 = 0$ we obtain

$$\langle \dot{V}_{x(0)}, \dot{x}(0) \rangle \leq 0,$$

because $V(x(0)) = 0$ and $V(x(t)) \leq 0$ for all $t \in [0,T]$.

Moreover, since $V(x(T)) = 0$, it holds that

$$0 \le \langle \dot V_{x(T)}, \dot x(T)\rangle = \langle \dot V_{Mx(0)}, N\dot x(0)\rangle,$$

by virtue of the boundary conditions in (1.1). Therefore, $v := \dot x(0)$ satisfies condition (4.13).

Using the same procedure as in the proof of Proposition 4.4, for $t_0 = 0$, $h_k \to 0^+$ and for $t_0 = T$, $\overline{h}_k \to 0^-$, respectively, we obtain the existence of a sequence of positive numbers $\{h_k^*\}_{k=1}^\infty$, $h_k^* \in (0, h_k)$, of a sequence of negative numbers $\{\overline{h}_k^*\}_{k=1}^\infty$, $\overline{h}_k^* \in (\overline{h}_k, 0)$ and of points $w_0 \in P(0, x(0), \dot x(0))$, $w_T \in P(T, x(T), \dot x(T))$ (P is defined by formula (4.7)) such that

$$\frac{\dot x(h_k^*) - \dot x(0)}{h_k^*} \to w_0, \quad \text{as } k \to \infty,$$

$$\frac{\dot x(T + \overline{h}_k^*) - \dot x(T)}{\overline{h}_k^*} \to w_T, \quad \text{as } k \to \infty.$$

By the same arguments as in the previous proof, we get

$$\liminf_{h\to 0^+} \frac{\langle \dot V_{x(0)+h\dot x(0)}, \dot x(0) + hw_0\rangle}{h} \le 0, \tag{4.16}$$

$$\liminf_{h\to 0^-} \frac{\langle \dot V_{x(T)+h\dot x(T)}, \dot x(T) + hw_T\rangle}{h} \le 0. \tag{4.17}$$

Moreover, using the boundary conditions in (1.1), the inequality (4.17) can be written in the form

$$\liminf_{h\to 0^-} \frac{\langle \dot V_{Mx(0)+hN\dot x(0)}, N\dot x(0) + hw_T\rangle}{h} \le 0. \tag{4.18}$$

Inequalities (4.16) and (4.18) are in a contradiction with conditions (4.14) and (4.15) which completes the proof. $\square$

Remark 4.7. Observe that Proposition 4.6 holds again, without any loss of generality, for the general upper-Carathéodory differential inclusion in (1.1), i.e. for $A = B \equiv 0$.

Definition 4.8. A C^1-function $V\colon E \to R$ with a locally Lipschitz–Fréchet derivative $\dot V$ which satisfies conditions (H1), (H2) and all assumptions in Propositions 4.2 or 4.6 is called a *bounding function* for problem (1.1).

5. Existence and localization results

Combining the continuation principle with the bound sets technique, we are ready to state the main result of the paper concerning the solvability and localization of a solution of the multivalued Floquet problem (1.1).

For this purpose, let us consider again the single-valued Floquet b.v.p. (2.9) which is equivalent to the first-order Floquet b.v.p. (2.10), provided ξ, $h(\,\cdot\,)$,

$C(\,\cdot\,)$ and $\widetilde{D}$ are defined by relations (2.11)–(2.14). Moreover, let $U(t,s)$ be the evolution operator associated with (2.15).

Theorem 5.1. *Consider the Floquet b.v.p.* (1.1). *Assume that conditions* (1_{i})–(1_{iii}) *are satisfied and that an open, convex set* $K \subset E$ *containing* 0 *exists such that* $M\partial K = \partial K$. *Furthermore, let the following conditions* (2_{i})–(2_{iv}) *be satisfied:*

- (2_{i}) $\widetilde{D} - U(T,0)$ *is invertible.*
- (2_{ii}) $\gamma(F(t,\Omega_1\times\Omega_2)) \le g(t)(\gamma(\Omega_1)+\gamma(\Omega_2))$, *for almost all* $t\in[0,T]$ *and each bounded* $\Omega_1,\Omega_2 \subset E$, *where* $g \in L^1([0,T],[0,\infty))$ *and* γ *is the Hausdorff measure of noncompactness in* E.
- (2_{iii}) *For every non-empty, bounded set* $\Omega\subset E\times E$, *there exists* $\nu_\Omega \in L^1([0,T],[0,\infty))$ *such that*
$$||F(t,x,y)|| \le \nu_\Omega(t),$$
for almost all $t\in[0,T]$ *and all* $(x,y)\in\Omega$.
- (2_{iv}) *The inequality*
$$4\mathrm{e}^{\int_0^T(1+||A(t)||+||B(t)||)\,dt}\big(4k\mathrm{e}^{\int_0^T(1+||A(t)||+||B(t)||)\,dt}+1\big)||g||_{L^1([0,T],[0,\infty))} < 1$$
holds, where k *is defined in* (2.17).

Finally, let there exist a function $V\in C^1(E,\mathbb{R})$ *with a locally Lipschitz Fréchet derivative* $\dot V$ *satisfying* (H1) *and* (H2), *jointly with condition* (4.2), *for all* $x\in\partial K$, $z\in E$ *and condition* (4.1), *for a suitable* $\varepsilon>0$, *all* $x\in\overline{K}\cap B(\partial K,\varepsilon)$, $t\in(0,T)$, $y\in E$, $\lambda\in(0,1)$ *and* $w\in\lambda F(t,x,y)-A(t)y-B(t)x$. *Then the Floquet b.v.p.* (1.1) *admits a solution whose values are located in* $\overline{K}$.

Proof. Let us define the closed set $S=S_1$ by
$$S := \{x\in AC^1([0,T],E) : x(T)=Mx(0),\ \dot x(T)=N\dot x(0)\}$$
and let the set Q of candidate solutions be defined as $Q:=C^1([0,T],\overline{K})$. Because of the convexity of K, the set Q is closed and convex.

For all $q\in Q$ and $\lambda\in[0,1]$, consider still the associated fully linearized problem
$$P(q,\lambda)\qquad \begin{cases} \ddot x(t)+A(t)\dot x(t)+B(t)x(t)\in\lambda F(t,q(t),\dot q(t)) & \text{for a.a. } t\in[0,T],\\ x(T)=Mx(0),\ \dot x(T)=N\dot x(0), \end{cases}$$
and denote by $\mathfrak{T}$ a solution mapping which assigns to each $(q,\lambda)\in Q\times[0,1]$ the set of solutions of $P(q,\lambda)$. We will show that the family of the above b.v.p.s $P(q,\lambda)$ satisfies all assumptions of Proposition 3.1.

In this case, $\varphi(t,x,\dot x) = F(t,x,\dot x) - A(t)\dot x - B(t)x$ which, together with the definition of $P(q,\lambda)$, ensures the validity of (3.2).

(i) In order to verify condition (a) in Proposition 3.1, we need to show that, for each $(q,\lambda) \in Q \times [0,1]$, the problem $P(q,\lambda)$ is solvable with a convex set of solutions. So, let $(q,\lambda) \in Q \times [0,1]$ be arbitrary and let $f_q(\,\cdot\,)$ be a strongly measurable selection of $F(\,\cdot\,, q(\,\cdot\,), \dot{q}(\,\cdot\,))$. Then, according to $(2_{\rm i})$, Lemma 2.7 and the equivalence, stated in Section 2, between the b.v.p. (2.7) and (2.9), the single-valued Floquet problem

$$\begin{cases} \ddot{x}(t) + A(t)\dot{x}(t) + B(t)x(t) = \lambda f_q(t), & \text{for a.a. } t \in [0,T], \\ x(T) = Mx(0), \ \dot{x}(T) = N\dot{x}(0) \end{cases}$$

admits a unique solution which is one of solutions of $P(q,\lambda)$. Thus, the set of solutions of $P(q,\lambda)$ is nonempty. The convexity of the solution sets follows immediately from the property $(1_{\rm ii})$ and the fact that problems $P(q,\lambda)$ are fully linearized.

(ii) Assuming that $H\colon [0,T] \times E \times E \times E \times E \times [0,1] \multimap E$ is defined by $H(t,x,y,q,r,\lambda) := \lambda F(t,q,r) - A(t)x - B(t)y$, condition (b) in Proposition 3.1 is ensured directly by assumption $(2_{\rm iii})$.

(iii) Since the verification of condition (c) in Proposition 3.1 is technically the most complicated, it will be splitted into two parts: $({\rm iii}_1)$ the quasi-compactness of the solution operator $\mathfrak{T}$, $({\rm iii}_2)$ the condensity of $\mathfrak{T}$ w.r.t. the monotone and non-singular (cf. Lemma 2.3) m.n.c. μ defined by (2.4).

Ad $({\rm iii}_1)$. Let us firstly prove that the solution mapping $\mathfrak{T}$ is quasi-compact. Since $C^1([0,T],E)$ is a metric space, it is sufficient to prove the sequential quasi-compactness of $\mathfrak{T}$. Hence, let us consider the sequences $\{q_n\}, \{\lambda_n\}, q_n \in Q$, $\lambda_n \in [0,1]$, for all $n \in \mathbb{N}$, such that $q_n \to q$ in $C^1([0,T],E)$ and $\lambda_n \to \lambda$. Moreover, let $x_n \in \mathfrak{T}(q_n,\lambda_n)$, for all $n \in \mathbb{N}$. Then there exists, for all $n \in \mathbb{N}$, $f_n(\,\cdot\,) \in F(\,\cdot\,, q_n(\,\cdot\,), \dot{q}_n(\,\cdot\,))$ such that

$$\ddot{x}_n(t) + A(t)\dot{x}_n(t) + B(t)x_n(t) = \lambda_n f_n(t), \quad \text{for a.a. } t \in [0,T], \tag{5.1}$$

and that $x_n(T) = Mx_n(0)$, $\dot{x}_n(T) = N\dot{x}_n(0)$.

Since $q_n \to q$ and $\dot{q}_n \to \dot{q}$, there exists a bounded $\Omega \subset E \times E$ such that $(q_n(t), \dot{q}_n(t)) \in \Omega$, for all $t \in [0,T]$ and $n \in \mathbb{N}$. Therefore, there exists, according to condition $(2_{\rm iii})$, $\nu_\Omega \in L^1([0,T],[0,\infty))$ such that $||f_n(t)|| \le \nu_\Omega(t)$, for every $n \in \mathbb{N}$ and almost all $t \in [0,T]$. According to the arguments below Remark 2.8,

$$||x_n(t)|| \le J \quad \text{and} \quad ||\dot{x}_n(t)|| \le J, \quad \text{for a.a. } t \in [0,T],$$

where

$$J := Z(Zk+1) \int_0^T \nu_\Omega(s)\, ds$$

and k, Z are defined by relations (2.17) and (2.21). Consequently, for almost all $t \in [0,T]$, we have

$$\begin{aligned}|\ddot{x}_n(t)| \le ||A(t)||\,||\dot{x}_n(t)|| + ||B(t)||\,||x_n(t)|| \\ + ||f_n(t)|| \le (||A(t)|| + ||B(t)||) \cdot J + \nu_\Omega(t).\end{aligned}$$

Thus, the sequences $\{x_n\}$ and $\{\dot{x}_n\}$ are bounded and $\{\ddot{x}_n\}$ is uniformly integrable.

The sequences $\{U_{ij}(t,s)f_n(s)\}$, $i,j \in \{1,2\}$, with $t \in (0,T]$, are uniformly integrable on $[0,t]$, because, according to (2.16),

$$||U_{ij}(t,s)f_n(s)|| \le Z\nu_\Omega(s), \tag{5.2}$$

for almost all $s \in [0,t]$ and all $n \in \mathbb{N}$.

Since the sequences $\{q_n\}$, $\{\dot{q}_n\}$ are converging, we obtain, in view of (2_{ii}),

$$\gamma(\{f_n(t)\}) \le g(t)(\gamma(\{q_n(t)\}) + \gamma(\{\dot{q}_n(t)\})) = 0, \quad \text{for a.a. } t \in [0,T],$$

which implies that $\{f_n(t)\}$ is relatively compact. For given $t \in (0,T]$, the sequences $\{U_{ij}(t,s)f_n(s)\}$, $i,j \in \{1,2\}$, are relatively compact as well, for a.a. $s \in [0,t]$, because, according to (2.1),

$$\gamma(\{U_{ij}(t,s)f_n(s)\}) \le ||U_{ij}(t,s)||\gamma(\{f_n(s)\}) = 0, \tag{5.3}$$

for all $i,j \in \{1,2\}$.

By means of (2.3) and (2.18),

$$\begin{aligned}\gamma(\{x_n(t)\}) \le{}& \gamma\biggl(\bigcup_{\lambda\in[0,1]} \lambda\biggl\{\int_0^t U_{12}(t,\tau)f_n(\tau)\,d\tau \\ &+ A_1(t)\int_0^T U_{12}(T,\tau)f_n(\tau)\,d\tau + A_2(t)\int_0^T U_{22}(T,\tau)f_n(\tau)\,d\tau\biggr\}\biggr) \\ \le{}& \gamma\biggl(\int_0^t U_{12}(t,\tau)f_n(\tau)\,d\tau \\ &+ A_1(t)\int_0^T U_{12}(T,\tau)f_n(\tau)\,d\tau + A_2(t)\int_0^T U_{22}(T,\tau)f_n(\tau)\,d\tau\biggr).\end{aligned}$$

By virtue of (2.1), (2.2), (5.2), (5.3) and the sub-additivity of γ, we finally arrive at

$$\begin{aligned}\gamma(\{x_n(t)\}) \le{}& \gamma\biggl(\int_0^t U_{12}(t,\tau)f_n(\tau)\,d\tau\biggr) + ||A_1(t)||\gamma\biggl(\int_0^T U_{12}(T,\tau)f_n(\tau)\,d\tau\biggr) \\ &+ ||A_2(t)||\gamma\biggl(\int_0^T U_{22}(T,\tau)f_n(\tau)\,d\tau\biggr) = 0.\end{aligned}$$

By similar reasonings, when using (2.19) instead of (2.18), we also get

$$\gamma(\{\dot{x}_n(t)\}) = 0$$

by which $\{x_n(t)\}$, $\{\dot{x}_n(t)\}$ are relatively compact, for almost all $t \in [0,T]$. Moreover, since x_n satisfies for all $n \in \mathbb{N}$ equation (5.1), $\{\ddot{x}_n(t)\}$ is relatively compact, for almost all $t \in [0,T]$. Thus, according to Lemma 2.5, there exist a subsequence of $\{\dot{x}_n\}$, for the sake of simplicity denoted in the same way as the sequence, and $x \in C^1([0,T],E)$ such that $\{\dot{x}_n\}$ converges to $\dot{x}$ in $C([0,T],E)$ and $\{\ddot{x}_n\}$ converges weakly to $\ddot{x}$ in $L^1([0,T],E)$. By similar arguments as in the proof of Proposition 3.1, we can obtain that $\ddot{x}(t)+A(t)\dot{x}(t)+B(t)x(t) \in \lambda F(t,q(t),\dot{q}(t))$, for almost all $t \in [0,T]$. Since S is closed and $x_n \in S$, for all n, we deduce that x satisfies the boundary conditions in (1.1). This already implies the quasi-compactness of $\mathfrak{T}$.

Ad (iii$_2$). In order to show that $\mathfrak{T}$ is μ-condensing, where μ is defined by (2.4), we will prove that any bounded subset $\Theta \subset Q$ such that $\mu(\mathfrak{T}(\Theta\times[0,1])) \geq \mu(\Theta)$ is relatively compact. Let $\{x_n\}_n \subset \mathfrak{T}(\Theta\times[0,1])$ be a sequence such that

$$\begin{aligned}\mu(&\mathfrak{T}(\Theta\times[0,1])) \\ &= \Big(\sup_{t\in[0,T]}[\gamma(\{x_n(t)\}_n)+\gamma(\{\dot{x}_n(t)\}_n)], \operatorname{mod}_C(\{x_n\}_n)+\operatorname{mod}_C(\{\dot{x}_n\}_n)\Big).\end{aligned}$$

According to (2.18) and (2.19), we can find $\{q_n\}_n \subset \Theta$, $\{f_n\}_n$ satisfying $f_n(t) \in F(t,q_n(t),\dot{q}_n(t))$, for almost al $t \in [0,T]$, and $\{\lambda_n\}_n \subset [0,1]$ such that, for all $t \in [0,T]$,

$$\begin{aligned}x_n(t) = \lambda_n\bigg(&A_1(t)\int_0^T U_{12}(T,\tau)f_n(\tau)\,d\tau \\ &+ A_2(t)\int_0^T U_{22}(T,\tau)f_n(\tau)\,d\tau\bigg) + \lambda_n\int_0^t U_{12}(t,\tau)f_n(\tau)\,d\tau\end{aligned} \tag{5.4}$$

and

$$\begin{aligned}\dot{x}_n(t) = \lambda_n\bigg(&A_3(t)\int_0^T U_{12}(T,\tau)f_n(\tau)\,d\tau \\ &+ A_4(t)\int_0^T U_{22}(T,\tau)f_n(\tau)\,d\tau\bigg) + \lambda_n\int_0^t U_{22}(t,\tau)f_n(\tau)\,d\tau.\end{aligned} \tag{5.5}$$

In view of (2$_{\rm ii}$), we have, for all $t \in [0,T]$,

$$\begin{aligned}\gamma(\{f_n(t),n\in\mathbb{N}\}) &\leq g(t)(\gamma(\{q_n(t),n\in\mathbb{N}\})+\gamma(\{\dot{q}_n(t),n\in\mathbb{N}\})) \\ &\leq g(t)\sup_{t\in[0,T]}(\gamma(\{q_n(t),n\in\mathbb{N}\})+\gamma(\{\dot{q}_n(t),n\in\mathbb{N}\})).\end{aligned}$$

Since $\{q_n\}_n \subset \Theta$ and Θ is bounded in $C^1([0,T],E)$, by means of (2$_{\rm iii}$), we get the existence of $\nu_\Theta \in L^1([0,T],[0,\infty))$ such that $|f_n(t)| \leq \nu_\Theta(t)$, for almost all $t \in [0,T]$ and all $n \in \mathbb{N}$. According to (2.16), this implies $|U_{i,j}(t)f_n(t)| \leq Z\nu_\Theta(t)$, for each $i,j=1,2$, almost all $t \in [0,T]$ and all $n \in \mathbb{N}$. Moreover, by virtue of (2.1), for each $(t,\tau)\in\Delta$, we have (here, the notation $\{\,\cdot\,, n\in\mathbb{N}\}$ means the same as $\{\,\cdot\,\}_n$ before)

$$\begin{aligned}\gamma(\{U_{i,j}(t,\tau)f_n(\tau),n\in\mathbb{N}\})&\\ &\le ||U_{ij}(t,\tau)||\gamma(\{f_n(\tau),n\in\mathbb{N}\})\le Z\gamma(\{f_n(\tau),n\in\mathbb{N}\})\\ &\le Zg(t)\sup_{t\in[0,T]}(\gamma(\{q_n(t),n\in\mathbb{N}\})+\gamma(\{\dot q_n(t),n\in\mathbb{N}\})).\end{aligned}$$

Applying the property (2.2), for each $t\in[0,T]$, we so obtain

$$\begin{aligned}\gamma\bigg(\bigg\{\int_0^t U_{1,2}(t,\tau)f_n(\tau)\,d\tau,n\in\mathbb{N}\bigg\}\bigg)&\\ &\le 2Z||g||_{L^1}\sup_{t\in[0,T]}(\gamma(\{q_n(t),n\in\mathbb{N}\})+\gamma(\{\dot q_n(t),n\in\mathbb{N}\})).\end{aligned}$$

By a similar reasoning, we arrive, for $i=1,2$, at

$$\begin{aligned}\gamma\bigg(\bigg\{\int_0^T U_{i,2}(T,\tau)f_n(\tau)\,d\tau,n\in\mathbb{N}\bigg\}\bigg)&\\ &\le 2Z||g||_{L^1}\sup_{t\in[0,T]}(\gamma(\{q_n(t),n\in\mathbb{N}\})+\gamma(\{\dot q_n(t),n\in\mathbb{N}\})).\end{aligned}$$

Therefore, according to (2.20), (5.4), properties (2.1), (2.3) and the subadditivity of γ, for all $t\in[0,T]$, we have that

$$\begin{aligned}\gamma(\{x_n(t),n\in\mathbb{N}\})&\le 2Z\|A_1(t)\|\|g\|_{L^1}\mathcal{S}+2Z\|A_2(t)\|\|g\|_{L^1}\mathcal{S}+2Z\|g\|_{L^1}\mathcal{S}\\ &=2Z\|g\|_{L^1}(4Zk+1)\mathcal{S},\end{aligned}$$

where $\mathcal{S}:=\sup_{t\in[0,T]}(\gamma(\{q_n(t),n\in\mathbb{N}\})+\gamma(\{\dot q_n(t),n\in\mathbb{N}\}))$.

The same estimate can be obtained, at each $t\in[0,T]$, for $\gamma(\{\dot x_n(t),n\in\mathbb{N}\})$, when starting from condition (5.5). Subsequently,

$$\gamma(\{x_n(t),n\in\mathbb{N}\})+\gamma(\{\dot x_n(t),n\in\mathbb{N}\})\le 4Z(4Zk+1)||g||_{L^1}\mathcal{S},$$

yielding

$$\sup_{t\in[0,T]}(\gamma(\{x_n(t),n\in\mathbb{N}\})+\gamma(\{\dot x_n(t),n\in\mathbb{N}\}))\le 4Z(4Zk+1)||g||_{L^1}\mathcal{S}. \tag{5.6}$$

Since $\mu(\mathfrak{T}(\Theta\times[0,1]))\ge\mu(\Theta)$ and $\{q_n\}_n\subset\Theta$, we so get

$$\begin{aligned}\sup_{t\in[0,T]}(\gamma(\{q_n(t),n\in\mathbb{N}\})+\gamma(\{\dot q_n(t),n\in\mathbb{N}\}))&\\ &\le \sup_{t\in[0,T]}(\gamma(\{x_n(t),n\in\mathbb{N}\})+\gamma(\{\dot x_n(t),n\in\mathbb{N}\}))\end{aligned}$$

and, in view of (5.6) and $(2_{\rm iv})$, we have that

$$\sup_{t\in[0,T]}(\gamma(\{q_n(t),n\in\mathbb{N}\})+\gamma(\{\dot q_n(t),n\in\mathbb{N}\}))=0.$$

Inequality (5.6) implies that

$$\sup_{t\in[0,T]} (\gamma(\{x_n(t), n\in\mathbb{N}\}) + \gamma(\{\dot{x}_n(t), n\in\mathbb{N}\})) = 0. \tag{5.7}$$

Now, we show that both the sequences $\{x_n\}$ and $\{\dot{x}_n\}$ are equi-continuous. Let $\widetilde{\Theta}\subset E$ be such that $q(t)\in\widetilde{\Theta}$ and $\dot{q}(t)\in\widetilde{\Theta}$ for $q\in\Theta$ and $t\in[0,T]$. Hence, reasoning as in the formulas after condition (5.1), we can show that, for all $n\in\mathbb{N}$,

$$|x_n(t)|\le Z(Zk+1)\int_0^T \nu_{\widetilde{\Theta}}(s)\,ds, \qquad |\dot{x}_n(t)|\le Z(Zk+1)\int_0^T \nu_{\widetilde{\Theta}}(s)\,ds,$$

where Z is defined by (2.21) and $\nu_{\widetilde{\Theta}}\in L^1([0,T],[0,\infty))$ comes from ($2_{\rm iii}$). By the arguments as in the formulas below (5.1), we get that $\{\ddot{x}_n\}_n$ is uniformly integrable. It implies that $\{\dot{x}_n\}$ is equi-continuous. Since $\{\dot{x}_n\}_n$ is bounded, $\{x_n\}$ is also equi-continuous. Therefore,

$$\mathrm{mod}_C(\{x_n\}) = \mathrm{mod}_C(\{\dot{x}_n\}) = 0.$$

In view of (5.7), we have obtained that

$$\mu(\mathfrak{T}(\Theta\times[0,1])) = (0,0).$$

Hence, also $\mu(\Theta)=(0,0)$ and since μ is regular, we have that Θ is relatively compact. Therefore, condition (c) in Proposition 3.1 holds.

(iv) For all $q\in Q$, the problem $P(q,0)$ has the trivial solution. According to Lemma 2.7 and the arguments below it, this is the only solution of $P(q,0)$, for all $q\in Q$. Since $0\in K$, condition (*iv*) in Proposition 3.1 is satisfied.

(v) Let $q_*\in Q$ be a solution of the b.v.p. $P(q_*,\lambda)$, for some $\lambda\in(0,1)$, i.e. a fixed point of the solution mapping $\mathfrak{T}$. In view of conditions (4.1), (4.2) (see Proposition 4.2), K is, for all $\lambda\in(0,1)$, a bound set for the problem

$$\begin{cases} \ddot{q}_*(t)+A(t)\dot{q}_*(t)+B(t)q_*(t)\in\lambda F(t,q_*(t),\dot{q}_*(t)), & \text{for a.a. } t\in[0,T],\\ x(T)=Mx(0),\ \dot{x}(T)=N\dot{x}(0). \end{cases}$$

This implies that $q_*\notin\partial Q$ which ensures condition (e) in Proposition 3.1. □

If the mapping $F(t,x,y)-A(t)y-B(t)x$ is globally u.s.c. in (t,x,y), then we are able to improve Theorem 5.1, when just replacing the arguments in Proposition 3.1 by those in Propositions 4.4 and 4.6 (cf. condition (e) in Proposition 3.1), in the following way.

Corollary 5.2. *Let us consider the Floquet b.v.p.* (1.1)*, where* $F\colon[0,T]\times E\times E\multimap E$ *is an upper semicontinuous mapping with nonempty, compact, convex values and* A *and* B *are continuous. Moreover, let condition* ($1_{\rm iii}$) *hold and let there exist a nonempty, open, convex set* $K\subset E$ *containing* 0 *such that* $M\partial K=\partial K$*, where* M *is invertible.*

Furthermore, let there exist a function $V \in C^1(E,\mathbb{R})$ *with a locally Lipschitz Fréchet derivative* $\dot{V}$ *satisfying* (H1) *and* (H2). *Moreover, let, for all* $x \in \partial K$, $t \in (0,T)$, $\lambda \in (0,1)$ *and* $y \in E$ *satisfying* (4.4), *condition* (4.5) *hold, for all* $w \in \lambda F(t,x,y) - A(t)y - B(t)x$.

At last, suppose that, for all $x \in \partial K$, $\lambda \in (0,1)$ *and* $y \in E$ *satisfying (4.13) at least one of conditions* (4.14), (4.15) *holds, for all* $w_1 \in \lambda F(0,x,y) - A(0)y - B(0)x$ *or for all* $w_2 \in \lambda F(T,Mx,Ny) - A(T)Ny - B(T)Mx$, *respectively.*

If conditions (2_{i})–(2_{iv}) *from Theorem 5.1 are satisfied, then the Floquet b.v.p.* (1.1) *admits a solution whose values are located in* $\overline{K}$.

Remark 5.3. Observe that the rather technical inequality in condition (2_{iv}) can be trivially satisfied in finite-dimensional spaces or for compact maps F.

6. Illustrative examples

It is known (see e.g. [19, Example 1.2.41(b), Remark 3.12.13]) that if E is a Banach space and $V(x) = ||x||^2/2 - R$, then $V\colon E \to \mathbb{R}$ is a proper convex function and $\partial V = \{x^* \in E^* \mid \langle x^*, x\rangle = ||x||_E^2 = ||x^*||_{E^*}^2\}$, for all $x \in E$, where ∂V is the subdifferential of V. If, in particular, E is a Hilbert space, then $\partial V(x) = x$.

Moreover, if V is Gâteaux differentiable at $x \in E$, then $\partial V(x) = \{V'(x)\}$ (see e.g. [19, Theorem 1.2.37]). The same is all the better true, provided V is Fréchet differentiable which is, for all $x \in E \setminus \{0\}$, equivalent with E to be *locally uniformly smooth*, i.e.

$$\lim_{\tau\to 0^+} \frac{1}{\tau}\sup\{||x+\tau y|| + ||x-\tau y|| - 2||x|| \mid ||y|| = 1\} = 0$$

(see e.g. [10], [11]).

If E is *uniformly smooth*, i.e. if there exists the limit

$$\lim_{\tau\to 0} \frac{1}{\tau}(||x+\tau y|| - ||x||),$$

uniformly for $x, y \in S_E$, where $S_E := \{x \in E \mid ||x|| = 1\}$ is the unit sphere which is, according to the well-known Smuljan theorem, equivalent with E^* to be *uniformly convex*, i.e.

$$\inf\left\{1 - \frac{1}{2}||x^*+y^*||_{E^*} \mid x^*, y^* \in S_{E^*},\ ||x^*-y^*||_{E^*} = \varepsilon\right\} > 0,$$

for every $\varepsilon > 0$ (see e.g. [10], [11]), then E is obviously locally uniformly smooth as well. Moreover, E is also reflexive (see again e.g. [10], [11]).

Thus, if E is uniformly smooth, then $V(x) = ||x||^2/2 - R$ must be Fréchet differentiable, for all $x \in E$, and $\dot{V}_x = V^{'}(x) = \{x^* \in X^* \mid \langle x^*, x\rangle = ||x||_E^2 = ||x^*||_{E^*}^2\}$, for $x \in E$. Observe that, despite the non-differentiability of $x \to ||x||$

at $x = 0$, the function V is entirely Fréchet differentiable in E (i.e. also at $x = 0$), because the square acts in its regularization. In fact, we have that

$$\begin{aligned}\lim_{h\to 0}\frac{\|V(h)-V(0)-\langle 0_E, h\rangle\|}{\|h\|} &= \lim_{h\to 0}\frac{|\|h\|^2/2 - R + R - 0|}{\|h\|}\\ &= \lim_{h\to 0}\frac{\|h\|^2/2}{\|h\|} = \lim_{h\to 0}\frac{1}{2}\|h\| = 0,\end{aligned}$$

where 0_E denotes the identically zero operator in E.

One can easily check that $\dot V_x$ is convex, i.e.

$$\dot V_{\lambda x_1+(1-\lambda)x_2} \le \lambda \dot V_{x_1} + (1-\lambda)\dot V_{x_2},$$

for all $x_1, x_2 \in E$ and $\lambda \in [0,1]$. We note that $\dot V_x$ is also locally Lipschitz continuous (see e.g. [19, Corollary 1.2.8]).

Example 6.1. Let E be a uniformly smooth Banach space and consider problem (1.1). Assume that conditions (1_{i})–(1_{iii}) and (2_{i})–(2_{iv}) are satisfied. Putting $K := \{x \in E \mid ||x|| < \sqrt{2R}\}$, let $M\partial K = \partial K$; for instance, let $M = N = \mathrm{id}$, for a periodic problem, or $M = N = -\mathrm{id}$, for an anti-periodic problem.

Taking $V(x) = ||x||^2/2 - R$, where $R > 0$ is a given constant in the definition of K, in view of the above considerations, we have that the locally Lipschitz continuous derivative $\dot V_x$ satisfies

$$\dot V_x = \{x^* \in E^* \mid \langle x^*, x\rangle = ||x||_E^2 = ||x^*||_{E^*}^2\}, \quad \text{for } x \in E.$$

One can readily check that conditions (H1), (H2) trivially hold. Furthermore, condition (4.1) takes the form

$$\limsup_{h\to 0^-}\frac{\langle (x+hy)^* - x^*, y\rangle}{h} + \langle (x+hy)^*, w\rangle > 0, \tag{6.1}$$

for a suitable $\varepsilon > 0$, all $x \in \overline{K} \cap B(\partial K, \varepsilon)$, $t \in (0,T)$, $y \in E$, $\lambda \in (0,1)$ and $w \in \lambda F(t,x,y) - A(t)y - B(t)x$.

If $M = N = \mathrm{id}$, then

$$\langle \dot V_{Mx}, Nz\rangle \cdot \langle \dot V_x, z\rangle = \langle \dot V_x, z\rangle \cdot \langle \dot V_x, z\rangle = \langle x^*, z\rangle^2 \ge 0$$

and, when $\langle x^*, z\rangle = 0$, then $\langle \dot V_{Mx}, Nz\rangle = \langle \dot V_x, z\rangle = 0$, by which condition (4.2) is satisfied.

It is easy to show that $\dot V_{-x} = -\dot V_x$, for all $x \in E$. Thus, if $M = N = -\mathrm{id}$, then

$$\langle \dot V_{Mx}, Nz\rangle \cdot \langle \dot V_x, z\rangle = -\langle \dot V_{-x}, z\rangle \cdot \langle \dot V_x, z\rangle = \langle \dot V_x, z\rangle \cdot \langle \dot V_x, z\rangle = \langle x^*, z\rangle^2 \ge 0;$$

so, as in the periodic case, when $\langle x^*, z\rangle = 0$, then $\langle \dot V_{-x}, -z\rangle = \langle \dot V_x, z\rangle = 0$. Hence, condition (4.2) is satisfied in the anti-periodic case as well.

In particular, if E is a Hilbert space, then condition (6.1) takes form,

$$\langle x, w\rangle + ||y||^2 > 0,$$

for a suitable $\varepsilon > 0$, all $x \in \overline{K} \cap B(\partial K, \varepsilon)$, $t \in (0, T)$, $y \in E$, $\lambda \in (0, 1)$ and $w \in \lambda F(t, x, y) - A(t)y - B(t)x$. Applying Theorem 5.1, problem (1.1) admits a solution whose values are located in $\overline{K}$.

For Marchaud inclusions, the application of Corollary 5.2 can be illustrated as follows.

Example 6.2. Let E be a uniformly smooth Banach space and consider problem (1.1), where this time F is an upper semicontinuous mapping and A, B are continuous. Assume that conditions (1_{iii}) and (2_{i})–(2_{iv}) are satisfied. Putting $K := \{x \in E \mid ||x|| < \sqrt{2R}\}$, let again $M\partial K = \partial K$.

For $V(x) = ||x||^2/2 - R$, conditions (H1), (H2) trivially hold, and with no change

$$\dot{V}_x = \{x^* \in X^* \mid \langle x^*, x\rangle = ||x||^2_E = ||x^*||^2_{E^*}\}, \quad \text{for } x \in E.$$

Conditions (4.4) and (4.5) take the form: for all $x \in \partial K$ and $y \in E$ satisfying

$$\langle x^*, y\rangle = 0,$$

the following inequality holds

$$\liminf_{h\to 0} \frac{1}{h}\langle (x + hy)^*, y\rangle + \langle (x + hy)^*, w\rangle > 0,$$

for all $t \in (0, T)$, $\lambda \in (0, 1)$ and $w \in \lambda F(t, x, y) - A(t)y - B(t)x$.

Furthermore, since for $M = N = \mathrm{id}$: $\langle \dot{V}_{Mx}, Ny\rangle = \langle \dot{V}_x, y\rangle = \langle x^*, y\rangle$, condition (4.13) is equivalent to

$$\langle x^*, y\rangle = 0, \quad \text{for all } x \in \partial K \text{ and } y \in E. \tag{6.2}$$

Since for $M = N = -\mathrm{id}$: $\langle \dot{V}_{Mx}, Ny\rangle = -\langle \dot{V}_{-x}, y\rangle = \langle \dot{V}_x, y\rangle$, condition (4.13) is also in this case equivalent to (6.2).

In view of

$$\frac{1}{h}\langle \dot{V}_{x+hy}, y + hw_1\rangle = \frac{1}{h}\langle (x + hy)^*, y\rangle + \langle (x + hy)^*, w_1\rangle,$$

condition (4.14) reads as

$$\liminf_{h\to 0^+} \frac{1}{h}\langle (x + hy)^*, y\rangle + \langle (x + hy)^*, w_1\rangle > 0,$$

for all $x \in \partial K$, $\lambda \in (0, 1)$, $y \in E$ and $w_1 \in \lambda F(0, x, y) - A(0)y - B(0)x$.

Finally, in the case when $M = N = \mathrm{id}$ or $M = N = -\mathrm{id}$ condition (4.15) takes the respective forms

$$\liminf_{h\to 0^-} \frac{1}{h}\langle (x+hy)^*, y\rangle + \langle (x+hy)^*, w_2\rangle > 0,$$
$$\liminf_{h\to 0^-} \frac{1}{h}\langle (x+hy)^*, y\rangle - \langle (x+hy)^*, w_2\rangle > 0,$$

for all $x \in \partial K$, $y \in E$, $\lambda \in (0,1)$ and $w_2 \in \lambda F(T,x,y) - A(T)y - B(T)x$ or $w_2 \in \lambda F(T,-x,-y) + A(T)y + B(T)x$.

In particular, if E is a Hilbert space and if $M = N = \mathrm{id}$, then conditions (4.4), (4.5), (4.13)–(4.15) reduce to: for all $x \in \partial K$, $y \in E$, $t \in (0,T)$ and $\lambda \in (0,1)$ satisfying

$$\langle x, y\rangle = 0, \tag{6.3}$$

the inequalities

$$\langle x, w\rangle + ||y||^2 > 0, \qquad \max\{\langle x, w_1\rangle + ||y||^2, \langle x, w_2\rangle + ||y||^2) > 0$$

hold, for all $w \in \lambda F(t,x,y) - A(t)y - B(t)x$, $w_1 \in \lambda F(0,x,y) - A(0)y - B(0)x$ and all $w_2 \in \lambda F(T,x,y) - A(T)y - B(T)x$.

On the other hand, for $M = N = -\mathrm{id}$, i.e. for anti-periodic problems in Hilbert spaces, conditions (4.4), (4.5), (4.13)–(4.15) take the form: for all $x \in \partial K$, $y \in E$, $t \in (0,T)$ and $\lambda \in (0,1)$ satisfying (6.3) the inequalities

$$\langle x, w\rangle + ||y||^2 > 0, \qquad \max\{\langle x, w_1\rangle + ||y||^2, -\langle x, w_2\rangle + ||y||^2) > 0$$

hold, for all $w \in \lambda F(t,x,y) - A(t)y - B(t)x$, $w_1 \in \lambda F(0,x,y) - A(0)y - B(0)x$ and all $w_2 \in \lambda F(T,-x,-y) + A(T)y + B(T)x$.

Applying Corollary 5.2, problem (1.1) admits a solution whose values are located in $\overline{K}$.

REMARK 6.3. Hilbert spaces are the best uniformly convex Banach spaces. Since they are self-adjoint, they are in particular reflexive and, according to the Smuljan theorem, uniformly smooth. That is also why illustrative examples in Hilbert spaces are, not only because of technically easy calculations, the most natural ones.

On the other hand, in uniformly smooth spaces which are not Hilbert, it depends on their concrete structure in order to express conditions in terms of the asterisque linear functionals, in Examples 6.1 and 6.2, explicitly.

Coming back to the stimulating example from introduction, we can now demonstrate how the main results apply to it.

Example 6.4. Consider again the problem in the Hilbert space $E := L^2(\Omega)$:

$$
(6.4)\qquad \begin{cases} \dfrac{\partial^2 u}{\partial t^2} + a\dfrac{\partial u}{\partial t} + \widetilde{B}u(t,\,\cdot\,) + \mathcal{B}||u(t,\,\cdot\,)||^{p^*-2}u = \varphi(t,u), \\ u(T,\,\cdot\,) = Mu(0,\,\cdot\,), \quad \dfrac{\partial u(T,\,\cdot\,)}{\partial t} = N\dfrac{\partial u(0,\,\cdot\,)}{\partial t}, \end{cases}
$$

where, for the sake of simplicity, we put $\widetilde{B} := b < 0$, where b is a constant,

$$
p^* := p(x) = \begin{cases} p_0 \in [3,\infty) & \text{for } ||x|| \le 1, \\ p_1 \in (1,2] & \text{for } ||x|| > 1, \end{cases}
$$

and the other symbols have the same meaning as above.

Let the constraint be also the same:

$$
u(t,\,\cdot\,) \in \overline{K} := \{e \in L^2(\Omega) \mid ||e|| \le r\}, \quad t \in [0,T].
$$

where $r > 0$ is a given constant.

Rewriting this problem into the form of (1.1), let us verify successively all the related conditions, in order to apply Theorem 5.1 and Corollary 5.2. One can readily check that $K \subset E$ is a nonempty, open, convex set containing 0 and that, for $M = N = \mathrm{id}$ or for $M = N = -\mathrm{id}$, the equality $M\partial K = \partial K$ trivially holds. Moreover, conditions (1_{i})–(1_{iii}), or their analogies in Corollary 5.2, are easily satisfied, provided $f\colon [0,T] \times E \to E$ is Carathéodory or continuous.

For $a \ge 0$, $b < 0$, the spectrum $\sigma(U(T,0))$ of the evolution operator U, associated with the homogeneous equation

$$
\ddot{x}(t) + a\dot{x}(t) + bx(t) = 0, \quad t \in [0,T],
$$

can be calculated as

$$
\sigma(U(T,0)) = \sigma(\mathrm{e}^{CT}).
$$

Moreover, it can be shown that

$$
\sigma(\mathrm{e}^{CT}) = \{\mathrm{e}^{\lambda_1 T}, \mathrm{e}^{\lambda_2 T}\},
$$

where $0 < \mathrm{e}^{\lambda_1 T} = \mathrm{e}^{(T/2)(a-\sqrt{a^2-4b})} < 1$, $\mathrm{e}^{\lambda_2 T} = \mathrm{e}^{(T/2)(a+\sqrt{a^2-4b})} > 1$.

Thus, the spectrum $\sigma(U(T,0))$ does not intersect the unit cycle which is, at least for $M = N = \mathrm{id}$ and $M = N = -\mathrm{id}$, equivalent with the invertibility of the operator $\widetilde{D} - U(T,0) = (M,N) - U(T,0)$, provided $\widetilde{D} - U(T,0)$ is still surjective (cf. [8], [19]).

Since the homogeneous equation $\ddot{x}(t) + a\dot{x}(t) + bx(t) = 0$ has constant coefficients, we can compute the linear operator e^{CT} and it is not difficult to show that it takes the following form

$$
\mathrm{e}^{CT} = \begin{pmatrix} c_1\mathrm{id}_E & c_2\mathrm{id}_E \\ c_3\mathrm{id}_E & c_4\mathrm{id}_E \end{pmatrix}, \quad c_1, c_2, c_3, c_4 \in \mathbb{R},
$$

implying that the 2×2 real matrix

$$\widehat{C} = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}$$

has the same eigenvalues as the linear operator e^{CT}. Moreover, $\pm\mathrm{id} - \mathrm{e}^{CT}$ is surjective if and only if $\pm\mathrm{id}_{\mathbb{R}^2\times\mathbb{R}^2} - \widehat{C}$ is so. Since, in our case, $\widetilde{D} - U(T,0) = \pm\mathrm{id}_{E\times E} - \mathrm{e}^{CT}$, one can check that, for $a \geq 0$, $b < 0$, we have

$$\det(\mathrm{id}_{\mathbb{R}^2\times\mathbb{R}^2} - \widehat{C}) = 1 - (\mathrm{e}^{\lambda_1 T} + \mathrm{e}^{\lambda_2 T}) + \mathrm{e}^{\lambda_1 T}\mathrm{e}^{\lambda_2 T} < 0,$$
$$\det(-\mathrm{id}_{\mathbb{R}^2\times\mathbb{R}^2} - \widehat{C}) = 1 + (\mathrm{e}^{\lambda_1 T} + \mathrm{e}^{\lambda_2 T}) + \mathrm{e}^{\lambda_1 T}\mathrm{e}^{\lambda_2 T} > 2,$$

which guarantees the surjectivity of $\widetilde{D} - U(T,0)$. Indeed, the function $\det(\lambda\mathrm{id} - \mathrm{e}^{CT}) = \lambda^2 - (\mathrm{e}^{\lambda_1 T} + \mathrm{e}^{\lambda_2 T})\lambda + \mathrm{e}^{\lambda_1 T}\mathrm{e}^{\lambda_2 T}$ is obviously strictly convex in λ with two zero points $\mathrm{e}^{\lambda_1 T}$, $\mathrm{e}^{\lambda_2 T}$ and one minimum at $(\mathrm{e}^{\lambda_1 T} + \mathrm{e}^{\lambda_2 T})/2 \in (\mathrm{e}^{\lambda_1 T}, \mathrm{e}^{\lambda_2 T})$.

Since, for $||x|| > 0$, we obtain the estimate (cf. [19, p. 263])

$$\begin{aligned} ||(\mathcal{B}||x||^{p^*-2}x)'|| &= \mathcal{B}||(p^*-2)||x||^{p^*-3}\frac{x}{||x||} + ||x||^{p^*-2}|| \\ &\leq \mathcal{B}(|p^*-2| \cdot ||x||^{p^*-3} + ||x||^{p^*-2}) \leq \mathcal{B}\max(p_0 - 1, -p_1 + 3), \end{aligned}$$

the mapping $x \to \mathcal{B}||x||^{p^*-2}x$ is Lipschitz with the constant $L := \mathcal{B}\max(p_0 - 1, -p_1 + 3) \geq \mathcal{B}$. If

$$\mathcal{B} < \frac{1}{\max(p_0 - 1, -p_1 + 3)} \quad (\leq 1),$$

then it is a contraction with the coefficient $L < 1$, and so condensing. Thus, condition (2_{ii}) reduces into $\gamma(f(t,\Omega)) \leq g(t)\gamma(\Omega)$, for almost all $t \in [0,T]$ and each bounded $\Omega \subset E$, where $g \in L^1([0,T],[0,\infty))$. Obviously, if f is compact or contractive in x, then (2_{ii}) trivially holds.

Let us have e.g. a growth estimate for f:

$$||f(t,x)|| \leq c_0(t) + c_1(t)||x||^m, \quad \text{for all } x \in E,$$

where $m \geq 0$, $c_0, c_1 \in L^1([0,T],[0,\infty))$ are suitable functions. Then, in view of the inequalities ($||x||^{p^*-2} \leq 1$)

$$||f(t,x) - \mathcal{B}||x||^{p^*-2}x|| \leq ||f(t,x)|| + \mathcal{B}||x|| \leq c_0(t) + c_1(t)||x||^m + \mathcal{B}||x||,$$

it is enough to take

$$\nu_\Omega(t) := c_0(t) + \omega^m c_1(t) + \omega\mathcal{B}, \quad \text{where } \omega := \sup_{x\in\Omega}||x||,$$

in order (2_{iii}) to be satisfied.

Condition (2_{iv}) simplifies into the inequality

$$4\mathrm{e}^{T(1+a-b)}(4k\mathrm{e}^{T(1+a-b)} + 1)||g||_{L^1([0,T],[0,\infty))} < 1, \tag{6.5}$$

where k was defined in (2.17). This inequality is satisfied, $||g||_{L^1([0,T],[0,\infty))}$ is sufficiently small and g is related only to f.

Now, defining $V(x) := ||x||^2/2 - r^2/2$, conditions (H1) and (H2) are trivially satisfied. Moreover, conditions (4.1) and (4.5), (4.14) yield the inequality $(-||x||^{p^*-2} \geq -1)$

$$\begin{aligned}\langle x, \lambda f(t,x) - \lambda\mathcal{B}||x||^{p^*-2}x - ay - bx\rangle + ||y||^2 \\ \geq -b||x||^2 + ||y||^2 - a\langle x,y\rangle + \lambda(\langle x, f(t,x)\rangle - \mathcal{B}||x||^2\rangle) > 0,\end{aligned} \tag{6.6}$$

for all $x \in \overline{K} \cap B(\partial K, \varepsilon)$, $y \in E$, $t \in (0,T)$, $\lambda \in (0,1)$, and for all $x \in \partial K$ (i.e. $||x|| = r > 0$), $y \in E$, $t \in [0,T)$, $\lambda \in (0,1)$, respectively.

If $a^2 \leq -4b$, $b < 0$, then

$$\begin{aligned} -b||x||^2 - a\langle x,y\rangle + ||y||^2 &\geq -b||x||^2 - a\langle x,y\rangle - \frac{a^2}{4b}||y||^2 \\ &\geq -b||x||^2 - a||x||\,||y|| - \frac{a^2}{4b}||y||^2 \\ &= \left(\sqrt{|b|}\;||x|| - \frac{a}{2\sqrt{|b|}}||y||\right)^2 \geq 0, \end{aligned}$$

and if $a^2 < -4b$, $||x|| > 0$, then we get $-b||x||^2 - a\langle x,y\rangle + ||y||^2 > 0$. Thus, if

$$\langle x, f(t,x)\rangle \geq \mathcal{B}||x||^2 \tag{6.7}$$

holds, where $x \in \overline{K} \cap B(\partial K, \varepsilon)$, $t \in (0,T)$ or, where $x \in \partial K$, $t \in [0,T)$, then for $a^2 < -4b$, the inequality (6.6) holds, on the respective sets. In the latter case, in view of condition (4.4), it is enough to take only $a^2 \leq -4b$, $b < 0$.

Otherwise, condition (6.7) can be obviously replaced by

$$(d - \mathcal{B})||x||^2 + \langle x, f(t,x)\rangle \geq 0, \tag{6.8}$$

provided $d > 0$ is a constant such that $a^2 < -4(b+d)$ or $a^2 \leq -4(b+d)$, respectively.

By the similar arguments, for $M = N = \mathrm{id}$, condition (4.15) can be (in view of (4.13)) satisfied, provided $a^2 \leq -4b, b < 0$ and $\langle x, f(T,x)\rangle \geq \mathcal{B}r^2$, where $x \in \partial K$, or if there exists a constant $d > 0$ such that $a^2 \leq -4(b+d)$ and $(d - \mathcal{B})r^2 + \langle x, f(T,x)\rangle \geq 0$, for $x \in \partial K$.

For $M = N = -\mathrm{id}$, condition (4.15) can be (in view of (4.13)) satisfied, provided $a^2 \leq -4b, b < 0$ and $-\langle x, f(T,-x)\rangle \geq \mathcal{B}r^2$, where $x \in \partial K$, or if there exists a constant $d > 0$ such that $a^2 \leq -4(b+d)$ and $(d - \mathcal{B})r^2 - \langle x, f(T,-x)\rangle \geq 0$, for $x \in \partial K$.

Summing up, for $M = N = \mathrm{id}$ or for $M = N = -\mathrm{id}$ together with $f(t,-x) \equiv -f(t,x)$, where $f \in C([0,T] \times E, E)$, conditions (4.5), (4.14), (4.15) are (in view of (4.4), (4.13)) satisfied, provided $a^2 \leq -4b, b < 0$ and condition (6.7) holds, for $x \in \partial K$, $t \in [0,T]$. If there exists $d > 0$ such that $a^2 \leq -4(b+d)$, then condition (6.7) can be replaced by (6.8), for $x \in \partial K$, $t \in [0,T]$. If f is Carathéodory, then it need not be odd (for $M = N = -\mathrm{id}$), but conditions (6.7) or (6.8) should hold, for $x \in \overline{K} \cap B(\partial K, \varepsilon)$, $t \in (0,T)$. Moreover, the class of Floquet boundary conditions with $M\partial K = \partial K$ can be larger than two particular cases above.

If, in particular, $a = 0$ and $b < 0$, then the only condition

$$\langle x, f(t,x)\rangle \geq (b + \mathcal{B})||x||^2$$

is sufficient (instead of (6.7) or (6.8)), on the respective sets.

Remark 6.5. Observe that, if $r \leq 1$ in the bound set $K_1 := \{e \in L^2(\Omega) \mid ||e|| < r\}$, then also the original problem with $p \in [3,\infty)$:

$$\left\{\begin{aligned} &\frac{\partial^2 u}{\partial t^2} + a\frac{\partial u}{\partial t} + \widetilde{B}u(t,\,\cdot\,) + \mathcal{B}||u(t,\,\cdot\,)||^{p-2}u = \varphi(t,u), \\ &x(T,\,\cdot\,) = Mx(0,\,\cdot\,), \quad \frac{\partial x(T,\,\cdot\,)}{\partial t} = N\frac{\partial x(0,\,\cdot\,)}{\partial t}, \end{aligned}\right. \tag{6.9}$$

admits, according to Theorem 5.1 and Corollary 5.2, the same solution $x(t) := u(t,\,\cdot\,) \in \overline{K}_1$, $t \in [0,T]$, as for (6.4), because $p^* = p_0 := p$, where $||x|| \leq 1$.

More precisely, problem (6.9), where $M = N = \mathrm{id}$ or $M = N = -\mathrm{id}$ together with $\varphi(t,-u) \equiv -\varphi(t,u)$, admits a (strong) solution $x(t) := u(t,\cdot)$ such that $x(t) \in \overline{K}_1$, $t \in [0,T]$, provided

(a) $a \geq 0$, $b < 0, 0 \leq \mathcal{B} < 1/(p-1)$, where $p \in [3,\infty)$,

(b) φ is Carathéodory (resp. continuous) and such that

$$|\varphi(t,\xi)| \leq \frac{c_0(t)}{\sqrt{|\Omega|+1}} + \frac{c_1(t)}{\sqrt{|\Omega|+1}}|\xi|^{2m}, \quad t \in [0,T],\ \xi \in \Omega,$$

where c_0, c_1 are suitable integrable coefficients

($\Rightarrow$ f is Carathéodory (resp. continuous) and such that $||f(t,x)|| \leq c_0(t) + c_1(t)||x||^m$, for all $x \in E$),

(c) $\varphi(t,\xi)$ is Lipschitz in ξ with a constant L (independent of t) such that

$$4\mathrm{e}^{T(1+a-b)}(4k\mathrm{e}^{T(1+a-b)} + 1)LT < 1 \quad \text{(cf. (6.5))} \tag{6.10}$$

($\Rightarrow$ f satisfies the γ-regularity condition, namely $\gamma(f(t,\widetilde{\Omega})) \leq L\gamma(\widetilde{\Omega})$, for almost all $t \in [0,T]$ and each bounded $\widetilde{\Omega} \subset E$, with $g(t) := L$ satisfying (6.5),

(d) condition (6.8) holds on the set $(0,T) \times \overline{K}_1 \cap B(\partial K, \varepsilon)$ (resp. on $[0,T] \times \partial K_1$), where $d \geq 0$ is a suitable constant such that $a^2 < -4(b+d)$ (resp. $a^2 \leq -4b(b+d)$).

Remark 6.6. It would be nice to express condition (d), as conditions (a)–(c), for function φ. Thus, for instance, the related equality $\sqrt{\int_\Omega x^2(\xi)\,d\xi} = r$ would, however, lead to the inequality

$$z\varphi(t,z) \geq (\mathcal{B} - d)z^2$$

required, for all $(t,z)\in[0,T]\times\mathbb{R}$. In this way, the information concerning the localization of solutions would be lost.

Remark 6.7. The most technical requirement (in nontrivial situations) is so the inequality (6.10) in condition (c). Nevertheless, the quotient

$$k := \|[\widetilde{D}-U(T,0)]^{-1}\| = \|[\pm\mathrm{id}-\mathrm{e}^{CT}]^{-1}\|_{E\times E}$$

in can be calculated as

$$k = k_0^{-1}\left\|\begin{matrix} \pm 1+\dfrac{\lambda_1\mathrm{e}^{\lambda_1 T}-\lambda_2\mathrm{e}^{\lambda_2 T}}{\lambda_2-\lambda_1} & \dfrac{\mathrm{e}^{\lambda_2 T}-\mathrm{e}^{\lambda_1 T}}{\lambda_2-\lambda_1} \\ \dfrac{\lambda_1\lambda_2(\mathrm{e}^{\lambda_1 T}-\mathrm{e}^{\lambda_2 T})}{\lambda_2-\lambda_1} & \pm 1+\dfrac{\lambda_1\mathrm{e}^{\lambda_2 T}-\lambda_2\mathrm{e}^{\lambda_1 T}}{\lambda_2-\lambda_1}\end{matrix}\right\|_{\mathbb{R}^2\times\mathbb{R}^2},$$

where

$$k_0^{-1} = [1\mp(\mathrm{e}^{\lambda_1 T}+\mathrm{e}^{\lambda_2 T})+\mathrm{e}^{\lambda_1 T+\lambda_2 T}]^{-1},$$
$$\lambda_1 = \frac{-a-\sqrt{a^2-4b}}{2},\quad \lambda_2 = \frac{-a+\sqrt{a^2-4b}}{2}.$$

For instance, for $a=0$, $b=-1$, we get $k\le(1+\mathrm{e}^{T})/(2+\mathrm{e}^{T}+\mathrm{e}^{-T})<1$; condition (6.10) can be then satisfied, when e.g. $L\le 1/T(16\mathrm{e}^{4T}+4\mathrm{e}^{2T})$.

7. Concluding remarks

Assuming, for $M=N=\mathrm{id}$, that $A(t)\equiv A(t+T)$ and $B(t)\equiv B(t+T)$ or, for $M=N=-\mathrm{id}$, that $A(t)\equiv -A(t+T)$ and $B(t)\equiv -B(t+T)$, the solutions of the homogeneous problem

$$\begin{cases} \ddot{x}(t)+A(t)\dot{x}(t)+B(t)x(t)=0, & \text{for a.a. } t\in[0,T],\\ x(T)=\pm x(0), \quad \dot{x}(T)=\pm\dot{x}(0), \end{cases}$$

where the sign plus in the b.v.p. refers to the case when A and B are T-periodic, while minus refers to the case when A and B are anti-periodic in $[0,T]$, can be obviously prolonged onto $(-\infty,\infty)$ in a T-periodic or a $2T$-periodic way, respectively.

Let the spectrum $\sigma(U(T,0))$ of $U(T,0)$ (or $\sigma(U(2T,0))$ of $U(2T,0)$) not intersect the unit circle, and so contain components lying in the interior or the exterior or in both of the unit circle.

In this context, $U(T,0)$ is called the *monodromy operator*. If $U(T,0)$ has a logarithm, that is if there is an operator S such that $U(T,0)=\mathrm{e}^S$, then its Floquet representation takes the form (cf. [8, Chapter V.1])

$$U(t,0)=R(t)\mathrm{e}^{-tT^{-1}\ln U(T,0)}\quad(\text{or } U(t,0)=R(t)\mathrm{e}^{-t(2T)^{-1}\ln U(2T,0)}),$$

where $R(t)\equiv R(t+T)$ (or $R(t)\equiv R(t+2T)$) is a suitable operator.

The condition imposed on the spectrum is equivalent (see e.g. [8, Theorem 2.1]) with the *regular exponential dichotomy* of the homogenous equation

$$\ddot{x}(t) + A(t)\dot{x}(t) + B(t)x(t) = 0 \tag{7.1}$$

which implies that the above T-periodic or $2T$-periodic prolongations either would tend to 0 or diverge to ∞, in the norm. Consider the inhomogeneous equation

$$\ddot{x}(t) + A(t)\dot{x}(t) + B(t)x(t) = f(t),$$

where $f \in L^1([0,T],E)$ is essentially bounded and such that $f(t) \equiv \pm f(t+T)$. It admits a unique entirely bounded solution

$$x(t) = \int_{-\infty}^{\infty} G(t,s)f(s)\,ds$$

whose first derivative

$$\dot{x}(t) = \int_{-\infty}^{\infty} \frac{\partial G(t,s)}{\partial t} f(s)\,ds$$

is entirely bounded as well. The symbol G means the *principal Green function* of (7.1) (see e.g. [8, Theorem IV.3.2]).

Since the spectral condition is, by the definition, also equivalent (cf. e.g. [19]) with (2_{i}), the bounded solution $x(\cdot)$ and its derivative $\dot{x}(\cdot)$ must be, according to Lemma 2.7, T-periodic ($2T$-periodic). If E is reflexive, then the T-periodicity or $2T$-periodicity of $x(\,\cdot\,)$ and $\dot{x}(\,\cdot\,)$ alternatively follows already from their boundedness on the half-line (see e.g. [16, Theorem II.114C]).

Thus, if f is essentially bounded, then for the solvability of T-periodic or $2T$-periodic problems, by means of the principal Green functions, condition (2_{i}) can be replaced by the spectral requirement on $U(T,0)$ or $U(2T,0)$, as indicated above.

Theorem 5.1 and Corollary 5.2 deal only with the localization of solutions, but not with their first derivatives. This is, however, not a disadvantage, because otherwise additional requirements occur. In such a case, it is more convenient to consider the equivalent first-order problems (see [5]).

The parameter set Q of candidate solutions was taken everywhere as $Q := C^1([0,T],\overline{K})$, but it is without any loss of generality to take it as $Q := AC^1([0,T],\overline{K})$. On the other hand, if Q is only taken as $Q := C([0,T],\overline{K})$, then the solution derivatives can behave still in a more liberal way. Nevertheless, it would be practically very delicate to employ this theoretical possibility.

Unlike in finite-dimensional spaces (cf. [4]), the localization of solution values in a nonconvex bound set K is always a difficult task because of a cumbersome application of degree arguments (cf. [2], [3, Chapter II.11]). Bound sets of the type $K_0 := \{w \in W^{2,2}(\Omega) \mid ||w|| < r \text{ and } \mathrm{Tr}(w) = 0 \text{ on } \partial\Omega\}$, where Ω is a nonempty, bounded domain in $\mathbb{R}^n$ with a Lipschitz boundary $\partial\Omega$, are convex but not open in $W^{2,2}(\Omega)$, and so not suitable for applications, too.

Moreover, in finite-dimensional spaces the diagonalization argument can be applied to guarantee sequentially entirely bounded solutions in given sets by means of results on compact intervals (see e.g. [3, Proposition III.1.37]). On the other hand, the compactness requirements in infinite-dimensional spaces (see e.g. [3, Proposition III.1.36]) allow us to employ e.g. appropriate results for Cauchy (initial value) problems, but not those obtained e.g. for periodic or anti-periodic problems. For first-order problems, this was solved in a sequential way (using the diagonalization arguments) in [5] and directly in [2]. Second-order problems, where e.g. some solutions should be entirely bounded and localized in a given set, but not necessarily their derivatives, will be treated by ourselves elsewhere.

References

[1] S. Aizicovici and M. Fečkan, *Forced symmetric oscillations of evolution equations*, Nonlin. Anal. **64** (2006), 1621–1640.

[2] J. Andres and R. Bader, *Asymptotic boundary value problems in Banach spaces*, J. Math. Anal. Appl. **247** (2002), 437–457.

[3] J. Andres and L. Górniewicz, *Topological Fixed Point Principles for Boundary Value Problems*, Topological Fixed Point Theory and Its Applications, vol. 1, Kluwer, Dordrecht, 2003.

[4] J. Andres, M. Kožušníková and L. Malaguti, *Bound sets approach to boundary value problems for vector second-order differential inclusions*, Nonlin. Anal. **71** (2009), 28–44.

[5] J. Andres, L. Malaguti and V. Taddei, *On boundary value problems in Banach spaces*, Dynam. Syst. Appl. **18** (2009), 275–302.

[6] J.-P. Aubin and A. Cellina, *Differential Inclusions*, Springer, Berlin, 1984.

[7] J. Chandra, V. Lakshmikantham and A. R. Mitchell, *Existence of solutions of boundary value problems for nonlinear second order systems in Banach space*, Nonlinear Anal. **2** (1978), 157–168.

[8] Ju. L. Daleckiĭ and M.G. Kreĭn, *Stability of Solutions of Differential Equations in Banach Space*, Translation of Mathematical Monographs, Amer. Math. Soc., Providence, R. I., 1974.

[9] K. Deimling, *Multivalued Differential Equations*, Walter de Gruyter, Berlin, 1992.

[10] R. Deville, G. Godefroy and V. Zizler, *Smoothness and Renorming in Banach Spaces*, Longman Scientific and Technical, Harlow, 1993.

[11] J. Diestel, *Geometry of Banach Spaces – Selected Topics* (1975), Springer, Berlin.

[12] R. Dragoni, J. W. Macki, P. Nistri and P. Zecca, *Solution Sets of Differential Equations in Abstract Spaces*, Pitman Research Notes in Mathematics Series, vol. 342, Longman, Harlow, 1996.

[13] S. Hu and N. S. Papageorgiou, *Handbook of Multivalued Analysis, Vol. I: Theory,*, Kluwer, Dordrecht, 1997.

[14] ———, *Handbook of Multivalued Analysis, Vol. II*, Kluwer, Dordrecht, 2000.

[15] M. I. Kamenskiĭ, V. V. Obukhovskiĭ and P. Zecca, *Condensing Multivalued Maps and Semilinear Differential Inclusions in Banach Spaces*, W. de Gruyter, Berlin, 2001.

[16] J. L. MASSERA AND J. J. SCHÄFFER, *Linear Differential Equations and Functional Spaces*, Academic Press, New York, 1966.

[17] J. MAWHIN, *Two point boundary value problems for nonlinear second order differential equations in Hilbert spaces*, Tôhoku Math. J. **32** (1980), 225–233.

[18] H. MÖNCH, *Boundary value problems for nonlinear differential equations of second order in Banach spaces*, Nonlinear Anal. TMA **4** (1980), 225–233.

[19] N. S. PAPAGEORGIOU AND S. TH. KYRITSI-YIALLOUROU, *Handbook of Applied Analysis*, Springer, Berlin, 2009.

[20] K. SCHMITT AND R. THOMPSON, *Boundary value problems for infinite systems of second-order differential equations*, J. Differential Equations **18** (1975), 277–295.

[21] K. SCHMITT AND P. VOLKMANN, *Boundary value problems for second order differential equations in convex subsets in a Banach space*, Trans. Amer. Math. Soc. **218** (1976), 397–405.

[22] G. SELL AND Y. YOU, *Dynamics of Evolutionary Equations*, Springer, Berlin, 2002.

[23] P. VOLKMANN, *Über die positive Invarianz einer abgeschlossenen Teilmenge eines Banachschen Raumes bezüglich der Differentialgleichung* $u' = f(t,u)$, J. Reine Angew. Math. **285** (1976), 59–65.

[24] P. VOLKMANN AND L. YIPING, *The positive invariant set of the differential equation on Banach space*, Ann. Differential Equations **14** (1998), 267–270.

[25] I. I. VRABIE, *Compactness Methods for Nonlinear Evolutions*, 2nd ed., Longman House, Burn Mill, Harlow, 1990.

[26] Z. WANG AND F. ZHANG, *Two points boundary value problems in Banach spaces*, Appl. Math. Mech. **17** (1996), 275–280.

Manuscript received July 23, 2010

JAN ANDRES AND MARTINA PAVLAČKOVÁ
Departmemt of Mathematic Analysis
and Applications of Mathematics
Faculty of Science
Palacký University
Tř. 17. listopadu 12
771 46 Olomouc, CZECH REPUBLIC

E-mail address: andres@inf.upol.cz, pavlackovam@centrum.cz

LUISA MALAGUTI
Department of Engineering Sciences and Methods
University of Modena and Reggio Emilia
Via G. Amendola, 2 – pad. Morselli
I-42122 Reggio Emilia, ITALY

E-mail address: luisa.malaguti@unimore.it