Topological Methods in Nonlinear Analysis
Journal of the Juliusz Schauder Center
Volume 37, 2011, 283–302

UNBOUNDED CONNECTED COMPONENT OF THE POSITIVE SOLUTIONS SET OF SOME SEMI-POSITONE PROBLEMS

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Abstract. In this paper, first we obtain some results for structure of positive solutions set of some nonlinear operator equation. Then using these results, we obtain some existence results for positive solutions of the nonlinear operator equation. The method to show our main results is the global bifurcation theory.

1. Introduction

Let E be a real Banach space which is ordered by a normal cone P, that is, $x \leq y$ if and only if $y - x \in P$. Here, a subset P of E is said to be a cone if it is closed, convex, invariant under multiplication by nonnegative real numbers, and if $P \cap (-P) = \emptyset$. We write $x < y$ if $x \leq y$ and $x \neq y$. Let θ denote the zero element of the real Banach space E, $e \in P \setminus \{\theta\}$, $\|e\| \leq 1$ and $Q = \{x \in P \mid x \geq \|x\|e\}$. It is easy to see that Q is also a cone of E.

In this paper we consider the following nonlinear operator equation

$$x = \lambda A x, \quad x \in P,\ \lambda \in \mathbb{R}^+, \tag{1.1$_\lambda$}$$

2010 *Mathematics Subject Classification.* 47H07, 47H10.

Key words and phrases. Connected component, semi-positone problems, structure of solutions set.

This paper is supported by NSFC10971179, Natural Science Foundation of Jiangsu Education Committee (09KJB110008) and Qing Lan Project.

where $\mathbb{R}^+ = [0, \infty)$, $\lambda > 0$ is a parameter, $A = KF$, K and F satisfy the following assumptions:

(H_1) $K: E \mapsto E$ is a linear completely continuous operator, $K: P \mapsto Q$; $F: P \mapsto E$ is a bounded and continuous operator.

(H_2) There exist $\omega_0 \in P$ and $\sigma_0 \geq 0$ such that $\omega_1 =: K\omega_0 \leq \sigma_0 e$ and

$$Fx \geq -\omega_0, \quad \text{for all } x \in P.$$

In the sequel of this paper we say that F is semi-positone whenever $\omega_0 \neq \theta$, and F is positone whenever $\omega_0 = \theta$. Semi-positone problems occur naturally in important applications. From an application viewpoint one is usually interested in the existence of positive solutions for semi-positone problems. Recall a population dynamics model given in [8], which leads to the study of steady states of semi-positone problems. Let $N(x,t)$ denote the population of a species which is harvested at a constant rate. The resulting population model is of the form:

$$\begin{aligned} &\frac{\partial N}{\partial t} = c\Delta N + (B - SN)N - H && \text{in } \Omega \times (0,\infty), \\ &N(x,0) = A && \text{in } \Omega, \\ &N(x,t) = 0 && \text{in } \partial\Omega \times [0,\infty), \end{aligned}$$

where Ω is a bounded domain in $\mathbb{R}^3$, c, B, S are positive constants, $H(x)$ denotes the quantity harvested per unit time, which is independent of time, and A denotes the initial population. This question is equivalent to find positive solutions of the semi-positone problem

$$\begin{aligned} c\Delta N + (B - SN)N - H &= 0 \quad \text{in } \Omega, \\ N &= 0 \quad \text{in } \partial\Omega. \end{aligned}$$

As pointed out by P. L. Lions in [17], semi-positone problems are mathematically very challenging. An existence result for positive solutions of some semi-positone problems was firstly given by Sun Jingxian in [18] by using the method of global bifurcation theory. However, the study of semi-positone problems was formally introduced by A. Castro and R. Shivaji [9]. During the last ten years finding positive solutions to semi-positone problems has been actively pursued and significant progress on semi-positone problems has taken place, see [1]–[13], [18], [20], [21] and the references therein. For instance, V. Anuradha et. al [3] considered the following Sturm–Liouville boundary value problem

$$(1.2_\lambda) \qquad \begin{cases} (p(t)u')' + \lambda f(t,u) = 0 \quad \text{for } 0 < t < 1, \\ au(r) - bu'(r) = 0, \quad cu(R) + du'(R) = 0 \end{cases}$$

where $a, b, c, d \geq 0$, $\lambda > 0$ was a parameter, $p \in C([r,R], \mathbb{R}^+)$, $p(t) > 0$ for all $t \in [r,R]$, the nonlinear term f were allowed to take negative value, and satisfied that

$$f(t,u) \geq -M_0, \quad \text{for all } (t,x) \in [r,R] \times \mathbb{R}^+$$

for some $M_0 > 0$. Under some super-linear conditions on the nonlinear term f, V. Anuradha et al in [3] proved that there exists $\lambda^* > 0$ such that (1.2_λ) has at least one positive solution for $0 < \lambda < \lambda^*$. The main methods of [3] is by using the fixed point index. For an overview of the semi-positone boundary value problems for ordinary differential equations we refer the reader to [8]. Obviously, the differential boundary value problem (1.2_λ) can be reduced to an nonlinear operator equations of the form (1.1_λ). To show the existence of positive solutions to semi-positone boundary value problems peoples employed various methods. For example, in [3] the authors employed degree theory, in [20] the authors employed variational methods while in [2] the results was obtained via bifurcation from infinity.

Let

$$\begin{aligned} S(P) &= \overline{\{(\lambda,x) \in \mathbb{R}^+ \times P \mid x \neq \theta,\ x = \lambda Ax\}}, \\ S(Q) &= \overline{\{(\lambda,x) \in \mathbb{R}^+ \times Q \mid x \neq \theta,\ x = \lambda Ax\}}. \end{aligned}$$

In this paper we will study the structure of the positive solutions set $S(P)$ by the global bifurcation theory. We will first give some results about the existence of unbounded connected component of the set $S(P)$. Then, as applications of the main results we will also give some results about the existence of positive solutions of the nonlinear operator equation (1.1_λ). Our main results of this paper generalize many results on semi-positone differential boundary value problems in the literature.

This paper is arranged as follows. We will give the main results of this paper in the Section 2. To illustrate applications of the main results of this paper, in Section 3 we will studied the connected component of positive solutions of some differential boundary value problems.

2. Main results

From [16, Theorem 18.1] we have the following lemma:

Lemma 2.1. *Let D be a closed convex subset of Banach space E. For every $\alpha > 0$ there exists a projection J_α onto D which satisfies*

$$\|x - J_\alpha x\| \leq (1+\alpha)\rho(x,D), \quad \text{for all } x \in E,$$

where $\rho(x,D)$ denotes the distance of x to D.

From [15, Lemma 29.1] we have:

Lemma 2.2. *Let X be a compact metric space. Assume that A and B are two disjoint closed subsets of X. Then either there exist a connected component of X meeting both A and B or $X = \Omega_A \cup \Omega_B$, where Ω_A, Ω_B are disjoint compact subsets of X containing A and B, respectively.*

The following lemma is well known as the generalized homotype invariant property of the fixed point index.

Lemma 2.3. *Let $\lambda_1, \lambda_2 \in \mathbb{R}^+$, $\lambda_1 < \lambda_2$, U be an open subset of $[\lambda_1, \lambda_2] \times Q$, $U(\lambda) = U \cap (\{\lambda\} \times Q)$. Assume that $A: \overline{U} \mapsto Q$ is a completely continuous operator such that $A(\lambda, x) \neq x$ for all $(\lambda, x) \in \partial U$. Then $i(A(\lambda, \cdot), U(\lambda), Q)$ is well defined and independent with $\lambda \in [\lambda_1, \lambda_2]$.*

From [15, Theorem 19.2], we have the following lemma:

Lemma 2.4 (Krein–Rutman). *Let E be a Banach space, $P \subset E$ a total cone and $K \in L(E)$ compact positive with the spectrum radii $r(K) > 0$. Then $r(K)$ is an eigenvalue with a positive eigenvector.*

First we assume that the cone P is a total cone and $r(K) > 0$. From Lemma 2.4, there exist $\phi \in P \setminus \{\theta\}$ and $h \in P^* \setminus \{\theta\}$, such that

$$K\phi = r(K)\phi, \quad K^* h = r(K)h. \tag{2.1}$$

Now let us list the following conditions which will be used in this section:

- (H_3) $\lim_{x \in D,\, \|x\| \to \infty} h(F(x))/h(x) = \infty$, where $D = \{x \in E \mid x \geq \|x\| e/2\}$ and h is defined by (2.1).
- (H_4) $\lim_{x \in D,\, \|x\| \to 0^+} h(F(x))/h(x) = \infty$ whenever $\omega_0 = \theta$, where h is defined by (2.1).
- (H_5) There exists a linear completely continuous operator $B: P \mapsto P$ such that $Be > \theta$, and $Ax \geq Bx$ for any $x \in Q$ whenever $\omega_0 = \theta$.

In the sequel, we will always regard that $\sigma_0 = 0$ whenever $\omega_0 = \theta$. For any $R > r \geq 0$, now let us introduce the following symbols for brevity.

$$\begin{aligned} M[r, R] &= \{(\lambda, x) \in \mathbb{R}^+ \times Q \mid r \leq \|x\| \leq R\}, \\ M[r, \infty) &= \{(\lambda, x) \in \mathbb{R}^+ \times Q \mid r \leq \|x\| < \infty\}, \\ M(r) &= \{(\lambda, x) \in \mathbb{R}^+ \times Q \mid \|x\| = r\}. \end{aligned}$$

We will say that a component C^* tends to $(0, \infty)$ if there exists $(\lambda, x_\lambda) \in C^*$ such that $\|x_\lambda\| \to \infty$ as $\lambda \to 0^+$; we say that C^* comes from $(0, \theta)$ if there exists $(\lambda, x_\lambda) \in C^*$ such that $\|x_\lambda\| \to 0$ as $\lambda \to 0^+$.

Now we give the main results of this paper.

THEOREM 2.5. *Let the cone P be a total cone and $r(K) > 0$, $h(e) > 0$, and $h(\omega_0) > 0$ when $\omega_0 > \theta$, where h is defined by* (2.1). *Assume that* (H_1)–(H_5) *hold. Then*

(a) *$S(P)$ possess an unbounded connected component C^* which tends to $(0, \infty)$ whenever F is semi-positone;*

(b) *$S(Q)$ possess an unbounded connected component C^* which comes from $(0, \theta)$ and tends to $(0, \infty)$ whenever F is positone.*

PROOF. We divide the proof into six steps.

Step 1. Let $\widetilde{X} = \{x \in Q | x \geq 4\sigma_0 e\}$, where σ_0 is defined as (H_2). Obviously, $\widetilde{X}$ is a closed convex set of E. Thus, $\widetilde{X}$ is a retraction of E. By Lemma 2.1, there exists a projection $J: E \mapsto \widetilde{X}$ such that for any $x \in Q$,

$$\|x - J(x)\| \leq 2\rho(x, \widetilde{X}). \tag{2.2}$$

Note that $4\sigma_0 e \in \widetilde{X}$, then by (2.2) we have for any $x \in Q$,

$$\|J(x)\| \leq \|x\| + \|x - J(x)\| \leq \|x\| + 2\|x - 4\sigma_0 e\| \leq 3\|x\| + 8\sigma_0\|e\|. \tag{2.3}$$

Let $g: \mathbb{R}^+ \mapsto \mathbb{R}^+$ be defined by

$$g(\lambda) = \begin{cases} 2\lambda & \text{for } \lambda \in [0, 1], \\ 2 & \text{for } \lambda \in [1, \infty) \end{cases} \tag{2.4}$$

and $\beta: \mathbb{R}^+ \times Q \mapsto E$ by

$$\beta(\lambda, x) = J(x) - g(\lambda)\omega_1, \quad (\lambda, x) \in \mathbb{R}^+ \times Q.$$

Consider the following operator equation

$$x = \lambda T(\lambda, x), \quad \text{for all } (\lambda, x) \in \mathbb{R}^+ \times Q, \tag{2.5_λ}$$

where $T(\,\cdot\,,\,\cdot\,): \mathbb{R}^+ \times Q \mapsto E$ is defined by

$$T(\lambda, x) = K[F(\beta(\lambda, x)) + 2\omega_0], \quad \text{for all } (\lambda, x) \in \mathbb{R}^+ \times Q.$$

For any $\lambda \in \mathbb{R}^+$ and $x \in Q$, by (2.4) and (H_2) we have

$$\beta(\lambda, x) = J(x) - g(\lambda)\omega_1 \geq 2\sigma_0 e \geq \theta,$$

and so

$$F(\beta(\lambda, x)) + 2\omega_0 \geq \omega_0 \geq \theta.$$

Then, by (H_1) we have for any $\lambda \in \mathbb{R}^+$ and $x \in Q$

$$T(\lambda, x) = K[F(\beta(\lambda, x)) + 2\omega_0] \in Q.$$

Note that $\sigma_0 = 0$ and $\widetilde{X} = Q$ whenever $\omega_0 = \theta$, then $\beta(\lambda, x) = x$ for any $\lambda \in \mathbb{R}^+$ and $x \in Q$ whenever $\omega_0 = \theta$. This means that (2.5_λ) is the same as (1.1_λ) whenever $\omega_0 = \theta$.

From (2.3) and (2.4), we see that $\beta(\,\cdot\,,\,\cdot\,)\colon\mathbb{R}^+\times Q$ is a continuous and bounded operator, and so $T(\,\cdot\,,\,\cdot\,)\colon\mathbb{R}^+\times Q\mapsto Q$ is a completely continuous operator. Let

$$L(Q)=\overline{\{(\lambda,x)\in\mathbb{R}^+\times Q|x\neq\theta,\ x=\lambda T(\lambda,x)\}}.$$

Then, $L(Q)=S(Q)$ whenever $\omega_0=\theta$.

Step 2. Next we will show that, for any $\overline{\lambda}>0$ with $([\overline{\lambda},\infty)\times Q)\cap L(Q)\neq\emptyset$, there exists $R_{\overline{\lambda}}>0$ such that

$$([\overline{\lambda},\infty)\times Q)\cap L(Q)\subset([\overline{\lambda},\infty)\times Q)\cap M[0,R_{\overline{\lambda}}]. \tag{2.6}$$

Let $\overline{\lambda}>0$ be such that $([\overline{\lambda},\infty)\times Q)\cap L(Q)\neq\emptyset$. Take $M_{\overline{\lambda}}=2(\overline{\lambda}r(K)h(e))^{-1}\|h\|+1$. Then, by ($\mathrm{H}_3$), there exists $R'_{\overline{\lambda}}>0$ such that for any $x\in D$ with $\|x\|\geq R'_{\overline{\lambda}}$,

$$h(F(x))\geq M_{\overline{\lambda}}h(x). \tag{2.7}$$

Let $R_{\overline{\lambda}}=\max\{R'_{\overline{\lambda}}+2\|\omega_1\|,4\sigma_0+2\|\omega_1\|\}$. For any $x\in Q$ with $\|x\|\geq R_{\overline{\lambda}}$, we have $x\geq\|x\|e\geq4\sigma_0e$. This implies that $x\in\widetilde{X}$, and so $J(x)=x$. Then, for any $x\in Q$ with $\|x\|\geq R_{\overline{\lambda}}$ we have

$$\begin{aligned}\beta(\lambda,x)&=J(x)-g(\lambda)\omega_1=x-g(\lambda)\omega_1\geq(\|x\|-2\sigma_0)e\\&\geq\frac12(\|x\|+2\|\omega_1\|)e\geq\frac12(\|x\|+g(\lambda)\|\omega_1\|)e\\&\geq\frac12\|x-g(\lambda)\omega_1\|e=\frac12\|J(x)-g(\lambda)\omega_1\|e=\frac12\|\beta(\lambda,x)\|e.\end{aligned}$$

This implies that $\beta(\lambda,x)\in D$. Also,

$$\|\beta(\lambda,x)\|=\|x-g(\lambda)\omega_1\|\geq\|x\|-2\|\omega_1\|\geq R'_{\overline{\lambda}}.$$

Thus, by (2.7) we have for any $\lambda\geq\overline{\lambda}$ and $x\in Q$ with $\|x\|\geq R_{\overline{\lambda}}$,

$$h(F(\beta(\lambda,x))\geq M_{\overline{\lambda}}h(\beta(\lambda,x)). \tag{2.8}$$

Hence, if there exist $\lambda'\geq\overline{\lambda}$ and $x'\in Q$ with $\|x'\|\geq R_{\overline{\lambda}}$ such that $(\lambda',x')\in L(Q)$, then we have by (2.8) that

$$\begin{aligned}\|x'\|\|h\|\geq h(x')&=h(\lambda'K(F(\beta(\lambda',x'))+2\omega_0))\geq h(\lambda'KF(\beta(\lambda',x')))\\&=\lambda'r(K)h(F(\beta(\lambda',x'))\geq\lambda'r(K)M_{\overline{\lambda}}h(\beta(\lambda',x'))\\&=\lambda'r(K)M_{\overline{\lambda}}h((x'-g(\lambda')\omega_1)\geq\lambda'r(K)M_{\overline{\lambda}}h((\|x'\|-2\sigma_0)e)\\&\geq\lambda'r(K)M_{\overline{\lambda}}h\bigg(\frac12\|x'\|e\bigg)\geq\frac{\overline{\lambda}}{2}r(K)M_{\overline{\lambda}}h(e)\|x'\|,\end{aligned}$$

and so

$$M_{\overline{\lambda}}\leq2(\overline{\lambda}r(K)h(e))^{-1}\|h\|,$$

which is a contradiction. So, (2.6) holds.

Step 3. Now we show that, there exists $\lambda_0>0$ such that

$$L(Q)\subset[0,\lambda_0]\times Q. \tag{2.9}$$

We will give the proof of (2.9) in two cases.

(3a) $\omega_0 > \theta$. If $([1,\infty)\times Q)\cap L(Q)\neq\emptyset$, by the arguments of Step 2, we see that there exists $R_1>0$ such that

$$([1,\infty)\times Q)\cap L(Q)\subset([1,\infty)\times Q)\cap M[0,R_1].$$

For any $(\lambda',x')\in([1,\infty)\times Q)\cap L(Q)$, we have

$$\begin{aligned}\|h\|R_1\geq\|h\|\|x'\|\geq h(x') &= \lambda' h(K(F(\beta(\lambda',x')+2\omega_0))\\ &\geq \lambda' h(K\omega_0)=\lambda' r(K)h(\omega_0),\end{aligned}$$

and so

$$\lambda'\leq R_1(h(\omega_0)r(K))^{-1}\|h\|.$$

Therefore, if we take

$$\lambda_0=\max\{1,R_1(h(\omega_0)r(K))^{-1}\|h\|\},$$

then (2.9) holds.

(3b) $\omega_0=\theta$. First we show that

$$([1,\infty)\times\{\theta\})\cap L(Q)=\emptyset. \tag{2.10}$$

Indeed, by (H_4), there exists $r_0>0$ such that for any $x\in Q$, $0<\|x\|\leq r_0$,

$$h(F(x))>\frac{4}{r(K)}h(x).$$

Thus, for any $\lambda\geq 1/2$ and $x\in Q$ with $0<\|x\|\leq r_0$ we have

$$h(\lambda KF(x))=\lambda r(K)h(F(x))\geq\frac{r(K)}{2}h(F(x))>h(x).$$

This implies that

$$\left(\left[\frac{1}{2},\infty\right)\times Q\right)\cap M(0,r_0]\cap\{(\lambda,x)\in\mathbb{R}^+\times Q|x\neq\theta,x=\lambda KFx\}=\emptyset.$$

Therefore, (2.10) holds. For any $(\lambda',x')\in([1,\infty)\times Q)\cap L(Q)$, by (2.10) we have $x'\neq\theta$. Then, by (H_5), we have

$$x'=\lambda' Ax'\geq\lambda' Bx'\geq\lambda'\|x'\|Be,$$

and so

$$\lambda'\leq(\tau\|Be\|)^{-1},$$

where $\tau>0$ is the normal constant of the cone P (a positive number τ is called the normal constant of P if τ is the infimum of the set of all numbers $\gamma>0$ such that $\theta\leq x\leq y$ implies $\|x\|\leq\gamma\|y\|$). Therefore, if we take $\lambda_0=\max\{1,(\tau\|Be\|)^{-1}\}$, then (2.9) holds.

Step 4. Next we will show that $L(Q)$ possess a connected component which comes from $(0,\theta)$ and tends to $(0,\infty)$. To prove this claim we will employ a method in [19]. Obviously, $(0,\theta)$ is a solution of (2.5_λ). Denote by C the connected component of $L(Q)$ which comes from $(0,\theta)$. Now we can show that C is unbounded. To show this we need to consider two cases:

(4a) There exists $\lambda'>0$ such that $T(\lambda',\theta)=\theta$. Assume on the contrary that C is bounded. Then there exists a bounded open neighbourhood U_0 of C in $\mathbb{R}^+\times Q$. Let $X=\mathrm{Cl}_{\mathbb{R}^+\times Q}U_0\cap L(Q)$, where $\mathrm{Cl}_{\mathbb{R}^+\times Q}U_0$ denotes the closure of U_0 in $\mathbb{R}^+\times Q$. Then X is a compact metric space. If $\partial_{\mathbb{R}^+\times Q}U_0\cap L(Q)\neq\emptyset$, then C and $\partial_{\mathbb{R}^+\times Q}U_0\cap L(Q)$ are two disjoint closed subsets of X. From the maximal connectedness of C, there doesn't exist connected subset of X which joints C and $\partial_{\mathbb{R}^+\times Q}U_0\cap L(Q)$. Then, by Lemma 2.2, there exist compact subsets Ω_1 and Ω_2 of X such that

$$X=\Omega_1\cup\Omega_2,\quad \Omega_1\cap\Omega_2=\emptyset,\quad C\subset\Omega_1,\quad \partial_{\mathbb{R}^+\times Q}U_0\cap L(Q)\subset\Omega_2.$$

Obviously, $\delta_0=\rho(\Omega_1,\Omega_2)>0$. Let U_1 be the $\delta_0/3$-neighbourhood of Ω_1 in $\mathbb{R}^+\times Q$, $U_2=U_1\cap U_0$. Obviously, $\partial_{\mathbb{R}^+\times Q}U_2\cap L(Q)=\emptyset$. Let

$$U=\begin{cases} U_0 & \text{when } \partial_{\mathbb{R}^+\times Q}U_0\cap L(Q)=\emptyset,\\ U_2 & \text{when } \partial_{\mathbb{R}^+\times Q}U_0\cap L(Q)\neq\emptyset.\end{cases}\tag{2.11}$$

Take $\overline{R}_0>0$ large enough such that $U\subset[0,\overline{R}_0]\times Q$ and $U\cap(\{\overline{R}_0\}\times Q)=\emptyset$. Since $(0,\theta)\in C\subset U$, $U\cap(\mathbb{R}^+\times\{\theta\})$ is an open subset of $\mathbb{R}^+\times\{\theta\}$, then we can take $\varepsilon_0>0$ with $\varepsilon_0<1$ small enough such that $[0,\varepsilon_0]\times\{\theta\}\subset U\cap(\mathbb{R}^+\times\{\theta\})$. From Lemma 2.3, we see that for any $\lambda\in[0,\varepsilon_0]$, $i(\lambda T(\lambda,\,\cdot\,),U(\lambda),Q)$ is well defined and independent with λ. Therefore,

$$i(\theta,U(0),Q)=i(\varepsilon_0T(\varepsilon_0,\,\cdot\,),U(\varepsilon_0),Q).\tag{2.12}$$

Since $(0,\theta)$ is the unique solution of (2.5_λ) in $U(0)$, then we easily have

$$i(\theta,U(0),Q)=1.\tag{2.13}$$

From (2.12) and (2.13), we have

$$i(\varepsilon_0T(\varepsilon_0,\,\cdot\,),U(\varepsilon_0),Q)=1.\tag{2.14}$$

Now we claim that

$$\lim_{x\in Q,\,\|x\|\to0^+}\frac{\lambda h(T(\lambda,x))}{h(x)}=\infty\quad\text{uniformly with } \lambda\in[\varepsilon_0,\infty).\tag{2.15}$$

Indeed, if F is semi-positone, then we have for any $\lambda\in[\varepsilon_0,\infty)$ and $x\in Q$

$$h(\lambda T(\lambda,x))=h(\lambda K[F(\beta(\lambda,x))+2\omega_0])\geq h(\varepsilon_0K\omega_0)=\varepsilon_0r(K)h(\omega_0).$$

So, (2.15) holds. If F is positone, then for any $\lambda\in[\varepsilon_0,\infty)$ and $x\in Q$

$$\lambda T(\lambda,x)=\lambda KF(\beta(\lambda,x))=\lambda KF(x).$$

Then (2.15) follows from (H_4). It follows from (2.15) that there exists $r_{\varepsilon_0} > 0$ such that for any $r' \in (0, r_{\varepsilon_0}]$, $0 < \|x\| \leq r'$, $x \in Q$ and $\lambda \in [\varepsilon_0, \infty)$,

$$h(\lambda T(\lambda, x)) \geq 2\varepsilon_0^{-1} h(x). \tag{2.16}$$

Take $u_0 \in Q \setminus \{\theta\}$. By (2.16) we can easily show that

$$x - \lambda T(\lambda, x) \neq t u_0,$$

for any $\lambda \in [\varepsilon_0, \infty)$, $t \geq 0$ and $x \in \partial B_{r'}$, where $B_{r'} = \{x \in Q \mid \|x\| < r'\}$. From (2.16), we have for any $\lambda \in [\varepsilon_0, \infty)$

$$i(\lambda T(\lambda, \,\cdot\,), B_{r'}, Q) = 0. \tag{2.17}$$

Let

$$\widetilde{r}_{\varepsilon_0} = \rho([0, \varepsilon_0] \times \{\theta\}, \partial_{\mathbb{R}^+ \times Q} U) > 0, \quad \widetilde{r}_0 = \min\left\{\frac{1}{2}\widetilde{r}_{\varepsilon_0}, \frac{1}{2} r_{\varepsilon_0}\right\},$$

$B_{\widetilde{r}_0} = \{x \in Q \mid \|x\| < \widetilde{r}_0\}$ and $U^* = U \setminus ([0, \overline{R}_0] \times B_{\widetilde{r}_0})$. It is easy to see that

$$(\partial_{\mathbb{R}^+ \times Q} U^* \cap ([\varepsilon_0, \overline{R}_0] \times Q)) \cap L(Q) = \emptyset,$$

and

$$\rho(\partial_{\mathbb{R}^+ \times Q} U^* \cap ([\varepsilon_0, \overline{R}_0] \times Q), [\varepsilon_0, \overline{R}_0] \times \{\theta\}) \geq \widetilde{r}_0 > 0.$$

By Lemma 2.3, $i(\lambda T(\lambda, \,\cdot\,), U^*(\lambda), Q)$ is well defined, and

$$i(\overline{R}_0 T(\overline{R}_0, \,\cdot\,), U^*(\overline{R}_0), Q) = i(\varepsilon_0 T(\varepsilon_0, \,\cdot\,), U^*(\varepsilon_0), Q).$$

Note that $U^*(\overline{R}_0) = \emptyset$, then we have

$$i(\varepsilon_0 T(\varepsilon_0, \,\cdot\,), U^*(\varepsilon_0), Q) = i(\overline{R}_0 T(\overline{R}_0, \,\cdot\,), U^*(\overline{R}_0), Q) = 0. \tag{2.18}$$

Obviously, $U(\varepsilon_0) = U^*(\varepsilon_0) \cup \overline{B}_{\widetilde{r}_0}$, $U^*(\varepsilon_0) \cap B_{\widetilde{r}_0} = \emptyset$. Then, by (2.17) and (2.18) we have

$$i(\varepsilon_0 T(\varepsilon_0, \,\cdot\,), U(\varepsilon_0), Q) = 0,$$

which is a contradiction of (2.14). Therefore, C is unbounded.

(4b) $T(\lambda, \theta) > \theta$ for any $\lambda > 0$. In the same way to prove the case (4a) we can prove that, if C is bounded, then there exists a bounded open set U of $\mathbb{R}^+ \times Q$ such that $\partial_{\mathbb{R}^+ \times Q} U \cap L(Q) = \emptyset$, and $C \subset U$ (see (2.11)). Take $\overline{R}_0 > 0$ large enough such that $U \subset [0, \overline{R}_0] \times Q$ and $U(\overline{R}_0) = \emptyset$. Note that (λ, θ) is not a solution of (2.5_λ) for any $\lambda \in (0, \infty)$. Therefore, $\partial_{\mathbb{R}^+ \times Q} U$ doesn't possess any solutions of (2.5_λ). Thus, by Lemma 2.3, we have for any $\lambda \in [0, \overline{R}_0]$

$$i(\lambda T(\lambda, \,\cdot\,), U(\lambda), Q) = i(\overline{R}_0 T(\overline{R}_0, \,\cdot\,), U(\overline{R}_0), Q) = 0. \tag{2.19}$$

In the same way as the case (4a) we can show that (2.13) also holds, which is a contradiction of (2.19). Therefore, C is an unbounded connected component of $L(Q)$.

Step 5. Now we see from the proof of Step 2 that, for any $\overline{\lambda} > 0$, if $C \cap ([\overline{\lambda}, \infty) \times Q) \neq \emptyset$, then there exists $R_{\overline{\lambda}} > 0$ such that

$$C \subset (([\overline{\lambda}, \infty) \times Q) \cap M[0, R_{\overline{\lambda}}]).$$

Therefore, C is an unbounded connected component of $L(Q)$ which comes from $(0, \theta)$ and tends to $(0, \infty)$. Note that $S(Q) = L(Q)$ when F is positone, if we let $C^* = C$, then the conclusion (b) holds.

To complete the proof of Theorem 2.5 now we need only to show the conclusion (a). Obviously, the projection of C in x-axis is an interval, denote it by $[0, \lambda_*]$. Let λ_0 be defined as Step 3. Then, we have $0 < \lambda_* \leq \lambda_0$. Assume without loss of generality that $\lambda_* > 1$ (in the same way we can show the case of $\lambda_* \leq 1$). Obviously, C is also an unbounded connected component of $([0, \lambda_*] \times Q) \cap L(Q)$. Let

$$D_1 = (\{1\} \times Q) \cap M[4\sigma_0, \infty), \ D_2 = M(4\sigma_0) \cap ([0, 1] \times Q)$$

and $X_1 = ([0, 1] \times Q) \cap M[4\sigma_0, \infty)$. Obviously, $C \cap (D_1 \cup D_2) \neq \emptyset$. For any $p \in C \cap (D_1 \cup D_2)$, denote by $E(p)$ the connected component of the metric space $C \cap X_1$ which passes the point p. Now we can prove that, there must exist a $p_0 \in C \cap (D_1 \cup D_2)$ such that $E(p_0)$ is an unbounded connected component of the metric space $C \cap X_1$. On the contrary, assume that $E(p)$ is bounded for any $p \in C \cap (D_1 \cup D_2)$. Then, for each $p \in C \cap (D_1 \cup D_2)$, in the same way as in the construction of U in (2.11) we can show that there exists a neighbourhood $U(p)$ of $E(p)$ in X_1 such that

$$\partial_{X_1} U(p) \cap C = \emptyset. \tag{2.20}$$

Obviously, the sets of $\{U(p) \cap (D_1 \cup D_2) | p \in C \cap (D_1 \cup D_2)\}$ is an open cover of the set $C \cap (D_1 \cup D_2)$ and $C \cap (D_1 \cup D_2)$ is a compact set. Thus, there exist finite subsets of $\{U(p) \cap (D_1 \cup D_2) | p \in C \cap (D_1 \cup D_2)\}$, say,

$$U(p_1) \cap (D_1 \cup D_2), U(p_2) \cap (D_1 \cup D_2), \ldots, U(p_{n_0}) \cap (D_1 \cup D_2)$$

which also is an open cover of $C \cap (D_1 \cup D_2)$, that is

$$\bigcup_{j=1}^{n_0} [U(p_j) \cap (D_1 \cup D_2)] \supset C \cap (D_1 \cup D_2).$$

Let $U = \bigcup_{j=1}^{n_0} U(p_j)$. Then U is an bounded open set of X_1. Since

$$\partial_{X_1} U \subset \bigcup_{j=1}^{n_0} \partial_{X_1} U(p_j),$$

then by (2.20) we have

$$\partial_{X_1} U \cap C = \emptyset. \tag{2.21}$$

From the arguments of Step 2, we see that for $\overline{\lambda} = 1$, there exists $R_1 > 4\sigma_0$, such that

$$L(Q) \cap ([1, \infty) \times Q) \subset ([1, \infty) \times Q) \cap M[0, R_1])$$

when $L(Q) \cap ([1, \infty) \times Q) \neq \emptyset$. Since U is a bounded set of X_1, then $U \cap D_1$ is a bounded subset of D_1. Take $R_2 > 0$ large enough such that $U \cap D_1 \subset M[4\sigma_0, R_2]$. Let $R_3 = R_1 + R_2$. Let

$$V_1 = M[0, 4\sigma_0) \cap ([0, \lambda_*] \times Q), \quad V_2 = ((1, \lambda_*] \times Q) \cap M[0, R_3)$$

and $V = V_1 \cup V_2 \cup U$. Let

$$S_1 = (M[4\sigma_0, R_3] \cap (\{1\} \times Q)) \setminus U, \quad S_2 = (M(4\sigma_0) \cap ([0, 1] \times Q)) \setminus U.$$

Obviously, V_1, V_2 and V are three bounded open sets of the metric space $[0, \lambda_*] \times Q$ and

$$\partial_{[0,\lambda_*]\times Q} V = (([1, \lambda_*] \times Q) \cap M(R_3)) \cup \partial_{X_1} U \cup S_1 \cup S_2.$$

Since $C \cap (D_1 \cup D_2) \subset U$, then $C \cap (S_1 \cup S_2) = \emptyset$. By (2.20) and the definition of R_3, we have $C \cap \partial_{[0,\lambda_*]\times Q} V = \emptyset$. Note the unboundedness of C, then $([0, \lambda_*] \times Q) \setminus \mathrm{Cl}_{[0,\lambda_*]\times Q} V$ is an nonempty open subset of C. Let

$$O_1 = C \cap V, \quad O_2 = (([0, \lambda_*] \times Q) \setminus \mathrm{Cl}_{[0,\lambda_*]\times Q} V) \cap C.$$

Then O_1 and O_2 are two nonempty open subsets of C, and $C = O_1 \cup O_2$, which is a contradiction of the connectedness of C. Therefore, there must exist $p_0 \in C \cap (D_1 \cup D_2)$ such that $E(p_0)$ is an unbounded connected component of $X_1 \cap C$.

Step 6. Let $\widetilde{g}\colon [0, 1] \times Q \mapsto [0, 1] \times E$ be defined by

$$\widetilde{g}(\lambda, x) = (\lambda, x - 2\lambda\omega_1), \quad \text{for all } (\lambda, x) \in [0, 1] \times Q.$$

Then $\widetilde{g}\colon [0, 1] \times Q \mapsto [0, 1] \times E$ is continuous, and so $O =: \widetilde{g}(E(p_0))$ is a connected subset of $[0, 1] \times E$. For any $(\lambda, y) \in O$, there exists $(\lambda, x) \in E(p_0)$ such that $y = x - 2\lambda\omega_1$. Then we have

$$\|y\| = \|x - 2\lambda\omega_1\| \geq \|x\| - 2\|\omega_1\|.$$

By the unboundedness of the set $E(p_0)$, we see that O is an unbounded connected subset of $[0, 1] \times E$. Now for any $(\lambda, y) \in O$ and the corresponding $(\lambda, x) \in E(p_0) \subset C$ such that $y = x - 2\lambda\omega_1$, since $\|x\| \geq 4\sigma_0$ and $x \in Q$, we have $J(x) = x$, and

$$y = x - 2\lambda\omega_1 = \beta(\lambda, x) \geq 2\sigma_0 e > \theta,$$

and so

$$x = \lambda K[F(x - 2\lambda\omega_1) + 2\omega_0] = \lambda K F(x - 2\lambda\omega_1) + 2\lambda\omega_1,$$

that is, $y = \lambda KF(y)$. Thus, (λ, y) is a solution of (1.1_λ). Therefore, $O \subset S(P)$. Denote by C^* the connected component of $S(P)$ which contains O. Then C^* is an unbounded connected component which tends to $(0, \infty)$. □

Remark 2.6. Obviously, the conclusion of Step 4 can also be deduced directly from Theorem 1 in [14] when $T(\lambda, \theta) > \theta$ for any $\lambda > 0$. Moreover, the conclusion (b) in Theorem 2.5 follows very easily from the second part of Theorem 1 in [14] if we assumed that B has positive spectral radius in (H_5) without the assumptions (H_3) and (H_4).

Now we apply Theorem 2.5 to show the existence of positive solutions of semi-positone problem (1.1_λ).

Corollary 2.7. *Assume that all conditions of Theorem* 2.5 *are satisfied. Then*

(a) *when F is positone, there exists $\lambda^* > 0$ such that for any $0 < \lambda \leq \lambda^*$, (1.1_λ) has at least two solutions;*
(b) *when F is semi-positone, there exists $\lambda^* > 0$ such that for any $0 < \lambda \leq \lambda^*$, (1.1_λ) has at least one solution.*

Proof. (a) When F is positone, by Theorem 2.5, $S(Q)(= L(Q))$ possess an unbounded connected component C^* which comes from $(0, \theta)$ and tends to $(0, \infty)$. Let λ_0 be defined by Step 3 of Theorem 2.5. Take $\delta_0 > 0$ small enough. Then, there exists $p_0 \in C^* \cap M(1 + \delta_0)$ such that the connected component of the metric space $C^* \cap M[1 + \delta_0, \infty)$ which passes p_0 is unbounded and tends to $(0, \infty)$, denote it by $E(p_0)$. Let

$$\lambda_*^1 = \sup\{\lambda \mid \text{there exists } x \in Q \setminus \{\theta\} \text{ such that } (\lambda, x) \in E(p_0)\}.$$

Denote by $E((0, \theta))$ the connected component of the metric space $C^* \cap M[0, 1]$ which passes $(0, \theta)$ and intersects with $M(1)$. Let

$$\lambda_*^2 = \sup\{\lambda \mid \text{there exists } x \in Q \setminus \{\theta\} \text{ such that } (\lambda, x) \in E((0, \theta))\}.$$

Let $\lambda^* = \min\{\lambda_*^1, \lambda_*^2\}$. Obviously, $\lambda^* > 0$. Then for each $\lambda \in (0, \lambda^*]$, (1.1_λ) has at least two solutions $(\lambda, x_{1,\lambda}^*) \in E((0, \theta))$ and $(\lambda, x_{2,\lambda}^*) \in E(p_0)$, such that $\|x_{1,\lambda}^*\| \leq 1$, $\|x_{2,\lambda}^*\| \geq 1 + \delta_0 > 1$. Obviously, $\|x_{1,\lambda}^*\| \to 0$ and $\|x_{2,\lambda}^*\| \to \infty$ as $\lambda \to 0$.

(b) When F is semi-positone, by Theorem 2.5, $S(P)$ possess an unbounded connected component C^* which tends to $(0, \infty)$. Let

$$\lambda_*^2 = \sup\{\lambda \mid \text{there exists } x \in P \setminus \{\theta\} \text{ such that } (\lambda, x) \in C^*\}.$$

Then, for each $\lambda \in (0, \lambda^*]$, (1.1_λ) has at least one solution. □

Theorem 2.8. *Let the cone P be a solid cone, $r(K) > 0$, $F(\theta) \in \overset{\circ}{P}$, h be defined by* (2.1)*, $h(e) > 0$, $h(\omega_0) > 0$ when $\omega_0 > \theta$. Assume that* (H_1)–(H_3) *hold. Then either*

(a) *$S(P)$ possess two connected components C_1^* and C_2^*, C_1^* tends to $(0,\infty)$ and C_2^* comes from $(0,\theta)$, or*
(b) *$S(P)$ possess an unbounded connected component C^* which comes from $(0,\theta)$ and tends to $(0,\infty)$.*

Proof. Since $F(\theta) \in \overset{\circ}{P}$, then there exists $\delta' > 0$ small enough such that $U(F(\theta),\delta') \subset P$. By the continuity of F, there exists $r_0 > 0$ small enough such that $F(U(\theta,r_0)\cap Q) \subset U(F(\theta),\delta'/4)$. Take $0 < \delta_0' \leq \delta'\|\omega_0\|^{-1}/2$. For any $x \in U(\theta,r_0)\cap Q$, we have

$$\|F(x) - \delta_0'\omega_0 - F(\theta)\| \leq \|F(x) - F(\theta)\| + \delta_0'\|\omega_0\| \leq \frac{\delta'}{4} + \frac{\delta'}{2} < \delta',$$

and so $F(x) - \delta_0'\omega_0 \in U(F(\theta),\delta') \subset P$. So,

$$F(x) \geq \delta_0'\omega_0. \tag{2.22}$$

Define $g\colon \mathbb{R}^+ \mapsto \mathbb{R}^+$ by (2.4). Let $R_0 = \max\{2\sigma_0, r_0+1\}$. Then we have for any $\lambda \in [0,1]$ and $x \in Q$, $\|x\| \geq R_0$

$$x - g(\lambda)\omega_1 \geq (\|x\| - 2\sigma_0)e \geq \theta.$$

By (H_2), we have

$$F(x - g(\lambda)\omega_1) + 2\omega_0 \geq \omega_0, \quad \text{for all } \lambda \in \mathbb{R}^+,\ x \in Q,\ \|x\| \geq R_0. \tag{2.23}$$

Let $\Omega_1 = \{x \in Q \mid \|x\| \leq r_0\} \cup \{x \in Q \mid \|x\| \geq R_0\}$. Define the operator $T\colon \mathbb{R}^+ \times \Omega_1 \mapsto E$ by

$$T(\lambda,x) = \begin{cases} F(x - g(\lambda)\omega_1) + 2\omega_0 & \text{for } (\lambda,x) \in \mathbb{R}^+ \times Q,\ \|x\| \geq R_0, \\ F(x) & \text{for } (\lambda,x) \in \mathbb{R}^+ \times Q,\ \|x\| \leq r_0. \end{cases}$$

Let $\delta_0 = \min\{1,\delta_0'\}$. Then, for any $(\lambda,x) \in \mathbb{R}^+ \times \Omega_1$, we have by (2.22) and (2.23)

$$T(\lambda,x) \geq \delta_0\omega_0. \tag{2.24}$$

Using the Extension Theorem of continuous operator, we see that there exists a continuous operator $\widetilde{T}\colon \mathbb{R}^+ \times Q \mapsto E$ such that

$$\widetilde{T}(\lambda,x) = T(\lambda,x), \quad \text{for all } (\lambda,x) \in \mathbb{R}^+ \times \Omega_1$$

and $\widetilde{T}(\mathbb{R}^+ \times Q) \subset \overline{\mathrm{co}}T(\mathbb{R}^+ \times \Omega_1)$, where $\overline{\mathrm{co}}T(\mathbb{R}^+ \times \Omega_1)$ denotes the convex closure of the set $T(\mathbb{R}^+ \times \Omega_1)$.

Next we will consider the following nonlinear operator equation

$$x = \lambda K\widetilde{T}(\lambda, x), \text{for all } (\lambda, x) \in \mathbb{R}^+ \times Q. \tag{2.25}$$

Let

$$L(Q) = \overline{\{(\lambda, x) \mid (\lambda, x) \in \mathbb{R}^+ \times Q \text{ is a solution of (2.25) and } x \neq \theta\}}.$$

Let $\overline{\lambda} > 0$ be fixed. Take $M_{\overline{\lambda}} \geq 4\|h\|/(\overline{\lambda} r(K)h(e))$. From ($\mathrm{H}_3$), there exists $R'_{\overline{\lambda}} > 0$ such that for any $x \in D$ with $\|x\| \geq R'_{\overline{\lambda}}$

$$h(F(x)) \geq M_{\overline{\lambda}} h(x). \tag{2.26}$$

Let $R_{\overline{\lambda}} \geq \max\{R'_{\overline{\lambda}} + 4\|\omega_1\|, R_0 + 2\|\omega_1\|, 4\sigma_0 + 4\|\omega_1\|\}$. Then we have for any $\lambda \geq \overline{\lambda}$ and $x \in Q$ with $\|x\| \geq R_{\overline{\lambda}}$

$$\begin{aligned} x - g(\lambda)\omega_1 &\geq x - 2\omega_1 \geq x - 2\sigma_0 e \geq (\|x\| - 2\sigma_0)e \\ &\geq \frac{1}{2}(\|x\| + 2\|\omega_1\|)e \geq \frac{1}{2}(\|x - g(\lambda)\omega_1\|)e. \end{aligned} \tag{2.27}$$

This implies that $x - g(\lambda)\omega_1 \in D$. On the other hand, for any $\lambda \geq \overline{\lambda}$ and $x \in Q$ with $\|x\| \geq R_{\overline{\lambda}}$, we have

$$\|x - g(\lambda)\omega_1\| \geq \|x\| - 2\|\omega_1\| \geq R'_{\overline{\lambda}}.$$

Thus, by (2.26) and (2.27) we have for any $\lambda \geq \overline{\lambda}$ and $x \in Q$ with $\|x\| \geq R_{\overline{\lambda}}$

$$\begin{aligned} h(\lambda K\widetilde{T}(\lambda, x)) &= \lambda h(KT(\lambda, x)) \geq \overline{\lambda} r(K)h(T(\lambda, x)) \\ &\geq \overline{\lambda} r(K)h(F(x - g(\lambda)\omega_1)) \geq \overline{\lambda} r(K) M_{\overline{\lambda}} h(x - g(\lambda)\omega_1) \\ &\geq \overline{\lambda} r(K) M_{\overline{\lambda}} h\left(\frac{1}{2}\|x\|e\right) = \frac{\overline{\lambda} M_{\overline{\lambda}} r(K)h(e)}{2}\|x\| \geq 2\|h\|\|x\| > h(x). \end{aligned}$$

This implies that

$$L(Q) \cap ([\overline{\lambda}, \infty) \times Q) \subset M[0, R_{\overline{\lambda}}] \tag{2.28}$$

whenever $L(Q) \cap ([\overline{\lambda}, \infty) \times Q) \neq \emptyset$.

Now we will show that, there exists $\lambda_0 > 0$ such that

$$L(Q) \subset [0, \lambda_0] \times Q. \tag{2.29}$$

If $L(Q) \cap ([1, \infty) \times Q) = \emptyset$, then (2.29) holds for $\lambda_0 = 1$. Assume that $L(Q) \cap ([1, \infty) \times Q) \neq \emptyset$. For any $(\lambda, x) \in L(Q) \cap ([1, \infty) \times Q)$, we have by (2.24)

$$x = \lambda K\widetilde{T}(\lambda, x) \geq \lambda\delta_0 K\omega_0 = \lambda\delta_0\omega_1. \tag{2.30}$$

From (2.28), we see that there exists $R_1 > 0$ such that

$$L(Q) \cap ([1, \infty) \times Q) \subset M[0, R_1]. \tag{2.31}$$

From (2.30) and (2.31), we have $\lambda \leq (\tau\delta_0\|\omega_1\|)^{-1}R_1$. Then (2.29) holds for

$$\lambda_0 = \max\{1, (\tau\delta_0\|\omega_1\|)^{-1}R_1\}.$$

Since

$$\widetilde{T}(\lambda, x) = T(\lambda, x) = F(x) \geq \delta_0'\omega_0$$

for any $\lambda > 0$, $x \in Q$ and $\|x\| \leq r_0$, then we have

$$\frac{h(K\widetilde{T}(\lambda, x))}{h(x)} = \frac{h(KF(x))}{h(x)} = \frac{r(K)h(F(x))}{h(x)} \geq \delta_0' r(K)\frac{h(\omega_0)}{h(x)} \to \infty$$

as $\|x\| \to 0^+$. A similar argument as Theorem 2.5 shows that, $L(Q)$ possess an unbounded connected component C which comes from $(0, \theta)$ and tends to $(0, \infty)$.

Now a similar argument as Step 5 of Theorem 2.5 yields that, there exists $\lambda_* > 0$ (assume without loss of generality that $\lambda_* \geq 1$) such that the projection of C on x-axis is $[0, \lambda_*]$. Let the sets D_1 and D_2, and $\widetilde{g}$ be defined as in Steps 5 and 6 of Theorem 2.5. Then there exists $p_0 \in D_1$ or $p_0 \in D_2$ such that the metric space $(([0,1] \times Q) \cap M[4\sigma_0, \infty)) \cap C$ possess an unbounded connected component which passes p_0 and tends to $(0, \infty)$, denote it by $E(p_0)$. Then a similar argument as in Step 6 of Theorem 2.5 yields that the metric space $S(P) \cap ([0,1] \times E)$ possess an unbounded connected subset O such that $O =: \widetilde{g}(E(p_0))$. Let C_1^* be the connected component of the metric space $S(P)$ which contains O. Then C_1^* is an unbounded connected component which tends to $(0, \infty)$.

Denote by $E((0, \theta))$ the connected component of the metric space $C \cap M[0, r_0]$ which passes $(0, \theta)$ and intersects with $M(r_0)$. Obviously, $E((0, \theta)) \subset S(P)$(In fact, $E((0, \theta)) \subset S(Q)$). Denote by C_2^* the connected component of the metric space $S(P)$ which contains $E((0, \theta))$. If $C_1^* \cap C_2^* \neq \emptyset$, then C_1^* and C_2^* are contained in a connected component, say C^*. Obviously, C^* is the connected component which comes from $(0, \theta)$ and tends to $(0, \infty)$. If $C_1^* \cap C_2^* = \emptyset$, then C_1^* and C_2^* are two different connected components of $S(P)$, C_2^* comes from $(0, \theta)$ and C_1^* tends to $(0, \infty)$, respectively. □

COROLLARY 2.9. *Assume all conditions of Theorem* 2.8 *hold. Then there exist* $\lambda^* > 0$ *and* $r_0 > 0$ *such that for any* $0 < \lambda < \lambda^*$, (1.1_λ) *has at least two solutions* $x_\lambda^{(1)}$ *and* $x_\lambda^{(2)}$ *with* $0 < \|x_\lambda^{(1)}\| \leq r_0 < \|x_\lambda^{(2)}\|$, *and*

$$\|x_\lambda^{(1)}\| \to 0, \quad \|x_\lambda^{(2)}\| \to \infty \quad (\lambda \to \infty).$$

In Theorems 2.5 and 2.8, in order to employ some conditions concerning h we assume that the cone is a total or solid cone, and $r(K) > 0$. However, theses conditions in many applications are not satisfied. Now we give some new conditions.

(H_6) $\lim_{x \in D,\, \|x\| \to \infty} \|KF(x)\|/\|x\| = \infty$, where $D = \{x \in E \mid x \geq \|x\|e/2\}$.

(H_7) $\lim_{x \in Q,\, \|x\| \to 0^+} \|KF(x)\|/\|x\| = \infty$ when $\omega_0 = \theta$.

By using the method of Theorems 2.5 and 2.8, we can prove the following Theorems 2.10 and 2.11 For simplicity we omit the proofs.

Theorem 2.10. *Assume that* (H_1), (H_2), (H_5)–(H_7) *hold. Then all conclusions of Theorem* 2.5 *hold.*

Theorem 2.11. *Assume that* (H_1), (H_6) *and* (H_7) *hold. Moreover, the cone* P *is a solid cone,* $\omega_0 > \theta$ *and* $F(\theta) \in \overset{\circ}{P}$. *Then all conclusions of Theorem* 2.8 *hold.*

Corollary 2.12. *Assume all conditions of Theorem* 2.10 *hold. Then all conclusions of Corollary* 2.7 *hold.*

Corollary 2.13. *Assume all conditions of Theorem* 2.11 *hold. Then all conclusions of Corollary* 2.9 *hold.*

3. Applications of abstract results in differential boundary value problems

In this section we will give some applications of Theorems 2.5, 2.8, 2.10, 2.11 in differential boundary value problems. We will study the connected component of the positive solutions set of the boundary value problem (1.2_λ).

Concerning the boundary value problem (1.2_λ), we make the following assumptions.

(A_1) There exists $M_0 \geq 0$ such that

$$f(t,x) \geq -M_0, \quad \text{for all } (t,x) \in [r,R] \times \mathbb{R}^+.$$

(A_2) There exists $[\alpha,\beta] \subset [r,R]$ such that

$$\lim_{x\to\infty} \frac{f(t,x)}{x} = \infty \quad \text{uniformly with } t \in [\alpha,\beta].$$

(A_3) When $M_0 = 0$, for any $[\alpha_1,\beta_1] \subset (r,R)$,

$$\lim_{x\to 0^+} \frac{f(t,x)}{x} = \infty \quad \text{uniformly with } t \in [\alpha_1,\beta_1].$$

(A_4) When $M_0 = 0$, there exists $b(t) \geq 0$ such that

$$f(t,x) \geq b(t)x \quad \text{for all } (t,x) \in [r,R] \times \mathbb{R}^+.$$

Let $E = C[0,1]$ be the well known real Banach space of all continuous functions on $[r,R]$ with the maximum norm $\|\cdot\|$, and $P = \{x \in E \mid x = x(t) \geq 0 \text{ for } t \in [r,R]\}$. Then, P is a solid cone of E. Let

$$G(t,s) = \begin{cases} \alpha^{-1}\left(b + a\displaystyle\int_r^s p^{-1}\right)\left(d + c\displaystyle\int_t^R p^{-1}\right) & \text{for } s \leq t, \\ \alpha^{-1}\left(b + a\displaystyle\int_r^t p^{-1}\right)\left(d + c\displaystyle\int_s^R p^{-1}\right) & \text{for } s > t, \end{cases}$$

where $\alpha = ad + bc + ac\int_r^R p^{-1}$ and $\int_r^R p^{-1} = \int_r^R p^{-1}(s)\,ds$. Let

$$e(t) = \min\left\{\frac{b + a\int_r^t p^{-1}}{b + a\int_r^R p^{-1}}, \frac{c + d\int_t^R p^{-1}}{c + d\int_r^R p^{-1}}\right\}, \quad t \in [r, R]$$

and $Q = \{x \in P \mid x = x(t) \geq \|x\|e(t),\ \text{for all } t \in [r, R]\}$. Then Q is also a cone of E.

Let us define the operators $K\colon E \mapsto E$ and $F\colon P \mapsto E$ by

$$\begin{aligned} (Kx)(t) &= \int_r^R G(t, s)x(s)\,ds, && \text{for all } x \in E,\ t \in [r, R], \\ (Fx)(t) &= f(t, x(t)), && \text{for all } x \in P,\ t \in [r, R]. \end{aligned}$$

Let $\omega_0(t) = M_0$ (for all $t \in [r, R]$) and

$$\omega_1(t) = M_0 \int_r^R G(t, s)\,ds, \quad t \in [r, R].$$

From the proof in [3] we have the following Lemmas 3.1 and 3.2.

Lemma 3.1. *$K\colon P \mapsto Q$ is a linear completely operator.*

Lemma 3.2. *There exists $\sigma_0 > 0$ such that $\omega_1 \leq \sigma_0 e$.*

Let

$$S(P) = \overline{\{(\lambda, x) \mid \lambda \in \mathbb{R}^+,\ x \neq \theta,\ x \in P,\ x \text{ is a solutions of } (1.2_\lambda)\}}.$$

Now first we will study the connected component of the metric space $S(P)$, then we study the multiple positive solutions of (1.2_λ).

Theorem 3.3. *Assume that* (A_1)–(A_4) *hold. Then:*

(a) *$S(P)$ possess an unbounded connected component C^* which tends to $(0, \infty)$ when $M_0 > 0$.*
(b) *$S(P)$ possess an unbounded connected component C^* which comes $(0, \theta)$ and tends to $(0, \infty)$ when $M_0 = 0$.*

Proof. Let us define the operator K and F, the elements ω_0 and ω_1 as above. It is easy to see that (H_1) and (H_2) hold. Now we show that (H_6) holds. By (A_2), for any $M > 0$, there exists $R_0 > 0$ such that $f(t, x) \geq Mx$ for any $x \geq R_0$ and $t \in [\alpha, \beta]$. Take $\overline{R}_0 = 2R_0(\min_{t\in[\alpha,\beta]} e(t))^{-1}$. Then we have, for any $x \in D$ with $\|x\| \geq \overline{R}_0$,

$$x(t) \geq \frac{1}{2}\|x\|e(t) \geq \frac{1}{2}\overline{R}_0 \min_{t\in[\alpha,\beta]} e(t) \geq R_0.$$

Then we have

$$f(t, x(t)) \geq Mx(t), \quad \text{for all } t \in [\alpha, \beta]$$

and so

$$\begin{aligned}(KFx)(t) &= \int_r^R G(t,s)[F(s,x(s)) + M_0]\,ds - \omega_1(t)\\ &\geq \int_\alpha^\beta G(t,s)[f(s,x(s)) + M_0]\,ds - \|\omega_1\|\\ &\geq \int_\alpha^\beta G(t,s)f(s,x(s))\,ds - \|\omega_1\|\\ &\geq M\int_\alpha^\beta G(t,s)x(s)\,ds - \|\omega_1\|\\ &\geq \frac{1}{2}M\int_\alpha^\beta G(t,s)e(s)\,ds\|x\| - \|\omega_1\|,\end{aligned}$$

for all $t \in [r,R]$. Thus, for $x \in D$ with $\|x\| \geq \overline{R}_0$, we have

$$\|KFx\| \geq \frac{1}{2}M\int_\alpha^\beta G(t,s)e(s)\,ds\|x\| - \|\omega_1\|, \quad \text{for all } t \in [\alpha,\beta].$$

Consequently,

$$\frac{\|KFx\|}{\|x\|} \geq \frac{1}{2}M\min_{t\in[\alpha,\beta]}\int_\alpha^\beta G(t,s)e(s)\,ds - \frac{\|\omega_1\|}{\|x\|}.$$

Note $\|\omega_1\|/\|x\| \to 0$ as $\|x\| \to \infty$, then we have

$$\lim_{x\in D,\,\|x\|\to\infty}\frac{\|KFx\|}{\|x\|} = \infty.$$

This means that (H_6) holds. Similar to the proof above we can show that (H_7) holds. By (A_4) we see that (H_5) holds. Now the conclusions follows from Theorem 2.10. □

Corollary 3.4. *Assume that all conditions of Theorem* 3.3 *hold. Then:*

(a) *When $M_0 > 0$, there exists $\lambda^* > 0$ such that (1.2_λ) has at least one positive solution for $0 < \lambda < \lambda^*$.*
(b) *When $M_0 = 0$, there exists $\lambda^* > 0$ such that (1.2_λ) has at least two positive solutions $x_\lambda^{(1)}$ and $x_\lambda^{(2)}$ for $0 < \lambda < \lambda^*$, $\|x_\lambda^{(1)}\| \leq 1 < \|x_\lambda^{(2)}\|$ and $\|x_\lambda^{(1)}\| \to 0$, $\|x_\lambda^{(2)}\| \to \infty$ as $\lambda \to 0^+$.*

Corollary 3.4 can be obtained by Corollary 2.12.

Theorem 3.5. *Assume that (A_1) and (A_2) hold. Moreover,*

$$f(t,0) > 0, \quad \textit{for all } t \in [r,R].$$

Then either

(a) *$S(P)$ possess an unbounded connected component C^* which comes from $(0,\theta)$ and tends to $(0,\infty)$, or*

(b) *$S(P)$ possess two unbounded connected components C_1^* and C_2^*, C_1^* comes from $(0,\theta)$ and C_2^* tends to $(0,\infty)$, respectively.*

Corollary 3.6. *Assume that all conditions of Theorem 3.5 hold. Then there exist $\lambda^* > 0$ and $r_0 > 0$ such that, for $0 < \lambda < \lambda^*$, (1.2_λ) has at least two solutions $x_\lambda^{(1)}$ and $x_\lambda^{(2)}$ such that*

$$0 < \|x_\lambda^{(1)}\| \leq r_0 < \|x_\lambda^{(2)}\|$$

and $\|x_\lambda^{(1)}\| \to 0$, $\|x_\lambda^{(2)}\| \to \infty$ as $\lambda \to 0^+$.

Remark 3.7. The conclusion (a) of Corollary 3.4 is the main result of [3]. Therefore, the abstract results of the Section 2 generalized the main results in [3]. Here we obtained the existence of positive solutions by employing a different method, that is the method of the global bifurcation theories.

Remark 3.8. We have employed some conditions concerning h defined by (2.1), see (H_3) and (H_4). In some applications to differential boundary value problems, we can easily check this kind of condition, see [21].

Remark 3.9. In our main results of this paper Theorems 2.5 and 2.8 we assumed that $K(P) \subset Q$. Many operators has this property, especially, for those operators which are related to ordinary differential boundary value problems. However, it is difficult to verify $K(P) \subset Q$ for those operators which are related to elliptic boundary value problems. Therefore, it's a question need to study further that what conditions can ensure our main results also valid for those operators which are related to elliptic boundary value problems.

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Manuscript received April 21, 2010

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