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FORCED OSCILLATIONS IN STRONGLY DAMPED BEAM EQUATION

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Abstract. It is proved that the extensible beam equation in Ball's model admits periodic solutions near equilibrium states if subject to external periodic force of high frequency. The approach is based on translation along trajectories, averaging method and homotopy invariants such as topological degree and fixed point index.

1. Introduction

We shall be concerned with a class of differential problems motivated by the following beam equation

$$\begin{aligned}(1.1)\quad & u_{tt}(x,t) + \alpha u_{txxxx}(x,t) + \beta u_t(x,t) + u_{xxxx}(x,t) \\ & - \left(a\int_0^l |u_\xi(\xi,t)|^2\,d\xi + b\right) u_{xx}(x,t) - \sigma\left(\int_0^l u_\xi(\xi,t)u_{\xi t}(\xi,t)\,d\xi\right) u_{xx}(x,t) \\ & = \varphi(x,\omega t) \quad \text{for } (x,t)\in(0,l)\times(0,\infty)\end{aligned}$$

under the boundary conditions

$$(1.2)\qquad u(0,t)=u(l,t)=0,\quad u_{xx}(0,t)=u_{xx}(l,t)=0\quad \text{for } t>0$$

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where $\alpha, \beta, \sigma, a, l > 0$, $b \in \mathbb{R}$, $\varphi\colon [0,l] \times [0,\infty) \to \mathbb{R}$ is time T-periodic for some $T > 0$ and $\omega > 0$ is a parameter. The problem (1.1)–(1.2) in terms of Ball's model (see [2], [3]) describes an extensible beam exposed to the external oscillating force φ. We are interested in the existence of periodic solutions forced by the time-dependent periodic term.

To study the problem we will put it in an abstract setting of evolution equations. Here we follow e.g. [11], [13], [15] or [9]. To this end define $A\colon D(A) \to X$ in $X := L^2(0,l)$ by

$$
\begin{gathered}
D(A) := \{u \in W^{4,2}(0,l) \mid u(0) = u(l) = 0,\ u''(0) = u''(l) = 0\}, \\
Au := u'''' \quad \text{for } u \in D(A).
\end{gathered}
\tag{1.3}
$$

It may be easily verified that the operator A is strictly positive, self-adjoint and has compact resolvent. This allows us to consider the fractional powers A^γ and spaces X^γ with the norms $|\cdot|_\gamma$ and the scalar products $(\,\cdot\,,\,\cdot\,)_\gamma$, for $\gamma \in \mathbb{R}$. It can be proved that $A^{1/2}u = -u''$ for $u \in X^{1/2}$ and $|A^{1/4}u|_0 = |u'|_0$ for each $u \in X^{1/4}$. Hence, the problem (1.1)–(1.2) can be transformed into an abstract form

$$
\ddot{u} + \alpha A\dot{u} + \beta\dot{u} + Au + (a|u|^2_{1/4} + b + \sigma(u,\dot{u})_{1/4})A^{1/2}u = f(\omega t)
\tag{1.4}
$$

where $f(t) := \varphi(\,\cdot\,,t)$. This can be formally rewritten as

$$
\begin{cases}
\dot{u} = v, \\
\dot{v} = -Au - \alpha Av - \beta v - (a|u|^2_{1/4} + b + \sigma(A^{1/2}u, v)_0)A^{1/2}u + f(\omega t)
\end{cases}
$$

and, subsequently, as an equation on $\mathbf{E} := X^{1/2} \times X^0$

$$
(\dot{u}(t), \dot{v}(t)) = \mathbf{A}(u(t), v(t)) + \mathbf{F}(\omega t, u(t), v(t))
\tag{1.5}
$$

where $\mathbf{A}\colon D(\mathbf{A}) \to \mathbf{E}$ is given by

$$
\begin{gathered}
D(\mathbf{A}) := \{(\overline{u}, \overline{v}) \in \mathbf{E} \mid \overline{u} + \alpha\overline{v} \in X^1,\ \overline{v} \in X^{1/2}\}, \\
\mathbf{A}(\overline{u}, \overline{v}) := (\overline{v}, -A(\overline{u} + \alpha\overline{v}) - \beta\overline{v}), \quad (\overline{u}, \overline{v}) \in D(\mathbf{A})
\end{gathered}
\tag{1.6}
$$

and $\mathbf{F}\colon [0,\infty) \times \mathbf{E} \to \mathbf{E}$ by

$$
\mathbf{F}(t, \overline{u}, \overline{v}) := (0, -(a|\overline{u}|^2_{1/4} + b + \sigma(A^{1/2}\overline{u}, \overline{v})_0)A^{1/2}\overline{u} + f(t)), \quad t \geq 0, (\overline{u}, \overline{v}) \in \mathbf{E}.
$$

It appears that (1.5) admits global unique solutions (understood in an appropriate sense). Therefore, the translation (along trajectories) operator $\Phi_{\overline{t}}\colon \mathbf{E} \to \mathbf{E}$, $\overline{t} > 0$, given by $\Phi_{\overline{t}}(\overline{u}, \overline{v}) := (u(\overline{t}), v(\overline{t}))$, where $(u,v)\colon [0,\infty) \to \mathbf{E}$ is the solution of (1.5) satisfying the initial value condition $(u(0), v(0)) = (\overline{u}, \overline{v})$, is correctly defined. Fixed points of the translation operator determine periodic solutions of (1.5).

The existence of fixed points will be established by the use of topological methods. Our approach is similar to that of [7] and consists of two main elements.

The first one is the Krasnsel'skiĭ type formula relating the fixed point index of the translation operator for an autonomous equation with the topological degree of the right hand side of that equation. The second key ingredient is the averaging formula stating that solutions of (1.5) converge as $\omega \to \infty$ to trajectories of the avareged equation

$$(\dot{u}, \dot{v}) = \mathbf{A}(u, v) + \widehat{\mathbf{F}}(u, v) \tag{1.7}$$

where $\widehat{\mathbf{F}}\colon \mathbf{E} \to \mathbf{E}$ is given by

$$\widehat{\mathbf{F}}(\overline{u}, \overline{v}) := \frac{1}{T}\int_0^T \mathbf{F}(\tau, \overline{u}, \overline{v})\, d\tau, \quad (\overline{u}, \overline{v}) \in \mathbf{E}.$$

It will be shown that for sufficiently large ω the fixed point index of the translation $\Phi_{T/\omega}$ operator is equal to the topological degree of $\mathbf{A} + \widehat{\mathbf{F}}$ (with respect to a proper open bounded set). By linearization, we shall deduce that, for large frequencies ω there is a periodic solution near each equilibrium point of (1.7). The following result is obtained.

Theorem 1.1. *Suppose that $|b| \neq j^2\pi^2/l^2$, for all $j \geq 1$, and $f\colon [0, \infty) \to X^0$ is a T-periodic Hölder continuous function such that*

$$\int_0^T f(\tau)\, d\tau = 0.$$

Let $k := [l\sqrt{-b}/\pi]$ if $b < -\pi^2/l^2$ and $k := 0$ if $b > -\pi^2/l^2$. Then there exists $\omega_0 > 0$ such that for all $\omega > \omega_0$ the equation (1.4) *admits at least $2k + 1$ (T/ω)-periodic solutions $u_j^{(\omega)}$, $j \in \{i \in \mathbb{Z} \mid |i| \leq k\}$, such that*

$$u_j^{(\omega)}(t) \to \overline{u}_j \quad \text{in } X^{1/2} \text{ as } \omega \to \infty, \text{ uniformly with respect to } t \in \mathbb{R},$$

and

$$\dot{u}_j^{(\omega)}(t) \to 0 \quad \text{in } X^0 \text{ as } \omega \to \infty, \text{ uniformly with respect to } t \in \mathbb{R},$$

for all integers with $|j| \leq k$, where $\overline{u}_j$, $j \in \{i \in \mathbb{Z} \mid |i| \leq k\}$, are equilibrium points of (1.4) *with $f \equiv 0$, i.e. $\overline{u}_0 = 0$, $\overline{u}_j = (l/2a)^{1/2}(-1 - bl^2/j^2\pi^2)^{1/2} \sin(j\pi x)$ if $0 < |j| \leq k$.*

Moreover, an advantage of this translation along trajectories approach is that one obtains, as an additional information from the proof, the fixed point indices of these periodic solutions (understood as indices of the translation operator for (1.5)). This information is related with their stability properties.

Theorem 1.1 will be a conclusion of more general results. In the whole paper we shall assume that X is a Hilbert space and A is an abstract strictly positive self-adjoint operator in X having compact resolvent.

The paper is organized as follows. In Section 2 we provide estimates in the spirit of [9] and a compactness result for the semigroup generated by $\mathbf{A}$, which will be used to show that the translation along trajectories is a k-set contraction with respect an appropriate measure of noncompactness. Section 3 is devoted to a suitable version of Krasnosel'skiĭ type formula. In Section 4 we develop an averaging rule for beam type equations. Finally, in Section 5 we apply the obtained theorems to the beam equation (1.4).

Notation. By $\mathbb{R}$ we denote the field of real numbers; by $[x]$ we mean the integer (or floor) part of $x \in \mathbb{R}$. $\mathbb{Z}$ stands for the set of all integers.

If X is a normed space with the norm $\|\cdot\|$ and $U \subset X$, then ∂U and $\overline{U}$ are the boundary and the closure of U, respectively. If $x_0 \in X$ and $r > 0$, then $B_X(x_0, r) = B(x_0, r) := \{x \in X \mid \|x - x_0\| < r\}$. The distance of $x \in X$ to a subset $M \subset X$ is defined by $d(x, M) := \inf\{\|x - z\| \mid z \in M\}$. By $\overline{\mathrm{conv}} M$ we mean the closed convex hull of M. By $L(X, X)$ we denote the algebra of all bounded linear operators on X with the operator norm $\|\cdot\|_{L(X,X)}$. We write I for the identity operator.

If $A: D(A) \to X$, $D(A) \subset X$, is a linear operator in X, then $\rho(A)$ is the resolvent set of A and $R(\lambda; A) := (\lambda I - A)^{-1}$ is the resolvent operator for $\lambda \in \rho(A)$. We write $\{S_A(t)\}_{t\geq 0}$ for the C_0 semigroup of bounded linear operators generated by A.

2. Estimates and compactness properties of the operator A

Throughout the paper, unless otherwise stated, we assume that $A: D(A) \to X$ is a strictly positive self-adjoint operator in a Hilbert space X such that the inverse operator $A^{-1}: X \to X$ is well defined and compact. In this case, the spectrum of A consists of a sequence of eigenvalues $(\lambda_k)_{k=1}^{\infty}$ such that $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots$ and $\lambda_k \to \infty$ as $k \to \infty$ and there exists a complete X-orthonormal sequence of eigenvectors $(e_k)_{k=1}^{\infty}$ such that $Ae_k = \lambda_k e_k$ for all $k \geq 1$. The fractional powers A^{γ} and the associated scale of fractional spaces X^{γ}, $\gamma \in \mathbb{R}$, can be considered.

Let $\alpha, \beta > 0$ be fixed and put $\mathbf{E} := X^{1/2} \times X^0$. Define $\mathbf{A}: D(\mathbf{A}) \to \mathbf{E}$ by

$$\begin{aligned} D(\mathbf{A}) &:= \{(\overline{u}, \overline{v}) \in X^{1/2} \times X^0 \mid \overline{u} + \alpha\overline{v} \in X^1,\ \overline{v} \in X^{1/2}\}, \\ \mathbf{A}(\overline{u}, \overline{v}) &:= (\overline{v}, -A(\overline{u} + \alpha\overline{v}) - \beta\overline{v}), \quad (\overline{u}, \overline{v}) \in D(\mathbf{A}). \end{aligned} \tag{2.1}$$

Let $\eta \geq 0$ and $(\,\cdot\,,\,\cdot\,)_\eta: \mathbf{E} \times \mathbf{E} \to \mathbb{R}$ be given by

$$((\overline{u}_1, \overline{v}_1), (\overline{u}_2, \overline{v}_2))_\eta := (1 + \alpha\eta)(\overline{u}_1, \overline{u}_2)_{1/2} + (\eta\overline{u}_1 + \overline{v}_1, \eta\overline{u}_2 + \overline{v}_2)_0$$

for any $(\overline{u}_1, \overline{v}_1), (\overline{u}_2, \overline{v}_2) \in \mathbf{E}$. It is easily seen that, for each $\eta > 0$, $(\,\cdot\,,\,\cdot\,)_\eta$ is a scalar product in $\mathbf{E}$ and that the induced norm $\|\cdot\|_\eta$ given by

$$\|(\overline{u}, \overline{v})\|_\eta := \sqrt{((\overline{u}, \overline{v}), (\overline{u}, \overline{v}))_\eta}, \quad (\overline{u}, \overline{v}) \in \mathbf{E}$$

is equivalent to the original norm $\|\cdot\|_{\mathbf{E}}$ associated with the scalar product $(\,\cdot\,,\,\cdot\,)_0$ (that is with $\eta := 0$).

PROPOSITION 2.1. *There exists $\eta_0 > 0$ such that, for each $\eta \in (0, \eta_0]$ there is $\omega = \omega(\eta) > 0$ such that the operator $\mathbf{A} + \omega I$ is m-dissipative in $\mathbf{E}$ endowed with the norm $\|\cdot\|_\eta$, i.e.*

$$(\mathbf{A}(\overline{u}, \overline{v}), (\overline{u}, \overline{v}))_\eta \le -\omega \|(\overline{u}, \overline{v})\|_\eta^2 \quad \textit{for each } (\overline{u}, \overline{v}) \in D(\mathbf{A}).$$

PROOF. By a direct computation, we see that, for any $(p, q) \in \mathbf{E}$, the equation $\mathbf{A}(\overline{u}, \overline{v}) = (p, q)$ admits a unique solution $(\overline{u}, \overline{v}) \in D(\mathbf{A})$, that is $\mathbf{A}$ is surjective. Further we note that, for any $(\overline{u}, \overline{v}) \in D(\mathbf{A})$ and $\eta \in (0, \beta]$, one has

$$\begin{aligned}
(\mathbf{A}(\overline{u}, \overline{v}), (\overline{u}, \overline{v}))_\eta &= (1 + \alpha\eta)(\overline{u}, \overline{v})_{1/2} + (\eta\overline{v} - A(\overline{u} + \alpha\overline{v}) - \beta\overline{v}, \eta\overline{u} + \overline{v})_0 \\
&= (1 + \alpha\eta)(\overline{u}, \overline{v})_{1/2} + \eta^2(\overline{u}, \overline{v})_0 + \eta|\overline{v}|_0^2 - \eta|\overline{u}|_{1/2}^2 - (\overline{u}, \overline{v})_{1/2} \\
&\quad - \alpha\eta(\overline{u}, \overline{v})_{1/2} - \alpha|\overline{v}|_{1/2}^2 - \beta\eta(\overline{u}, \overline{v})_0 - \beta|\overline{v}|_0^2 \\
&= -\eta|\overline{u}|_{1/2}^2 - (\beta - \eta)|\overline{v}|_0^2 - \eta(\beta - \eta)(\overline{u}, \overline{v})_0 \\
&\le -\eta|\overline{u}|_{1/2}^2 - (\beta - \eta)|\overline{v}|_0^2 + \eta(\beta - \eta)\left(\frac{\lambda_1}{2\beta}|\overline{u}|_0^2 + \frac{\beta}{2\lambda_1}|\overline{v}|_0^2\right) \\
&\le -\eta|\overline{u}|_{1/2}^2 - (\beta - \eta)|\overline{v}|_0^2 + \eta(\beta - \eta)\left(\frac{1}{2\beta}|\overline{u}|_{1/2}^2 + \frac{\beta}{2\lambda_1}|\overline{v}|_0^2\right) \\
&= -\eta\left(\frac{1}{2} + \frac{\eta}{2\beta}\right)|\overline{u}|_{1/2}^2 - (\beta - \eta)(1 - \eta\beta/2\lambda_1)|\overline{v}|_0^2.
\end{aligned}$$

Hence, for $\eta_0 := \min\{\beta/2, \lambda_1/\beta\}$, the assertion holds. □

REMARK 2.2. Proposition 2.1 along with the Lumer–Philips theorem implies that $\mathbf{A}$ generates a C_0 semigroup $\{S_{\mathbf{A}}(t)\}_{t\ge 0}$ of bounded linear operators on $\mathbf{E}$. If η_0 is as in Proposition 2.1, then

$$\|S_{\mathbf{A}}(t)(\overline{u}, \overline{v})\|_{\eta_0} \le e^{-\omega t}\|(\overline{u}, \overline{v})\|_{\eta_0}, \quad \text{for all } (\overline{u}, \overline{v}) \in \mathbf{E},$$

with $\omega = \omega(\eta_0)$.

The operator $-\mathbf{A}$ is sectorial (see [13] and Appendix of [9]). Therefore the C_0 semigroup $\{S_{\mathbf{A}}(t)\}_{t\ge 0}$ generated by $\mathbf{A}$ is analytic. In consequence, for any $(\overline{u}, \overline{v}) \in \mathbf{E}$, there exists $(u, v)\colon [0, \infty) \to \mathbf{E}$ with $(u(0), v(0)) = (\overline{u}, \overline{v})$ such that, for all $t > 0$, $(u(t), v(t)) \in D(\mathbf{A})$, (u, v) is differentiable at t into $\mathbf{E}$ and

$$(\dot{u}(t), \dot{v}(t)) = \mathbf{A}(u(t), v(t)).$$

In order to study compactness properties of the translation operator associated to the beam equation, we derive proper estimates for the semigroup generated by $\mathbf{A}$. First, recall the following standard result.

Lemma 2.3. *Let E be a Banach space, A be the generator of a C_0 semigroup $\{S_A(t)\}_{t\geq 0}$ of bounded linear operators on E and $f\colon[0,\overline{t}]\to E$, with $\overline{t}>0$, be continuous. If $u\colon[0,\overline{t}]\to E$ is continuous and such that, for each $t\in(0,\overline{t}]$, $u(t)\in D(A)$, $\dot u(t)$ exists and $\dot u(t)=Au(t)+f(t)$, then, for all $t\in[0,\overline{t}]$,*

$$u(t)=S_A(t)u(0)+\int_0^t S_A(t-s)f(s)\,ds.$$

Let $P_{X^{1/2}}\colon\mathbf{E}\to X^{1/2}$ and $P_{X^0}\colon\mathbf{E}\to X^0$ be the projections on the first and second coordinate, respectively.

Lemma 2.4. *If*

$$\lambda_1\alpha+\beta-\alpha^{-1}>\alpha^{-1}-\frac{|\beta-1/\alpha|}{\alpha(\lambda_1\alpha-2\alpha^{-1}+\beta)}>0,$$

then

$$|P_{X^{1/2}}S_{\mathbf{A}}(t)(\overline{u},\overline{v})|_{1/2}\leq e^{-\rho t}|\overline{u}|_{1/2}+\frac{Ce^{-\rho t}}{(\lambda_1\alpha-2\alpha^{-1}+\beta)^{1/2}}\|(\overline{u},\overline{v})\|_{\mathbf{E}}$$

and

$$\begin{aligned}|P_{X^0}S_{\mathbf{A}}(t)(\overline{u},\overline{v})|_0\leq C\bigg(&e^{-(\lambda_1\alpha+\beta-\alpha^{-1})t}+\frac{e^{-\rho t}}{\lambda_1^{1/2}}\\&+\frac{e^{-\rho t}}{\lambda_1^{1/2}(\lambda_1\alpha+\beta-\alpha^{-1}-\rho)}+\frac{e^{-\rho t}}{\lambda_1^{1/2}(\lambda_1\alpha-2\alpha^{-1}+\beta)}\\&\quad+\frac{e^{-\rho t}}{\lambda_1^{1/2}(\lambda_1\alpha+\beta-\alpha^{-1}-\rho)(\lambda_1\alpha-2\alpha^{-1}+\beta)^{1/2}}\bigg)\|(\overline{u},\overline{v})\|_{\mathbf{E}}\end{aligned}$$

with

$$\rho:=\alpha^{-1}-\frac{|\beta-1/\alpha|}{\alpha(\lambda_1\alpha-2\alpha^{-1}+\beta)}$$

and $C>0$ independent of A.

Proof. We will follow the proof of Lemma 2.2 of [9]. Let $(u,v)\colon[0,\infty)\to\mathbf{E}$ be given by $(u(t),v(t))=S_{\mathbf{A}}(t)(\overline{u},\overline{v})$ for fixed $(\overline{u},\overline{v})\in\mathbf{E}$, $t\geq 0$. Clearly, (u,v) is a solution of

$$\begin{cases}\dot u=v & \text{for } t>0,\\ \dot v=-A(u+\alpha v)-\beta v & \text{for } t>0.\end{cases}$$

If we define $z\colon[0,\infty)\to X^0$ by $z(t):=u(t)+\alpha v(t)$, $t\geq 0$, then (u,z) is a solution of

$$\begin{cases}\dot u=-(1/\alpha)u+(1/\alpha)z & \text{for } t>0,\\ \dot z=-\widetilde{A}z+\kappa u & \text{for } t>0,\\ u(0)=\overline{u},\quad z(0)=\overline{u}+\alpha\overline{v},\end{cases}$$

where $\kappa:=\beta-1/\alpha$ and $\widetilde{A}:=\alpha A+\kappa I$. Note that $S_{-\widetilde{A}}(t)=e^{-\kappa t}S_{-A}(\alpha t)$ and

$$\|S_{-\widetilde{A}}(t)\|_{L(X^{1/2},X^{1/2})}\leq e^{-(\lambda_1\alpha+\kappa)t}=e^{-(\lambda_1\alpha+\beta-1/\alpha)t}.$$

By Lemma 2.3, we get, for any $t > 0$,

$$\begin{aligned} u(t) &= e^{-t/\alpha}\overline{u} + \frac{1}{\alpha}\int_0^t e^{-\alpha^{-1}(t-s)} z(s)\,ds \\ &= e^{-t/\alpha}\overline{u} + \frac{1}{\alpha}\int_0^t e^{-\alpha^{-1}(t-s)} \Big(S_{-\widetilde{A}}(s)\overline{z} + \kappa\int_0^s S_{-\widetilde{A}}(s-r)u(r)\,dr\Big)\,ds \\ &= e^{-t/\alpha}\overline{u} + \frac{1}{\alpha}\int_0^t e^{-\alpha^{-1}(t-s)} S_{-\widetilde{A}}(s)\overline{z}\,ds \\ &\quad + \frac{\kappa}{\alpha}\int_0^t\int_0^s e^{-\alpha^{-1}(t-s)} S_{-\widetilde{A}}(s-r)u(r)\,dr\,ds. \end{aligned}$$

Hence

$$|u(t)|_{1/2} \le e^{-\alpha^{-1}t}|\overline{u}|_{1/2} + \frac{1}{\alpha}\,\Theta_1 + \frac{|\kappa|}{\alpha}\,\Theta_2 \tag{2.2}$$

where

$$\Theta_1 := \int_0^t e^{-\alpha^{-1}(t-s)} |S_{-\widetilde{A}}(s)\overline{z}|_{1/2}\,ds$$

and

$$\Theta_2 := \int_0^t\int_0^s e^{-\alpha^{-1}(t-s)} |S_{-\widetilde{A}}(s-r)u(r)|_{1/2}\,dr\,ds.$$

We estimate

$$\begin{aligned} \Theta_1 &= \int_0^t e^{-\alpha^{-1}(t-s)} e^{-\kappa s} |A^{1/2} S_{-A}(\alpha s)\overline{z}|_0 ds \\ &\le \int_0^t e^{-\alpha^{-1}(t-s)} e^{-\kappa s} C_{1/2}\,\frac{e^{-\lambda_1\alpha s}}{(\alpha s)^{1/2}}|\overline{z}|_0 ds \\ &= \frac{C_{1/2}\,e^{-\alpha^{-1}t}}{\alpha^{1/2}}\int_0^t s^{-1/2} e^{-(\lambda_1\alpha-\alpha^{-1}+\kappa)s}|\overline{z}|_0 ds \\ &\le \frac{C_{1/2}\,e^{-\alpha^{-1}t}}{\alpha^{1/2}(\lambda_1\alpha-\alpha^{-1}+\kappa)^{1/2}}|\overline{z}|_0\int_0^\infty \tau^{-1/2}e^{-\tau}\,d\tau \\ &= \frac{C_{1/2}\sqrt{\pi}}{\alpha^{1/2}(\lambda_1\alpha-\alpha^{-1}+\kappa)^{1/2}}\cdot e^{-\alpha^{-1}t}|\overline{z}|_0, \end{aligned}$$

where $C_{1/2} > 0$ is a constant independent of A, and

$$\begin{aligned} \Theta_2 &\le \int_0^t\int_0^s e^{-\alpha^{-1}(t-s)} e^{-(\lambda_1\alpha+\kappa)(s-r)}|u(r)|_{1/2}\,dr\,ds \\ &= e^{-\alpha^{-1}t}\int_0^t e^{(\lambda_1\alpha+\kappa)r}|u(r)|_{1/2}\bigg(\int_r^t e^{-(\lambda_1\alpha-\alpha^{-1}+\kappa)s}\,ds\bigg)\,dr \\ &\le e^{-\alpha^{-1}t}\int_0^t e^{(\lambda_1\alpha+\kappa)r}|u(r)|_{1/2}\frac{e^{-(\lambda_1\alpha-\alpha^{-1}+\kappa)r}}{\lambda_1\alpha-\alpha^{-1}+\kappa}\,dr \\ &\le \frac{e^{-\alpha^{-1}t}}{\lambda_1\alpha-\alpha^{-1}+\kappa}\int_0^t e^{\alpha^{-1}r}|u(r)|_{1/2}\,dr. \end{aligned}$$

Hence combining the obtained estimates and (2.2), we get, for all $t \geq 0$,

$$e^{\alpha^{-1}t}|u(t)|_{1/2} \leq |\overline{u}|_{1/2} + C_1\,|\overline{z}|_0 + \frac{|\kappa|}{\alpha(\lambda_1\alpha - \alpha^{-1} + \kappa)} \int_0^t e^{\alpha^{-1}r}|u(r)|_{1/2}\,dr$$

where

$$C_1 := \frac{C_{1/2}\sqrt{\pi}}{\alpha^{3/2}(\lambda_1\alpha - \alpha^{-1} + \kappa)^{1/2}}.$$

And this, by use of the Gronwall inequality, gives

$$|u(t)|_{1/2} \leq \big(|\overline{u}|_{1/2} + C_1\,|\overline{z}|_0\big)\,e^{-\rho t}, \quad \text{for all } t \geq 0.$$

Next we get

$$\begin{aligned}
|z(t)|_0 &\leq |S_{-\widetilde{A}}(t)\overline{z}|_0 + |\kappa| \int_0^t |S_{-\widetilde{A}}(t-s)u(s)|_0\,ds \\
&\leq e^{-(\lambda_1\alpha+\beta-\alpha^{-1})t}|\overline{z}|_0 + |\kappa| \int_0^t e^{-(\lambda_1\alpha+\beta-\alpha^{-1})(t-s)}|u(s)|_0\,ds
\end{aligned}$$

and further

$$\begin{aligned}
&\int_0^t e^{-(\lambda_1\alpha+\beta-\alpha^{-1})(t-s)}|u(s)|_0\,ds \\
&\quad \leq \lambda_1^{-1/2} \int_0^t e^{-(\lambda_1\alpha+\beta-\alpha^{-1})(t-s)}|u(s)|_{1/2}\,ds \\
&\quad \leq \lambda_1^{-1/2} \int_0^t e^{-(\lambda_1\alpha+\beta-\alpha^{-1})(t-s)}(|\overline{u}|_{1/2} + C_1\,|\overline{z}|_0)e^{-\rho s}\,ds \\
&\quad \leq \lambda_1^{-1/2}(|\overline{u}|_{1/2} + C_1\,|\overline{z}|_0)e^{-(\lambda_1\alpha+\beta-\alpha^{-1})t} \int_0^t e^{(\lambda_1\alpha+\beta-\alpha^{-1}-\rho)s}\,ds \\
&\quad \leq \frac{e^{-\rho t}}{\lambda_1^{1/2}(\lambda_1\alpha+\beta-\alpha^{-1}-\rho)}(|\overline{u}|_{1/2} + C_1\,|\overline{z}|_0),
\end{aligned}$$

which implies the assertion as $|v(t)|_0 \leq \alpha^{-1}|z(t)|_0 + \alpha^{-1}\lambda_1^{-1/2}|u(t)|_{1/2}$, $t \geq 0$. □

Let X_n, $n \geq 1$, be the space spanned by $e_1, \ldots, e_n$ and $\widetilde{X}_n^{1/2}$ and $\widetilde{X}_n^0$ its orthogonal complements in $X^{1/2}$ and X^0, respectively.

Lemma 2.5. *For each $n \geq 1$ and $t \geq 0$,*

$$S_{\mathbf{A}}(t)(X_n \times X_n) \subset X_n \times X_n \quad \textit{and} \quad S_{\mathbf{A}}(t)(\widetilde{X}_n^{1/2} \times \widetilde{X}_n^0) \subset \widetilde{X}_n^{1/2} \times \widetilde{X}_n^0.$$

Proof. By Proposition 2.1, one has $(0,\infty) \subset \rho(\mathbf{A})$. Therefore the Euler exponential formula states

$$S_{\mathbf{A}}(t)(\overline{u},\overline{v}) = \lim_{k\to\infty}(I - (t/k)\mathbf{A})^{-k}(\overline{u},\overline{v})$$

for all $t > 0$ and $(\overline{u},\overline{v}) \in \mathbf{E}$. A straightforward computation shows that the closed spaces $X_n \times X_n$ and $\widetilde{X}_n^{1/2} \times \widetilde{X}_n^0$ are invariant with respect to $(I - (t/k)\mathbf{A})^{-1}$. Hence, the assertion. □

Lemma 2.6. *There exist $n_0 \geq 1$ and a constant $\widetilde{C} > 0$ such that, for any $n \geq n_0$, $\overline{v} \in \widetilde{X}_n^0$ and $t \geq 0$,*

$$|P_{X^{1/2}} S_{\mathbf{A}}(t)(0, \overline{v})|_{1/2} \leq \frac{\widetilde{C} e^{-\rho_n t}}{\lambda_{n+1}^{1/2}} |\overline{v}|_0$$

and

$$|P_{X^0} S_{\mathbf{A}}(t)(0, \overline{v})|_0 \leq \widetilde{C}\bigg(e^{-(\lambda_n \alpha + \beta - \alpha^{-1})t} + \frac{e^{-\rho_n t}}{\lambda_{n+1}^{1/2}}\bigg)|\overline{v}|_0$$

with

$$\rho_n := \alpha^{-1} - \frac{|\beta - 1/\alpha|}{\alpha(\lambda_{n+1}\alpha - 2\alpha^{-1} + \beta)}.$$

Proof. Let A_n, $n \geq 1$, be the part of the operator A in $\widetilde{X}_n^0$, i.e. $A_n \colon D(A_n) \to \widetilde{X}_n^0$, $D(A_n) := \widetilde{X}_n^0 \cap D(A)$, $A_n \overline{u} := A\overline{u}$, $\overline{u} \in D(A_n)$. Now apply Lemma 2.4 for A_n and $\mathbf{A}$ defined on $\mathbf{E} := \widetilde{X}_n^{1/2} \times \widetilde{X}_n^0$ by (1.6) with $A := A_n$. The first eiegenvalue of A_n is λ_{n+1} and, since $\lambda_n \to \infty$, for suffieciently large n, the assumptions of Lemma 2.4 are fulfilled and the desired estimates follow. □

Proposition 2.7. *For any bounded $V \subset X^0$ and $t > 0$, the set $S_{\mathbf{A}}(t)(\{0\} \times V)$ is relatively compact.*

Proof. Choose $R > 0$ so that $V \subset B_{X^0}(0, R)$ and take any $t > 0$ and $\{(\overline{v}_k)\}_{k \geq 1} \subset V$. It is convenient to use the following formula

$$\gamma(\{S_{\mathbf{A}}(t)(0, \overline{v}_k)\}_{k \geq 1}) = \lim_{n \to \infty} \Big(\sup_{k \geq 1} d(S_{\mathbf{A}}(t)(0, \overline{v}_k), X_n^{1/2} \times X_n^0)\Big)$$

(see e.g. [10]) where γ stands for the Hausdorff measure of noncompactness in $\mathbf{E}$. Due to Lemma 2.5, one gets

$$\begin{aligned} d(S_{\mathbf{A}}(t)(0, \overline{v}_k), X_n \times X_n) &\leq \|S_{\mathbf{A}}(t)(0, \overline{v}_k) - S_{\mathbf{A}}(t)(0, P_n \overline{v}_k)\|_{\mathbf{E}} \\ &\leq \|S_{\mathbf{A}}(t)(0, Q_n \overline{v}_k)\|_{\mathbf{E}} \leq |P_{X^{1/2}} S_{\mathbf{A}}(t)(0, Q_n \overline{v}_k)|_{1/2} + |P_{X^0} S_{\mathbf{A}}(t)(0, Q_n \overline{v}_k)|_0 \end{aligned}$$

where $P_n \colon X_n^0 \to X_n$ and $Q_n \colon X_n^0 \to \widetilde{X}_n^0$ are the orthogonal projections. Further, application of Lemma 2.6, gives, for large n,

$$|P_{X^{1/2}} S_{\mathbf{A}}(t)(0, Q_n \overline{v}_k)|_{1/2} \leq \frac{\widetilde{C} e^{-\rho_n t}}{\lambda_{n+1}^{1/2}} |\overline{v}_k|_0 \leq \frac{\widetilde{C} R}{\lambda_{n+1}^{1/2}} \to 0 \quad \text{as } n \to \infty$$

and

$$|P_{X^0} S_{\mathbf{A}}(t)(0, Q_n \overline{v}_k)|_0 \leq \bigg(e^{-(\lambda_n \alpha + \beta - \alpha^{-1})t} + \frac{1}{\lambda_{n+1}^{1/2}}\bigg) R \to 0 \quad \text{as } n \to \infty.$$

This implies that $\gamma(\{S_{\mathbf{A}}(t)(0, \overline{v}_k)\}_{k \geq 1}) = 0$ and the proof is completed. □

3. Krasnosel'skiĭ type formula

We shall derive a Krasnosel'skiĭ type result for the autonomous equation

$$(\dot{u},\dot{v}) = \mathbf{A}(u,v) + (0, F(u,v)) \tag{3.1}$$

where $\mathbf{A}$ is given by (2.1) and $F\colon X^{1/2}\times X^0 \to X^0$. We shall consider *mild solutions* of the system. Recall that a continuous function $(u,v)\colon [t_0, t_0+r)\to \mathbf{E}$, $t_0\in\mathbb{R}$ and $r>0$, is a mild solution of (3.1) provided

$$(u(t),v(t)) = S_{\mathbf{A}}(t-t_0)(u(t_0),v(t_0)) + \int_{t_0}^{t} S_{\mathbf{A}}(t-s)(0,F(u(s),v(s)))\,ds$$

for all $t\in[t_0,t_0+r)$.

At this point let us remark that $\mathbf{A}$ is invertible and

$$\mathbf{A}^{-1}(0,\overline{v}) = (-A^{-1}\overline{v},0), \quad \overline{v}\in X^0, \tag{3.2}$$

which, due to the compactness of $A^{-1}\colon X^0\to X^{1/2}$, means that the superposition $\mathbf{A}^{-1}(0,F)$ is a completely continuous map of $\mathbf{E}$.

Theorem 3.1. *Assume that $F\colon X^{1/2}\times X^0\to X^0$ is locally Lipschitz, bounded on bounded sets and such that*

$$\begin{cases} \text{for any } t>0 \text{ and } (\overline{u},\overline{v})\in\mathbf{E}, \text{ the equation (3.1)} \\ \text{admits a unique solution } (u(\,\cdot\,;\overline{u},\overline{v}),v(\,\cdot\,;\overline{u},\overline{v}))\colon[0,t]\to\mathbf{E} \\ \text{satisfying } (u(0;\overline{u},\overline{v}),v(0;\overline{u},\overline{v}))=(\overline{u},\overline{v}) \end{cases} \tag{3.3}$$

and

$$\begin{cases} \text{for any } R>0 \text{ and } t>0, \text{ there exists } C=C(R,t)>0 \\ \text{such that } \|(u(s;\overline{u},\overline{v}),v(s;\overline{u},\overline{v}))\|_{\mathbf{E}}\le C, \text{ for all } s\in[0,t], \\ \text{whenever } \|(\overline{u},\overline{v})\|_{\mathbf{E}}\le R. \end{cases} \tag{3.4}$$

Let $\Phi_t\colon\mathbf{E}\to\mathbf{E}$, $t>0$, be the translation operator for (3.1), i.e. $\Phi_t(\overline{u},\overline{v}) = (u(t;\overline{u},\overline{v}),v(t;\overline{u},\overline{v}))$, $(\overline{u},\overline{v})\in\mathbf{E}$. If $\mathbf{U}\subset\mathbf{E}$ is open bounded and

$$\mathbf{A}(\overline{u},\overline{v}) + (0,F(\overline{u},\overline{v}))\ne 0 \quad \text{for all } (\overline{u},\overline{v})\in\partial\mathbf{U}\cap D(\mathbf{A}),$$

then there exists $\overline{t}>0$ such that, for all $t\in(0,\overline{t}]$, $\Phi_t(\overline{u},\overline{v})\ne(\overline{u},\overline{v})$, for any $(\overline{u},\overline{v})\in\partial\mathbf{U}$, and

$$\operatorname{ind}(\Phi_t,\mathbf{U}) = \deg(I+\mathbf{A}^{-1}(0,F),\mathbf{U})$$

where ind *stands for the fixed point index due to Sadovskiĭ (see* [1] *and also* [14]) *and* deg *is the Leray–Schauder topological degree with respect to* 0.

Before passing to the proof, we recall three general results, which are stated here for easy reference.

PROPOSITION 3.2 (see Proposition 2.7 of [6]). *Let A be the generator of a C_0 semigroup of bounded linear operators on a Banach space E. Suppose that $\{\overline{u}_n\}_{n\geq 1} \subset E$ is relatively compact and $\{w_n\}_{n\geq 1} \subset L^1([0,l],E)$, $l > 0$, is uniformly integrable* (*i.e. for any $\varepsilon > 0$, there is $\delta > 0$ such that, for any measurable $J \subset [0,l]$ with the Lebesgue measure $|J| \leq \delta$ and any $n \geq 1$, one has $\int_J \|w_n(t)\| dt \leq \varepsilon$*). *Define $u_n\colon [0,l] \to E$, $n \geq 1$, by*

$$u_n(t) := S_A(t)\overline{u}_n + \int_0^t S_A(t-s)w_n(s)\,ds, \quad t \in [0,l].$$

Then the following conditions are equivalent

(a) *$\{u_n(t)\}_{n\geq 1}$ is relatively compact for each $t \in [0,l]$;*
(b) *$\{u_n\}_{n\geq 1}$ is relatively compact in the space $C([0,l],E)$* (*with the uniform convergence norm*).

PROPOSITION 3.3 (see Proposition 2.2 of [7]). *Let A be the generator of a C_0 semigroup of bounded linear operators on a Banach space E and let $F\colon [0,\infty) \times E \times P \to E$, where P is a compact metric space of parameters, be continuous and locally Lipschitz in the second variable uniformly with respect to parameter. Suppose that, for each $\overline{u} \in E$ and $\mu \in P$, there exists a unique mild solution $u(\,\cdot\,,\overline{u},\mu)\colon [0,\infty) \to E$ of*

$$\dot{u}(t) = Au(t) + F(t,u(t),\mu)$$

and that, for any relatively compact $V \subset E$ and $t > 0$, the set

$$\{u(t;\overline{u},\mu) \mid \overline{u} \in V,\ \mu \in P\}$$

is relatively compact. Then $u(t_n;\overline{u}_n,\mu_n) \to u(t;\overline{u},\mu)$ as $n \to \infty$ whenever $t_n \to t$ in $[0,\infty)$, $\overline{u}_n \to \overline{u}$ in E and $\mu_n \to \mu$ in P as $n \to \infty$.

THEOREM 3.4 (see Theorem 5.2 of [8]). *Suppose $A\colon D(A) \to E$ is a strongly m-dissipative operator in a separable Banach space E and $F\colon E \to E$ is a locally Lipschitz compact map. Let $\Phi_t\colon E \to E$, $t > 0$, be the translation along trajectories operator for*

$$\dot{u}(t) = Au(t) + F(u(t)), \quad t > 0.$$

If $U \subset E$ is open bounded and $A\overline{u} + F(\overline{u}) \neq 0$ for all $\overline{u} \in \partial U \cap D(A)$, then there exists $t_0 > 0$ such that, for each $t \in (0,t_0]$, $\Phi_t(\overline{u}) \neq \overline{u}$, for all $\overline{u} \in \partial U$, and

$$\operatorname{ind}(\Phi_t,U) = \deg(I + A^{-1}F,U).$$

PROOF OF THEOREM 3.1. The idea of the proof is to reduce the problem to the case with a finite dimensional nonlinearity, to which Theorem 3.4 applies. We shall proceed in a few steps.

Step 1. Note that, by (3.4), for a fixed $\widetilde{t} > 0$, we get $C > 0$ such that

$$\|(u(t;\overline{u},\overline{v}), v(t;\overline{u},\overline{v}))\|_{\mathbf{E}} \leq C \quad \text{for all } t \in [0,\widetilde{t}] \text{ and } (\overline{u},\overline{v}) \in \overline{\mathbf{U}}.$$

Let $\widetilde{F}: X^{1/2} \times X^0 \to X^0$ be given by

$$\widetilde{F}(\overline{u},\overline{v}) := \rho(\|(\overline{u},\overline{v})\|_{\mathbf{E}}) F(\overline{u},\overline{v}), \quad \text{for } (\overline{u},\overline{v}) \in X^{1/2} \times X^0$$

where $\rho: [0,\infty) \to \mathbb{R}$ is a locally Lipschitz function such that $\rho(s) = 1$ if $s \in [0,C]$ and $\rho(s) = 0$ if $s > 2C$. Then $\widetilde{F}$ is locally Lipschitz, bounded and one can easily see that, for each $(\overline{u},\overline{v}) \in \overline{\mathbf{U}}$, the mild solution $(u(\cdot;\overline{u},\overline{v}), v(\cdot;\overline{u},\overline{v})): [0,\widetilde{t}] \to \mathbf{E}$ is also a mild solution of

$$(\dot{u},\dot{v}) = \mathbf{A}(u,v) + (0,\widetilde{F}(u,v)), \quad t \in [0,\widetilde{t}].$$

Therefore the translations along trajectories operators of (3.3) and the above equation coincide (for times from $[0,\widetilde{t}]$).

Thus, we may assume, without loss of generality, that F is bounded.

Step 2. Let $F_n: X^{1/2} \times X^0 \times [0,1] \to X^0$ be given by

$$F_n(\overline{u},\overline{v},\mu) := (\mu I + (1-\mu)P_n)F(\overline{u},\overline{v}), \quad \overline{u} \in X^{1/2},\ \overline{v} \in X^0,\ \mu \in [0,1],$$

where $P_n: X^0 \to X_n$ is the orthogonal projection. Clearly F_n, $n \geq 1$, are bounded and locally Lipschitz.

We claim that there exists $n_0 \geq 1$ such that, for each $n \geq n_0$,

$$\mathbf{A}(\overline{u},\overline{v}) + (0, F_n(\overline{u},\overline{v},\mu)) \neq 0 \quad \text{for all } (\overline{u},\overline{v}) \in \partial\mathbf{U} \cap D(\mathbf{A}) \text{ and } \mu \in [0,1].$$

To see this, suppose to the contrary that there exist $(\overline{u}_k,\overline{v}_k) \in \partial\mathbf{U} \cap D(\mathbf{A})$, $\mu_k \in [0,1]$ and integers $n_k \geq 1$, $k \geq 1$, such that $\mathbf{A}(\overline{u}_k,\overline{v}_k) + (0, F_{n_k}(\overline{u}_k,\overline{v}_k,\mu_k)) = 0$, for all $k \geq 1$, and $n_k \to \infty$ as $k \to \infty$. This implies that $(\overline{u}_k,\overline{v}_k) = -\mathbf{A}^{-1}(0, F_{n_k}(\overline{u}_k,\overline{v}_k,\mu_k))$ and since $(F_{n_k}(\overline{u}_k,\overline{v}_k,\mu_k))$ is bounded, we get a subsequence of $((\overline{u}_k,\overline{v}_k))$, denoted again by $((\overline{u}_k,\overline{v}_k))$, convergent to some $(\overline{u},\overline{v}) \in \partial\mathbf{U}$. We may also assume that $\mu_k \to \mu$ for some $\mu \in [0,1]$. Since

$$F_{n_k}(\overline{u}_k,\overline{v}_k,\mu_k) = (\mu_k I + (1-\mu_k)P_{n_k})F(\overline{u}_k,\overline{v}_k) \to F(\overline{u},\overline{v}),$$

we infer that

$$(\overline{u},\overline{v}) = -\mathbf{A}^{-1}(0, F(\overline{u},\overline{v})),$$

i.e. $\mathbf{A}(\overline{u},\overline{v}) + (0, F(\overline{u},\overline{v})) = 0$, a contradiction proving the claim.

Hence, using the homotopy property, we get

$$\deg(I + \mathbf{A}^{-1}(0,F)) = \deg(I + \mathbf{A}^{-1}(0,P_nF)) \quad \text{for all } n \geq n_0. \tag{3.5}$$

Step 3. For $t > 0$ and $n \geq 1$, define $\Theta_t^{(n)}: \overline{\mathbf{U}} \times [0,1] \to \mathbf{E}$ by

$$\Theta_t^{(n)}(\overline{u},\overline{v},\mu) := (u(t), v(t))$$

where $(u,v)\colon [0,\infty) \to \mathbf{E}$ is the mild solution of

$$(\dot{u},\dot{v}) = \mathbf{A}(u,v) + (0, F_n(u,v,\mu)), \quad (u(0),v(0)) = (\overline{u},\overline{v}). \tag{3.6}$$

We claim that, for any $t>0$ and $n \geq 1$,

$$\gamma_{\eta_0}(\Theta_t^{(n)}(\mathbf{V}\times[0,1])) \leq e^{-\omega t}\gamma_{\eta_0}(\mathbf{V}) \quad \text{for all } \mathbf{V}\subset \overline{\mathbf{U}} \tag{3.7}$$

where $\eta_0 > 0$ and $\omega = \omega(\eta_0) > 0$ come from Proposition 2.1 and γ_{η_0} is the Hausdorff measure of noncompactness in $\mathbf{E}$ endowed with the norm $\|\cdot\|_{\eta_0}$. To verify it, note that, since F is bounded, there is $C>0$ such that

$$|F_n(\overline{u},\overline{v},\mu)|_0 \leq C \quad \text{for all } (\overline{u},\overline{v})\in\mathbf{E},\ \mu\in[0,1], n\geq 1. \tag{3.8}$$

Fixing an arbitrary $\varepsilon\in(0,t)$ and using the Duhamel formula give

$$\begin{aligned}\Theta_t^{(n)}(\overline{u},\overline{v},\mu) &= S_{\mathbf{A}}(t)(\overline{u},\overline{v}) + \int_0^t S_{\mathbf{A}}(t-s)(0,F_n(\Theta_s^{(n)}(\overline{u},\overline{v},\mu),\mu))ds\\ &= S_{\mathbf{A}}(t)(\overline{u},\overline{v}) + \int_0^{t-\varepsilon} S_{\mathbf{A}}(t-\varepsilon-s)S_{\mathbf{A}}(\varepsilon)(0,F_n(\Theta_s^{(n)}(\overline{u},\overline{v},\mu),\mu))\,ds\\ &\quad+\int_{t-\varepsilon}^t S_{\mathbf{A}}(t-s)(0,F_n(\Theta_s^{(n)}(\overline{u},\overline{v},\mu),\mu))\,ds\end{aligned}$$

for any $n\geq 1$, $t>0$, $(\overline{u},\overline{v})\in\mathbf{E}$ and $\mu\in[0,1]$. Furthermore, (3.8) implies

$$\int_0^{t-\varepsilon} S_{\mathbf{A}}(t-\varepsilon-s)S_{\mathbf{A}}(\varepsilon)(0,F_n(\Theta_s^{(n)}(\overline{u},\overline{v},\mu),\mu))\,ds \in (t-\varepsilon)\cdot\overline{\operatorname{conv}}\,\widetilde{\mathbf{V}}$$

with $\widetilde{\mathbf{V}} := \{S_{\mathbf{A}}(\tau)(\overline{u},\overline{v}) \mid \tau\in[0,t-\varepsilon],\ (\overline{u},\overline{v})\in\widetilde{\mathbf{V}}_1\}$ and $\widetilde{\mathbf{V}}_1 := S_{\mathbf{A}}(\varepsilon)(\{0\}\times B_{X^0}(0,C))$. In view of Proposition 2.7, $\widetilde{\mathbf{V}}_1$ is relatively compact and, consequently, so is $\widetilde{\mathbf{V}}$. Therefore, using Proposition 2.1 with Remark 2.2 and (3.8), one gets, for any $\varepsilon\in(0,t)$,

$$\begin{aligned}\gamma_{\eta_0}\bigg(\bigcup_{n\geq 1}\Theta_t^{(n)}(\mathbf{V}\times[0,1])\bigg) &\leq e^{-\omega t}\gamma_{\eta_0}(\mathbf{V}) + \varepsilon C + \gamma_{\eta_0}((t-\varepsilon)\cdot\overline{\operatorname{conv}}\,\widetilde{\mathbf{V}})\\ &\leq e^{-\omega t}\gamma_{\eta_0}(\mathbf{V}) + \varepsilon C.\end{aligned}$$

Since $\varepsilon\in(0,t)$ can be taken arbitrarily small, it yields

$$\gamma_{\eta_0}\bigg(\bigcup_{n\geq 1}\Theta_t^{(n)}(\mathbf{V}\times[0,1])\bigg) \leq e^{-\omega t}\gamma_{\eta_0}(\mathbf{V}) \tag{3.9}$$

and, in particular, (3.7).

Now the continuity of $\Theta_t^{(n)}$, for $t>0$ and $n\geq 1$, follows from Proposition 3.3.

Step 4. There exist $t_1 > 0$ and $n_1 \geq n_0$ such that, for all $n\geq n_1$ and $t\in(0,t_1]$,

$$\Theta_t^{(n)}(\overline{u},\overline{v},\mu) \neq (\overline{u},\overline{v}) \quad \text{for any } (\overline{u},\overline{v})\in\partial\mathbf{U},\ \mu\in[0,1]. \tag{3.10}$$

Suppose to the contrary that there exist a sequence of integers (n_k), (t_k) in $(0,\infty)$, $((\overline{u}_k,\overline{v}_k))$ in $\partial\mathbf{U}$ and (μ_k) in $[0,1]$ such that

$$\Theta_{t_k}^{(n_k)}(\overline{u}_k,\overline{v}_k,\mu_k)=(\overline{u}_k,\overline{v}_k),\quad \text{for each } k\ge 1,$$

$t_k\to 0$ and $n_k\to\infty$ as $k\to\infty$. Let $N\ge 1$ be arbitrary and put $s_k(N):=([N/t_k]+1)t_k-N$, $k\ge 1$. Clearly, for all $k\ge 1$

$$(\overline{u}_k,\overline{v}_k)=\Theta_{([N/t_k]+1)t_k}^{(n_k)}(\overline{u}_k,\overline{v}_k,\mu_k)=\Theta_N^{(n_k)}(\Theta_{s_k(N)}^{(n_k)}(\overline{u}_k,\overline{v}_k,\mu_k),\mu_k),$$

therefore $\{(\overline{u}_k,\overline{v}_k)\}_{k\ge1}\subset\bigcup_{k\ge1}\Theta_N^{(n_k)}(\mathbf{W}\times[0,1])$ with

$$\mathbf{W}:=\{\Theta_s^{(n)}(\overline{u},\overline{v},\mu)\mid(\overline{u},\overline{v})\in\overline{\mathbf{U}},\ \mu\in[0,1],\ n\ge1,\ s\in[0,\widetilde{t}]\}$$

where $\widetilde{t}:=\sup_{k\ge1}t_k\ge\sup_{k\ge1}s_k(N)$. Observe that (3.8) implies that $\mathbf{W}$ is bounded. This, in view of (3.9), yields

$$\gamma_{\eta_0}(\{(\overline{u}_k,\overline{v}_k)\}_{k\ge1})\le e^{-\omega N}\gamma_{\eta_0}(\mathbf{W}).$$

Since N was arbitrary, it entails $\gamma_{\eta_0}(\{(\overline{u}_k,\overline{v}_k)\}_{k\ge1})=0$, which allows us to assume that $(\overline{u}_k,\overline{v}_k)\to(\overline{u}_0,\overline{v}_0)$ in $\mathbf{E}$ for some $(\overline{u}_0,\overline{v}_0)\in\partial\mathbf{U}$ and that $\mu_k\to\mu_0$ in $[0,1]$. Let $(u_k,v_k)\colon[0,\infty)\to\mathbf{E}$, $k\ge1$, be the mild solution of (3.6) with $n=n_k$, $\mu=\mu_k$ and $(u_k(0),v_k(0))=(\overline{u}_k,\overline{u}_k)$. Clearly, it follows from (3.9) that $\{(u_k(t),v_k(t))\}_{k\ge1}$ is relatively compact for any $t\ge0$. Consequently, by use of Proposition 3.2, we may assume that $((u_k,v_k))$ converges uniformly on bounded intervals to some $(u,v)\colon[0,\infty)\to\mathbf{E}$. Letting $k\to\infty$ in the formula

$$(u_k(t),v_k(t))=S_{\mathbf{A}}(t)(\overline{u}_k,\overline{v}_k)+\int_0^t S_{\mathbf{A}}(t-s)(0,F_{n_k}(u_k(s),v_k(s),\mu_k))\,ds$$

for all $t\ge0$, we get

$$(u(t),v(t))=S_{\mathbf{A}}(t)(\overline{u},\overline{v})+\int_0^t S_{\mathbf{A}}(t-s)(0,F(u(s),v(s)))\,ds\quad\text{for all } t\ge0,$$

i.e. (u,v) is a mild solution of (3.1). Since (u_k,v_k) are t_k-periodic, $k\ge1$, and $((u_k,v_k))$ converges uniformly on bounded subsets, one has, for each $t>0$ and $k\ge1$,

$$\begin{aligned}\|(u_k(t),v_k(t))-(\overline{u}_0,\overline{v}_0)\|_{\mathbf{E}}\le\|(u_k(t),v_k(t))-(u_k([t/t_k]t_k),v_k([t/t_k]t_k))\|_{\mathbf{E}}\\ +\|(\overline{u}_k,\overline{v}_k)-(\overline{u}_0,\overline{v}_0)\|_{\mathbf{E}}\end{aligned}$$

and, letting $k\to\infty$, we infer that (u,v) is a constant function equal to $(\overline{u}_0,\overline{v}_0)$. Consequently, since a constant mild solution must be an equilibrium, we see that $(\overline{u}_0,\overline{v}_0)\in D(\mathbf{A})$ and $\mathbf{A}(\overline{u}_0,\overline{v}_0)+(0,F(\overline{u}_0,\overline{v}_0))=0$, which contradicts the assumption and proves the claim.

Hence, by (3.10) and the homotopy invariance of fixed point index, for all $n \geq n_1$ and $t \in (0, t_1]$,

$$\operatorname{ind}(\Phi_t, \mathbf{U}) = \operatorname{ind}(\Theta_t^{(n)}(\,\cdot\,, 1), \mathbf{U}) = \operatorname{ind}(\Theta_t^{(n)}(\,\cdot\,, 0), \mathbf{U}) = \operatorname{ind}(\Phi_t^{(n)}, \mathbf{U}) \tag{3.11}$$

where $\Phi_t^{(n)}$ is the translation along trajectories operator for the equation

$$(\dot{u}, \dot{v}) = \mathbf{A}(u, v) + (0, P_n F(u, v)).$$

Step 5. Fix $n \geq n_1$. Since $P_n F$ is completely continuous we can use Theorem 3.4 to obtain $t_0 \in (0, t_1]$ such that, for all $t \in (0, t_0]$,

$$\operatorname{ind}(\Phi_t^{(n)}, \mathbf{U}) = \deg(I + \mathbf{A}^{-1}(0, P_n F), \mathbf{U}).$$

Finally, by combining it with (3.11) and (3.5), one finally has

$$\operatorname{ind}(\Phi_t, \mathbf{U}) = \operatorname{ind}(\Phi_t^{(n)}, U) = \deg(I + \mathbf{A}^{-1}(0, P_n F), \mathbf{U}) = \deg(I + \mathbf{A}^{-1}(0, F), \mathbf{U}),$$

which completes the proof. □

4. Averaging principle for periodic solutions

The following abstract averaging principle is a version of Henry's result from [12].

Theorem 4.1. *Let A be the generator of a C_0 semigroup of bounded linear operators on a Banach space E and let $G\colon [0, \infty) \times E \times P \to E$, where P is a compact metric space of parameters, be continuous and locally Lipschitz in the second variable and such that, for some $T > 0$,*

$$G(t + T, \overline{u}, \mu) = G(t, \overline{u}, \mu) \quad \text{for all } t \geq 0,\ \overline{u} \in E,\ \mu \in P.$$

Suppose that

$$\begin{cases} \text{for each } \overline{u} \in E,\ \mu \in P \text{ and } \lambda > 0, \text{ there is a unique mild solution} \\ u(\,\cdot\,; \overline{u}, \mu, \lambda)\colon [0, \infty) \to E \text{ of } \dot{u}(t) = Au(t) + G(t/\lambda, u(t), \mu),\ u(0) = \overline{u}; \end{cases} \tag{4.1}$$

$$\begin{cases} \text{the set } \{u(s; \overline{u}, \mu, \lambda) \mid s \in [0, t],\ \overline{u} \in V,\ \mu \in P,\ \lambda > 0\} \text{ is bounded} \\ \text{for any bounded } V \subset E \text{ and } t \geq 0; \end{cases} \tag{4.2}$$

$$\begin{cases} \text{the set } \{u(t; \overline{u}, \mu, \lambda) \mid \overline{u} \in V_0,\ \mu \in P,\ \lambda > 0\} \text{ is relatively compact} \\ \text{for any relatively compact set } V_0 \subset E \text{ and } t \geq 0, \end{cases} \tag{4.3}$$

and for $\widehat{G}\colon E \times P \to E$, given by $\widehat{G}(\overline{u}, \mu) := (1/T) \int_0^T G(\tau, \overline{u}, \mu)\, d\tau$, $\overline{u} \in E$, $\mu \in P$, assume that

$$\begin{cases} \text{for each } \overline{u} \in E \text{ and } \mu \in P \text{ the problem} \\ \qquad \dot{u}(t) = Au(t) + \widehat{G}(\overline{u}, \mu), \quad u(0) = \overline{u} \\ \text{admits a unique mild solution } \widehat{u}(\,\cdot\,; \overline{u}, \mu)\colon [0, \infty) \to E. \end{cases} \tag{4.4}$$

Then $u(t_n;\overline{u}_n,\mu_n,\lambda_n)\to\widehat{u}(t,\overline{u},\mu)$ *whenever* $t_n\to t$ *in* $[0,\infty)$, $\overline{u}_n\to\overline{u}$ *in* E, $\mu_n\to\mu$ *in* P *and* $\lambda_n\to 0^+$ *as* $n\to\infty$.

Proof. We shall deduce the result from Theorem 3.2 of [7] where G is required to have sublinear growth. Indeed, take $l>\sup_{n\geq 1}t_n$ and let $R>0$ be such that $\{\overline{u}_n\}_{n\geq 1}\subset B(0,R)$. In view of (4.2), there exists $C>0$ such that

$$\|u(t;\overline{u},\mu,\lambda)\|\leq C\quad \text{for all } t\in[0,l],\ \overline{u}\in B(0,R),\ \mu\in P,\ \lambda>0.$$

Define $G_0\colon[0,\infty)\times E\times P\to E$ by $G_0(t,\overline{u},\mu):=\rho(\|\overline{u}\|)G(t,\overline{u},\mu)$ where $\rho\colon[0,\infty)\to\mathbb{R}$ is a locally Lipschitz function such that $\rho(s)=1$ if $s\in[0,C]$ and $\rho(s)=0$ if $s>2C$. Clearly, for any $\overline{u}\in B(0,R)$, $\mu\in P$ and $\lambda>0$, the mild solution of

$$\begin{cases}\dot{u}(t)=Au(t)+G_0(t/\lambda,u(t),\mu), & t\in[0,l],\\ u(0)=\overline{u}\end{cases}$$

coincides with $u(\,\cdot\,;\overline{u},\mu,\lambda)_{|[0,l]}$. Hence, the theorem has been reduced to the case when G is bounded and Theorem 3.1 of [7] applies. □

Now pass to

$$(\dot{u},\dot{v})=\mathbf{A}(u,v)+(0,F(t/\lambda,u,v,\varepsilon))\tag{4.5}$$

where $\mathbf{A}$ is the operator defined by (2.1), $\lambda>0$ and $\varepsilon\in[0,1]$ are parameters and $F\colon[0,\infty)\times X^{1/2}\times X^0\times[0,1]\to X^0$.

Theorem 4.2. *Assume that F is continuous, locally Lipschitz in the second and third variables uniformly with respect to t and ε, $F([0,\infty)\times\mathbf{V}\times[0,1])$ is bounded for any bounded $\mathbf{V}\subset\mathbf{E}$, and that for some $T>0$*

$$F(t+T,\overline{u},\overline{v},\varepsilon)=F(t,\overline{u},\overline{v},\varepsilon)\quad \textit{for all } (\overline{u},\overline{v})\in X^{1/2}\times X^0,\ t\geq 0,\ \varepsilon\in[0,1].$$

Let $G\colon[0,\infty)\times\mathbf{E}\times[0,1]\times[0,1]\to X^0$ *be defined by*

$$G(t,\overline{u},\overline{v},\mu,\varepsilon):=(1-\mu)F(t,\overline{u},\overline{v},\varepsilon)+\mu\widehat{F}(\overline{u},\overline{v},\varepsilon),\quad t\geq 0,\ (\overline{u},\overline{v})\in\mathbf{E},\ \mu,\varepsilon\in[0,1],$$

with $\widehat{F}\colon X^{1/2}\times X^0\times[0,1]\to X^0$ *given by*

$$\widehat{F}(\overline{u},\overline{v},\varepsilon):=\frac{1}{T}\int_0^T F(t,\overline{u},\overline{v},\varepsilon)\,dt,\quad (\overline{u},\overline{v})\in X^{1/2}\times X^0,\ \varepsilon\in[0,1].$$

Moreover, suppose that conditions (4.1) *and* (4.2) *are satisfied with* $A:=\mathbf{A}$ *and* $G:=(0,G)$ *and let* $\Phi_t^{(\lambda,\varepsilon)}\colon\mathbf{E}\to\mathbf{E}$ *denote the translation along trajectories for the equation* (4.5). *If* $\mathbf{U}\subset\mathbf{E}$ *is open, bounded and*

$$\mathbf{A}(\overline{u},\overline{v})+(0,\widehat{F}(\overline{u},\overline{v},0))\neq 0\quad \textit{for all } (\overline{u},\overline{v})\in\partial\mathbf{U}\cap D(\mathbf{A}),$$

then there exist $\lambda_0,\varepsilon_0>0$ *such that, for all* $\lambda\in(0,\lambda_0]$ *and* $\varepsilon\in(0,\varepsilon_0]$,

$$\operatorname{ind}(\Phi_{\lambda T}^{(\lambda,\varepsilon)},\mathbf{U})=\deg(I+\mathbf{A}^{-1}(0,\widehat{F}(\,\cdot\,,0)),\mathbf{U}).$$

PROOF. Consider the equation

$$(\dot{u}(t), \dot{v}(t)) = \mathbf{A}(u(t), v(t)) + (0, G(t/\lambda, u(t), v(t), \mu, \varepsilon)). \tag{4.6}$$

We claim that the following property holds:

$$\begin{cases} \text{if } (\lambda_n) \text{ in } (0,\infty),\ (\mu_n), (\varepsilon_n) \text{ in } [0,1],\ (t_n) \text{ in } (0,\infty) \\ \text{and mild solutions } (u_n, v_n)\colon [0,\infty) \to \mathbf{E} \text{ of (4.6)} \\ \text{with } \lambda = \lambda_n,\ \mu = \mu_n \text{ and } \varepsilon = \varepsilon_n,\ n \ge 1, \text{ are such that} \\ \{(u_n(t), v_n(t)) \mid t \in [0, t_n],\ n \ge 1\} \text{ is bounded and } t_n \to \infty \\ \text{then the set } \{(u_n(t_n), v_n(t_n)\}_{n\ge 1} \text{ is relatively compact.} \end{cases} \tag{4.7}$$

Without loss of generality we may assume that (t_n) is increasing. First, by the Duhamel formula, we have

$$\begin{aligned} (u_n(t_n), v_n(t_n)) &= S_{\mathbf{A}}(t_N)(u_n(t_n - t_N), v_n(t_n - t_N)) \\ &\quad + \int_{t_n - t_N}^{t_n - \delta} S_{\mathbf{A}}(t_n - \delta - s) S_{\mathbf{A}}(\delta)(0, G(s/\lambda_n, u_n(s), v_n(s), \mu_n, \varepsilon_n))\, ds \\ &\quad + \int_{t_n - \delta}^{t_n} S_{\mathbf{A}}(t_n - s)(0, G(s/\lambda_n, u_n(s), v_n(s), \mu_n, \varepsilon_n))\, ds \end{aligned}$$

whenever $n \ge N \ge 1$ and $\delta \in (0, t_N)$. Choose $R > 0$ so that $\|(u_n(s), v_n(s))\|_{\mathbf{E}} \le R$ for all $s \in [0, t_n]$, $n \ge 1$. Then there is $C > 0$ such that

$$|G(s/\lambda_n, u_n(s), v_n(s), \mu_n, \varepsilon_n)|_0 \le C \quad \text{for all } s \in [0, t_n] \text{ and } n \ge 1$$

and that

$$\int_{t_n - t_N}^{t_n - \delta} S_{\mathbf{A}}(t_n - \delta - s) S_{\mathbf{A}}(\delta)(0, G(s/\lambda_n, u_n(s), v_n(s), \mu_n, \varepsilon_n))\, ds \in (t_N - \delta) \cdot \overline{\operatorname{conv}}\, \mathbf{W}$$

with $\mathbf{W} := \{S_{\mathbf{A}}(\tau)(\overline{u}, \overline{v}) \mid \tau \in [0, t_N],\ (\overline{u}, \overline{v}) \in \widetilde{\mathbf{W}}\}$ where $\widetilde{\mathbf{W}} := S_{\mathbf{A}}(\delta)(\{0\} \times B_{X^0}(0, C))$. And since in view of Proposition 2.7, $\widetilde{\mathbf{W}}$ is relatively compact, so must be $\mathbf{W}$. Therefore, if $\eta_0 > 0$ and $\omega = \omega(\eta_0) > 0$ are determined by Proposition 2.1, then

$$\gamma_{\eta_0}(\{(u_n(t_n), v_n(t_n))\}_{n \ge N}) \le e^{-\omega t_N} \gamma_{\eta_0}(B_{\mathbf{E}}(0, R)) + \delta C$$

and, as $\delta \in (0, t_N)$ may be taken arbitrarily small, we infer that

$$\gamma_{\eta_0}(\{(u_n(t_n), v_n(t_n))\}_{n \ge N}) \le e^{-\omega t_N} \gamma_{\eta_0}(B_{\mathbf{E}}(0, R)).$$

Summing up, we have, for any $N \ge 1$,

$$\gamma_{\eta_0}(\{(u_n(t_n), v_n(t_n))\}_{n \ge 1}) = \gamma_{\eta_0}(\{(u_n(t_n), v_n(t_n))\}_{n \ge N}) \le e^{-\omega t_N} \gamma_{\eta_0}(B_{\mathbf{E}}(0, R)),$$

which implies $\gamma_{\eta_0}(\{(u_n(t_n), v_n(t_n))\}_{n \ge 1}) = 0$ and completes the proof of (4.7).

For $\lambda > 0$ and $t > 0$ define $\Theta_t^{(\lambda)}\colon \mathbf{E} \times [0,1] \times [0,1] \to \mathbf{E}$ by

$$\Theta_t^{(\lambda)}(\overline{u}, \overline{v}, \mu, \varepsilon) := (u(t), u(t))$$

where $(u,v)\colon [0,\infty)\to \mathbf{E}$ is the unique mild solution of (4.6) with $(u(0),v(0)) = (\overline{u},\overline{v})$. Clearly, in view of the assumptions, $\Theta_t^{(\lambda)}$ is well defined. Using the arguments that we employed while proving the property (4.7), it can be shown that

$$\gamma_{\eta_0}\bigg(\bigcup_{\lambda>0}\Theta_t^{(\lambda)}(\mathbf{V}\times[0,1]\times[0,1])\bigg)\leq e^{-\omega t}\gamma_{\eta_0}(\mathbf{V})$$

for any bounded $\mathbf{V}\subset\mathbf{E}$ and $t>0$. This entails the assumption (4.3) (with $A:=\mathbf{A}$, $G:=(0,G)$) and, due to Proposition 3.3, the continuity of $\Theta_t^{(\lambda)}$.

Now we claim that

$$\text{(4.8)}\qquad \begin{cases} \text{there exist } \lambda_0,\varepsilon_0>0 \text{ such that, for any } \lambda\in(0,\lambda_0], \\ \Theta_{\lambda T}^{(\lambda)}(\overline{u},\overline{v},\mu,\varepsilon)\neq(\overline{u},\overline{v}) \text{ for all } (\overline{u},\overline{v})\in\partial\mathbf{U},\ \mu\in[0,1] \text{ and } \varepsilon\in[0,\varepsilon_0]. \end{cases}$$

Suppose to the contrary that there are (λ_n) in $(0,\infty)$, $((\overline{u}_n,\overline{v}_n))$ in $\partial\mathbf{U}$, (μ_n) and (ε_n) in $[0,1]$ such that $\lambda_n\to 0^+$, $\varepsilon_n\to 0$ and

$$\Theta_{\lambda_n T}^{(\lambda_n)}(\overline{u}_n,\overline{v}_n,\mu_n,\varepsilon_n)=(\overline{u}_n,\overline{v}_n)\quad\text{for all } n\geq 1.$$

Let $(u_n,v_n)\colon[0,\infty)\to\mathbf{E}$, $n\geq 1$, be $\lambda_n T$-periodic solutions of (4.6) with $\lambda=\lambda_n$, $\mu=\mu_n$, $\varepsilon=\varepsilon_n$ and $(u_n(0),v_n(0))=(\overline{u}_n,\overline{v}_n)$. Since (4.2) holds and (u_n,v_n) are periodic, there exists $\widetilde{C}>0$ such that

$$\|(u_n(s),v_n(s))\|_{\mathbf{E}}\leq\widetilde{C}\quad\text{for all } s\geq 0 \text{ and } n\geq 1.$$

Moreover, putting $l_n:=n[1/\lambda_n T]$ and using the $\lambda_n T$-periodicity, we get, for each $n\geq 1$, $(u_n(l_n\lambda_n T),v_n(l_n\lambda_n T))=(\overline{u}_n,\overline{v}_n)$. Since $l_n\lambda_n T\to\infty$ as $n\to\infty$, in view of the property (4.7), $((\overline{u}_n,\overline{v}_n))$ contains a convergent subsequence. Further assume that $(\overline{u}_n,\overline{v}_n)\to(\overline{u}_0,\overline{v}_0)$ and $\mu\to\mu_0$ as $n\to\infty$ for some $(\overline{u}_0,\overline{v}_0)\in\partial\mathbf{U}$ and $\mu_0\in[0,1]$. Take any $t>0$ and observe that application of Theorem 4.1 gives $(u_n([t/\lambda_n T]\lambda_n T),v_n([t/\lambda_n T]\lambda_n T))\to(\widehat{u}(t),\widehat{v}(t))$ as $n\to\infty$ where $(\widehat{u},\widehat{v})\colon[0,\infty)\to\mathbf{E}$ is the mild solution of

$$(\dot{u},\dot{v})=\mathbf{A}(u,v)+(0,\widehat{G}(u,v,\mu_0,0)),\quad (u(0),v(0))=(\overline{u}_0,\overline{v}_0).$$

On the other hand, $(u_n([t/\lambda_n T]\lambda_n T),v_n([t/\lambda_n T]\lambda_n T))=(\overline{u}_n,\overline{v}_n)\to(\overline{u}_0,\overline{v}_0)$ for $n\geq 1$. This means that $(\widehat{u},\widehat{v})$ is a constant mild solution equal to $(\overline{u}_0,\overline{v}_0)$. In consequence, $(\overline{u}_0,\overline{v}_0)\in\partial\mathbf{U}\cap D(\mathbf{A})$ and $\mathbf{A}(\overline{u}_0,\overline{v}_0)+(0,\widehat{G}(\overline{u}_0,\overline{v}_0,\mu_0,0))=0$, i.e. $\mathbf{A}(\overline{u}_0,\overline{v}_0)+(0,\widehat{F}(\overline{u}_0,\overline{v}_0,0))=0$, a contradiction proving that (4.8) holds.

Thus, the homotopy invariance of the fixed point index applied to $\Theta_{\lambda T}^{(\lambda)}$ with $\lambda\in(0,\lambda_0]$ and $\varepsilon\in[0,\varepsilon_0]$, leads to

$$\text{(4.9)}\qquad \begin{aligned}\operatorname{ind}(\Phi_{\lambda T}^{(\lambda,\varepsilon)},\mathbf{U})&=\operatorname{ind}(\Theta_{\lambda T}^{(\lambda)}(\,\cdot\,,0,\varepsilon),\mathbf{U})=\operatorname{ind}(\Theta_{\lambda T}^{(\lambda)}(\,\cdot\,,1,\varepsilon),\mathbf{U})\\&=\operatorname{ind}(\Theta_{\lambda T}^{(\lambda)}(\,\cdot\,,1,0),\mathbf{U})=\operatorname{ind}(\widehat{\Phi}_{\lambda T},\mathbf{U})\end{aligned}$$

where $\widehat{\Phi}_t\colon \mathbf{E} \to \mathbf{E}$ stands for the translation along trajectories operator for

$$(\dot{u}, \dot{v}) = \mathbf{A}(u, v) + (0, \widehat{F}(u, v, 0)).$$

Now Theorem 3.1 states that

$$\operatorname{ind}(\widehat{\Phi}_{\lambda T}, \mathbf{U}) = \deg(I + \mathbf{A}^{-1}(0, \widehat{F}(\,\cdot\,, 0)), \mathbf{U})$$

for sufficiently small $\lambda > 0$. This together with (4.9) ends the proof. □

Finally, let us also make the following observation to be used in the next section.

THEOREM 4.3. *Assume that F is as in Theorem 4.2 and that the conditions (4.1) and (4.2) are satisfied with $A := \mathbf{A}$ and $G := (0, F)$. Suppose that $(u_n, v_n)\colon [0, \infty) \to \mathbf{E}$, $n \geq 1$, are $\lambda_n T$-periodic mild solutions of (4.5) with $\lambda = \lambda_n$ and $\varepsilon = \varepsilon_n$, $n \geq 1$, such that $\{(u_n(0), v_n(0))\}_{n\geq 1}$ is bounded and $\lambda_n \to 0^+$, $\varepsilon_n \to 0$ as $n \to \infty$, then there are a subsequence $((u_{n_k}, v_{n_k}))$ of $((u_n, v_n))$ and $(\overline{u}, \overline{v}) \in D(\mathbf{A})$ such that $\mathbf{A}(\overline{u}, \overline{v}) + (0, \widehat{F}(\overline{u}, \overline{v}, 0)) = 0$ and*

$$(u_{n_k}(t), v_{n_k}(t)) \to (\overline{u}, \overline{v}) \quad \text{in } \mathbf{E}, \text{ uniformly for } t \geq 0.$$

PROOF. It can be easily deduced by inspection of the proof of the property (4.8) with $\mu_n := 0$, $n \geq 1$ and $\varepsilon_n \to 0$ as $n \to \infty$. □

5. Periodic solutions for the beam equation

Now we shall deal with the equation

$$\ddot{u} + \alpha A\dot{u} + \beta\dot{u} + Au + (g(|u|_{1/4}^2) + \sigma(u, \dot{u})_{1/4})A^{1/2}u = f(\omega t) \tag{5.1}$$

where A is as before, $\omega > 0$ is a parameter, $g\colon [0, \infty) \to \mathbb{R}$ is continuous with $C > 0$ such that

$$\int_0^\rho g(s)\, ds > -C, \quad \text{for any } \rho > 0, \tag{5.2}$$

$f\colon [0, \infty) \to X^0$ is a T-periodic Hölder function, i.e. $f(t + T) = f(t)$ for all $t \geq 0$ and there exist $L > 0$ and $\theta > 0$ such that

$$|f(t) - f(s)|_0 \leq L|t - s|^\theta \quad \text{for all } t, s \geq 0.$$

Since our intention is to use the averaging from the previous section, we need to consider the parameterized equation

$$\ddot{u} + \alpha A\dot{u} + \beta\dot{u} + Au + (g(|u|_{1/4}^2) + \sigma(u, \dot{u})_{1/4})A^{1/2}u = (1 - \mu)f(t/\lambda) + \mu\widehat{f} \tag{5.3}$$

with $\widehat{f} := (1/T)\int_0^T f(s)\, ds$. It is clear that it coincides with (5.1) if $\lambda = \omega^{-1}$ and $\mu = 0$. The equation (5.3) can be formally rewritten as

$$(\dot{u}(t), \dot{v}(t)) = \mathbf{A}(u(t), v(t)) + (0, G(t/\lambda, u(t), v(t), \mu)) \tag{5.4}$$

where $\mathbf{A}$ is defined by (1.6) and $G\colon [0,\infty)\times\mathbf{E}\times[0,1]\to X^0$ is given by

$$G(t,\overline{u},\overline{v},\mu) := -(g(|\overline{u}|_{1/4}^2)+\sigma(A^{1/2}\overline{u},\overline{v})_0)A^{1/2}\overline{u}+(1-\mu)f(t)+\mu\widehat{f}, \qquad (\overline{u},\overline{v})\in\mathbf{E},\ t\geq 0.$$

As mild solutions of (5.3) we take mild solutions of (5.4). However, as we mentioned in Section 2, $-\mathbf{A}$ is sectorial and, as a result, (5.4) will admit also (pointwise) solutions in the sense of Henry [12] (see also [4]). We speak of a solution of (5.3) or (5.4) meaning a solution in the above sense. In particular, if $(u,v)\colon[t_0,t_0+r)\to\mathbf{E}$, with $t_0\in\mathbb{R}$ and $r>0$, is a solution of (5.4), then $u\in C([t_0,t_0+r),X^{1/2})$, $v\in C([t_0,t_0+r),X^0)$, and, for all $t>t_0$, $u(t)+\alpha v(t)\in X^1$, $v(t)\in X^{1/2}$, (u,v) is differentiable into $\mathbf{E}$ at t and

$$\begin{aligned}\dot{u}(t)&=v(t),\\ \dot{v}(t)&=-A(u(t)+\alpha v(t))-\beta v(t)+G(t/\lambda,u(t),v(t),\mu).\end{aligned}$$

Obviously, in view of Lemma 2.3, any such a solution of (5.4) is also a mild solution.

Below we strongly use these implications of the sectoriality of $-\mathbf{A}$ to show that (5.4) has global existence property.

Proposition 5.1.

(a) *For any $\lambda>0$ and $(\overline{u},\overline{v})\in\mathbf{E}$, there exists a unique solution $(u,v)=(u(\,\cdot\,;\overline{u},\overline{v},\mu,\lambda),v(\,\cdot\,;\overline{u},\overline{v},\mu,\lambda))\colon[0,\infty)\to\mathbf{E}$ of* (5.4) *satisfying the initial value condition $(u(0),v(0))=(\overline{u},\overline{v})$.*

(b) *For any $(\overline{u},\overline{v})\in\mathbf{E}$, $\mu\in[0,1]$, $\lambda>0$ and $t\geq 0$,*

$$|u(t;\overline{u},\overline{v},\mu,\lambda)|_{1/2}^2+|v(t;\overline{u},\overline{v},\mu,\lambda)|_0^2 \leq |\overline{u}|_{1/2}^2+|\overline{v}|_0^2+C+\int_0^{|\overline{u}|_{1/4}^2} g(s)\,ds+\frac{t}{4\beta}\sup_{s\in[0,T]}|f(s)|_0^2.$$

In particular, for each $R>0$ and $t>0$, the set

$$\{(u(s;\overline{u},\overline{v},\mu,\lambda),v(s;\overline{u},\overline{v},\mu,\lambda))\mid s\in[0,t],\ (\overline{u},\overline{v})\in B_{\mathbf{E}}(0,R),\ \mu\in[0,1],\ \lambda>0\}$$

is bounded.

Proof. (a) Note that G has the following property: for any $(t,\overline{u},\overline{v})\in[0,\infty)\times\mathbf{E}$, there exist $L>0$ and $\delta>0$ such that

$$|G(t_1,\overline{u}_1,\overline{v}_1,\mu)-G(t_2,\overline{u}_2,\overline{v}_2,\mu)|_0\leq L(|t_1-t_2|^\theta+\|(\overline{u}_1,\overline{v}_1)-(\overline{u}_2,\overline{v}_2)\|_{\mathbf{E}})$$

whenever $t_1,t_2\in(t-\delta,t+\delta)\cap[0,\infty)$, $(\overline{u}_1,\overline{v}_1),(\overline{u}_2,\overline{v}_2)\in B_{\mathbf{E}}((\overline{u},\overline{v}),\delta)$, $\mu\in[0,1]$. Therefore, in view of Theorem 3.3.3 of [12], for any $(\overline{u},\overline{v})\in\mathbf{E}$, $\lambda>0$ and

$\mu \in [0,1]$, we get a unique solution $(u,v)\colon [0,\overline{t}) \to \mathbf{E}$ of (5.4). Suppose that (u,v) is a maximal solution. Then, for $t \in (0,\overline{t})$,

$$\begin{aligned}(\dot u(t),u(t))_{1/2} =&(u(t),v(t))_{1/2},\\ (\dot v(t),v(t))_0 =&-(u(t),v(t))_{1/2}-\alpha|v(t)|_{1/2}^2-\beta|v(t)|_0^2\\ &-(g(|u(t)|_{1/4}^2)+\sigma(A^{1/2}u(t),v(t))_0)(u(t),v(t))_{1/2}\\ &+((1-\mu)f(t/\lambda)+\mu\widehat{f},v(t))_0.\end{aligned}$$

Addition of both equalities yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(|u(t)|_{1/2}^2+|v(t)|_0^2) =& -\alpha|v(t)|_{1/2}^2-\beta|v(t)|_0^2-g(|u(t)|_{1/4}^2)(u(t),v(t))_{1/4}\\ &-\sigma|(u(t),v(t))_{1/4}|^2+((1-\mu)f(t/\lambda)+\mu\widehat{f},v(t))_0\end{aligned}$$

and, further, by taking $\Psi\colon[0,\infty)\to\mathbb{R}$ given by $\Psi(\rho):=\int_0^\rho g(s)\,ds$, $\rho\ge 0$, one has

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(|u(t)|_{1/2}^2+|v(t)|_0^2+\Psi(|u(t)|_{1/4}^2))\\ &\quad= -\alpha|v(t)|_{1/2}^2-\beta|v(t)|_0^2-\sigma|(A^{1/2}u(t),v(t))_0|^2+((1-\mu)f(t/\lambda)+\mu\widehat{f},v(t))_0\\ &\quad\le -\beta|v(t)|_0^2+\beta|v(t)|_0^2+(1/4\beta)|(1-\mu)f(t/\lambda)+\mu\widehat{f}|_0^2\le(1/4\beta)\sup_{s\in[0,T]}|f(s)|_0^2.\end{aligned}$$

After integration, we obtain

$$\begin{aligned}(5.5)\qquad &|u(t)|_{1/2}^2+|v(t)|_0^2\\ &\quad\le|\overline{u}|_{1/2}^2+|\overline{v}|_0^2+\Psi(|\overline{u}|_{1/4}^2)-\Psi(|u(t)|_{1/4}^2)+(\overline{t}/4\beta)\sup_{s\in[0,T]}|f(s)|_0^2\\ &\quad\le|\overline{u}|_{1/2}^2+|\overline{v}|_0^2+\Psi(|\overline{u}|_{1/4}^2)+C+(\overline{t}/4\beta)\sup_{s\in[0,T]}|f(s)|_0^2.\end{aligned}$$

Hence, $\limsup_{t\to\overline{t}}\|(u(t),v(t))\|_{\mathbf{E}}<\infty$ and, in view of Theorem 3.3.4 of [12], we see that $\overline{t}=\infty$, i.e. assertion (a) holds.

Assertion (b) readily follows from (5.5). □

The above proposition will allow us to employ Theorem 4.2 in search of periodic solutions. In the remainder of the paper we shall focus on a special case of the beam equation

$$(5.6)\qquad \ddot u+\alpha A\dot u+\beta\dot u+Au+(a|u|_{1/4}^2+b+\sigma(u,\dot u)_{1/4})A^{1/2}u=f(\omega t)+\varepsilon f_0(\omega t)$$

with fixed $a>0$ and $b\in\mathbb{R}$, parameters $\omega,\varepsilon>0$ and T-periodic Hölder continuous $f,f_0\colon[0,\infty)\to X^0$. In this special case $g\colon[0,\infty)\to\mathbb{R}$ is given by $g(s):=a\ s+b$, for $s\ge 0$. Define $\mathbf{F}_0\colon\mathbf{E}\to\mathbf{E}$ by

$$\mathbf{F}_0(\overline{u},\overline{v}):=(0,-(a|\overline{u}|_{1/4}^2+b+\sigma(A^{1/2}\overline{u},\overline{v})_0)A^{1/2}\overline{u}).$$

Proposition 5.2. *Assume that*

$$\widehat{f} = 0 \quad \text{and} \quad |b| \neq \lambda_j^{1/2} \quad \text{for all } j \geq 1 \tag{5.7}$$

and let k be either the integer such that $\lambda_k^{1/2} < -b < \lambda_{k+1}^{1/2}$ or $k = 0$ if $b > -\lambda_1^{1/2}$.

(a) *Under the above assumptions, the solution set of the equation*

$$\mathbf{A}(\overline{u}, \overline{v}) + \mathbf{F}_0(\overline{u}, \overline{v}) = 0 \tag{5.8}$$

consists of $(\overline{u}_j, 0)$, $j \in \{i \in \mathbb{Z} \mid |i| \leq k\}$, where

$$\overline{u}_j := \begin{cases} 0 & \text{if } j = 0, \\ a^{-1/2}(-1 - b/\lambda_j^{1/2})^{1/2} e_j & \text{if } 1 \leq |j| \leq k, \end{cases}$$

and $e_j := -e_{-j}$, $\lambda_j := \lambda_{-j}$ for integers $j < 0$.

(b) *For sufficiently small $r > 0$,*

$$\deg(I + \mathbf{A}^{-1}\mathbf{F}_0, B_{\mathbf{E}}((0,0), r)) = (-1)^{k_0},$$

where k_0 is the integer such that $\lambda_{k_0}^{1/2} < b < \lambda_{k_0+1}^{1/2}$ and $k_0 = 0$ if $b < \lambda_1^{1/2}$, and

$$\deg(I + \mathbf{A}^{-1}\mathbf{F}_0, B_{\mathbf{E}}((\overline{u}_j, 0), r)) = (-1)^{j+1} \quad \text{if } 0 < |j| \leq k.$$

Proof. (a) Suppose that $(\overline{u}, \overline{v}) \in \mathbf{E}$ is a solution of (5.8). Then

$$\overline{v} = 0 \quad \text{and} \quad A(\overline{u} + \alpha\overline{v}) = -(a|\overline{u}|_{1/4}^2 + b + \sigma(A^{1/2}\overline{u}, \overline{v})_0)A^{1/2}\overline{u},$$

which implies $A^{1/2}\overline{u} = -(a|\overline{u}|_{1/4}^2 + b)\overline{u}$. Clearly, either $\overline{u} = 0$ or there exists $j \geq 1$ such that $-\lambda_j^{1/2} = a\,|\overline{u}|_{1/4}^2 + b$ and $\overline{u} = se_j$ for some $s \in \mathbb{R}$. Hence either $\overline{u} = 0$ or $1 \leq |j| \leq k$ and

$$a\lambda_j^{1/2} s^2 + b = -\lambda_j^{1/2}.$$

Such s exists provided $b < -\lambda_j^{1/2}$ and

$$s = \pm a^{-1/2}(-1 - b/\lambda_j^{1/2})^{1/2}.$$

This implies (a).

(b) The Frêchet derivative of $\mathbf{F}_0$ at $(\overline{u}, 0)$ is determined by the formula

$$\mathbf{F}_0'(\overline{u}, 0)(\phi, \psi) = (0, -(2a(\overline{u}, \phi)_{1/4} + \sigma(A^{1/2}\overline{u}, \psi)_0)A^{1/2}\overline{u} - (a|\overline{u}|_{1/4}^2 + b)A^{1/2}\phi)$$

for all $\overline{u} \in X^{1/2}$ and $(\phi, \psi) \in \mathbf{E}$. This together with (3.2) gives

$$\begin{aligned}\mathbf{A}^{-1}\mathbf{F}_0'(\overline{u}, 0)(\phi, \psi) \\ = ((2a(\overline{u}, \phi)_{1/4} + \sigma(A^{1/2}\overline{u}, \psi)_0)A^{-1/2}\overline{u} + (a|\overline{u}|_{1/4}^2 + b)A^{-1/2}\phi, 0)\end{aligned}$$

for $(\phi, \psi) \in \mathbf{E}$. In particular

$$\mathbf{A}^{-1}\mathbf{F}_0'(0, 0)(\phi, \psi) = (bA^{-1/2}\phi, 0).$$

And it is easy to verify that $I + \mathbf{A}^{-1}\mathbf{F}_0'(0,0)$ has no negative eigenvalues if $b < \lambda_1^{1/2}$ and k_0 negative eigenvalues $1 - b/\lambda_j$, $j = 1, \ldots, k_0$, if $k_0 > 1$ (counted algebraically). The linearization formula for the Leray–Schauder theorem gives the desired formula. When $k > 0$ and $\lambda_k^{1/2} < -b < \lambda_{k+1}^{1/2}$, then

$$\mathbf{A}^{-1}\mathbf{F}_0'(\overline{u}_j, 0)(\phi, \psi) = ((2a(e_j, \phi)_0 + \sigma(e_j, \psi)_0)s_j^2 e_j + (a|\overline{u}_j|^2 + b)A^{-1/2}\phi, 0)$$

for all $(\phi, \psi) \in \mathbf{E}$ and $j \in \{i \in \mathbb{Z} \mid 0 < |i| \leq k\}$. It can be verified by direct computation that $I + \mathbf{A}^{-1}\mathbf{F}_0'(\overline{u}_j, 0)$ has $|j| - 1$ negative eigenvalues $1 - (\lambda_j/\lambda_i)^{1/2}$, $i = 1, \ldots, |j| - 1$. Hence, due to the linearization formula for the Leray–Schauder degree, their indices are equal to $(-1)^{|j|-1} = (-1)^{j+1}$. □

Theorem 5.3. *Under the condition* (5.7), *there exist* $\varepsilon_0 > 0$ *and* $\omega_0 > 0$ *such that for all* $\varepsilon \in [0, \varepsilon_0]$ *and* $\omega \geq \omega_0$, *the equation* (5.6) *admits at least* $2k + 1$ (T/ω)*-periodic solutions* $u_j^{(\omega,\varepsilon)}$, $j \in \{i \in \mathbb{Z} \mid |i| \leq k\}$, *such that*

$$u_j^{(\omega,\varepsilon)}(t) \to \overline{u}_j \quad \text{in } X^{1/2} \text{ as } \omega \to \infty,\ \varepsilon \to 0,\ \text{uniformly with respect to } t \in \mathbb{R},$$

and

$$\dot{u}_j^{(\omega,\varepsilon)}(t) \to 0 \quad \text{in } X^0 \text{ as } \omega \to \infty,\ \varepsilon \to 0 \text{ uniformly with respect to } t \in \mathbb{R},$$

for all $j \in \{i \in \mathbb{Z} \mid |i| \leq k\}$. *The topological indices of solutions are equal to the topological indices of the corresponding equilibria – see Proposition* 5.2.

Proof. In view of Proposition 5.2, there is $r > 0$ such that $B_{\mathbf{E}}((\overline{u}_j, 0), r)$, $j \in \{i \in \mathbb{Z} \mid |i| \leq k\}$, are pairwise disjoint and

$$\deg(I + \mathbf{A}^{-1}\mathbf{F}_0, B_{\mathbf{E}}((\overline{u}_j, 0), r)) \neq 0, \quad j \in \{i \in \mathbb{Z} \mid |i| \leq k\}.$$

Let $F\colon [0, \infty) \times X^{1/2} \times X^0 \times [0,1] \to X^0$ be given by

$$F(t, \overline{u}, \overline{v}, \varepsilon) := -(a|\overline{u}|_{1/4}^2 + b + \sigma(A^{1/2}\overline{u}, \overline{v})_0)A^{1/2}\overline{u} + f(t) + \varepsilon f_0(t),$$

for $t \geq 0$, $(\overline{u}, \overline{v}) \in X^{1/2} \times X^0$, $\varepsilon \in [0,1]$. Clearly, in view of Proposition 5.1, (5.6) satisfies assumptions of Theorem 4.2. Hence, there are $\varepsilon_0 > 0$ and $\lambda_0 > 0$ such that for all $\varepsilon \in [0, \varepsilon_0]$ and $\lambda \in (0, \lambda_0]$

$$\operatorname{ind}(\Phi_{\lambda T}^{(\lambda,\varepsilon)}, B_{\mathbf{E}}((\overline{u}_j, 0), r)) = \deg(I + \mathbf{A}^{-1}\mathbf{F}_0, B_{\mathbf{E}}((\overline{u}_j, 0), r)) \neq 0$$

for each $j \in \{i \in \mathbb{Z} \mid |i| \leq k\}$ where $\Phi_t^{(\lambda,\varepsilon)} : \mathbf{E} \to \mathbf{E}$ is the translation along trejectories (by time $t > 0$) for the equation

$$(\dot{u}, \dot{v}) = \mathbf{A}(u, v) + (0, F(t/\lambda, \overline{u}, \overline{v}, \varepsilon)). \tag{5.9}$$

By the existence property of the fixed point index, for each $\varepsilon \in [0, \varepsilon_0]$ and $\omega > \omega_0 := 1/\lambda_0$ one obtains at least $2k+1$ periodic solutions $(u_j^{(\omega,\varepsilon)}, v_j^{(\omega,\varepsilon)})\colon [0,\infty) \to \mathbf{E}$, $j \in \{i \in \mathbb{Z} \mid |i| \leq k\}$, of (5.9) with $\lambda = \omega^{-1}$ such that

$$(u_j^{(\omega,\varepsilon)}(0), v_j^{(\omega,\varepsilon)}(0)) = (u_j^{(\omega,\varepsilon)}(T/\omega), v_j^{(\omega,\varepsilon)}(T/\omega)) \in B_{\mathbf{E}}((\overline{u}_j, 0), r)$$

for $j \in \{i \in \mathbb{Z} \mid |i| \le k\}$. Now the assertion follows from Theorem 4.3 and the fact that $B_{\mathbf{E}}((\overline{u}_j, 0), r)$, $j \in \{i \in \mathbb{Z} \mid |i| \le k\}$ are pairwise disjoint. □

It clear that to deduce Theorem 1.1 we take A given by (1.3). Then $\lambda_j = j^4\pi^4/l^4$ and $e_j(x) = (l/2)^{1/2}\sin(j\pi x/l)$, $x \in [0, l]$, $j = 0, 1, \ldots$.

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