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EXISTENCE OF PERIODIC SOLUTIONS FOR p-LAPLACIAN NEUTRAL FUNCTIONAL EQUATION WITH MULTIPLE DEVIATING ARGUMENTS

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Abstract. By using the theory of coincidence degree and some refined analysis techniques, we study a general kind of periodic solutions to p-Laplacian neutral functional differential equation with multiple deviating arguments. A general analysis method to tackle with such equations is formed. Some new and universal results on the existence of periodic solutions are obtained, meanwhile, some known results in the literatures are improved. An example is provided as an application to our theorems.

1. Introduction

Throughout this paper, $1 < p < \infty$ is a fixed real number. The conjugate exponent of p is denoted by q, i.e. $1/p + 1/q = 1$. Let $\varphi_p\colon \mathbb{R} \to \mathbb{R}$ be defined by $\varphi_p(u) = |u|^{p-2}u$. Then φ_p is a homeomorphism of $\mathbb{R}$ with the inverse φ_q. In this paper, we study the existence of periodic solutions for p-Laplacian neutral functional differential equation with multiple deviating arguments

$$(1.1)\quad (\varphi_p((x(t) - cx(t-\sigma))'))' = h(x'(t)) + f(x(t))x'(t) + \sum_{j=1}^{n} \beta_j(t)g(x(t-\tau_j(t))) + p(t),$$

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where $f, g, h \in C(\mathbb{R}, \mathbb{R})$; $\beta_j(t)$, $\tau_j(t)$ $(j = 1, \ldots, n)$, $p(t)$ are continuous periodic functions defined on $\mathbb{R}$ with period $T > 0$, $\sigma, c \in \mathbb{R}$ are constants such that $|c| \neq 1$.

The problem of periodic solutions of ordinary differential equation was widely studied, see [1]–[3], [9]. In recent years, there are many results about periodic solutions for some types of second-order differential equations with deviating arguments ([6]–[8], [11], [13] and the references therein). For example, in [11], the author considered a kind of Rayleigh equation with a deviating argument in the following form

$$x''(t) + h(x'(t)) + g(x(t - \tau(t)) = p(t), \tag{1.2}$$

where h, g, p and τ are real continuous functions defined on $\mathbb{R}$, τ and p are periodic with period 2π. Under the conditions that $h(0) = 0$, $\int_0^T p(t)\,dt = 0$ and some other assumptions, the author studied the existence of periodic solution of equation (1.2). In [7], the authors considered the following equation with a deviating argument

$$(x(t) + cx(t - r))'' + h(x'(t)) + g(x(t - \tau(t)) = p(t). \tag{1.3}$$

Also, the authors assumed that $h(0) = 0$ and $\int_0^T p(t)\,dt = 0$. By using Mawhin continuation theorem, some results on the existence of periodic solutions are obtained. But the corresponding problem of p-Laplacian differential equation with multiple deviating arguments has been studied less often. The possible reason for this is that the differential operator $(\varphi_p(u))' = (|u|^{p-2}u)'$ for $p \neq 2$ is no longer linear, therefore the coincidence degree can not be applied directly. In [13], the authors studied the following equation

$$(\varphi_p((x(t) - cx(t - \sigma))'))' = f(x(t))x'(t) + \sum_{j=1}^{n} \beta_j(t)g(x(t - \tau_j(t))) + p(t), \tag{1.4}$$

in which the condition $\overline{p} = \int_0^T p(t)\,dt < 0$ must be satisfied. That is, the result can not be applied in the case of $\overline{p} = \int_0^T p(t)\,dt \geq 0$. Further, the condition imposed on function g is that $g(x) > 0$ for all $x \in \mathbb{R}$. Hence the result is far from ideal. Moreover, clearly equations (1.2), (1.3) and (1.4) are special cases of (1.1). Therefore, it is necessary and meaningful to study equation (1.1).

The primary purpose of this paper is to provide a general analysis technique to deal with such equations as (1.1), and meanwhile , we establish some general criteria to guarantee the existence of T-periodic solutions for equation (1.1) by using coincidence degree theory. For example, our Theorem 3.11 covers the corresponding theorem in [11] and the results in this article are also new when h is bounded. Secondly, the methods used to estimate a priori bounds of periodic solution are different from the corresponding ones in [6], [7], [13]. We integrate some novel analysis techniques to estimate the priori bounds. Finally, the significance of this paper is that the conditions $\int_0^T p(t)\,dt = 0$, $h(0) = 0$ and the boundedness of h are not required, which are required assumptions of some papers in the literature.

In the sequel, we will use the following notations.

The L^p-norm in $L^p([0,T],\mathbb{R}^n)$ is defined by

$$\|x\|_p = \bigg(\sum_{i=1}^n \int_0^T |x_i(t)|^p\,dt\bigg)^{1/p}.$$

The L^∞-norm in $L^\infty([0,T],\mathbb{R}^n)$ is

$$\|x\|_\infty = \max_{1\le i\le n} \|x_i\|_\infty,$$

where $\|x_i\|_\infty = \text{ess sup}_{t\in[0,T]} |x_i(t)|$, $(i = 1,\ldots,n)$.

2. Some preliminary lemmas

For the moment, we make the following notations: $T > 0$ is a constant, $C_T = \{x \in C(\mathbb{R},\mathbb{R}) \mid x(t+T) = x(t)\}$ with the norm $\|x\|_\infty = \max_{t\in[0,T]} |x(t)|$, $Z = \{z = (x(\,\cdot\,), y(\,\cdot\,))^T \in C(\mathbb{R},\mathbb{R}^2) \mid z(t+T) = z(t)\}$ with the norm $\|z\| = \max\{\|x\|_\infty, \|y\|_\infty\}$, and $X = \{w = (u(\,\cdot\,), v(\,\cdot\,))^T \in C^1(\mathbb{R},\mathbb{R}^2) \mid w(t+T) = w(t)\}$ with the norm $\|w\|_1 = \max\{\|w\|_\infty, \|w'\|_\infty\}$. Clearly, X, Z are Banach spaces.

We also define operators A and L in the following form, respectively:

$$\begin{aligned} A\colon C_T &\to C_T, &\quad (Ax)(t) &= x(t) - cx(t-\sigma), \\ L\colon D(L) \subset X &\to Z, &\quad (Lz)(t) &= \begin{pmatrix} (Ax)'(t) \\ y'(t) \end{pmatrix}, \end{aligned}$$

where $z = (x(\,\cdot\,), y(\,\cdot\,))^T$.

Lemma 2.1 (see [7]). *If $|c| \neq 1$, then A has a unique continuous bounded inverse and satisfies the following properties:*

(a) $\displaystyle\int_0^{2\pi} |(A^{-1}x)(t)|\,dt \le \frac{1}{|1-|c||} \int_0^{2\pi} |x(t)|\,dt$, *for all $x \in C_{2\pi}$;*

(b) $Ax'' = (Ax)''$, *for all $x \in C^2_{2\pi} := \{x \mid x \in C^2(\mathbb{R},\mathbb{R}),\ x(t+2\pi) = x(t)\}$.*

In order to use coincidence degree theory to study the existence of T-periodic solutions for (1.1), we rewrite (1.1) in the following form:

$$\text{(2.1)}\qquad \begin{cases} (Ax)'(t) = \varphi_q(y(t)) = |y(t)|^{q-2}y(t), \\ y'(t) = h(x'(t)) + f(x(t))x'(t) + \displaystyle\sum_{j=1}^n \beta_j(t)g(x(t-\tau_j(t))) + p(t). \end{cases}$$

One can easily know that if $z(t) = (x(t), y(t))^T$ is a T-periodic solution of (2.1), then $x(t)$ is a T-periodic solution of (1.1). By Lemma 2.1, we obtain that

$$\operatorname{Ker} L = \mathbb{R}^2 \quad \text{and} \quad \operatorname{Im} L = \biggl\{ z \in Z : \int_0^T z(t)\,dt = 0 \biggr\}.$$

So L is a Fredholm operator with index zero.

Letting $P: X \to \operatorname{Ker} L$ and $Q: Z \to \operatorname{Im} Q \subset \mathbb{R}^2$ be defined by

$$Pw = \begin{pmatrix} (Au)(0) \\ v(0) \end{pmatrix}, \quad w \in X; \qquad Qz = \frac{1}{T} \int_0^T z(s)\,ds, \quad z \in Z,$$

we get $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Ker} Q = \operatorname{Im} L$.

Letting $K_p: \operatorname{Im} L \to D(L) \cap \operatorname{Ker} P$ denote the inverse of $L_{D(L)\cap \operatorname{Ker} P}$, we have

$$\begin{aligned} (K_p z)(t) &= \begin{pmatrix} (A^{-1}Fx)(t) \\ (Fy)(t) \end{pmatrix}, & z(t) &= (x(t), y(t))^T, \\ (Fx)(t) &= \int_0^t x(s)\,ds, & (Fy)(t) &= \int_0^t y(s)\,ds. \end{aligned} \tag{2.2}$$

We introduce the nonlinear operator $N: X \to Z$ as follows:

$$(Nz)(t) = \begin{pmatrix} \varphi_q(y(t)) \\ h(x'(t)) + f(x(t))x'(t) + \sum_{j=1}^n \beta_j(t) g(x(t - \tau_j(t))) + p(t) \end{pmatrix}. \tag{2.3}$$

Lemma 2.2. *The nonlinear operator $N: X \to Z$ is L-compact on $\overline{\Omega}$, where Ω is an open bounded subset of X.*

Proof. From (2.2) and (2.3), we can easily see that $QN(\overline{\Omega}) \subset \mathbb{R}^2$ is bounded, and N is continuous and bounded. A^{-1} is bounded due to Lemma 2.1. Using Ascoli–Arzela theorem, one can examine that K_p is completely continuous. Hence $QN(\overline{\Omega})$ and $K_p(I - Q)N(\overline{\Omega})$ are relatively compact subset in Z and X, respectively, i.e. N is L-compact on $\overline{\Omega}$. □

Lemma 2.3 (see [5]). *Let $g \in C_T^0, \tau \in C_T^1$ and $\tau'(t) < 1$, for all $t \in [0, T]$. Then $g(\mu(t)) \in C_T^0$ and $\mu(t + T) = \mu(t) + T$, for all $t \in [0, T]$, where $\mu(t)$ is the inverse function of $t - \tau(t)$.*

Let $W = W^{1,p}([0, T], \mathbb{R}^n)$ be the Sobolev space.

Lemma 2.4 (see [12]). *Suppose $u \in W$ and $u(0) = u(T) = 0$, then*

$$\|u\|_\infty \le \left(\frac{T}{2}\right)^{1/q} \|u'\|_p, \quad \text{and} \quad \|u\|_p \le \frac{T}{\pi_p} \|u'\|_p,$$

where

$$\pi_p = 2 \int_0^{(p-1)^{1/p}} \frac{ds}{(1 - s^p/(p-1))^{1/p}} = \frac{2\pi(p-1)^{1/p}}{p \sin(\pi/p)}.$$

Lemma 2.5 (see [4]). *Let X and Z be two Banach spaces, $L: D(L) \subset X \to Z$ be a Fredholm operator with index zero, $\Omega \subset X$ be an open bounded set, and $N: \overline{\Omega} \to Z$ be L-compact and such that*

(a) $Lx \neq \lambda Nx$ *for all* $(x, \lambda) \in [D(L) \cap \partial\Omega] \times (0, 1)$*;*
(b) $Nx \notin \operatorname{Im} L$ *for all* $x \in \operatorname{Ker} L \cap \partial\Omega$*;*
(c) $\deg(JQN, \Omega \cap \operatorname{Ker} L, 0) \neq 0$*, where* $J: \operatorname{Im} Q \to \operatorname{Ker} L$ *is an isomorphism.*

Then the equation $Lx = Nx$ has at least one solution in $D(L) \cap \overline{\Omega}$.

3. Main results and their proofs

Throughout this paper, we assume $\tau_j \in C_T^1$ and $\tau_j'(t) < 1$, for all $t \in [0, T]$ $(j = 1, \dots, n)$. So the function $t - \tau_j(t)$ has a unique inverse denoted by $\mu_j(t)$ $(j = 1, \dots, n)$. We also denote

$$\overline{x} = \frac{1}{T}\int_0^T x(s)ds, \qquad \widetilde{x} = \frac{1}{T}\int_0^T |x(s)|ds, \quad \text{for all } x \in C_T,$$
$$\Gamma(t) = \sum_{j=1}^n \frac{\beta_j(\mu_j(t))}{1 - \tau_j'(\mu_j(t))}, \qquad \Gamma_1(t) = \sum_{j=1}^n \frac{|\beta_j(\mu_j(t))|}{1 - \tau_j'(\mu_j(t))}.$$

In order to state our main results conveniently, we make some basic hypotheses and notations.

(H1) $\Gamma(t) > 0$, for all $t \in \mathbb{R}$.

There exist constants $r_1 \geq 0$, $r_2 > 0$,$r_3 \geq 0$, $r_4 > 0$, $K_1 \geq 0$, $K_2, K_3 \in \mathbb{R}$ and $D > 0$ such that

(H2) $|h(x)| \leq r_1 T\overline{\Gamma}|x|^{p-1} + K_1 T\overline{\Gamma}$, for all $x \in \mathbb{R}$,
(H3) $xg(x) > 0$, for $|x| > D$, and $g(x) < -K_3$ for $x < -D$ and $K_3 \geq TK_1 + \overline{p}/\overline{\Gamma}$; and also $g(x) > K_2 + r_1 r_2 x^{p-1}$ for $x > D$, finally, we have $K_2 + r_1 r_2 D^{p-1} \geq TK_1 - \overline{p}/\overline{\Gamma}$,
(H4) $\limsup_{|x|\to\infty} |F(x)|/|x|^{p-1} \leq r_3$, where $F(x) = \int_0^x f(s)\,ds$.
(H5) $\lim_{x\to-\infty} |g(x)|/|x|^{p-1} = r_1 r_4$.

Further, we denote

$$\alpha(z) = \left(\frac{T}{2}\right)^{1/q} + T^{(q-1)/p}\left(\max\left\{\frac{1}{r_2}, \frac{r_1}{r_1 r_4 - z}\right\}\right)^{q-1} \operatorname{Sgn} r_1,$$
$$d(z) = \frac{T}{\pi_p} + \left(T \max\left\{\frac{1}{r_2}, \frac{r_1}{r_1 r_4 - z}\right\}\right)^{q-1} \operatorname{Sgn} r_1,$$
$$m_1(z) := m_1(\alpha, d, z) = \min\{\alpha(z)T^{1/p}, d(z)\},$$
$$m_2(z) := m_2(\alpha, d, z) = \min\{T^{1/q}\alpha^{p-1}(z), d^{p-1}(z)\}.$$

Finally, we set

$$A_p(\alpha,d,z)=r_1\frac{(1+|c|)}{|1-|c||}T\overline{\Gamma}\bigg[m_1(\alpha,d,z)+\alpha T^{1/p}\bigg\|\frac{\Gamma_1}{\Gamma}\bigg\|_\infty\bigg] + \frac{|c|}{|1-|c||}(r_3+z)m_2(\alpha,d,z)+2\alpha\|\Gamma\|_p\bigg\|\frac{\Gamma_1}{\Gamma}\bigg\|_\infty\frac{(1+|c|)}{|1-|c||}(r_1r_4+z)d^{p-1}.$$

Now, let us begin with the main results.

Theorem 3.1. *Suppose that the conditions* (H1)–(H5) *are fulfilled, then equation* (1.1) *has at least one T-periodic solution if* $A_p(\alpha(0),d(0),0)<1$.

Proof. We set $\Omega_1=\{z\in X : Lz=\lambda Nz,\ \lambda\in(0,1)\}$. Then for each $z=(x(t),y(t))^T\in\Omega_1$, one can see that $x(t)$ must satisfy the following equation:

$$\begin{cases} (Ax)'(t) = \lambda\varphi_q(y(t))=\lambda|y(t)|^{q-2}y(t), \\ y'(t) = \lambda h(x'(t))+\lambda f(x(t))x'(t)+\lambda\displaystyle\sum_{j=1}^n\beta_j(t)g(x(t-\tau_j(t)))+\lambda p(t). \end{cases} \tag{3.1}$$

Let $z(t)=(x(t),y(t))^T$ be a T-periodic solution of (3.1) for some $\lambda\in(0,1)$. One can know that $x=x(t)$ is a T-periodic solution of the following equation:

$$\bigg(\varphi_p\bigg(\frac{1}{\lambda}(Ax)'(t)\bigg)\bigg)'=\lambda h(x'(t))+\lambda f(x(t))x'(t)+\lambda\sum_{j=1}^n\beta_j(t)g(x(t-\tau_j(t)))+\lambda p(t),$$

i.e.

$$(\varphi_p((Ax)'(t)))'=\lambda^p h(x'(t))+\lambda^p f(x(t))x'(t) + \lambda^p\sum_{j=1}^n\beta_j(t)g(x(t-\tau_j(t)))+\lambda^p p(t). \tag{3.2}$$

Integrating both sides of (3.2) over $[0,T]$, we obtain

$$\int_0^T h(x'(t))\,dt+\int_0^T\sum_{j=1}^n\beta_j(t)g(x(t-\tau_j(t)))\,dt+T\overline{p}=0. \tag{3.3}$$

It is easy to see that

$$\begin{aligned}\int_0^T\beta_j(t)g(x(t-\tau_j(t)))\,dt&=\int_{-\tau_j(0)}^{T-\tau_j(T)}\frac{\beta_j(\mu_j(s))}{1-\tau_j'(\mu_j(s))}g(x(s))\,ds\\ &=\int_0^T\frac{\beta_j(\mu_j(s))}{1-\tau_j'(\mu_j(s))}g(x(s))\,ds\quad(j=1,\dots,n),\end{aligned}$$

since $\beta_j(\mu_j(t))/(1-\tau_j'(\mu_j(t)))\in C_T$ $(j=1,\dots,n)$ in view of Lemma 2.3. Thus we derive from (3.3) that

$$\int_0^T[h(x'(t))+\Gamma(t)g(x(t))]\,dt+T\overline{p}=0. \tag{3.4}$$

By using integral mean value theorem, there exists a point $\xi_1 \in [0,T]$ such that $\int_0^T \Gamma(t)g(x(t))\,dt = T\overline{\Gamma}g(x(\xi_1))$. Therefore (3.4) becomes

$$T\overline{\Gamma}g(x(\xi_1)) = -\int_0^T h(x'(t))\,dt - T\overline{p}. \tag{3.5}$$

Now, suppose that $x(\xi_1) > D$, then from (3.5), (H2) and (H3), we deduce that

$$\begin{aligned} K_2 + r_1 r_2 (x(\xi_1))^{p-1} < g(x(\xi_1)) &= -\frac{1}{T\overline{\Gamma}}\int_0^T h(x'(t))\,dt - \frac{\overline{p}}{\overline{\Gamma}} \\ &\le \frac{1}{T\overline{\Gamma}}\int_0^T |h(x'(t))|\,dt - \frac{\overline{p}}{\overline{\Gamma}} \\ &\le \int_0^T [r_1|x'(t)|^{p-1} + K_1]\,dt - \frac{\overline{p}}{\overline{\Gamma}} \\ &\le r_1 T^{1/p}\|x'\|_p^{p-1} + TK_1 - \frac{\overline{p}}{\overline{\Gamma}}. \end{aligned} \tag{3.6}$$

Case of $r_1 = 0$. It follows from (H3) and (3.6) that

$$TK_1 - \frac{\overline{p}}{\overline{\Gamma}} \le K_2 < TK_1 - \frac{\overline{p}}{\overline{\Gamma}},$$

which is a contradiction. Hence we get $x(\xi_1) \le D$.

If $x(\xi_1) < -D$, then we have from (H3) and (3.5) that

$$TK_1 + \frac{\overline{p}}{\overline{\Gamma}} \le K_3 < -g(x(\xi_1)) \le TK_1 + \frac{\overline{p}}{\overline{\Gamma}},$$

which is also a contradiction. Summing up the above arguments, we conclude that

$$|x(\xi_1)| \le D, \quad \text{when } r_1 = 0. \tag{3.7}$$

Case of $r_1 > 0$. On the one hand, we deduce from (3.6) that

$$|x(\xi_1)|^{p-1} < \frac{T^{1/p}}{r_2}\|x'\|_p^{p-1} + \frac{(TK_1 - K_2)\overline{\Gamma} - \overline{p}}{r_1 r_2 \overline{\Gamma}} := \frac{T^{1/p}}{r_2}\|x'\|_p^{p-1} + C_1, \tag{3.8}$$

here and in the following, C_i denoting some appropriate constants, which are independent of λ.

On the other hand, it is easy to check that $A_p(\alpha(z), d(z), z)$ is continuous on $(0, r_1 r_4)$ with respect to z. Since $A_p(\alpha(0), d(0), 0) < 1$, there exists a constant $\varepsilon > 0$ such that $A_p(\alpha(\varepsilon), d(\varepsilon), \varepsilon) < 1$. For such a small $\varepsilon > 0$, in view of assumption (H5), we find that there must be a constant $\rho_1 > D$, which is independent of λ, such that

$$\begin{aligned} &\frac{|g(x)|}{|x|^{p-1}} > (r_1 r_4 - \varepsilon) > 0 \quad \text{for } x < -\rho_1,\ r_1 > 0, \\ &\frac{|g(x)|}{|x|^{p-1}} < (r_1 r_4 + \varepsilon) \qquad \text{for } x < -\rho_1,\ r_1 \ge 0. \end{aligned} \tag{3.9}$$

If $x(\xi_1) < -\rho_1$, then from (3.9), (3.5) and (H2), we can derive that

$$(r_1 r_4 - \varepsilon)|x(\xi_1)|^{p-1} < |g(x(\xi_1))| \le r_1 T^{1/p} \|x'\|_p^{p-1} + TK_1 + \frac{|\overline{p}|}{\overline{\Gamma}},$$

i.e.

$$|x(\xi_1)|^{p-1} \le \frac{r_1 T^{1/p}}{r_1 r_4 - \varepsilon} \|x'\|_p^{p-1} + C_2. \tag{3.10}$$

From (3.7), (3.8) and (3.10), it is easy to see in either case of $r_1 = 0$ or case of $r_1 > 0$ that

$$|x(\xi_1)|^{p-1} \le \max\left\{ D^{p-1}, \frac{T^{1/p}}{r_2}\|x'\|_p^{p-1} + C_1, \frac{r_1 T^{1/p}}{r_1 r_4 - \varepsilon}\|x'\|_p^{p-1} + C_2 \right\} \le d_1 \|x'\|_p^{p-1} + e_1,$$

where $d_1 = \max\{T^{1/p}/r_2, r_1 T^{1/p}/(r_1 r_4 - \varepsilon)\}$, $e_1 = D^{p-1} + C_1 + C_2$. Without loss of generality, we can assume that $e_1 > 0$. Thus from the above inequality we obtain

$$|x(\xi_1)| \le (d_1 \|x'\|_p^{p-1} + e_1)^{1/(p-1)}. \tag{3.11}$$

For the sake of concision and standing out the ideas, the following two lemmas are formulated. The proof will be continued after the two lemmas.

Lemma 3.2. *We have the following relations*

$$\|x\|_p \le d(\varepsilon)\|x'\|_p + e\|x'\|_p^{2-p} + \rho, \tag{3.12}$$

$$\|x\|_\infty \le \alpha(\varepsilon)\|x'\|_p + \gamma\|x'\|_p^{2-p} + \rho, \tag{3.13}$$

where e, γ, ρ are defined in the proof, which are independent of λ.

Proof. By elementary analysis, there is a constant $\delta > 0$, which is independent of λ, such that

$$(1+x)^k < 1 + (1+k)x, \quad k \ge 0, \ \text{for all } x \in (0, \delta]. \tag{3.14}$$

If $\|x'\|_p = 0$, then from (3.11) we have $|x(\xi_1)| < e_1^{\frac{1}{p-1}}$.

If $\|x'\|_p > 0$, then we know

$$(d_1 \|x'\|_p^{p-1} + e_1)^{1/(p-1)} = d_1^{1/(p-1)} \|x'\|_p \left(1 + \frac{e_1}{d_1 \|x'\|_p^{p-1}}\right)^{1/(p-1)}. \tag{3.15}$$

If $e_1/(d_1\|x'\|_p^{p-1}) > \delta$, then $\|x'\|_p < (e_1/(d_1\delta))^{1/(p-1)}$. So from (3.11) we have $|x(\xi_1)| < (e_1/\delta + e_1)^{1/(p-1)}$.

If $e_1/(d_1\|x'\|_p^{p-1}) \le \delta$, then $\|x'\|_p \ge (e_1/(d_1\delta))^{1/(p-1)}$. Thus from (3.11), (3.14) and (3.15), we derive that

$$|x(\xi_1)| < d_1^{1/(p-1)} \|x'\|_p + d_1^{(2-p)/(p-1)} \frac{p e_1}{p-1} \|x'\|_p^{2-p}. \tag{3.16}$$

Now, for any fixed point $\xi \in [0,T]$, on the one hand, we have

$$\|x\|_\infty \le |x(\xi)| + \int_0^T |x'(s)|\,ds \le |x(\xi)| + T^{1/q}\|x'\|_p.$$

On the other hand, putting $u(t) = x(t+\xi) - x(\xi)$, then $u(0) = u(T) = 0$ and $u \in W^{1,p}([0,T],\mathbb{R})$. By Lemma 2.4, one can see

$$\begin{aligned}\|x\|_p &\le \bigg(\int_0^T (|u(t)| + |x(\xi)|)^p\,dt\bigg)^{1/p} \le \bigg(\int_0^T |u(t)|^p\,dt\bigg)^{1/p} + T^{1/p}|x(\xi)|\\ &\le \frac{T}{\pi_p}\|u'\|_p + T^{1/p}|x(\xi)| = \frac{T}{\pi_p}\|x'\|_p + T^{1/p}|x(\xi)|\end{aligned}$$

and

$$\|x\|_\infty \le \|u\|_\infty + |x(\xi)| \le \left(\frac{T}{2}\right)^{1/q}\|x'\|_p + |x(\xi)|.$$

From the above estimates, we sum up that

$$\|x\|_p \le \frac{T}{\pi_p}\|x'\|_p + T^{1/p}|x(\xi)| \tag{3.17}$$

$$\|x\|_\infty \le \left(\frac{T}{2}\right)^{1/q}\|x'\|_p + |x(\xi)|. \tag{3.18}$$

We denote $\rho_2 = \max\{\rho_1, e_1^{1/(p-1)}, (e_1/\delta + e_1)^{1/p-1}\}$ and always assume that $\|x'\|_p \ge (e_1/(d_1\delta))^{1/(p-1)}$. Obviously, ρ_2 is independent of λ. Thus from (3.7), (3.16)–(3.18), we conclude in ether case of $r_1 = 0$ or case of $r_1 > 0$ that

$$\begin{aligned}\|x\|_p &\le \left(\frac{T}{\pi_p} + T^{1/p}d_1^{q-1}\right)\|x'\|_p + d_1^{(2-p)/(p-1)}T^{1/p}\frac{pe_1}{p-1}\|x'\|_p^{2-p} + \rho\\ &= d(\varepsilon)\|x'\|_p + e\|x'\|_p^{2-p} + \rho\\ \|x\|_\infty &< \left[\left(\frac{T}{2}\right)^{1/q} + d_1^{1/(p-1)}\right]\|x'\|_p + d_1^{(2-p)/(p-1)}\frac{pe_1}{p-1}\|x'\|_p^{2-p} + \rho\\ &= \alpha(\varepsilon)\|x'\|_p + \gamma\|x'\|_p^{2-p} + \rho,\end{aligned}$$

where

$$e = d_1^{(2-p)/(p-1)}T^{1/p}\frac{pe_1}{p-1}, \quad \gamma = d_1^{(2-p)/(p-1)}\frac{pe_1}{p-1}, \quad \rho = \max\{\rho_2 T^{1/p}, \rho_2\}.$$

We state that d_1 is understood as zero when $r_1 = 0$. This completes the lemma.□

Lemma 3.3. *There is a appropriate constant $M > D$, which is independent of λ, such that*

$$\|(x,y)\|_1 = \max\{\|x\|_\infty, \|y\|_\infty, \|x'\|_\infty, \|y'\|_\infty\} < M.$$

Proof. Multiplying the two sides of (3.2) by $(Ax)(t)$ and integrating them over $[0,T]$, we get

$$
\begin{aligned}
(3.19)\qquad -\int_0^T |(Ax)'(t)|^p\,dt &= \lambda^p \int_0^T h(x'(t))[x(t)-cx(t-\sigma)]\,dt \\
&+ \lambda^p \int_0^T f(x(t))x'(t)[x(t)-cx(t-\sigma)]\,dt \\
&+ \lambda^p \int_0^T \sum_{j=1}^n \beta_j(t) g(x(t-\tau_j(t)))[x(t)-cx(t-\sigma)]\,dt \\
&+ \lambda^p \int_0^T p(t)[x(t)-cx(t-\sigma)]\,dt.
\end{aligned}
$$

In view of $\int_0^T f(x(t))x'(t)x(t)\,dt = 0$ and (b) of Lemma 2.1, we obtain from (3.19) that

$$
\begin{aligned}
(3.20)\qquad \int_0^T |(Ax')(t)|^p\,dt &= -\lambda^p \int_0^T h(x'(t))[x(t)-cx(t-\sigma)]\,dt \\
&+ c\lambda^p \int_0^T f(x(t))x'(t)x(t-\sigma)\,dt \\
&- \lambda^p \int_0^T \sum_{j=1}^n \beta_j(t) g(x(t-\tau_j(t))) \Big[x(t)-cx(t-\sigma)]\,dt \\
&- \lambda^p \int_0^T p(t)[x(t)-cx(t-\sigma)\Big]\,dt \\
&\leq \bigg|\int_0^T h(x'(t))[x(t)-cx(t-\sigma)]\,dt\bigg| + |c|\bigg|\int_0^T f(x(t))x'(t)x(t-\sigma)\,dt\bigg| \\
&+ (1+|c|)\|x\|_\infty \bigg[\sum_{j=1}^n \int_0^T \frac{|\beta_j(\mu_j(s))|}{1-\tau_j'(\mu_j(s))} g(x(s))\,ds + T\widetilde{p}\bigg] \\
&= \bigg|\int_0^T h(x'(t))[x(t)-cx(t-\sigma)]\,dt\bigg| + |c|\bigg|\int_0^T f(x(t))x'(t)x(t-\sigma)\,dt\bigg| \\
&+ (1+|c|)\|x\|_\infty \bigg[\int_0^T \Gamma_1(s) g(x(s))\,ds + T\widetilde{p}\bigg].
\end{aligned}
$$

On the one hand, from (H2) and (3.12), we have

$$
\begin{aligned}
\bigg|\int_0^T h(x'(t))x(t)\,dt\bigg| &\leq T\overline{\Gamma}\bigg(r_1 \int_0^T |x'(t)|^{p-1}|x(t)|\,dt + K_1 \int_0^T |x(t)|\,dt\bigg) \\
&\leq T\overline{\Gamma}(r_1\|x'\|_p^{p-1}\|x\|_p + K_1 T^{1/q}\|x\|_p) \\
&\leq r_1 d(\varepsilon) T\overline{\Gamma}\|x'\|_p^p + C_3\|x'\|_p + C_4\|x'\|_p^{p-1} + C_5\|x'\|_p^{2-p} + C_6.
\end{aligned}
$$

Hence, we get that

$$\begin{aligned}(3.21)\quad &\left|\int_0^T h(x'(t))[x(t)-cx(t-\sigma)]\,dt\right| \\ &\le \left|\int_0^T h(x'(t))x(t)\,dt\right| + |c|\left|\int_0^T h(x'(t))x(t-\sigma)\,dt\right| \\ &\le (1+|c|)r_1 d(\varepsilon)T\overline{\Gamma}\|x'\|_p^p + C_7\|x'\|_p + C_8\|x'\|_p^{p-1} + C_9\|x'\|_p^{2-p} + C_{10}.\end{aligned}$$

It follows from (3.13) that

$$(3.22)\qquad \|x\|_\infty\|x'\|_p^{p-1} \le \alpha(\varepsilon)\|x'\|_p^p + \gamma\|x'\|_p + \rho\|x'\|_p^{p-1}.$$

On the other hand, from (3.13), (3.22) and (H2), we have

$$\begin{aligned}(3.23)\qquad &\left|\int_0^T h(x'(t))[x(t)-cx(t-\sigma)]\,dt\right| \\ &\le (1+|c|)T\overline{\Gamma}(r_1T^{1/p}\|x'\|_p^{p-1} + K_1)\|x\|_\infty \\ &\le (1+|c|)r_1\alpha(\varepsilon)T^{(p+1)/p}\overline{\Gamma}\|x'\|_p^p \\ &\quad + C_{11}\|x'\|_p + C_{12}\|x'\|_p^{p-1} + C_{13}\|x'\|_p^{2-p} + C_{14}.\end{aligned}$$

Now set $m_1(\varepsilon) := m_1(\alpha(\varepsilon), d(\varepsilon), \varepsilon) = \min\{\alpha(\varepsilon)T^{1/p}, d(\varepsilon)\}$. Combing (3.21) with (3.23), we conclude that

$$\begin{aligned}(3.24)\quad \left|\int_0^T h(x'(t))[x(t)-cx(t-\sigma)]\,dt\right| &\le r_1 m_1(\varepsilon)(1+|c|)T\overline{\Gamma}\|x'\|_p^p \\ &\quad + C_{15}\|x'\|_p + C_{16}\|x'\|_p^{p-1} + C_{17}\|x'\|_p^{2-p} + C_{18}.\end{aligned}$$

Let $E_1 = \{t\in[0,T] : x(t) > \rho\}$, $E_2 = \{t\in[0,T] : x(t) < -\rho\}$ and $E_3 = \{t \in [0,T] : |x(t)| \le \rho\}$. Then from (3.4), we have that

$$(3.25)\qquad \left(\int_{E_1} + \int_{E_2} + \int_{E_3}\right)\Gamma(t)g(x(t))\,dt \le \int_0^T |h(x'(t))|\,dt + T|\overline{p}|.$$

As $\int_{E_1} |\Gamma(t)g(x(t))|\,dt = \int_{E_1}\Gamma(t)g(x(t))\,dt$ and from (3.9) we have

$$\int_{E_2} |\Gamma(t)g(x(t))|\,dt < (r_1r_4+\varepsilon)\|\Gamma\|_p\|x\|_p^{p-1}.$$

It follows from (3.25) that

$$\begin{aligned}(3.26)\quad &\int_{E_1}|\Gamma(t)g(x(t))|\,dt \\ &\le \int_{E_2}|\Gamma(t)g(x(t))|\,dt + \int_{E_3}|\Gamma(t)g(x(t))|\,dt + \int_0^T |h(x'(t))|dt + T|\overline{p}| \\ &< (r_1r_4+\varepsilon)\|\Gamma\|_p\|x\|_p^{p-1} + g_\rho T\overline{\Gamma} + \int_0^T |h(x'(t))|\,dt + T|\overline{p}|,\end{aligned}$$

where $g_\rho = \max_{|x|\le\rho}|g(x)|$. From (3.26), (3.5), (3.12) and (H2), we deduce that

$$
\begin{aligned}
(3.27)\qquad \int_0^T \Gamma_1(t)|g(x(t))|\,dt &= \int_0^T \frac{\Gamma_1(t)}{\Gamma(t)}\Gamma(t)|g(x(t))|\,dt\\
&\le 2\bigg\|\frac{\Gamma_1}{\Gamma}\bigg\|_\infty\bigg(\int_{E_2}+\int_{E_3}\bigg)\Gamma(t)|g(x(t))|\,dt\\
&\quad+\bigg\|\frac{\Gamma_1}{\Gamma}\bigg\|_\infty\int_0^T|h(x'(t))|\,dt+\bigg\|\frac{\Gamma_1}{\Gamma}\bigg\|_\infty T|\overline{p}|\\
&\le\bigg\|\frac{\Gamma_1}{\Gamma}\bigg\|_\infty\{2[(r_1r_4+\varepsilon)\|\Gamma\|_p\|x\|_p^{p-1}+g_\rho T\overline{\Gamma}]\\
&\quad+T\overline{\Gamma}[r_1T^{1/p}\|x'\|_p^{p-1}+TK_1]+T|\overline{p}|\}\\
&\le 2\bigg\|\frac{\Gamma_1}{\Gamma}\bigg\|_\infty(r_1r_4+\varepsilon)\|\Gamma\|_p(d(\varepsilon)\|x'\|_p+e\|x'\|_p^{2-p}+\rho)^{p-1}\\
&\quad+r_1\|\bigg\|\frac{\Gamma_1}{\Gamma}\bigg\|_\infty T\overline{\Gamma}T^{1/p}\|x'\|_p^{p-1}+C_{19}.
\end{aligned}
$$

We consider two cases.

Case 1. If $p\ge 2$, then we have

$$e\|x'\|_p^{2-p}+\rho\le e\bigg(\frac{e_1}{d_1\delta}\bigg)^{(2-p)/(p-1)}+\rho:=e_2.$$

Using the idea to prove (3.16), we know that there is $\delta_1>0$, which is independent of λ, such that

$$(d(\varepsilon)\|x'\|_p+e\|x'\|_p^{2-p}+\rho)^{p-1}<d^{p-1}(\varepsilon)\|x'\|_p^{p-1}+pe_2d^{p-2}(\varepsilon)\|x'\|_p^{p-2},$$

when

$$\|x'\|_p\ge\max\bigg\{\bigg(\frac{e_1}{d_1\delta}\bigg)^{1/(p-1)},\frac{e_2}{d(\varepsilon)\delta_1}\bigg\}.$$

Case 2. If $1<p<2$, then $0<p-1<1$. Applying the well-known inequality

$$(a+b)^k\le a^k+b^k,\quad\text{for all } a\ge 0,\ b\ge 0,\ 0<k\le 1,$$

we get

$$(d(\varepsilon)\|x'\|_p+e\|x'\|_p^{2-p}+\rho)^{p-1}<d^{p-1}(\varepsilon)\|x'\|_p^{p-1}+e^{p-1}\|x'\|_p^{(2-p)(p-1)}+\rho^{p-1}.$$

Summing up Cases 1 and 2, when

$$\|x'\|_p\ge\max\bigg\{\bigg(\frac{e_1}{d_1\delta}\bigg)^{1/(p-1)},\frac{e_2}{d(\varepsilon)\delta_1}\bigg\},$$

we conclude in either case of $p\ge 2$ or case of $1<p<2$ that

$$
\begin{aligned}
(3.28)\quad &(d(\varepsilon)\|x'\|_p+e\|x'\|_p^{2-p}+\rho)^{p-1}\\
&\qquad\le d^{p-1}(\varepsilon)\|x'\|_p^{p-1}+C_{20}\|x'\|_p^{p-2}+C_{21}\|x'\|_p^{(2-p)(p-1)}+C_{22}.
\end{aligned}
$$

Now, substituting (3.28) into (3.27), we obtain that

$$\text{(3.29)}\quad \int_0^T \Gamma_1(t)|g(x(t))|\,dt \le \left\|\frac{\Gamma_1}{\Gamma}\right\|_\infty [2\|\Gamma\|_p(r_1r_4+\varepsilon)d^{p-1}(\varepsilon) + r_1T^{(p+1)/p}\overline{\Gamma}]\|x'\|_p^{p-1} + C_{23}\|x'\|_p^{p-2} + C_{24}\|x'\|_p^{(2-p)(p-1)} + C_{25}.$$

From assumption (H4), without loss of generality, we can assume that

$$\text{(3.30)}\quad |F(x)| \le (r_3+\varepsilon)|x|^{p-1}, \quad \text{for } |x|>\rho.$$

As

$$\text{(3.31)}\quad \left|\int_0^T f(x(t))x'(t)x(t-\sigma)\,dt\right| = \left|\int_0^T F(x(t))x'(t-\sigma)\,dt\right| \le \int_0^T |F(x(t))x'(t-\sigma)|\,dt = \int_{E_1\cup E_2} |F(x(t))x'(t-\sigma)|\,dt + \int_{E_3} |F(x(t))x'(t-\sigma)|\,dt,$$

from (3.12), (3.30), (3.31) and (3.28), we deduce that

$$\begin{aligned}\text{(3.32)}\quad |c|\left|\int_0^T f(x(t))x'(t)x(t-\sigma)\,dt\right| &\le |c|(r_3+\varepsilon)\int_0^T |x(t)|^{p-1}x'(t-\sigma)|\,dt + |c|F_\rho\int_0^T |x'(t-\sigma)|\,dt \\ &\le |c|(r_3+\varepsilon)\|x\|_p^{p-1}\bigg(\int_{-\sigma}^{T-\sigma} |x'(t)|^p\,dt\bigg)^{1/p} + |c|F_\rho\int_{-\sigma}^{T-\sigma}|x'(t)|\,dt \\ &\le |c|(r_3+\varepsilon)\|x'\|_p(d(\varepsilon)\|x'\|_p + e\|x'\|_p^{2-p}+\rho)^{p-1} + |c|F_\rho T^{1/q}\|x'\|_p, \\ &\le |c|(r_3+\varepsilon)d^{p-1}(\varepsilon)\|x'\|_p^p \\ &\quad + C_{26}\|x'\|_p^{p-1} + C_{27}\|x'\|_p^{(2-p)(p-1)+1} + C_{28}\|x'\|_p,\end{aligned}$$

where $F_\rho = \max_{|x|\le\rho}|F(x)|$.

Similarly as the above proof, we can derive from (3.31) and (3.13) that

$$|c|\left|\int_0^T f(x(t))x'(t)x(t-\sigma)\,dt\right| \le |c|(r_3+\varepsilon)T^{1/q}\|x'\|_p(\alpha(\varepsilon)\|x'\|_p + \gamma\|x'\|_p^{2-p}+\rho)^{p-1} + |c|F_\rho T^{\frac{1}{q}}\|x'\|_p,$$

and when

$$\|x'\|_p \ge \max\bigg\{\bigg(\frac{e_1}{d_1\delta}\bigg)^{1/(p-1)}, \frac{e_2'}{\alpha(\varepsilon)\delta_1}\bigg\}, \quad \text{here } e_2' = \gamma\bigg(\frac{e_1}{d_1\delta}\bigg)^{(2-p)/(p-1)} + \rho,$$

we have

$$(\alpha(\varepsilon)\|x'\|_p + \gamma\|x'\|_p^{2-p}+\rho)^{p-1} \le \alpha^{p-1}(\varepsilon)\|x'\|_p^{p-1} + C_{29}\|x'\|_p^{p-2} + C_{30}\|x'\|_p^{(2-p)(p-1)} + C_{31}.$$

Thus, we obtain that

$$(3.33)\quad |c|\left|\int_0^T f(x(t))x'(t)x(t-\sigma)\,dt\right| \le |c|(r_3+\varepsilon)\alpha^{p-1}(\varepsilon)T^{1/q}\|x'\|_p^p + C_{32}\|x'\|_p^{p-1} + C_{33}\|x'\|_p^{(2-p)(p-1)+1} + C_{34}\|x'\|_p.$$

Let

$$m_2(\varepsilon) := m_2(\alpha(\varepsilon), d(\varepsilon), \varepsilon) = \min\{T^{1/q}\alpha^{p-1}(\varepsilon), d^{p-1}(\varepsilon)\}.$$

Then when $\|x'\|_p \ge \max\{(e_1/(d_1\delta))^{1/(p-1)}, e_2/(d(\varepsilon)\delta_1), e_2'/(\alpha(\varepsilon)\delta_1)\}$, we conclude from (3.32) and (3.33) that

$$(3.34)\quad |c|\left|\int_0^T f(x(t))x'(t)x(t-\sigma)\,dt\right| \le |c|(r_3+\varepsilon)m_2(\varepsilon)\|x'\|_p^p + C_{35}\|x'\|_p^{p-1} + C_{36}\|x'\|_p^{(2-p)(p-1)+1} + C_{37}\|x'\|_p.$$

Finally, substituting (3.13), (3.22), (3.24), (3.29) and (3.34) into (3.20), we sum up that

$$\begin{aligned}(3.35)\quad &\int_0^T |(Ax')(t)|^p\,dt \\ &\le r_1 m_1(\varepsilon)(1+|c|)T\overline{\Gamma}\|x'\|_p^p + C_{15}\|x'\|_p + C_{16}\|x'\|_p^{p-1} + C_{17}\|x'\|_p^{2-p} + C_{18} \\ &\quad + |c|(r_3+\varepsilon)m_2(\varepsilon)\|x'\|_p^p + C_{35}\|x'\|_p^{p-1} + C_{36}\|x'\|_p^{(2-p)(p-1)+1} \\ &\quad + C_{37}\|x'\|_p + (1+|c|)\|x\|_\infty\bigg\{\left\|\frac{\Gamma_1}{\Gamma}\right\|_\infty [2(r_1r_4+\varepsilon)\|\Gamma\|_p d^{p-1}(\varepsilon) \\ &\quad + r_1 T^{(p+1)/p}\overline{\Gamma}]\|x'\|_p^{p-1} + C_{23}\|x'\|_p^{p-2} + C_{24}\|x'\|_p^{(2-p)(p-1)} + C_{25} + T\widetilde{p}\bigg\} \\ &\le a_0\|x'\|_p^p + C_{38}\|x'\|_p^{p-1} + C_{39}\|x'\|_p^{p-2} + C_{40}\|x'\|_p + C_{41}\|x'\|_p^{(2-p)(p-1)+1} \\ &\quad + C_{42}\|x'\|_p^{p(2-p)} + C_{43}\|x'\|_p^{(2-p)(p-1)} + C_{44}\|x'\|_p^{2-p} + C_{45},\end{aligned}$$

where

$$\begin{aligned}a_0 = {}& r_1(1+|c|)T\overline{\Gamma}\left[m_1(\varepsilon) + \alpha(\varepsilon)T^{1/p}\left\|\frac{\Gamma_1}{\Gamma}\right\|_\infty\right] + |c|(r_3+\varepsilon)m_2(\varepsilon) \\ &+ 2\alpha(\varepsilon)(1+|c|)(r_1r_4+\varepsilon)\|\Gamma\|_p\left\|\frac{\Gamma_1}{\Gamma}\right\|_\infty d^{p-1}(\varepsilon).\end{aligned}$$

By using the first part of Lemma 2.1, we have

$$\int_0^T |x'(t)|^p\,dt = \int_0^T |(A^{-1}Ax')(t)|^p\,dt \le \frac{\int_0^T |(Ax)'(t)|^p\,dt}{|1-|c||},$$

i.e.

$$(3.36)\qquad \int_0^T |(Ax)'(t)|^p\,dt \ge |1-|c||\int_0^T |x'(t)|^p\,dt.$$

It follows from (3.35) and (3.36) that

$$(3.37)\quad |1-|c|| \|x'\|_p^p \le a_0\|x'\|_p^p + C_{38}\|x'\|_p^{p-1} + C_{39}\|x'\|_p^{p-2} + C_{40}\|x'\|_p + C_{41}\|x'\|_p^{(2-p)(p-1)+1} + C_{42}\|x'\|_p^{p(2-p)} + C_{43}\|x'\|_p^{(2-p)(p-1)} + C_{44}\|x'\|_p^{2-p} + C_{45}.$$

Notice that $a_0 = |1-|c||A_p(\alpha(\varepsilon), d(\varepsilon), \varepsilon) < |1-|c||$ and $p > 1$. Hence (3.37) implies that there is a constant $M_1 > 0$ such that $\|x'\|_p \le M_1$. Let

$$M_2 = \max\left\{\left(\frac{e_1}{d_1\delta}\right)^{1/(p-1)}, \frac{e_2}{d(\varepsilon)\delta_1}, \frac{e_2'}{\alpha(\varepsilon)\delta_1}, M_1\right\}.$$

Thus from (3.13) we have

$$\|x\|_\infty \le \alpha(\varepsilon)M_2 + \gamma M_2^{2-p} + \rho := R_1.$$

By the first equation of (3.1), we have $\int_0^T \varphi_q(y(t))\,dt = 0$, which implies that there is a constant $t_0 \in [0,T]$ such that $y(t_0) = 0$. So $\|y\|_\infty \le \int_0^T |y'(t)|\,dt$. It follows from the second equation of (3.1) that

$$y'(t) = \lambda h(x'(t)) + \lambda f(x(t))x'(t) + \lambda\sum_{j=1}^n \beta_j(t)g(x(t-\tau_j(t))) + \lambda p(t).$$

Therefore, we conclude that

$$\begin{aligned}\|y\|_\infty \le{}& \int_0^T |h(x'(t))|\,dt + \int_0^T |f(x(t))x'(t)|\,dt \\ &+ \sum_{j=1}^n \int_0^T |\beta_j(t)g(x(t-\tau_j(t)))|\,dt + \int_0^T |p(t)|\,dt \\ ={}& \int_0^T |h(x'(t))|\,dt + \int_0^T |f(x(t))x'(t)|\,dt + \int_0^T \Gamma_1(t)|g(x(t))|\,dt + T\widetilde{p} \\ \le{}& T\overline{\Gamma}[r_1T^{1/p}\|x'\|_p^{p-1} + TK_1] + f_{R_1}T^{1/q}\|x'\|_p + g_{R_1}T\|\Gamma_1\|_\infty + T\widetilde{p} \\ \le{}& T\overline{\Gamma}[r_1T^{1/p}M_2^{p-1} + TK_1] + f_{R_1}T^{1/q}M_2 + g_{R_1}T\|\Gamma_1\|_\infty + T\widetilde{p} := R_2,\end{aligned}$$

where $f_{R_1} = \max_{|x|\le R_1}|f(x)|$, $g_{R_1} = \max_{|x|\le R_1}|g(x)|$. Now, again by the equation (3.1), we obtain that

$$\|x'\|_\infty = \|A^{-1}(\lambda\varphi_q(y(\,\cdot\,)))\|_\infty \le \frac{\|y\|_\infty^{q-1}}{|1-|c||} \le \frac{R_2^{q-1}}{|1-|c||} := R_3.$$

and

$$\|y'\|_\infty \le h_{R_3} + f_{R_1}R_3 + g_{R_1}\sum_{j=1}^n \|\beta_j\|_\infty + \|p\|_\infty := R_4,$$

where $h_{R_3} = \max_{|x|\le R_3}|h(x)|$. Now, let $M \ge \max\{R_1, R_2, R_3, R_4\} + 1$. Then the lemma is proved. □

Continuation of the proof of Theorem 3.1. Next, we shall illustrate that all the conditions of Lemma 2.5 are satisfied. To this end, we put $\Omega = \{z = (x,y)^T \in X : \|(x,y)\|_1 < M\}$, where M is derived from Lemma 3.3. Then $Lx \neq \lambda Nx$ for all $(x,\lambda) \in [D(L)\cap\partial\Omega]\times(0,1)$, i.e. the condition (a) of Lemma 2.5 is fulfilled. And for each $z = (x,y)^T \in \partial\Omega \cap KerL$, we have

$$QNz = \frac{1}{T}\int_0^T \begin{pmatrix} |y|^{q-2}y \\ h(0) + \sum_{j=1}^n \beta_j(t)g(x) + p(t) \end{pmatrix} dt.$$

Notice that

$$\begin{aligned}\int_0^T \sum_{j=1}^n \beta_j(t)\,dt &= \sum_{j=1}^n \int_{-\tau_j(0)}^{T-\tau_j(T)} \frac{\beta_j(\mu_j(s))}{1-\tau_j'(\mu_j(s))}\,ds \\ &= \int_0^T \sum_{j=1}^n \frac{\beta_j(\mu_j(s))}{1-\tau_j'(\mu_j(s))}\,ds = T\overline{\Gamma}.\end{aligned}$$

Hence

$$QNz = \begin{pmatrix} |y|^{q-2}y \\ h(0) + \overline{\Gamma}g(x) + \overline{p} \end{pmatrix}, \quad z = (x,y)^T \in \partial\Omega \cap \operatorname{Ker} L. \tag{3.38}$$

We prove that $QNz \neq 0$, for all $z \in \partial\Omega\cap\operatorname{Ker} L$. In fact, if $QNz = 0$, then from (3.38) we have $y = 0$, $x = M$ or $x = -M$. For the case of $x = M$, on the one hand, it follows from (3.38) and (H2) that

$$g(M) = -\frac{h(0)}{\overline{\Gamma}} - \frac{\overline{p}}{\overline{\Gamma}} \leq TK_1 - \frac{\overline{p}}{\overline{\Gamma}}, \tag{3.39}$$

on the other hand, notice that $M > D$, thus from (H3), we obtain

$$g(M) > K_2 + r_1r_2D^{p-1}. \tag{3.40}$$

Together with (3.39) and (3.40) yield that $K_2 + r_1r_2D^{p-1} < TK_1 - \overline{p}/\overline{\Gamma}$, which contradicts (H3). For the case of $x = -M$, we have from (3.38), (H2) and (H3) that

$$|g(-M)| = -g(-M) = \frac{h(0)}{\overline{\Gamma}} + \frac{\overline{p}}{\overline{\Gamma}} \leq TK_1 + \frac{\overline{p}}{\overline{\Gamma}}, \tag{3.41}$$

meanwhile, we also obtain from (H3) that

$$|g(-M)| = -g(-M) > K_3. \tag{3.42}$$

Clearly, (3.41) is incompatible with (3.42). Therefore, we sum up that $QNz \neq 0$, i.e. $Nz \notin \operatorname{Im} L$, for all $z \in \partial\Omega\cap\operatorname{Ker} L$. So the condition (b) of Lemma 2.5 is fulfilled. It is remained to check that the condition (c) of Lemma 2.5 is also fulfilled. To this purpose, we define the isomorphism $J\colon \operatorname{Im} Q \to \operatorname{Ker} L$ as follows:

$$J(x,y)^T = (y,x)^T. \tag{3.43}$$

Let

$$H(z,\mu) = \mu z + (1-\mu)JQNz, \quad (z,\mu) \in \Omega \times [0,1]. \tag{3.44}$$

Then for any $(z,\mu) \in (\partial\Omega \cap \operatorname{Ker} L) \times [0,1]$, we have from (3.44), (3.43) and (3.38) that

$$H(z,\mu) = \begin{pmatrix} \mu x + (1-\mu)[h(0) + \overline{\Gamma} g(x) + \overline{p}] \\ [\mu + (1-\mu)|y|^{q-2}]y \end{pmatrix},$$

for all $(z,\mu) \in (\partial\Omega \cap \operatorname{Ker} L) \times [0,1]$. If $H(z,\mu) = 0$, then $y = 0$, $x = M$ or $x = -M$.

When $x = M$, it follows from (3.40) and (H3) that

$$h(0)+\overline{\Gamma} g(M)+\overline{p} > h(0)+\overline{\Gamma}(K_2+r_1r_2D^{p-1})+\overline{p} \geq h(0)+T\overline{\Gamma}K_1-\overline{p}+\overline{p} \geq 0. \tag{3.45}$$

Similarly when $x = -M$, again by (H3), we obtain that

$$h(0) + \overline{\Gamma} g(-M) + \overline{p} \leq T\overline{\Gamma}K_1 + \overline{\Gamma} g(-M) + \overline{p} \leq \overline{\Gamma} g(-M) + K_3\overline{\Gamma} < 0. \tag{3.46}$$

From (3.45) and (3.46), one can easily deduce that $H(z,\mu) \neq 0$, for all $(z,\mu) \in (\partial\Omega \cap \operatorname{Ker} L) \times [0,1]$, which illustrates that $H(z,\mu)$ is a homotopic mapping. Hence

$$\begin{aligned} \deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} &= \deg\{H(\,\cdot\,,0), \Omega \cap \operatorname{Ker} L, 0\} \\ &= \deg\{H(\,\cdot\,,1), \Omega \cap \operatorname{Ker} L, 0\} = \deg\{I, \Omega \cap \operatorname{Ker} L, 0\} \neq 0. \end{aligned}$$

So the condition (c) of Lemma 2.5 is also satisfied. Recalling that we have proved in Lemma 2.2 that $N: X \to Z$ is L-compact on $\overline{\Omega}$. By Lemma 2.5, we obtain that the equation $Lz = Nz$ has at least one solution $z(t) = (x(t), y(t))^T$ on $\overline{\Omega} \cap D(L)$, i.e. equation (1.1) has at least one T-periodic solution $x(t)$ on $\overline{\Omega} \cap D(L)$. □

Corollary 3.4. *In addition to the conditions* (H1), (H2), (H4) *and* (H5), *we also suppose there exist constants* $r_0 > 0$, $s > 0$ *and* $D_1 > 0$ *such that*

- (a) $xg(x) > 0$, *for* $|x| > D_1$, *and* $g(x) < -K_3$ *for* $x < -D_1$ *and* $K_3 \geq TK_1 + \overline{p}/\overline{\Gamma}$; *and also* $K_2 + r_1r_2D_1^{p-1} \geq TK_1 - \overline{p}/\overline{\Gamma}$,
- (b) $\lim_{x\to\infty} g(x)/x^{p-1+s} = r_0$, *or* $\lim_{x\to\infty} g(x)/x^{p-1+s} = \infty$.

Then equation (1.1) *has at least one T-periodic solution if* $A_p(\alpha(0), d(0), 0) < 1$.

Proof. Indeed, it follows from (b) that

$$\lim_{x\to\infty} \frac{g(x) - K_2 - r_1r_2x^{p-1}}{x^{p-1+s}} > \frac{r_0}{3} > 0,$$

which yields that there is a constant $D > D_1$ such that

$$g(x) - K_2 - r_1r_2x^{p-1} > \frac{r_0}{3}x^{p-1+s} \quad \text{for } x > D,$$

i.e. the hypothesis (H3) holds, thus the result follows from Theorem 3.1. □

Now, we modify the assumptions (H3) and (H5) as follows.

(H3') $xg(x) > 0$, for $|x| > D$, and $g(x) > K_3$ for $x > D$ and $K_3 \geq TK_1 + \overline{p}/\overline{\Gamma}$; and also $g(x) < -K_2 - r_1 r_2 |x|^{p-1}$ for $x < -D$, finally, we have $K_2 + r_1 r_2 D^{p-1} \geq TK_1 - \overline{p}/\overline{\Gamma}$,

(H5') $\lim_{x\to\infty} |g(x)|/|x|^{p-1} = r_1 r_4$.

Similarly argument as in the proof of Theorem 3.1 and Corollary 3.4, we have the following results.

Theorem 3.5. *Suppose that the conditions* (H1), (H2), (H3'), (H4) *and* (H5') *are fulfilled, then equation* (1.1) *has at least one T-periodic solution if* $A_p(\alpha(0), d(0), 0) < 1$.

Corollary 3.6. *In addition to the conditions* (H1), (H2), (H4) *and* (H5'), *we also suppose there exist constants $r_0 > 0$, $s > 0$ and $D_1 > 0$ such that*

(a) $xg(x) > 0$ *for* $|x| > D_1$, $g(x) > K_3$ *for* $x > D_1$ *and* $K_3 \geq TK_1 + \overline{p}/\overline{\Gamma}$; *and also* $K_2 + r_1 r_2 D_1^{p-1} \geq TK_1 - \overline{p}/\overline{\Gamma}$,

(b) $\lim_{x\to-\infty} |g(x)|/|x|^{p-1+s} = r_0$, *or* $\lim_{x\to-\infty} |g(x)|/|x|^{p-1+s} = \infty$.

Then equation (1.1) *has at least one T-periodic solution if* $A_p(\alpha(0), d(0), 0) < 1$.

Remark 3.7. If $r_1 = 0$ and $c = 0$, then the condition $A_p(\alpha(0), d(0), 0) = 0 < 1$ naturally holds, which illustrates that the assumptions (H1)–(H5) can guarantee the existence of T-periodic solution for (1.1).

If $h \equiv 0$ and $\overline{p} = 0$, then the assumption (H3) becomes the very simple form as follows.

(H3') $xg(x) > 0$ for $|x| > D$, $g(x) < -K_3$ for $x < -D$ and $K_3 \geq 0$; and also $g(x) > K_2$ for $x > D$, finally, we have $K_2 \geq 0$.

It is known that (H3') is a usual and basic assumption in the related papers, which shows that our hypothesis (H3) is not stringent but rational.

If only $r_1 = 0$, then we have

$$A_p(\alpha(0), d(0), 0) = \frac{r_3|c|}{|1-|c||} m_1(\alpha(0), d(0), 0) = \frac{r_3|c|T^{p-1}}{|1-|c||} \min\left\{\frac{1}{2^{(p-1)/q}}, \frac{1}{\pi_p^{p-1}}\right\}.$$

We point out that in such a special case we can use another analysis technique to deal with it, which may be more refined than the method used in Theorem 3.1. We would like to study a more large class of functions, precisely, we relax assumption (H5) as follows:

(H5'') $\limsup_{x\to-\infty} |g(x)|/|x|^{p-1} \leq r_4$, here $r_4 \geq 0$.

For convenience, we denote

$$B_p(z) = \frac{T^{1/q}}{|1-|c||}\left[|c|(r_3 + z) + 2(1+|c|)T\overline{\Gamma}\left\|\frac{\Gamma_1}{\Gamma}\right\|_\infty (r_4 + z)\right]^{1/p}.$$

We derive the following result.

Theorem 3.8. *Assume that* (H1)–(H4) *and* (H5") *are satisfied for* $r_1 = 0$, *then equation* (1.1) *has at least one* T*-periodic solution if* $B_p(0) < 1$.

Proof. We can derive that there is a constant $\rho > D$, which is independent of λ, such that

$$\|x\|_\infty \leq \rho + \int_0^T |x'(t)|\,dt := \rho + \|x'\|. \tag{3.47}$$

It follows from (3.47) and (H2) that

$$\left|\int_0^T h(x'(t))[x(t) - cx(t-\sigma)]\,dt\right| \leq T^2K_1(1+|c|)\overline{\Gamma}(\rho + \|x'\|). \tag{3.48}$$

Again by (3.4) and (H2), we obtain that

$$\int_0^T \Gamma(t)g(x(t))\,dt \leq T[TK_1\overline{\Gamma} - \overline{p}]. \tag{3.49}$$

We deduce from (3.47) and (H5") that

$$\int_{E_2} \Gamma(t)|g(x(t))|\,dt < (r_4 + \varepsilon)T\overline{\Gamma}(\rho + \|x'\|)^{p-1}, \tag{3.50}$$

here and in the following, $\varepsilon > 0$ satisfying $B_p(\varepsilon) < 1$. As

$$(\rho + \|x'\|)^{p-1} < \|x'\|^{p-1} + p\rho\|x'\|^{p-2}, \quad \text{when } \|x'\| \geq \frac{\rho}{\delta_1}, \tag{3.51}$$

it follows from (3.50) and (3.51) that

$$\int_{E_2} \Gamma(t)|g(x(t))|\,dt < (r_4 + \varepsilon)T\overline{\Gamma}\|x'\|^{p-1} + p\rho(r_4 + \varepsilon)T\overline{\Gamma}\|x'\|^{p-2}. \tag{3.52}$$

One can deduce from (3.49) and (3.52) that

$$\int_{E_1} \Gamma(t)|g(x(t))|\,dt \leq (r_4 + \varepsilon)T\overline{\Gamma}\|x'\|^{p-1} + p\rho(r_4 + \varepsilon)T\overline{\Gamma}\|x'\|^{p-2} + C_{46}. \tag{3.53}$$

We conclude from (3.52) and (3.53) that

$$\int_0^T \Gamma(t)|g(x(t))|\,dt \leq 2(r_4 + \varepsilon)T\overline{\Gamma}\|x'\|^{p-1} + 2p\rho(r_4 + \varepsilon)T\overline{\Gamma}\|x'\|^{p-2} + C_{47}. \tag{3.54}$$

Thus, it follows from (3.54) that

$$\begin{aligned}
&\int_0^T \Gamma_1(t)|g(x(t))|\,dt \\
&\quad = \int_0^T \frac{\Gamma_1(t)}{\Gamma(t)}\Gamma(t)|g(x(t))|\,dt \leq \left\|\frac{\Gamma_1}{\Gamma}\right\|_\infty \int_0^T \Gamma(t)|g(x(t))|\,dt \\
&\quad \leq 2(r_4 + \varepsilon)T\overline{\Gamma}\left\|\frac{\Gamma_1}{\Gamma}\right\|_\infty \|x'\|^{p-1} + 2p\rho(r_4 + \varepsilon)T\overline{\Gamma}\left\|\frac{\Gamma_1}{\Gamma}\right\|_\infty \|x'\|^{p-2} + C_{48}.
\end{aligned} \tag{3.55}$$

We obtain from (3.31) and (3.51)that

$$
\begin{aligned}
(3.56)\quad |c|\bigg|\int_0^T f(x(t))x'(t)x(t-\sigma)\,dt\bigg|
&\le |c|(r_3+\varepsilon)\int_0^T |x(t)|^{p-1}|x'(t-\sigma)|\,dt + |c|F_\rho\int_0^T |x'(t-\sigma)|\,dt\\
&\le |c|(r_3+\varepsilon)(\rho+\|x'\|)^{p-1}\|x'\| + |c|F_\rho\|x'\|\\
&\le |c|(r_3+\varepsilon)\|x'\|^p + |c|(r_3+\varepsilon)p\rho\|x'\|^{p-1} + |c|F_\rho\|x'\|.
\end{aligned}
$$

Summing up (3.47), (3.48), (3.55) and (3.56), we infer that

$$
\begin{aligned}
(3.57)\quad \int_0^T |(Ax')(t)|^p\,dt &\le T^2K_1(1+|c|)\overline{\Gamma}(\rho+\|x'\|) + |c|(r_3+\varepsilon)\|x'\|^p\\
&\quad + |c|(r_3+\varepsilon)p\rho\|x'\|^{p-1} + |c|F_\rho\|x'\|\\
&\quad + (1+|c|)\|x\|_\infty\bigg\{2(r_4+\varepsilon)T\overline{\Gamma}\bigg\|\frac{\Gamma_1}{\Gamma}\bigg\|_\infty\|x'\|^{p-1}\\
&\quad + 2p\rho(r_4+\varepsilon)T\overline{\Gamma}\bigg\|\frac{\Gamma_1}{\Gamma}\bigg\|_\infty\|x'\|^{p-2} + C_{48} + T\widetilde{p}\bigg\}.\\
&\le b_0'\|x'\|^p + C_{49}\|x'\|^{p-1} + C_{50}\|x'\|^{p-2} + C_{51}\|x'\| + C_{52},
\end{aligned}
$$

where $b_0' = |c|(r_3+\varepsilon)+2(1+|c|)T\overline{\Gamma}\|\Gamma_1/\Gamma\|_\infty(r_4+\varepsilon)$. The first part of Lemma 2.1 implies that

$$
\begin{aligned}
(3.58)\quad \int_0^T |x'(t)|\,dt &= \int_0^T |(A^{-1}Ax')(t)|\,dt\\
&\le \frac{\int_0^T |(Ax)'(t)|\,dt}{|1-|c||} \le \frac{T^{1/q}(\int_0^T |(Ax)'(t)|^p\,dt)^{1/p}}{|1-|c||}.
\end{aligned}
$$

Notice that $0<1/p<1$. Using the well-known inequality

$$
(a+b)^k \le a^k + b^k, \quad \text{for all } a\ge 0,\ b\ge 0,\ 0<k\le 1,
$$

we deduce from (3.57) and (3.58) that

$$
\begin{aligned}
(3.59)\quad \|x'\| &\le \frac{T^{1/q}}{|1-|c||}\{b_0'\|x'\|^p + C_{49}\|x'\|^{p-1} + C_{50}\|x'\|^{p-2} + C_{51}\|x'\| + C_{52}\}^{1/p}\\
&\le b_0\|x'\| + C_{53}\|x'\|^{1/q} + C_{54}\|x'\|^{(p-2)/p} + C_{55}\|x'\|^{1/p} + C_{56},
\end{aligned}
$$

where

$$
b_0 = \frac{T^{1/q}}{|1-|c||}\bigg[|c|(r_3+\varepsilon) + 2(1+|c|)T\overline{\Gamma}\bigg\|\frac{\Gamma_1}{\Gamma}\bigg\|_\infty(r_4+\varepsilon)\bigg]^{1/p}.
$$

Notice that $b_0 = B_p(\varepsilon) < 1$, $p>1$ and $q>1$. Hence (3.59) implies that there is a constant $M_3>0$ such that $\|x'\|\le M_3$. Let $M_4=\max\{\rho/\delta_1, M_3\}$. The remained proof follows along the lines of Theorem 3.1, and hence we omit it. □

Remark 3.9. One can easily see that the analysis technique used in Theorem 3.8 can not be applied in the case of Theorem 3.1.

We remark that the analysis technique can be applied in the following equation:

$$(\varphi_p((x(t)-cx(t-\sigma))'))' = h(x'(t)) + f(x(t))x'(t) + g(x(t-\tau(t))) + p(t), \tag{3.60}$$

where h, f, g, p are defined in the same way as above, τ is only required to be a continuous T-periodic function. In this situation, the hypotheses (H2) and (H3) correspondingly become

- (H2') $|h(x)| \le r_1T|x|^{p-1} + K_1T$, for all $x \in \mathbb{R}$,
- (H3") $xg(x) > 0$, for $|x| > D$, and $g(x) < -K_3$ for $x < -D$ and $K_3 \ge TK_1 + \overline{p}$; and also $g(x) > K_2 + r_1r_2x^{p-1}$ for $x > D$, finally, we have $K_2 + r_1r_2D^{p-1} \ge TK_1 - \overline{p}$.

And $A_p(\alpha, d, z)$ becomes

$$\begin{aligned} A_p'(\alpha,d,z) = {} & r_1\frac{(1+|c|)}{|1-|c||}T[m_1(\alpha,d,z) + \alpha T^{1/p}] \\ & + \frac{|c|}{|1-|c||}(r_3+z)m_2(\alpha,d,z) + 2T\frac{(1+|c|)}{|1-|c||}(r_1r_4+z)\alpha^p. \end{aligned}$$

In the similar way as in the proof of Theorems 3.1, we have the following result.

Theorem 3.10. *Assume that* (H2'), (H3"), (H4) *and* (H5) *hold, then equation* (3.60) *has at least one T-periodic solution if* $A_p'(\alpha(0), d(0), 0) < 1$.

Proof. In this setting, (3.4) reduces to

$$\int_0^T [h(x'(t)) + g(x(t-\tau(t)))]\,dt + T\overline{p} = 0, \tag{3.61}$$

which implies that there is a $\xi_2 \in [0,T]$ such that

$$Tg(x(\xi_2 - \tau(\xi_2))) = -\int_0^T h(x'(t))\,dt - T\overline{p}. \tag{3.62}$$

Clearly, there are an integer number k and $\xi_1' \in [0,T]$ such that $x({\xi_1}') = x(\xi_2 - \tau(\xi_2) - kT) = x(\xi_2 - \tau(\xi_2))$, which yields (3.62) that

$$Tg(x(\xi_1')) = -\int_0^T h(x'(t))\,dt - T\overline{p}. \tag{3.63}$$

Now, let ξ_1 be replaced by ξ_1' and $\Gamma(t) = \Gamma_1(t) \equiv 1$ in Theorem 3.1. Once we give a priori estimate on $\int_0^T |g(x(t-\tau(t)))(Ax)(t)|\,dt$. Then the remained proof follows along the lines of Theorem 3.1.

To this end, we put $E'_1 = \{t \in [0,T] : x(t-\tau(t)) > \rho\}$, $E'_2 = \{t \in [0,T] : x(t-\tau(t)) < -\rho\}$ and $E'_3 = \{t \in [0,T] : |x(t-\tau(t))| \le \rho\}$. Then from (3.9) we have

$$\int_{E'_2} |g(x(t-\tau(t)))|\,dt \le (r_1 r_4 + \varepsilon)T\|x\|_\infty^{p-1}. \tag{3.64}$$

We deduce from (3.61) and (3.64) that

$$\int_{E'_1} |g(x(t-\tau(t)))|\,dt \le (r_1 r_4 + \varepsilon)T\|x\|_\infty^{p-1} + r_1 T^{(p+1)/p}\|x'\|_p^{p-1} + C_{57}. \tag{3.65}$$

Thus, we obtain from (3.61), (3.64) and (3.65) that

$$\int_0^T |g(x(t-\tau(t)))|\,dt \le 2(r_1 r_4 + \varepsilon)T\|x\|_\infty^{p-1} + r_1 T^{(p+1)/p}\|x'\|_p^{p-1} + C_{58}. \tag{3.66}$$

From (3.13), (3.22), (3.66) and the ideas used to prove (3.28), when $\|x'\|_p$ is suitably large, we conclude that

$$\begin{aligned}\int_0^T &|g(x(t-\tau(t)))(Ax)(t)|\,dt \\ &\le (1+|c|)\alpha(\varepsilon)\Big[2(r_1 r_4+\varepsilon)T\alpha^{p-1}(\varepsilon) + r_1 T^{(p+1)/p}\Big]\|x'\|_p^p \\ &\quad + C_{59}\|x'\|_p^{p-1} + C_{60}\|x'\|_p^{p-2} + C_{61}\|x'\|_p + C_{62}\|x'\|_p^{(2-p)(p-1)+1} \\ &\quad + C_{63}\|x'\|_p^{p(2-p)} + C_{64}\|x'\|_p^{(2-p)(p-1)} + C_{65}\|x'\|_p^{2-p} + C_{66}.\end{aligned}$$

□

Together with the ideas used to prove Theorems 3.10 and 3.8, we have the following result.

Theorem 3.11. *Assume that* (H2'), (H3"), (H4) *and* (H5") *hold for* $r_1 = 0$, *then equation* (3.60) *has at least one T-periodic solution if* $B'_p(0) < 1$, *where*

$$B'_p(z) = \frac{T^{1/q}}{|1-|c||}[|c|(r_3+z) + 2T(1+|c|)(r_4+z)]^{1/p}.$$

Remark 3.12. If $r_3 = 0$, then $B'_p(0) = T(2r_4)^{1/p}/|1-|c||$. Under the conditions of Theorem 1 in [11], we have $r_4 = 0$, thus $B'_p(0) = 0 < 1$ and all the assumptions of Theorem 3.11 are satisfied. So Theorem 3.11 generalizes the corresponding theorem in [11]. It should be pointed out that, in the the particular case that $f \equiv 0$ and $p = 2$ in equation (3.60), another analysis method can deal with it, see [7]. But it is not difficult to see that the analysis method is invalid in either case of $p \ne 2$ or case of $f \not\equiv 0$.

4. An example

As an application, we consider the following equation

$$(\varphi_2((x(t)-2x(t-\pi))'))' = h(x'(t))+f(x(t))x'(t)+\beta(t)g(x(t-\tau(t)))+p(t), \tag{4.1}$$

where $h(x) = r_1 x$, $f(x) = r_3 x\sin x$, $\beta(t) = 1 + (4/5)\sin t$, $\tau(t) = (4/5)\cos t$, $p(t) = 4\sin t + 1$ and

$$g(x) = r_1 r_4 \begin{cases} x^3 & \text{if } x \geq 0, \\ x & \text{if } x < 0. \end{cases}$$

We have $T = 2\pi$, $p = 2$, $\overline{p} = 1$, $c = 2$,

$$F(x) = -r_3 x\cos x + r_3 \sin x$$

and

$$\Gamma_1(t) = \Gamma(t) = \frac{1 + (4/5)\sin\mu(t)}{1 + (4/5)\sin\mu(t)} = 1 > 0,$$

for all $t \in [0, 2\pi]$, $\overline{\Gamma} = 1$, $\|\Gamma_1/\Gamma\|_\infty = 1$, $\|\Gamma\|_2 = \sqrt{2\pi}$, where $\mu(t)$ denotes the inverse function of $t - (4/5)\cos t$.

Corresponding to Theorem 3.1, we can choose $K_1 = 0$, $K_2 \geq 0$, $K_3 \geq 1$ and $r_2 = r_4 = 1$, then the assumptions (H1)–(H5) hold.

A simple calculation yields that

$$\alpha(0) = (\sqrt{2}\mathrm{Sgn}\, r_1 + 1)\sqrt{\pi}, \quad d(0) = 2(\pi\mathrm{Sgn}\ r_1 + 1),$$
$$m_1(\alpha(0), d(0), 0) = m_2(\alpha(0), d(0), 0) = 2(\pi\mathrm{Sgn}\, r_1 + 1)$$

and

$$\begin{aligned} A_p(\alpha(0), d(0), 0) = {} & 6\pi[4\pi\mathrm{Sgn}\, r_1 + \sqrt{2}\pi + 2]r_1 + 4(\pi\mathrm{Sgn}\, r_1 + 1)r_3 \\ & + 12\sqrt{2}\pi(\sqrt{2}\mathrm{Sgn}\, r_1 + 1)(\pi\mathrm{Sgn}\, r_1 + 1)r_1. \end{aligned} \tag{4.2}$$

According to (4.2), so long as we choose $r_1 \geq 0$, $r_3 \geq 0$, such that

$$[6\pi(4\pi + \sqrt{2}\pi + 2) + 12(2 + \sqrt{2})\pi(\pi + 1)]r_1 + 4(\pi + 1)r_3 < 1.$$

Then all the hypotheses of Theorem 3.1 are satisfied, hence equation (4.1) has at least one 2π-periodic solution.

Remark 4.1. Particularly, setting $r_1 = 0$, then the above result can not be obtained by the Theorem of [13], since, on the one hand, $\overline{p} = 1 > 0$, on the other hand, the condition $g(x) > 0$ is not satisfied, which indicates that Theorem 3.1 is essentially new in such case. More importantly, the conditions that $\int_0^T p(t)\,dt = 0$, $h(0) = 0$ and the boundedness of h in Theorem 3.1 are not required.

Remark 4.2. Clearly, the case of $xg(x) < 0$ can be studied similarly.

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