Topological Methods in Nonlinear Analysis
Journal of the Juliusz Schauder Center
Volume 37, 2011, 225–234

EIGHT POSITIVE PERIODIC SOLUTIONS TO THREE SPECIES NON-AUTONOMOUS LOTKA–VOLTERRA COOPERATIVE SYSTEMS WITH HARVESTING TERMS

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Abstract. By using Mawhin's continuation theorem of coincidence degree theory and linear inequality, we establish the existence of eight positive periodic solutions for three species non-autonomous Lotka-Volterra cooperative systems with harvesting terms. An example is given to illustrate the effectiveness of our results.

1. Introduction

The three species Lotka–Volterra cooperative model with harvesting terms is described as follows (see [4], [5]):

$$\dot{x_i}(t) = x_i(t)\bigg(a_i - b_i x_i(t) + \sum_{j=1, j\neq i}^{3} c_{ij} x_j(t)\bigg) - h_i, \quad i = 1, 2, 3,$$

where $x_i(t)$ $(i = 1, 2, 3)$ is the density function of the ith species; a_i and b_i are all positive constants and denote the intrinsic growth rates, death rates, respectively; $c_{ij} > 0$ stand for the cooperative rate between the ith species and the jth species; h_i, $i = 1, 2, 3$ is the ith species harvesting terms standing for the

2010 *Mathematics Subject Classification.* 34C25; 92D25.
Key words and phrases. Eight positive periodic solutions, non-autonomous Lotka–Volterra cooperative system, coincidence degree, Harvesting term.

harvests. Since realistic models require taking into account the effect of changing environment we will consider the following nonautonomous model

$$\dot{x_i}(t) = x_i(t)\bigg(a_i(t) - b_i(t)x_i(t) + \sum_{j=1,j\neq i}^{3} c_{ij}(t)x_j(t)\bigg) - h_i(t), \quad i = 1,2,3. \tag{1.1}$$

In addition, the effects of a periodically varying environment are important for evolutionary theory as the selective forces on systems in a fluctuating environment differ from those in a stable environment. Therefore, the assumptions of periodicity of the parameters are a way of incorporating the periodicity of the environment (e.g. seasonal effects of weather, food supplies, mating habits, etc.), which leads us to assume that $a_i(t)$, $b_i(t)$, $c_{ij}(t)$ and $h_i(t)$ $(i,j = 1,2,3)$ are all positive continuous ω-periodic functions.

A very basic and important problem in the study of a population growth model with a periodic environment is the global existence and stability of a positive periodic solution, which plays a similar role as a globally stable equilibrium does in an autonomous model. Also, only a few results concerning the existence of positive periodic solutions of system (1.1) can be found in the literature. This motivates us to investigate the existence of a positive periodic or multiple positive periodic solutions for system (1.1). In fact, it is more likely for some biological species to take on multiple periodic change regulations and have multiple local stable periodic phenomena. Therefore it is essential for us to investigate the existence of multiple positive periodic solutions for population models. Our main purpose of this paper is by using Mawhin's continuation theorem of coincidence degree theory [2], to establish the existence of eight positive periodic solutions for system (1.1). For the work concerning the multiple existence of periodic solutions of periodic population models which was done using coincidence degree theory, we refer to [1], [3], [6].

The organization of the rest of this paper is as follows. In Section 2, by employing the continuation theorem of coincidence degree theory and linear inequality, we establish the existence of eight positive periodic solutions of system (1.1). In Section 3, an example is given to illustrate the effectiveness of our results.

2. Existence of eight positive periodic solutions

In this section, by using Mawhin's continuation theorem and linear inequality, we shall show the existence of positive periodic solutions of (1.1). To do so, we need to make some preparations.

Let X and Z be real normed vector spaces. Let $L\colon \operatorname{Dom} L \subset X \to Z$ be a linear mapping and $N\colon X\times[0,1] \to Z$ be a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < \infty$

and $\operatorname{Im} L$ is closed in Z. If L is a Fredholm mapping of index zero, then there exist continuous projectors $P: X \to X$ and $Q: Z \to Z$ such that $\operatorname{Im} P = \operatorname{Ker} L$ and $\operatorname{Ker} Q = \operatorname{Im} L = \operatorname{Im}(I - Q)$, and $X = \operatorname{Ker} L \oplus \operatorname{Ker} P$, $Z = \operatorname{Im} L \oplus \operatorname{Im} Q$. It follows that $L|_{\operatorname{Dom} L \cap \operatorname{Ker} P}: (I - P)X \to \operatorname{Im} L$ is invertible and its inverse is denoted by K_P. If Ω is a bounded open subset of X, the mapping N is called L-compact on $\overline{\Omega}\times[0,1]$, if $QN(\overline{\Omega}\times[0,1])$ is bounded and $K_P(I-Q)N: \overline{\Omega}\times[0,1] \to X$ is compact. Because $\operatorname{Im} Q$ is isomorphic to Ker L, there exists an isomorphism $J: \operatorname{Im} Q \to \operatorname{Ker} L$.

The Mawhin's continuous theorem [2, p. 40] reads as follows.

Lemma 2.1 ([2]). *Let L be a Fredholm mapping of index zero and let N be L-compact on $\overline{\Omega} \times [0,1]$. Assume that:*

(a) *for each $\lambda \in (0,1)$, every solution x of $Lx = \lambda N(x,\lambda)$ is such that $x \notin \partial\Omega \cap \operatorname{Dom} L$;*
(b) $QN(x,0)x \neq 0$ *for each $x \in \partial\Omega \cap \operatorname{Ker} L$;*
(c) $\deg(JQN(x,0), \Omega \cap \operatorname{Ker} L, 0) \neq 0$.

Then $Lx = Nx$ has at least one solution in $\overline{\Omega} \cap \operatorname{Dom} L$.

For the sake of convenience, we denote by

$$f^l = \min_{t\in[0,\omega]} f(t), \quad f^M = \max_{t\in[0,\omega]} f(t), \quad \overline{f} = \frac{1}{\omega}\int_0^\omega f(t)\,dt,$$

respectively, here $f(t)$ is a continuous ω-periodic function. In this paper, matrix $A = (a_{ij}) > 0$ means that each elements $a_{ij} > 0$.

For simplicity, we need to introduce some notations as follows.

$$l_i^\pm = \frac{a_i^l \pm \sqrt{(a_i^l)^2 - 4b_i^M h_i^M}}{2b_i^M}, \quad q_i^l = \min\{b_i^l, h_i^l\}, \quad i = 1,2,3;$$

$$D = \begin{pmatrix} q_1^l & -c_{12}^M & -c_{13}^M \\ -c_{21}^M & q_2^l & -c_{23}^M \\ -c_{31}^M & -c_{32}^M & q_3^l \end{pmatrix}, \quad D^{-1}\begin{pmatrix} a_1^M \\ a_2^M \\ a_3^M \end{pmatrix} := \begin{pmatrix} H_1^+ \\ H_2^+ \\ H_3^+ \end{pmatrix}.$$

Throughout this paper, we need the following assumptions.

(H_1) $a_i^l > 2b_i^M$ and $h_i^M > b_i^M$, $i = 1,2,3$;
(H_2) $|D| > 0$, $q_i^l q_j^l - c_{ij}^M c_{ji}^M \geq 0$ $(i \neq j)$ and $H_i^+ > a_i^l / b_i^M$, $i,j = 1,2,3$.

Lemma 2.2. *Suppose that $|A| > 0$ and $a_{ii}a_{jj} - a_{ij}a_{ji} \geq 0$, $i,j = 1,2,3$, then $AX < B$ implies $X < A^{-1}B$, where*

$$A = \begin{pmatrix} a_{11} & -a_{12} & -a_{13} \\ -a_{21} & a_{22} & -a_{23} \\ -a_{31} & -a_{32} & a_{33} \end{pmatrix}, \quad a_{ij} > 0 \quad (i,j = 1,2,3),$$

$X = (x_1, x_2, x_3)^T \in \mathbb{R}^3$ *and* $B = (b_1, b_2, b_3)^T \in \mathbb{R}^3$.

Proof. In fact, there exists a positive vector $\varepsilon_0 = (\varepsilon_1, \varepsilon_2, \varepsilon_3)^T \in \mathbb{R}^3$ such that $AX - B + \varepsilon_0 = 0$, which implies that $X - A^{-1}B + A^{-1}\varepsilon_0 = 0$. Since

$$A^{-1} = \frac{1}{|A|}\begin{pmatrix} a_{22}a_{33} - a_{23}a_{32} & a_{12}a_{33} + a_{13}a_{32} & a_{12}a_{23} + a_{13}a_{22} \\ a_{21}a_{33} + a_{23}a_{31} & a_{11}a_{33} - a_{13}a_{31} & a_{11}a_{23} + a_{13}a_{21} \\ a_{21}a_{32} + a_{22}a_{31} & a_{11}a_{32} + a_{12}a_{31} & a_{11}a_{22} - a_{12}a_{21} \end{pmatrix} > 0,$$

$A^{-1}\varepsilon_0 > 0$. Thus, we obtain $X < A^{-1}B$. $\square$

Theorem 2.3. *Assume that* (H_1) *and* (H_2) *hold. Then system* (1.1) *has at least eight positive* ω*-periodic solutions.*

Proof. By the substitution

$$x_i(t) = \exp\{u_i(t)\}, \quad i = 1, 2, \ldots, n, \tag{2.1}$$

system (1.1) can be reformulated as

$$\dot{u}_i(t) = a_i(t) - b_i(t)e^{u_i(t)} + \sum_{j=1, j\neq i}^{3} c_{ij}(t)e^{u_j(t)} - h_i(t)e^{-u_i(t)}, \quad i = 1, 2, 3. \tag{2.2}$$

Let $X = Z = \{u = (u_1, u_2, u_3)^T \in C(\mathbb{R}, \mathbb{R}^3) : u(t + \omega) = u(t)\}$ and define

$$\|u\| = \sum_{i=1}^{3} \max_{t\in[0,\omega]} |u_i(t)|, \quad u \in X \text{ or } Z.$$

Equipped with the above norm $\|\cdot\|$, X and Z are Banach spaces. Let

$$N(u, \lambda) = \begin{pmatrix} a_1(t) - b_1(t)e^{u_1(t)} + \lambda(c_{12}(t)e^{u_2(t)} + c_{13}(t)e^{u_3(t)}) - h_1(t)e^{-u_1(t)} \\ a_2(t) - b_2(t)e^{u_2(t)} + \lambda(c_{21}(t)e^{u_1(t)} + c_{23}(t)e^{u_3(t)}) - h_2(t)e^{-u_2(t)} \\ a_3(t) - b_3(t)e^{u_3(t)} + \lambda(c_{31}(t)e^{u_1(t)} + c_{32}(t)e^{u_2(t)}) - h_n(t)e^{-u_3(t)} \end{pmatrix}_{3\times 1},$$

for $u \in X$, $Lu = \dot{u} = du(t)/dt$. We put

$$Pu = \frac{1}{\omega}\int_0^\omega u(t)\,dt, \quad u \in X, \qquad Qz = \frac{1}{\omega}\int_0^\omega z(t)\,dt, \quad z \in Z.$$

Thus it follows that $\operatorname{Ker} L = \mathbb{R}^3$, $\operatorname{Im} L = \{z \in Z : \int_0^\omega z(t)\,dt = 0\}$ is closed in Z, $\dim \operatorname{Ker} L = 3 = \operatorname{codim} \operatorname{Im} L$, and P, Q are continuous projectors such that

$$\operatorname{Im} P = \operatorname{Ker} L, \qquad \operatorname{Ker} Q = \operatorname{Im} L = \operatorname{Im}(I - Q).$$

Hence, L is a Fredholm mapping of index zero. Furthermore, the generalized inverse (to L) $K_P\colon \operatorname{Im} L \to \operatorname{Ker} P \cap \operatorname{Dom} L$ is given by

$$K_P(z) = \int_0^t z(s)\,ds - \frac{1}{\omega}\int_0^\omega \int_0^t z(s)\,ds\,dt.$$

Then

$$QN(u,\lambda)=\begin{pmatrix}\frac{1}{\omega}\int_0^{\omega}F_1(s,\lambda)\,ds\\ \frac{1}{\omega}\int_0^{\omega}F_2(s,\lambda)\,ds\\ \frac{1}{\omega}\int_0^{\omega}F_3(s,\lambda)\,ds\end{pmatrix}_{3\times 1}$$

and

$$\begin{aligned}&K_P(I-Q)N(u,\lambda)\\ &=\begin{pmatrix}\int_0^t F_1(s,\lambda)\,ds-\frac{1}{\omega}\int_0^{\omega}\int_0^t F_1(s,\lambda)\,ds\,dt+\left(\frac{1}{2}-\frac{t}{\omega}\right)\int_0^{\omega}F_1(s,\lambda)\,ds\\ \int_0^t F_2(s,\lambda)\,ds-\frac{1}{\omega}\int_0^{\omega}\int_0^t F_2(s,\lambda)\,ds\,dt+\left(\frac{1}{2}-\frac{t}{\omega}\right)\int_0^{\omega}F_2(s,\lambda)\,ds\\ \int_0^t F_3(s,\lambda)\,ds-\frac{1}{\omega}\int_0^{\omega}\int_0^t F_3(s,\lambda)\,ds\,dt+\left(\frac{1}{2}-\frac{t}{\omega}\right)\int_0^{\omega}F_3(s,\lambda)\,ds\end{pmatrix}_{3\times 1},\end{aligned}$$

where

$$F_i(s,\lambda)=a_i(s)-b_i(s)e^{u_i(s)}+\lambda\sum_{j=1,j\neq i}^{3}c_{ij}(s)e^{u_j(s)}-h_i(s)e^{-u_i(s)},\quad i=1,2,3.$$

Obviously, QN and $K_P(I-Q)N$ are continuous. It is not difficult to show that $K_P(I-Q)N(\overline{\Omega})$ is compact for any open bounded set $\Omega\subset X$ by using the Arzela–Ascoli theorem. Moreover, $QN(\overline{\Omega})$ is clearly bounded. Thus, N is L-compact on $\overline{\Omega}$ with any open bounded set $\Omega\subset X$.

In order to use Lemma 2.1, we have to find at least eight appropriate open bounded subsets in X. Considering the operator equation $Lu=\lambda N(u,\lambda),\lambda\in(0,1)$, we have

$$\dot{u_i}(t)=\lambda\left(a_i(t)-b_i(t)e^{u_i(t)}+\lambda\sum_{j=1,j\neq i}^{3}c_{ij}(t)e^{u_j(t)}-h_i(t)e^{-u_j(t)}\right),\tag{2.3}$$

for $i=1,2,3$. Assume that $u\in X$ is an ω-periodic solution of system (2.3) for some $\lambda\in(0,1)$. Then there exist $\xi_i,\eta_i\in[0,\omega]$ such that

$$u_i(\xi_i)=\max_{t\in[0,\omega]}u_i(t),\quad u_i(\eta_i)=\min_{t\in[0,\omega]}u_i(t),\quad i=1,2,3.$$

It is clear that

$$\dot{u_i}(\xi_i)=0,\quad \dot{u_i}(\eta_i)=0,\quad i=1,2,3.$$

From this and (2.3), we have

$$a_i(\xi_i)-b_i(\xi_i)e^{u_i(\xi_i)}+\lambda\sum_{j=1,j\neq i}^{3}c_{ij}(\xi_i)e^{u_j(\xi_i)}-h_i(\xi_i)e^{-u_i(\xi_i)}=0,\tag{2.4}$$

for $i=1,2,3$, and

$$a_i(\eta_i)-b_i(\eta_i)e^{u_i(\eta_i)}+\lambda\sum_{j=1,j\neq i}^{3}c_{ij}(\eta_j)e^{u_j(\eta_j)}-h_i(\eta_i)e^{-u_i(\eta_i)}=0,\tag{2.5}$$

for $i = 1, 2, 3$. (2.4) and (2.5) give

$$\begin{pmatrix} q_1^l & -c_{12}^M & -c_{13}^M \\ -c_{21}^M & q_2^l & -c_{23}^M \\ -c_{31}^M & -c_{32}^M & q_3^l \end{pmatrix} \begin{pmatrix} e^{u_1(\xi_1)} \\ e^{u_2(\xi_2)} \\ e^{u_3(\xi_3)} \end{pmatrix} < \begin{pmatrix} a_1^M \\ a_2^M \\ a_3^M \end{pmatrix} \tag{2.6}$$

and

$$\begin{pmatrix} q_1^l & -c_{12}^M & -c_{13}^M \\ -c_{21}^M & q_2^l & -c_{23}^M \\ -c_{31}^M & -c_{32}^M & q_3^l \end{pmatrix} \begin{pmatrix} e^{-u_1(\eta_1)} \\ e^{-u_2(\eta_2)} \\ e^{-u_3(\eta_3)} \end{pmatrix} < \begin{pmatrix} a_1^M \\ a_2^M \\ a_3^M \end{pmatrix}, \tag{2.7}$$

respectively. By assumption (H_2) and Lemma 2.2, we obtain

$$\begin{pmatrix} e^{u_1(\xi_1)} \\ e^{u_2(\xi_2)} \\ e^{u_3(\xi_3)} \end{pmatrix} < \begin{pmatrix} q_1^l & -c_{12}^M & -c_{13}^M \\ -c_{21}^M & q_2^l & -c_{23}^M \\ -c_{31}^M & -c_{32}^M & q_3^l \end{pmatrix}^{-1} \begin{pmatrix} a_1^M \\ a_2^M \\ a_3^M \end{pmatrix} := \begin{pmatrix} H_1^+ \\ H_2^+ \\ H_3^+ \end{pmatrix} \tag{2.8}$$

and

$$\begin{pmatrix} e^{-u_1(\eta_1)} \\ e^{-u_2(\eta_2)} \\ e^{-u_3(\eta_3)} \end{pmatrix} < \begin{pmatrix} q_1^l & -c_{12}^M & -c_{13}^M \\ -c_{21}^M & q_2^l & -c_{23}^M \\ -c_{31}^M & -c_{32}^M & q_3^l \end{pmatrix}^{-1} \begin{pmatrix} a_1^M \\ a_2^M \\ a_3^M \end{pmatrix} := \begin{pmatrix} H_1^+ \\ H_2^+ \\ H_3^+ \end{pmatrix}, \tag{2.9}$$

respectively, which imply that

$$\begin{pmatrix} u_1(\xi_1) \\ u_2(\xi_2) \\ u_3(\xi_3) \end{pmatrix} < \begin{pmatrix} \ln H_1^+ \\ \ln H_2^+ \\ \ln H_3^+ \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} u_1(\eta_1) \\ u_2(\eta_2) \\ u_3(\eta_3) \end{pmatrix} > \begin{pmatrix} -\ln H_1^+ \\ -\ln H_2^+ \\ -\ln H_3^+ \end{pmatrix}, \tag{2.10}$$

respectively. Moreover, according to (2.4), we have

$$b_i^M e^{u_i(\xi_i)} + h_i^M e^{-u_i(\xi_i)} > a_i^l, \quad i = 1, 2, 3,$$

or equivalently,

$$b_i^M e^{2u_i(\xi_i)} - a_i^l e^{u_i(\xi_i)} + h_i^M > 0, \quad i = 1, 2, 3,$$

which imply that

$$\begin{pmatrix} u_1(\xi_1) \\ u_2(\xi_2) \\ u_3(\xi_3) \end{pmatrix} > \begin{pmatrix} \ln l_1^+ \\ \ln l_2^+ \\ \ln l_3^+ \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} u_1(\xi_1) \\ u_2(\xi_2) \\ u_3(\xi_3) \end{pmatrix} < \begin{pmatrix} \ln l_1^- \\ \ln l_2^- \\ \ln l_3^- \end{pmatrix}. \tag{2.11}$$

Similarly, by (2.5), we get

$$\begin{pmatrix} u_1(\eta_1) \\ u_2(\eta_2) \\ u_3(\eta_3) \end{pmatrix} > \begin{pmatrix} \ln l_1^+ \\ \ln l_2^+ \\ \ln l_3^+ \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} u_1(\eta_1) \\ u_2(\eta_2) \\ u_3(\eta_3) \end{pmatrix} < \begin{pmatrix} \ln l_1^- \\ \ln l_2^- \\ \ln l_3^- \end{pmatrix}. \tag{2.12}$$

By assumptions (H_1) and (H_2), we have

$$-\ln H_i^+ < \ln l_i^- < \ln l_i^+ < \ln H_i^+, \quad i = 1, 2, 3. \tag{2.13}$$

From (2.10)–(2.13) we obtain

$$-\ln H_i^+ < u_i(t) < \ln l_i^- \quad \text{or} \quad \ln l_i^+ < u_i(t) < \ln H_i^+, \quad i = 1, 2, 3.$$

For convenience, we denote

$$G_i = (-\ln H_i^+, \ln l_i^-), \quad H_i = (\ln l_i^+, \ln H_i^+), \quad i = 1, 2, 3.$$

Clearly, $l_i^\pm$ and $H_i^+, i = 1, 2, 3$ are independent of λ. For each $i = 1, 2, 3$, we choose an interval between two intervals G_i and H_i and denote it as Δ_i, then define the set

$$\{u = (u_1, u_2, u_3)^T \in X : u_i(t) \in \Delta_i,\ t \in \mathbb{R},\ i = 1, 2, 3\}.$$

Obviously, the number of the above sets is eight. We denote these sets as Ω_k, $k = 1, \dots, 8$. Ω_k, $k = 1, \dots, 8$ are bounded open subsets of X, $\Omega_i \cap \Omega_j = \phi$, $i \neq j$. Thus Ω_k $(k = 1, \dots, 8)$ satisfies the requirement (a) in Lemma 2.1.

Now we show that (b) of Lemma 2.1 holds, i.e. we prove when $u \in \partial\Omega_k \cap \operatorname{Ker} L = \partial\Omega_k \cap \mathbb{R}^3, QN(u, 0) \neq (0, 0)^T$, $k = 1, \dots, 8$. If it is not true, then when $u \in \partial\Omega_k \cap \operatorname{Ker} L = \partial\Omega_k \cap \mathbb{R}^3$, $i = 1, \dots, 8$, constant vector $u = (u_1, u_2, u_3)^T$ with $u \in \partial\Omega_k$, $k = 1, \dots, 8$, satisfies

$$\int_0^\omega a_i(t)\,dt - \int_0^\omega b_i(t)e^{u_i}\,dt - \int_0^\omega h_i(t)e^{-u_i}\,dt = 0, \quad i = 1, \dots, n.$$

In view of the mean value theorem of calculous, there exist 3 points t_i $(i = 1, 2, 3)$ such that

$$a_i(t_i) - b_i(t_i)e^{u_i} - h_i(t_i)e^{-u_i} = 0, \quad i = 1, 2, 3.$$

Following the arguments of (2.6)–(2.12), we have

$$-\ln H_i^+ < u_i(t_i) < \ln l_i^- \quad \text{or} \quad \ln l_i^+ < u_i(t_i) < \ln H_i^+, \quad i = 1, 2, 3.$$

Then u belongs to one of $\Omega_k \cap \mathbb{R}^3$, $k = 1, \dots, 8$. This contradicts the fact that $u \in \partial\Omega_k \cap \mathbb{R}^3$, $k = 1, \dots, 8$. Thus condition (b) in Lemma 2.1 is satisfied. Finally, we show that (c) in Lemma 2.1 holds. Note that the system of algebraic equations:

$$a_i(t_i) - b_i(t_i)e^{x_i} - h_i(t_i)e^{-x_i} = 0, \quad i = 1, 2, \dots, n,$$

has eight distinct solutions since (H_1) and (H_2) hold,

$$(x_1^*, x_2^*, x_3^*) = (\ln \widehat{x}_1, \ln \widehat{x}_2, \ln \widehat{x}_3),$$

where

$$x_i^\pm = \frac{a_i(t_i) \pm \sqrt{(a_i(t_i))^2 - 4b_i(t_i)h_i(t_i)}}{2b_i(t_i)}, \quad \widehat{x}_i = x_i^- \quad \text{or} \quad \widehat{x}_i = x_i^+, \quad i = 1, 2, 3.$$

It is easy to verify that

$$-\ln H_i^+ < \ln x_i^- < \ln l_i^- < \ln l_i^+ < \ln x_i^+ < \ln H_i^+, \quad i = 1, 2, 3.$$

Therefore, (x_1^*, x_2^*, x_3^*) uniquely belongs to the corresponding Ω_k. Since $\operatorname{Ker} L = \operatorname{Im} Q$, we can take $J = I$. A direct computation gives, for $k = 1, \ldots, 8$,

$$\deg\{JQN(u,0), \Omega_k \cap \operatorname{Ker} L, (0,0)^T\} = \operatorname{sign}\bigg[\prod_{i=1}^{3}\bigg(-b_i(t_i)x_i^* + \frac{h_i(t_i)}{x_i^*}\bigg)\bigg].$$

Since $a_i(t_i) - b_i(t_i)x_i^* - h_i(t_i)/x_i^* = 0$, $i = 1, 2, 3$, then

$$\deg\{JQN(u,0), \Omega_k \cap \operatorname{Ker} L, (0,0)^T\} = \operatorname{sign}\bigg[\prod_{i=1}^{3}(a_i(t_i) - 2b_i(t_i)x_i^*)\bigg] = \pm 1,$$

for $k = 1, \ldots, 8$.

So far, we have proved that Ω_k $(k = 1, \ldots, 8)$ satisfies all the assumptions in Lemma 2.1. Hence, system (2.2) has at least eight different ω-periodic solutions. Thus by (2.1) system (1.1) has at least eight different positive ω-periodic solutions. This completes the proof of Theorem 2.1. □

Remark 2.4. In [3], the authors investigated the existence of four positive periodic solutions to a Lotka–Volterra cooperative system with harvesting terms by a complicated proof process. However, the same results will be obtained very simply by our method used in this paper.

3. An example

Now, let us consider the following three species cooperative system with harvesting terms:

$$\tag{3.1} \begin{cases} \dot{x}(t) = x(t)\bigg(3 + \sin t - \dfrac{4+\sin t}{10}x(t) + \dfrac{1}{5}y(t) + \dfrac{1}{5}z(t)\bigg) - \dfrac{9+\cos t}{15}, \\ \dot{y}(t) = y(t)\bigg(3 + \cos t - \dfrac{5+\cos t}{10}y(t) + \dfrac{1}{5}x(t) + \dfrac{1}{5}z(t)\bigg) - \dfrac{3+\cos t}{5}, \\ \dot{z}(t) = z(t)\bigg(3 + \sin 2t - \dfrac{7+\sin 2t}{10}z(t) + \dfrac{1}{5}x(t) + \dfrac{1}{5}y(t)\bigg) - \dfrac{8+\cos 2t}{10}. \end{cases}$$

In this case,

$$\begin{aligned} &a_1(t) = 3 + \sin t, && b_1(t) = \frac{4+\sin t}{10}, && h_1(t) = \frac{9+\cos t}{15}, \\ &a_2(t) = 3 + \cos t, && b_2(t) = \frac{5+\cos t}{10}, && h_2(t) = \frac{3+\cos t}{5}, \\ &a_3(t) = 3 + \sin 2t, && b_3(t) = \frac{7+\sin 2t}{10}, && h_3(t) = \frac{8+\cos 2t}{10}, \end{aligned}$$

and

$$c_{12}(t) = c_{21}(t) = c_{13}(t) = c_{31}(t) = c_{23}(t) = c_{32}(t) = \frac{1}{5}.$$

Since

$$\begin{pmatrix} a_1^l \\ a_2^l \\ a_3^l \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} b_1^l \\ b_2^l \\ b_3^l \end{pmatrix} = \begin{pmatrix} \frac{3}{10} \\ \frac{2}{5} \\ \frac{3}{5} \end{pmatrix}, \quad \begin{pmatrix} h_1^l \\ h_2^l \\ h_3^l \end{pmatrix} = \begin{pmatrix} \frac{8}{15} \\ \frac{2}{5} \\ \frac{7}{10} \end{pmatrix}, \quad \begin{pmatrix} q_1^l \\ q_2^l \\ q_3^l \end{pmatrix} = \begin{pmatrix} \frac{3}{10} \\ \frac{2}{5} \\ \frac{3}{5} \end{pmatrix},$$

$$\begin{pmatrix} h_1^M \\ h_2^M \\ h_3^M \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ \frac{4}{5} \\ \frac{9}{10} \end{pmatrix} > \begin{pmatrix} b_1^M \\ b_2^M \\ b_3^M \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{3}{5} \\ \frac{4}{5} \end{pmatrix},$$

$$D = \begin{pmatrix} \frac{3}{10} & -\frac{1}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{2}{5} & -\frac{1}{5} \\ -\frac{1}{5} & -\frac{1}{5} & \frac{3}{5} \end{pmatrix}, \qquad |D| = \frac{1}{250} > 0,$$

then

$$\begin{pmatrix} a_1^l \\ a_2^l \\ a_3^l \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} > \begin{pmatrix} 2b_1^M \\ 2b_2^M \\ 2b_3^M \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{6}{5} \\ \frac{8}{5} \end{pmatrix},$$

$$\begin{pmatrix} q_1^l q_2^l \\ q_1^l q_3^l \\ q_2^l q_3^l \end{pmatrix} = \begin{pmatrix} \frac{3}{25} \\ \frac{9}{50} \\ \frac{6}{25} \end{pmatrix} > \begin{pmatrix} c_{12}^M c_{21}^M \\ c_{13}^M c_{31}^M \\ c_{23}^M c_{32}^M \end{pmatrix} = \begin{pmatrix} \frac{1}{25} \\ \frac{1}{25} \\ \frac{1}{25} \end{pmatrix},$$

$$D^{-1} = \begin{pmatrix} 50 & 40 & 30 \\ 40 & 35 & 25 \\ 30 & 25 & 20 \end{pmatrix},$$

$$\begin{pmatrix} H_1^+ \\ H_2^+ \\ H_3^+ \end{pmatrix} = D^{-1} \begin{pmatrix} a_1^M \\ a_2^M \\ a_3^M \end{pmatrix} = \begin{pmatrix} 480 \\ 400 \\ 300 \end{pmatrix} > \begin{pmatrix} \frac{a_1^l}{b_1^M} \\ \frac{a_2^l}{b_2^M} \\ \frac{a_3^l}{b_3^M} \end{pmatrix} = \begin{pmatrix} 4 \\ \frac{10}{3} \\ \frac{5}{2} \end{pmatrix}.$$

Therefore, all conditions of Theorem 2.1 are satisfied. By Theorem 2.1, system (3.1) has at least eight positive 2π-periodic solutions.

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Manuscript received July 20, 2009

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