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AN EXTENSION OF LEGGETT–WILLIAMS NORM-TYPE THEOREM FOR COINCIDENCES AND ITS APPLICATION

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Abstract. In this paper, several versions extension of Leggett–Williams norm-type theorem for coincidences are given and proved to obtain the positive solutions of the operator equation $Mx = Nx$, where M is a quasi-linear operator and N is nonlinear. Moreover, as an application, the existence of positive solutions for multi-point boundary value problem with a p-Laplacian is obtained by one of those theorems.

1. Introduction

In [9], D. O'Regan and M. Zima proved the Leggett–Williams norm-type theorem for the abstract equation $Lx = Nx$ with L a noninvertible linear operator, which has become a useful tool in finding positive solutions to differential equations boundary value problems at resonance. Those theorems based upon the famous Mawhin's continuous theorem [8] and the properties of cones in Banach spaces and Leray–Schauder degree for completely continuous operators. In 2004, W. Ge and J. Ren extended Mawhin's theorem to nonlinear operator in [5]. Motivated by above works, in this paper, we give and prove the extension of the

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Leggett–Williams norm-type theorems in Section 2 and introduce an application to a kind of multi-point boundary value problem with a p-Laplacian in Section 3.

2. Norm type theorems for quasi-linear operator

For the convenience of the reader, we first recall some of fundamental facts on quasi-linear operator and cone theory in Banach spaces. Let X, Z be real Banach spaces. Consider a continuous mapping $M\colon \operatorname{dom} M \subset X \to Z$. Assume that:

(1) M is a quasi-linear operator, i.e. $\operatorname{Im} M$ is a closed subset of Z and $\operatorname{Ker} M$ is linearly homeomorphic to $\mathbb{R}^n$, $n < \infty$.

From the condition (1), there exist continuous projection $P\colon X \to X_1$ and semi-projection $Q\colon Z \to Z_1$ such that $\operatorname{Im} P = \operatorname{Ker} M$ and $\operatorname{Ker} Q = \operatorname{Im} M$ (see [2], [4], [5]). Moreover, since $\dim \operatorname{Im} Q = \operatorname{codim} \operatorname{Im} M$, there exists an isomorphism $J\colon \operatorname{Im} Q \to \operatorname{Ker} M$ with $J(\theta) = \theta$. Let $\Omega \subset X$ be an open and bounded set with the origin $\theta \in \Omega$. Suppose $N_\lambda\colon \overline{\Omega} \to Z$, $\lambda \in [0,1]$ is a continuous operator, denote N_1 by N. Let $\Sigma_\lambda = \{x \in \overline{\Omega} : Mx = N_\lambda x\}$. Assume N_λ is M-compact in $\overline{\Omega}$, that is,

(2) there is a vector subspace Z_1 of Z with $\dim Z_1 = \dim X_1$ and an operator $R\colon \overline{\Omega} \times [0,1] \to X_2$ being continuous and compact such that, for $\lambda \in [0,1]$,

$$(I - Q)N_\lambda(\overline{\Omega}) \subset \operatorname{Im} M \subset (I - Q)Z, \tag{2.1}$$

$$QN_\lambda x = \theta,\ \lambda \in (0,1) \iff QNx = \theta, \tag{2.2}$$

$$R(\,\cdot\,,0) \text{ is the zero operator and } R(\,\cdot\,,\lambda)|_{\Sigma_\lambda} = (I - P)|_{\Sigma_\lambda}, \tag{2.3}$$

$$M[P + R(\,\cdot\,,\lambda)] = (I - Q)N_\lambda. \tag{2.4}$$

Define $S_\lambda\colon \overline{\Omega} \cap \operatorname{dom} M \to X$, $\lambda \in [0,1]$ by

$$S_\lambda = P + R(\,\cdot\,,\lambda) + JQN. \tag{2.5}$$

Then the equation $Mx = N_\lambda x$ has a solution $x \in \overline{\Omega}$ if and only if x is a fixed point of S_λ for all $\lambda \in [0,1]$ (see [5]). Set $S = S_1$.

Let C be a cone in X, it is well known that C induces a partial order in X by

$$x \preceq y \iff y - x \in C.$$

Moreover, for every $u \in C \setminus \{\theta\}$ there exists a positive number $\sigma(u)$ such that

$$||x + u|| \geq \sigma(u)||x|| \quad \text{for all } x \in C \tag{2.6}$$

(see [10]). Let $\gamma\colon X \to C$ be a retraction, that is, a continuous mapping such that $\gamma(x) = x$ for all $x \in C$.

Theorem 2.1. *Let C be a cone in X. If Ω_1, Ω_2 are open bounded subsets of X with $\overline{\Omega}_1 \subset \Omega_2$ and $C \cap (\overline{\Omega}_2 \setminus \Omega_1) \neq \emptyset$. Assume that* (1), (2) *and the following conditions hold:*

(3) $||N_\lambda x|| < ||Mx||$ *for all* $x \in C \cap \partial\Omega_2 \cap \operatorname{dom} L$ *and* $\lambda \in (0,1)$,

(4) $\deg_B\{[I - (P + JQN)\gamma]|_{\operatorname{Ker} M}, \operatorname{Ker} M \cap \Omega_2, \theta\} \neq 0$, *where* $\deg_B$ *is the Brouwer degree,*

(5) *there exists* $u_0 \in C \setminus \{\theta\}$ *such that* $||x|| \leq \sigma(u_0)||Sx||$ *for* $x \in C(u_0) \cap \partial\Omega_1$, *where* $C(u_0) = \{x \in C : \mu u_0 \preceq x$ *for some* $\mu > 0\}$ *and* $\sigma(u_0)$ *is such that* $||x + u_0|| \geq \sigma(u_0)||x||$ *for every* $x \in C$,

(6) $S_\lambda \circ \gamma(\overline{\Omega}_2 \setminus \Omega_1) \subset C$ *for* $\lambda \in [0,1]$.

Then the equation $Mx = Nx$ *has a solution in* $C \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Proof. Without loss of generality, we suppose that $Mx = Nx$ has no solution in $C \cap (\partial\Omega_1 \cup \partial\Omega_2)$. It remains to prove that S has a fixed point in $C \cap (\Omega_2 \setminus \overline{\Omega}_1)$. For $\lambda \in [0,1]$ and $x \in \overline{\Omega}_2$, define

$$\widetilde{S}(\lambda, x) = (P + JQN)\gamma x + R(\gamma x, \lambda). \tag{2.7}$$

The continuity of P, J and γ together with the condition (2) imply that $\widetilde{S}$ is compact on $[0,1] \times \overline{\Omega}_2$ (see [4], [5]). It is similar to the proof of Theorem 1 in [9], we first prove that

$$x \neq \mu u_0 + (1+\mu)\widetilde{S}(1,x) \quad \text{for } x \in \partial\Omega_1, \mu \geq 0. \tag{2.8}$$

Otherwise, suppose that there exist $x_0 \in \partial\Omega_1$ and $\mu_0 > 0$ such that

$$x_0 = \mu_0 u_0 + (1+\mu_0)\widetilde{S}(1, x_0).$$

In view of (6), one gets $x_0 \in C(u_0)$. In addition,

$$\begin{aligned} ||x_0|| &= ||\mu_0 u_0 + (1+\mu_0)\widetilde{S}(1,x_0)|| \\ &\geq \sigma(u_0)(1+\mu_0)||\widetilde{S}(1,x_0)|| > \sigma(u_0)||\widetilde{S}(1,x_0)|| = \sigma(u_0)||Sx_0||, \end{aligned}$$

which contradicts (5). In addition, $x \neq \widetilde{S}(1,x)$ for $x \in \partial\Omega_1$ because we assume that $Mx = Nx$ has no solution on $C \cap (\partial\Omega_1 \cup \partial\Omega_2)$. So (2.8) is satisfied.

Notice that $\widetilde{S}(1,\cdot)$ is compact on $\overline{\Omega}_2 \setminus \Omega_1$. Thus, by Dugundji extension theorem (see [4], [9]), there exists a compact operator $F\colon \overline{\Omega}_2 \to X$ such that $F|_{\overline{\Omega}_2 \setminus \Omega_1} = \widetilde{S}(1,\cdot)$ and $F(\overline{\Omega}_2) \subset \overline{\operatorname{conv}}\, \widetilde{S}(1,\cdot)(\overline{\Omega}_2 \setminus \Omega_1)$. The assumption (6) implies that $\widetilde{S}(1,\cdot)(\overline{\Omega}_2 \setminus \Omega_1) \subset C$. Notice that C is closed and convex, then $F(\overline{\Omega}_2) \subset C$, which leads to

$$\inf\{||u_0 + Fx|| : x \in \overline{\Omega}_1\} > 0.$$

If it is not the case, then for each $n \in \mathbb{N}$, take $x_n \in \overline{\Omega}_1$ such that $||u_0 + Fx_n|| < 1/n$. Thus, $\lim_{n\to\infty} ||u_0 + Fx_n|| = 0$. As a result, $\lim_{n\to\infty} Fx_n = -u_0$. In view of $\{Fx_n\} \subset C$ and C is closed, we have $-u_0 \in C$, which is impossible.

In the following, choose $\mu^* > 0$ satisfying

$$\mu^* > \frac{\sup\{||x|| + ||\widetilde{S}(1,x)|| : x \in \overline{\Omega}_1\}}{\inf\{||u_0 + Fx|| : x \in \overline{\Omega}_1\}}. \tag{2.9}$$

Define the homotopy mapping $H\colon [0,1] \times \overline{\Omega}_1 \to X$ by

$$H(t,x) = \widetilde{S}(1,x) + t\mu^*(u_0 + Fx).$$

Since $Fx = \widetilde{S}(1,x)$ for $x \in \partial\Omega_1$, $H(t,x) = t\mu^* u_0 + (1+t\mu^*)\widetilde{S}(1,x)$. (2.8) implies that $x \neq H(t,x)$ for all $t \in [0,1]$ and $x \in \partial\Omega_1$. Then, by homotopy invariance of Leray–Schauder degree we can obtain

$$\deg\{I - H(1,\,\cdot\,), \Omega_1, \theta\} = \deg\{I - H(0,\,\cdot\,), \Omega_1, \theta\}.$$

We claim that $\deg\{I - H(1,\,\cdot\,), \Omega_1, \theta\} = 0$. Otherwise, if $\deg\{I - H(1,\,\cdot\,), \Omega_1, \theta\} \neq 0$, then there exists $x_0 \in \Omega_1$ such that

$$x_0 = \widetilde{S}(1,x_0) + \mu^*(u_0 + Fx_0),$$

which yields

$$\mu^* \leq \frac{||x_0|| + ||\widetilde{S}(1,x_0)||}{||u_0 + Fx_0||},$$

contrary to (2.9). Then, we can obtain

$$\deg\{I - \widetilde{S}(1,\,\cdot\,), \Omega_1, \theta\} = 0. \tag{2.10}$$

Next, we show that

$$\deg\{I - \widetilde{S}(1,\,\cdot\,), \Omega_2, \theta\} \neq 0. \tag{2.11}$$

To do this, we first show that $x \neq \widetilde{S}(\lambda,x)$ for $x \in \partial\Omega_2$. Clearly, $x \neq \widetilde{S}(1,x)$ for $x \in \partial\Omega_2$, and $x \neq \widetilde{S}(0,x)$ for $x \in \partial\Omega_2$ from (4) (since the Brouwer degree is defined) and (2). Suppose that there exist $\lambda_0 \in (0,1)$ and $x_0 \in \partial\Omega_2$ such that $x_0 = \widetilde{S}(\lambda_0,x_0)$, that is,

$$x_0 = (P + JQN)\gamma x_0 + R(\gamma x_0, \lambda_0).$$

In view of (6), $x_0 \in C$. Thus, $x_0 = (P+JQN)x_0 + R(x_0,\lambda_0)$, which is equivalent to $Mx_0 = N_{\lambda_0}x_0$, contrary to (3). As a result, we have

$$\begin{aligned}\deg\{I - \widetilde{S}(1,\,\cdot\,), \Omega_2, \theta\} &= \deg\{I - \widetilde{S}(0,\,\cdot\,), \Omega_2, \theta\}\\ &= \deg\{I - (P + JQN)\gamma, \Omega_2, \theta\}\\ &= \deg_B([I - (P + JQN)\gamma]|_{\operatorname{Ker} M}, \operatorname{Ker} M \cap \Omega_2, \theta\} \neq 0.\end{aligned}$$

Thus, (2.11) holds. From (2.10), (2.11) and the additivity of Leray–Schauder degree, we obtain

$$\deg\{I - \widetilde{S}(1,\,\cdot\,), \Omega_2 \setminus \overline{\Omega}_1, \theta\} \neq 0,$$

which implies that S has a fixed point in the set $C \cap (\Omega_2 \setminus \overline{\Omega}_1)$. This proof is completed. $\square$

Remark 2.2. If $M = L$ is a linear operator and $N_\lambda = \lambda N$ for N is L-compact, let $R(x,\lambda) = \lambda K_p(I-Q)Nx$, then

$$\widetilde{S}(\lambda, x) = (P + JQN)\gamma x + \lambda K_p(I-Q)N\gamma x = \widetilde{\Psi}(\lambda, x),$$

where L_p, K_p and $\widetilde{\Psi}$ are the same as defined in [9]. So Theorem 2.1 extends Leggett–Williams norm-type theorem for coincidences.

Remark 2.3. If $N_\lambda = \lambda N$ for $\lambda \in (0,1)$ satisfies the conditions in Leggett–Williams norm-type theorem for coincidences, then (3) holds immediately. However, when N_λ is M-compact, $||N_\lambda x|| < ||Nx||$ is crucial since N and λ are no longer linear relation. In (6), due to the same reason that $R(\,\cdot\,,\lambda)$ is nonlinear on λ, $\lambda = 1$ is not enough, it is necessary to let $\lambda \in [0,1]$. Meanwhile, when $\lambda = 0$, $S_0 \circ \gamma(\partial\Omega_2) = (P + JQN)\gamma(\partial\Omega_2)$, so we omit the condition $(P + JQN)\gamma(\partial\Omega_2) \subset C$ comparing to the original theorem in [9].

Notice that the condition (3) can be replaced by

(7) $Mx \neq N_\lambda x$ for all $x \in C \cap \partial\Omega_2 \cap \operatorname{dom} M$ and $\lambda \in (0,1)$.

Therefore, the following theorem holds.

Theorem 2.4. *Let the assumptions of Theorem* 2.1 *be satisfied with* (3) *replaced by* (7). *Then the equation $Mx = Nx$ has at least one solution in $C \cap (\overline{\Omega}_2 \setminus \Omega_1)$.*

In the following, we consider the existence of positive solutions to the equation $Mx = Nx$ with $R(\lambda, \cdot\,)$ satisfying a k-set contractive assumption. Recall that a map $T\colon D \subset X \to X$, is said to be k-set contraction if it is continuous and bounded and there exists $k \geq 0$ such that $\alpha(TA) \leq k\alpha(A)$ for every bounded subset A of D, where α is the Kuratowski measure of noncompactness (see [1]).

Theorem 2.5. *Let C be a cone in X. If Ω_1, Ω_2 are open bounded subsets of X with $\overline{\Omega}_1 \subset \Omega_2$ and $C \cap (\overline{\Omega}_2 \setminus \Omega_1) \neq \emptyset$. Assume that the assumptions of Theorem* 2.1 *are satisfied with* (2) *and* (6) *replaced by*

(8) *the assumptions of* (2) *hold with $R(\lambda, \cdot\,)\colon X \to X_2$ is not compact but a k-set contraction on every bounded subset of X with $k < 1$,*

(9) $S_\lambda \circ \gamma(\overline{\Omega}_2) \subset C$,

respectively. Then the equation $Mx = Nx$ has a solution in $C \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Proof. As in the proof of Theorem 2.1, consider $\widetilde{S}$ defined by (2.7) for $\lambda \in [0,1]$ and $x \in \overline{\Omega}_2$. The condition 2° and the continuity of P, J and γ imply that $\widetilde{S}$ is a k-set contraction on $[0,1] \times \overline{\Omega}_2$. If $k = 0$, the assertion follows from

Theorem 2.1. For the case $k \neq 0$, choose a positive constant L satisfying

$$L < \frac{1-k}{k} \frac{\inf\{||u_0 + L\widetilde{S}(1,x)|| : x \in \overline{\Omega}_1\}}{\sup\{||x|| + ||\widetilde{S}(1,x)|| : x \in \overline{\Omega}_1\}}.$$

It can be shown that such an L exists (see [9]). Next, we claim that

$$x \neq \mu u_0 + (1+\mu L)\widetilde{S}(1,x) \quad \text{for } x \in \partial\Omega_1, \mu \geq 0. \tag{2.12}$$

In fact, if there exist $x_0 \in \partial\Omega_1$ and $\mu_0 > 0$ such that

$$x_0 = \mu_0 u_0 + (1+\mu_0 L)\widetilde{S}(1,x_0).$$

It follows from (9) that $x_0 \in C(u_0)$. Moreover,

$$\begin{aligned}||x_0|| &= ||\mu_0 u_0 + (1+\mu_0 L)\widetilde{S}(1,x_0)|| \geq \sigma(u_0)(1+\mu_0 L)||\widetilde{S}(1,x_0)|| \\ &> \sigma(u_0)||\widetilde{S}(1,x_0)|| = \sigma(u_0)||Sx_0||,\end{aligned}$$

which contradicts (5). Obviously, $x \neq \widetilde{S}(1,x)$ for $x \in \partial\Omega_1$. Hence (2.12) holds.

Let $\mu^* > 0$ such that

$$\frac{1-k}{kL} > \mu^* > \frac{\sup\{||x|| + ||\widetilde{S}(1,x)|| : x \in \overline{\Omega}_1\}}{\inf\{||u_0 + L\widetilde{S}(1,x)|| : x \in \overline{\Omega}_1\}}. \tag{2.13}$$

Set $k_1 = (\mu^* L + 1)k$. Obviously, $k_1 \in (0,1)$. For $t \in [0,1]$ and $x \in \overline{\Omega}_1$, consider

$$H(t,x) = t\mu^* u_0 + (t\mu^* L + 1)\widetilde{S}(1,x).$$

Clearly, $H\colon [0,1] \times \overline{\Omega}_1 \to C$ and H is a k_1-set contractive map. Moreover, from (2.12) we obtain $x \neq H(t,x)$ for all $t \in [0,1]$ and $x \in \partial\Omega_1$. Thus,

$$\deg\{I - H(1,\,\cdot\,), \Omega_1, \theta\} = \deg\{I - H(0,\,\cdot\,), \Omega_1, \theta\}.$$

We can prove that $\deg\{I - H(1,\,\cdot\,), \Omega_1, \theta\} = 0$ (see [9]). Then,

$$\deg\{I - \widetilde{S}(1,\,\cdot\,), \Omega_1, \theta\} = 0. \tag{2.14}$$

As in the proof of Theorem 2.1 we can also show that

$$\deg\{I - \widetilde{S}(1,\,\cdot\,), \Omega_2, \theta\} \neq 0. \tag{2.15}$$

From (2.14), (2.15) we obtain

$$\deg\{I - \widetilde{S}(1,\,\cdot\,), \Omega_2 \setminus \overline{\Omega}_1, \theta\} \neq 0,$$

and the assertion follows. □

Next, we consider the case of $R(\,\cdot\,,\lambda)$ being a condensing mapping. Recall that a map $T\colon D \subset X \to X$ is said to be condensing if it is continuous and for any bounded set $S \subset D$ with $\alpha(S) > 0$, $T(S)$ is also bounded and $\alpha(T(S)) \leq \alpha(S)$.

Theorem 2.6. *Let C be a cone in X. If Ω_1, Ω_2 are open bounded subsets of X with $\overline{\Omega}_1 \subset \Omega_2$ and $C \cap (\overline{\Omega}_2 \setminus \Omega_1) \neq \emptyset$. Assume that the assumptions of Theorem 2.5 are satisfied with (8) and (5) replaced by*

(10) *the assumptions of (2) hold with $R(\lambda, \cdot): X \to X_2$ is not compact but a condensing mapping on every bounded subset of X,*

(11) *there exist $u_0 \in C \setminus \{\theta\}$ such that $||x|| < \sigma(u_0)||Sx||$ for $x \in C(u_0) \cap \partial\Omega_1$,*

respectively. Then the equation $Mx = Nx$ has a solution in $C \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Proof. As in the proof of Theorem 2.5, consider $\widetilde{S}$ given by (2.7) for $\lambda \in [0,1]$ and $x \in \overline{\Omega}_2$. In view of the assumptions, $\widetilde{S}$ is a condensing mapping on $[0,1] \times \overline{\Omega}_2$. Similar to the proof of Theorem 4 in [9], we can prove that for all $\mu > 0$ and $x \in \partial\Omega_1$,

$$x \neq \mu u_0 + \widetilde{S}(1,x). \tag{2.16}$$

Let $\mu^* > 0$ be such that

$$\mu^* > \frac{\sup\{||x|| + ||\widetilde{S}(1,x)|| : x \in \overline{\Omega}_1\}}{||u_0||}. \tag{2.17}$$

For $t \in [0,1]$ and $x \in \overline{\Omega}_1$, define

$$H(t,x) = t\mu^* u_0 + \widetilde{S}(1,x).$$

Obviously, $H: [0,1] \times \overline{\Omega}_1 \to C$ and H is a condensing mapping. From (2.16) we get $H(t,x) \neq x$ for $(t,x) \in [0,1] \times \partial\Omega_1$. So,

$$\deg\{I - H(1,\cdot), \Omega_1, \theta\} = \deg\{I - H(0,\cdot), \Omega_1, \theta\}.$$

We can verify that $\deg\{I - H(1,\cdot), \Omega_1, \theta\} = 0$ (see [9]). Then,

$$\deg\{I - \widetilde{S}(1,\cdot), \Omega_1, \theta\} = 0.$$

The rest of the proof follows as before. □

Remark 2.7. In particular, when $Mu = (\phi_p(u'))'$, where $\phi_p(s) = |s|^{p-2} \cdot s$, $p > 1$, then the operators $M = (d/dt)(\phi_p((d/dt)\cdot))$ is a quasi-linear operator. The same is for $M^* = (d/dt)(\phi((d/dt)\cdot))$, where $\phi: \mathbb{R} \to \mathbb{R}$ is a homeomorphism with $\phi(0) = 0$, $\phi(\pm\infty) = \pm\infty$. Here M and M^* are said to be a p-Laplacian and a p-Laplacian-like, respectively. Theorem 2.1 can be used to discuss those kind of equations.

3. Positive solutions for multi-point BVP with a p-Laplacian

As an application of Theorem 2.1, we consider the following multi-point boundary value problem:

$$(\phi_p(x'(t)))' + f(t, x(t)) = 0, \quad t \in (0,1), \tag{3.1}$$

$$x(0) = \sum_{i=1}^{m-2} \alpha_i x(\xi_i), \quad x'(1) = 0, \tag{3.2}$$

where $\phi_p(s) = |s|^{p-2} \cdot s$, $p > 1$, $1/p + 1/q = 1$, $\phi_p^{-1} = \phi_q$, $0 < \xi_1 < \ldots < \xi_{m-2} < 1$, $\alpha_i \geq 0$, $i = 1, \ldots, m-2$, and $\sum_{i=1}^{m-2} \alpha_i = 1$.

When $p = 2$, G. Infante and M. Zima [7] studied the existence of positive solutions to

$$\begin{cases} -x''(t) = f(t, x(t)) & \text{for } t \in (0,1), \\ x'(0) = 0, \quad x(1) = \displaystyle\sum_{i=1}^{m-2} \alpha_i x(\eta_i). \end{cases}$$

with $\alpha_i \geq 0$ and $\sum_{i=1}^{m-2} \alpha_i = 1$ by Leggett–Williams norm-type theorem for coincidences.

For the case $p \neq 2$, H. Feng et al [2] considered the multi-point BVP with one dimension p-Laplacian

$$\begin{cases} (\phi_p(x'(t)))' = f(t, x(t), x'(t)) & \text{for } t \in (0,1), \\ x(0) = \displaystyle\sum_{i=1}^{n} \alpha_i x(\xi_i), \quad x(1) = \sum_{i=1}^{n} \alpha_i x(\eta_i), \end{cases}$$

with $\sum_{i=1}^{m-2} \alpha_i = 1$. The authors obtained the existence of at least one symmetric solution by using the Ge-Mawhin's continuation theorem.

In [11], Y. Zhu et al considered the second order multi-point BVP with p-Laplacian

$$\begin{cases} (\phi_p(u'(t)))' = f(t, u(t), u'(t)) & \text{for } t \in (0,1), \\ u'(0) = 0, \quad u(1) = \displaystyle\sum_{i=1}^{m-2} \alpha_i u(\xi_i), \end{cases}$$

with $\sum_{i=1}^{m-2} \alpha_i = 1$ and obtained the existence of at least one solution by the coincidence degree theory of Mawhin.

From above works, we can see that under the condition $\sum_{i=1}^{m-2} \alpha_i = 1$, the authors obtained positive solutions in [7], when the differential operator is linear, while in [2] and [11], the differential operator involved is p-Laplacian, but those results can's ensure the solutions to be positive. This is crucial since only positive solutions are useful for many applications. In this section, we will develop the results in [2], [7], [11].

In order to prove the existence result, we present here a definition.

DEFINITION 3.1. We say that the function $f\colon [0,1] \times \mathbb{R} \to \mathbb{R}$ satisfies the L^1-Carathéodory conditions, if

(A1) for each $u \in \mathbb{R}$, the mapping $t \mapsto f(t,u)$ is Lebesgue measurable,
(A2) for almost every $t \in [0,1]$, the mapping $u \mapsto f(t,u)$ is continuous on $\mathbb{R}$,
(A3) for each $r > 0$, there exists $\alpha_r \in L^1[0,1]$ satisfying $\alpha_r(t) > 0$ on $[0,1]$ such that

$$|u| \le r \;\Rightarrow\; |f(t,u)| \le \alpha_r(t).$$

Now, we state our main result on the existence of positive solutions for the BVP (3.1)–(3.2).

THEOREM 3.2. *Assume that*

(H1) $f\colon [0,1] \times \mathbb{R}^+ \to \mathbb{R}$ *satisfies the L^1-Carathéodory conditions,*
(H2) *there exists $B > 0$, $\kappa \in (0,1]$ and $c_2 \ge c_1 > 0$ satisfying $q(q-1)c_1 \ge (2q+1)(c_2-c_1)^{q-1}$ such that $f(t,B) < 0$, $c_1 \le f(t,0) \le c_2$ for $t \in [0,1]$ and*

$$-\kappa x < f(t,x) < \kappa x \quad \text{for } (t,x) \in [0,1] \times (0,B],$$

(H3) *there exist $b \in (0,B)$, $\rho \in (0,1]$, $\delta \in (0,1)$ and $g \in L^1[0,1]$, $g(t) \ge 0$ on $[0,1]$, $h_1 \in C([0,1] \times (0,b], \mathbb{R}^+)$, $h_2 \in C([0,1] \times (0,b^{q-1}], \mathbb{R}^+)$ such that $f(t,x) \ge g(t)[h_1(t,x) + h_2(t,x^{q-1})]$ for $(t,x) \in [0,1] \times (0,b]$. $h_1(t,x)/x^\rho$ is non-increasing on $(0,b]$ and $h_2(t,x)/x$ non-increasing on $(0,b^{q-1}]$ with*

$$\int_{\xi_{m-2}}^1 g(s)\frac{h_1(s,b)}{b}\,ds \ge \phi_p(\Gamma)\frac{1-\delta}{\delta^\rho} \tag{3.3}$$

and

$$\int_{\xi_{m-2}}^1 g(s)\frac{h_2(s,b^{q-1})}{b^{q-1}}\,ds \ge \phi_p(\Gamma)\frac{3\cdot 2^{q-2}\kappa^{q-1}}{\delta^{q-1}}, \tag{3.4}$$

where

$$\Gamma = \bigg(1 - \sum_{i=1}^{m-2}\alpha_i(1-\xi_i)^q\bigg)\bigg/\bigg(q\sum_{i=1}^{m-2}\alpha_i\xi_i\bigg).$$

Then the BVP (3.1)–(3.2) *has at least one positive solution on* $[0,1]$ *provided*

$$\kappa\phi_p\bigg(\frac{1}{\Gamma}\bigg) + \frac{3}{2}\kappa^{q-1}\phi_q\bigg(1 + \phi_p\bigg(\frac{1}{\Gamma}\bigg)\bigg)B^{q-2} \le 1. \tag{3.5}$$

PROOF. Consider the Banach spaces $X = C[0,1]$ and $Z = L^1[0,1]$ with the usual sup norm $\|\cdot\|_\infty$ and Lebesgue norm $\|\cdot\|_1$, respectively.

Define $M\colon \operatorname{dom} M \to Z$ and $N_\lambda\colon \overline{\Omega} \to Z$ with

$$\operatorname{dom} M = \bigg\{x \in X : x, \phi_p(x') \in \mathrm{AC}[0,1], \\ x(0) = \sum_{i=1}^{m-2} \alpha_i x(\xi_i),\ x'(1) = 0,\ (\phi_p(x'))' \in L^1[0,1]\bigg\}$$

by $Mx(t) = -(\phi_p(x'(t)))'$ and $N_\lambda x(t) = \lambda f(t, x(t))$, $t \in [0,1]$, respectively. Then

$$\operatorname{Ker} M = \{x \in \operatorname{dom} M : x(t) \equiv c \text{on } [0,1]\}$$

and

$$\operatorname{Im} M = \bigg\{z \in Z : \sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \phi_q\bigg(\int_s^1 z(\tau)\,d\tau\bigg)\,ds = 0\bigg\}. \tag{3.6}$$

Clearly, $\dim \operatorname{Ker} M = 1$ and $\operatorname{Im} M$ is closed. So (1) holds.

Define the projection $P\colon X \to X_1$ by $(Px)(t) = \int_0^1 x(s)\,ds$, and semi-projection $Q\colon Z \to Z_1$ by

$$(Qz)(t) = \phi_p\bigg(q\bigg/\bigg(1 - \sum_{i=1}^{m-2} \alpha_i (1-\xi_i)^q\bigg)\bigg) \\ \cdot \phi_p\bigg(\sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \phi_q\bigg(\int_s^1 z(\tau)\,d\tau\bigg)\,ds\bigg). \tag{3.7}$$

Clearly, $\operatorname{Im} P = \operatorname{Ker} M$, $\operatorname{Ker} Q = \operatorname{Im} M$.

Let $\Omega \subset X$ be an open and bounded subset with $\theta \in \Omega$. For for all $x \in \overline{\Omega}$, it is easy to know that $Q[(I-Q)N_\lambda(x)] = \theta$. So $(I-Q)N_\lambda(x) \in \operatorname{Ker} Q = \operatorname{Im} M$. For for all $z \in \operatorname{Im} M$, one gets $Qz = 0$. Thus, $z = z - Qz = (I-Q)z \in (I-Q)Z$. Therefore, (2.1) holds. Obviously (2.2) holds.

Define $R\colon \overline{\Omega} \times [0,1] \to X_2$ by

$$R(x,\lambda)(t) = -\int_0^1 r(t,s)\phi_q\bigg(\int_s^1 \lambda(f(\tau, x(\tau)) - (Qf)(\tau))\,d\tau\bigg)\,ds, \tag{3.8}$$

where X_2 is the complement space of $X_1 = \operatorname{Ker} M$ in X and

$$r(t,s) = \begin{cases} 1-s & \text{for } 0 \le t \le s \le 1, \\ -s & \text{for } 0 \le s \le t \le 1. \end{cases}$$

It is clearly that $R(\,\cdot\,,0) = \theta$. Since f satisfies the L^1-Carathéodory conditions, Arzela–Ascoli theorem implies that R is relatively compact and the continuity of R on $\overline{\Omega}$ follows from the Lebesgue dominated convergence theorem.

For $x \in \Sigma_\lambda$, we have $\lambda f(t,x(t)) = -(\phi_p(x'(t)))' \in \operatorname{Im} M = \operatorname{Ker} Q$. So

$$\begin{aligned}R(x,\lambda)(t) &= -\int_0^1 r(t,s)\phi_q\bigg(\int_s^1 \lambda(f(\tau,x(\tau)) - (Qf)(\tau)\Big)\,d\tau\bigg)\,ds\\ &= \int_0^1 r(t,s)\phi_q\bigg(\int_s^1 (\phi_p(x'(\tau)))'\,d\tau\bigg)\,ds\\ &= x(t) - \int_0^1 x(s)\,ds = [(I-P)x](t),\end{aligned}$$

which implies (2.3). For $x \in \overline{\Omega}$, we have

$$\begin{aligned}M[Px + R(x,\lambda)](t) = M\bigg[&\int_0^1 x(s)\,ds\\ &- \int_0^1 r(t,s)\phi_q\bigg(\int_s^1 \lambda(f(\tau,x(\tau)) - (Qf)(\tau))\,d\tau\bigg)\,ds\bigg]\\ = \lambda[&f(t,x(t)) - Qf(t,x(t))] = [((I-Q)N_\lambda)(x)](t),\end{aligned}$$

which yields (2.4). Therefore, N_λ is M-compact in $\overline{\Omega}$, that is, (2) is satisfied.

Next, consider the cone

$$C = \{x \in X : x(t) \geq 0 \text{ on } [0,1]\}.$$

Let $\Omega_1 = \{x \in X : \delta||x||_\infty < |x(t)| < b \text{ on } [0,1]\}$, $\Omega_2 = \{x \in X : ||x||_\infty < B\}$. Clearly, Ω_1 and Ω_2 are bounded and open sets and

$$\overline{\Omega}_1 = \{x \in X : \delta||x||_\infty \leq |x(t)| \leq b \text{ on } [0,1]\} \subset \Omega_2$$

(see [9]). Moreover, $C \cap (\overline{\Omega}_2 \setminus \Omega_1) \neq \emptyset$. Let $J = I$ and $(\gamma x)(t) = |x(t)|$ for $x \in X$.

In order to show (3), suppose that there exist $x_0 \in C \cap \partial\Omega_2 \cap \operatorname{dom} M$ and $\lambda_0 \in (0,1)$ such that $Mx_0 = N_{\lambda_0}x_0$, that is, $-(\phi_p(x_0'(t)))' = \lambda_0 f(t,x_0(t))$ for all $t \in [0,1]$. For $t_1 \in (0,1]$ such that $x_0(t_1) = B$. This gives

$$0 \geq (\phi_p(x_0'(t_1)))' = -\lambda_0 f(t,B),$$

which contradicts (H2). For the case $t_1 = 0$, $x_0(0) = B$, from the boundary condition $x_0(0) = \sum_{i=1}^{m-2}\alpha_i x_0(\xi_i)$ and $\sum_{i=1}^{m-2}\alpha_i = 1$, we can see $x_0(\xi_i) = B$, $\xi_i \in (0,1)$, $i = 1,\dots,m-2$. Then $0 \geq (\phi_p(x_0'(\xi_i)))' = -\lambda_0 f(t,B)$, which is also in contradiction to (H2).

To prove (4), consider $x \in \operatorname{Ker} M \cap \overline{\Omega}_2$. Then $x(t) \equiv c$ on $[0,1]$. Define

$$\begin{aligned}H(c,\lambda) = c - \lambda|c| - \lambda\phi_p\bigg(q\Big/\bigg(1 - \sum_{i=1}^{m-2}\alpha_i(1-\xi_i)^q\bigg)\bigg)&\\ \cdot\,\phi_p\bigg(\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}\phi_q\bigg(\int_s^1 f(\tau,|c|)\,d\tau\bigg)\,ds\bigg),&\end{aligned}$$

where $c \in [-B, B]$ and $\lambda \in [0,1]$. Suppose $H(c,\lambda) = 0$, in view of (H2), we obtain

$$\begin{aligned} c = \lambda|c| + \lambda\phi_p\bigg(q\bigg/\bigg(1-\sum_{i=1}^{m-2}\alpha_i(1-\xi_i)^q\bigg)\bigg) \\ \cdot\,\phi_p\bigg(\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}\phi_q\bigg(\int_s^1 f(\tau,|c|)\,d\tau\bigg)\,ds\bigg) \geq \lambda|c| - \lambda\kappa|c| = \lambda(1-\kappa)|c| \geq 0. \end{aligned}$$

Hence, $H(c,\lambda) = 0$ implies $c \geq 0$. Furthermore, if $H(B,\lambda) = 0$, we would have

$$\begin{aligned} 0 \leq B(1-\lambda) = \lambda\phi_p\bigg(q\bigg/\bigg(1-\sum_{i=1}^{m-2}\alpha_i(1-\xi_i)^q\bigg)\bigg) \\ \cdot\,\phi_p\bigg(\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}\phi_q\bigg(\int_s^1 f(\tau,B)\,d\tau\bigg)\,ds\bigg), \end{aligned}$$

which contradicts (H2) for $\lambda \in (0,1]$. Obviously, if $\lambda = 0$, then $B = 0$, which is impossible. Thus, $H(c,\lambda) \neq 0$ for $c \in \operatorname{Ker} M \cap \partial\Omega_2$ and $\lambda \in [0,1]$. Therefore,

$$\deg_B\{H(\,\cdot\,,1), \operatorname{Ker} M \cap \Omega_2, \theta\} = \deg_B\{H(\,\cdot\,,0), \operatorname{Ker} M \cap \Omega_2, \theta\}.$$

However,

$$\deg_B\{H(\,\cdot\,,0), \operatorname{Ker} M \cap \Omega_2, \theta\} = \deg_B\{I, \operatorname{Ker} M \cap \Omega_2, \theta\} = 1.$$

Then

$$\deg_B\{[I-(P+JQN)\gamma]_{\operatorname{Ker} M}, \operatorname{Ker} M\cap\Omega_2, \theta\} = \deg_B\{H(\,\cdot\,,1), \operatorname{Ker} M\cap\Omega_2, \theta\} \neq 0.$$

Next, we prove (6). For $x \in \overline{\Omega}_2 \setminus \Omega_1$ and $t \in [0,1]$, in the case $||x||_\infty > 0$, in view of (H2),

$$\begin{aligned} (P+JQN)\gamma x(t) &= \int_0^1 |x(s)|\,ds + \phi_p\bigg(q\bigg/\bigg(1-\sum_{i=1}^{m-2}\alpha_i(1-\xi_i)^q\bigg)\bigg) \\ &\quad\cdot\,\phi_p\bigg(\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}\phi_q\bigg(\int_s^1 f(\tau,|x(\tau)|)\,d\tau\bigg)\,ds\bigg) \\ &> \int_0^1 |x(s)|\,ds + \phi_p\bigg(q\bigg/\bigg(1-\sum_{i=1}^{m-2}\alpha_i(1-\xi_i)^q\bigg)\bigg) \\ &\quad\cdot\,\phi_p\bigg(\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}\phi_q\bigg(\int_s^1 (-\kappa|x(\tau)|)\,d\tau\bigg)\,ds\bigg) \\ &\geq \int_0^1 |x(s)|\,ds - \kappa\phi_p\bigg(\frac{1}{\Gamma}\bigg)\int_0^1 |x(s)|\,ds \\ &= \bigg(1-\kappa\phi_p\bigg(\frac{1}{\Gamma}\bigg)\bigg)\int_0^1 |x(s)|\,ds \end{aligned} \tag{3.9}$$

and

$$\begin{aligned}
R(\gamma x,\lambda)(t) &= -\int_0^1 r(t,s)\phi_q\biggl(\int_s^1 \lambda(f(\tau,|x(\tau)|)-(Qf)(\tau))\,d\tau\biggr)\,ds\\
&= -\int_0^1 (1-s)\phi_q\biggl(\int_s^1 \lambda(f(\tau,|x(\tau)|)-(Qf)(\tau))\,d\tau\biggr)\,ds\\
&\quad+\int_0^t \phi_q\biggl(\int_s^1 \lambda(f(\tau,|x(\tau)|)-(Qf)(\tau))\,d\tau\biggr)\,ds\\
&> -\frac{3}{2}\phi_q\biggl(\int_0^1 (\kappa|x(s)|+Q(\kappa|x(s)|))\,ds\biggr)\\
&= -\frac{3}{2}\kappa^{q-1}\phi_q\biggl(\int_0^1 |x(s)|\,ds+\phi_p\biggl(q\Big/\biggl(1-\sum_{i=1}^{m-2}\alpha_i(1-\xi_i)^q\biggr)\biggr)\\
&\quad\cdot\phi_p\biggl(\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}\phi_q\biggl(\int_s^1 |x(\tau)|\,d\tau\biggr)\,ds\biggr)\biggr)\\
&\geq -\frac{3}{2}\kappa^{q-1}\phi_q\biggl(1+\phi_p\biggl(\frac{1}{\Gamma}\biggr)\biggr)\phi_q\biggl(\int_0^1 |x(s)|\,ds\biggr)\\
&\geq -\frac{3}{2}\kappa^{q-1}\phi_q\biggl(1+\phi_p\biggl(\frac{1}{\Gamma}\biggr)\biggr)B^{q-2}\int_0^1 |x(s)|\,ds.
\end{aligned}$$

From (3.5), we get

$$\begin{aligned}
(S_\lambda\circ\gamma x)(t) &= (P+JQN)\gamma x(t)+R(\gamma x,\lambda)(t)\\
&\geq\biggl[1-\kappa\phi_p\biggl(\frac{1}{\Gamma}\biggr)-\frac{3}{2}\kappa^{q-1}\phi_q\biggl(1+\phi_p\biggl(\frac{1}{\Gamma}\biggr)\biggr)B^{q-2}\biggr]\int_0^1 |x(s)|\,ds\geq 0.
\end{aligned}$$

In the case $x(t)\equiv 0$ on $[0,1]$, from (H2),

$$\begin{aligned}
&(S_\lambda\circ\gamma x)(t)\\
&=\phi_p\biggl(q\Big/\biggl(1-\sum_{i=1}^{m-2}\alpha_i(1-\xi_i)^q\biggr)\biggr)\phi_p\biggl(\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}\phi_q\biggl(\int_s^1 f(\tau,0)\,d\tau\biggr)\,ds\biggr)\\
&\quad-\int_0^1(1-s)\phi_q\biggl(\int_s^1\lambda(f(\tau,0)-Q(f(\tau,0)))\,d\tau\biggr)\,ds\\
&\quad+\int_0^t\phi_q\biggl(\int_s^1\lambda(f(\tau,0)-Q(f(\tau,0)))\,d\tau\biggr)\,ds\\
&\geq\phi_p\biggl(q\Big/\biggl(1-\sum_{i=1}^{m-2}\alpha_i(1-\xi_i)^q\biggr)\biggr)\phi_p\biggl(\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}\phi_q\biggl(\int_s^1 c_1\,d\tau\biggr)\,ds\biggr)\\
&\quad-\int_0^1(1-s)\phi_q\biggl(\int_s^1\lambda(c_2-Q(c_1))\,d\tau\biggr)\,ds+\int_0^t\phi_q\biggl(\int_s^1\lambda\Bigl(c_1-Q(c_2)\Bigr)\,d\tau\biggr)\,ds\\
&\geq c_1-\frac{(c_2-c_1)^{q-1}}{q+1}-(c_2-c_1)^{q-1}/q\geq 0.
\end{aligned}$$

As a result, $S_\lambda\circ\gamma(\overline{\Omega}_2\setminus\Omega_1)\subset C$.

It remains to show that (5) is satisfied. Take $u_0(t) \equiv 1$ on $[0,1]$. Then $u_0 \in C \setminus \{\theta\}$, $C(u_0) = \{x \in C : x(t) > 0 \text{ on } [0,1]\}$ and choose $\sigma(u_0) = 1$. Let $x \in C(u_0) \cap \partial\Omega_1$, then $x(t) > 0$, $0 < ||x||_\infty \leq b$ and $x(t) \geq \delta ||x||_\infty$ on $[0,1]$. Therefore, from (H3) and (H2), we obtain for $x \in C(u_0) \cap \partial\Omega_1$,

$$\begin{aligned}
&(JQNx)(t)\\
&= \phi_p\Big(q\Big/\Big(1 - \sum_{i=1}^{m-2} \alpha_i (1-\xi_i)^q\Big)\Big)\phi_p\Big(\sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \phi_q\Big(\int_s^1 f(\tau, x(\tau))\,d\tau\Big)\,ds\Big)\\
&\geq \phi_p\Big(q\Big/\Big(1 - \sum_{i=1}^{m-2} \alpha_i (1-\xi_i)^q\Big)\Big)\\
&\quad \cdot \phi_p\Big(\sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \phi_q\Big(\int_s^1 g(\tau)(h_1(\tau, x(\tau)) + h_2(\tau, x^{q-1}(\tau)))\,d\tau\Big)\,ds\Big)\\
&\geq \phi_p\Big(\frac{1}{\Gamma}\Big)\Big[\int_{\xi_{m-2}}^1 g(s) h_1(s, x(s))\,ds + \int_{\xi_{m-2}}^1 g(s) h_2(s, x^{q-1}(s))\,ds\Big]\\
&= \phi_p\Big(\frac{1}{\Gamma}\Big)\Big[\int_{\xi_{m-2}}^1 g(s)\frac{h_1(s, x(s)}{x^\rho(s)} x^\rho(s)\,ds + \int_{\xi_{m-2}}^1 g(s)\frac{h_2(s, x^{q-1}(s))}{x^{q-1}(s)} x^{q-1}(s)\,ds\Big]\\
&\geq \phi_p\Big(\frac{1}{\Gamma}\Big)\Big[\delta^\rho ||x||_\infty^\rho \int_{\xi_{m-2}}^1 g(s)\frac{h_1(s,b)}{b^\rho}\,ds + \delta^{q-1}||x||_\infty^{q-1}\int_{\xi_{m-2}}^1 g(s)\frac{h_2(s, b^{q-1})}{b^{q-1}}\,ds\Big]\\
&\geq \phi_p\Big(\frac{1}{\Gamma}\Big)[(1-\delta)||x||_\infty + 3\cdot 2^{q-2}\kappa^{q-1}||x||_\infty^{q-1}]\phi_p(\Gamma)\\
&= (1-\delta)||x||_\infty + 3\cdot 2^{q-2}\kappa^{q-1}||x||_\infty^{q-1}.
\end{aligned}$$

and

$$\begin{aligned}
R(\gamma x, \lambda)(t) &= -\int_0^1 r(t,s)\phi_q\Big(\int_s^1 \lambda(f(\tau, x(\tau)) - (Qf)(\tau))\,d\tau\Big)\,ds\\
&> -\frac{3}{2}\phi_q\Big(\int_0^1 (\kappa x(s) + Q(\kappa x(s)))\,ds\Big)\\
&\geq -\frac{3}{2}\phi_q(\kappa||x||_\infty + Q(\kappa||x||_\infty)) = -3\cdot 2^{q-2}\kappa^{q-1}||x||_\infty^{q-1}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&(Sx)(t) = (P + JQN)x(t) + R(\gamma x, \lambda)(t)\\
&\geq \delta||x||_\infty + (1-\delta)||x||_\infty + 3\cdot 2^{q-2}\kappa^{q-1}||x||_\infty^{q-1} - 3\cdot 2^{q-2}\kappa^{q-1}||x||_\infty^{q-1} = ||x||_\infty.
\end{aligned}$$

Thus, $||x|| \leq \sigma(u_0)||Sx||$ for all $x \in C(u_0) \cap \partial\Omega_1$. Theorem 2.1 implies that the equation $Mx = Nx$ has at least one solution $x^* \in C \cap (\overline{\Omega}_2 \setminus \Omega_1)$ on $[0,1]$ and the assertion follows. □

REMARK 3.3. Note that with the projection $P(x) = x(0)$ as in [2], [11], condition (6) of Theorem 2.1 is no longer satisfied.

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