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POSITIVE SOLUTIONS FOR A $2n$-ORDER BOUNDARY VALUE PROBLEM INVOLVING ALL DERIVATIVES OF ODD ORDERS

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Abstract. We are concerned with the existence, multiplicity and uniqueness of positive solutions for the $2n$-order boundary value problem

$$\begin{cases} (-1)^n u^{(2n)} = f(t, u, u', -u''', \dots, \\ \qquad\qquad (-1)^{i-1} u^{(2i-1)}, \dots, (-1)^{n-1} u^{(2n-1)}), \\ u^{(2i)}(0) = u^{(2i+1)}(1) = 0, \quad i = 0, \dots, n-1. \end{cases}$$

where $n \geq 2$ and $f \in C([0,1] \times \mathbb{R}_+^{n+1}, \mathbb{R}_+)$ ($\mathbb{R}_+ := [0,\infty)$) depends on u and all derivatives of odd orders. Our main hypotheses on f are formulated in terms of the linear function $g(x) := x_1 + 2\sum_{i=2}^{n+1} x_i$. We use fixed point index theory to establish our main results, based on a priori estimates achieved by utilizing some integral identities and an integral inequality. Finally, we apply our main results to establish the existence, multiplicity and uniqueness of positive symmetric solutions for a Lidostone problem involving an open question posed by P. W. Eloe in 2000.

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1. Introduction

We are concerned with the existence, multiplicity and uniqueness of positive solutions for the $2n$-order value boundary problem

$$\tag{1.1} \begin{cases} (-1)^n u^{(2n)} = f(t, u, u', -u''', \dots, \\ \qquad\qquad (-1)^{i-1} u^{(2i-1)}, \dots, (-1)^{n-1} u^{(2n-1)}), \\ u^{(2i)}(0) = u^{(2i+1)}(1) = 0, \quad i = 0, \dots, n-1, \end{cases}$$

where $n \geq 2$, and $f \in C([0,1] \times \mathbb{R}_+^{n+1}, \mathbb{R}_+)$ ($\mathbb{R}_+ := [0, \infty)$) depends on u and all derivatives of odd orders. By a positive solution of (1.1) we mean a function $u \in C^{2n}[0,1]$ that solves (1.1) and satisfies $u(t) > 0$ for all $t \in (0,1]$.

In recent years, the so-called Lidstone problem

$$\tag{1.2} \begin{cases} (-1)^n u^{(2n)} = f(t, u, -u'', \dots, (-1)^{n-1} u^{(2n-2)}), \\ u^{(2i)}(0) = u^{(2i)}(1) = 0, \quad i = 0, \dots, n-1, \end{cases}$$

has been extensively studied; see [1]–[7], [9]–[14], [17], [19]–[23], [26]–[28] and the references cited therein. In [24], the author studied the existence and uniqueness of positive solutions for the generalized Lidstone problem

$$\tag{1.3} \begin{cases} (-1)^n u^{(2n)} = f(t, u, -u'', \dots, (-1)^{n-1} u^{(2n-2)}), \\ \alpha_0 u^{(2i)}(0) - \beta_0 u^{(2i+1)}(0) = 0 \quad (i = 0, \dots, n-1), \\ \alpha_1 u^{(2i)}(1) + \beta_1 u^{(2i+1)}(1) = 0 \quad (i = 0, \dots, n-1), \end{cases}$$

where $f \in C([0,1] \times \mathbb{R}_+^n, \mathbb{R}_+)$, and α_j, β_j ($j = 0, 1$) are nonnegative constants with $\alpha_0\alpha_1 + \alpha_0\beta_1 + \alpha_1\beta_0 > 0$. The main results obtained in [24] are presented in terms of spectral radii of some linear integral operators associated with the nonlinearity f and the boundary conditions in (1.3), and thus can be viewed as extensions of the corresponding *optimal* results on second order differential equations, due to Z. Liu and F. Li in 1996 (see [16]). This means that, owing to the symmetry brought about by derivatives of even orders, (1.2) and (1.3) have much in common with the Dirichlet problem and the Sturm–Liouville problem for second-order ordinary differential equations.

To the best of our knowledge, (1.1) is an untreated topic in the literature, and it involves an open question posed by P. W. Eloe in [13]. Interestingly, we find, by observing some integral identities, (1.1) does possess some kind of symmetry, which enables us to formulate our main hypotheses on the nonlinearity f in terms of a linear function $g(x) := x_1 + 2\sum_{i=2}^{n+1} x_i$ on $\mathbb{R}_+^{n+1}$. This means that with extending the corresponding results in [16], [25], our main results are optimal in some sense. Our basic strategy in tackling (1.1) is to first use the method of order reduction to transform (1.1) into a boundary value problem for an integro-differential equation and then seek positive solutions of the resulting problem. We use fixed point index theory to develop our work here based on a priori estimates achieved by utilizing some integral identities and an integral inequality. In deriving a priori estimates of positive solutions for the case of f growing superlinearly at infinity, we need a condition of Berstein–Nagumo type; see [8] and [18]. This is an essential difference between (1.1) and (1.3) (see [24], [25]).

The plan of this paper is as follows. Sections 2 contains some integral identities and an integral inequality, which play an important role in the proofs of our main results. Our main results are stated and proved in Section 3. Section 4 contains some examples that illustrate our main results. As a byproduct of our main results, in Section 5, finally, we apply our main results to establish the existence, multiplicity and uniqueness of positive symmetric solutions for a Lidostone problem that involves an open question posed by P. W. Eloe in 2000 (see [13]).

2. Preliminaries

We assume the following condition in this section and the next section.

(H1) $f \in C([0,1] \times \mathbb{R}_+^{n+1}, \mathbb{R}_+)$ ($\mathbb{R}_+ := [0,\infty)$).

Let

$$E = C^1[0,1], \quad \|u\| = \max\{\|u\|_0, \|u'\|_0\},$$

where $\|w\|_0 = \max\{|w(t)| : 0 \le t \le 1\}$. Furthermore, let

$$P := \{u \in E : u(t) \ge 0,\ u'(t) \ge 0,\ \text{for all } t \in [0,1]\}.$$

Then $(E, \|\cdot\|)$ is a real Banach space and P is a cone on it.

Let

$$k(t,s) := \min\{t,s\}, (Tu)(t) := \int_0^1 k(t,s)u(s)\,ds.$$

Now let $v := (-1)^{n-1}u^{(2n-2)}$, and note (1.1) is equivalent to the boundary value problem for the integro-differential equation

$$\begin{cases} -v'' = f(t, T^{n-1}v, (T^{n-1}v)', (T^{n-2}v)', \dots, (Tv)', v'), \\ v(0) = v'(1) = 0. \end{cases}$$

Furthermore, the above problem is equivalent to

$$v(t) = \int_0^1 k(t,s) f(s, (T^{n-1}v)(s), (T^{n-1}v)'(s), (T^{n-2}v)'(s), \\ \dots, (Tv)'(s), v'(s))\,ds.$$

Define the operator A by

$$(Av)(t) := \int_0^1 k(t,s) f(s, (T^{n-1}v)(s), (T^{n-1}v)'(s), (T^{n-2}v)'(s), \\ \dots, (Tv)'(s), v'(s))\,ds.$$

Note (H1) implies that $A: P \to P$ is a completely continuous operator.

Lemma 2.1. *If $v \in P$, then*

$$\int_0^1 ((Tv)(t) + 2(Tv)'(t))te^t\,dt = \int_0^1 v(t)te^t\,dt. \tag{2.1}$$

Proof. Notice $(Tv)(0) = 0$. Integrating by parts, we obtain

$$\int_0^1 (Tv)(t)te^t\,dt = \int_0^1 (Tv)'(t)(1-t)e^t\,dt$$

and thus

$$\int_0^1 ((Tv)(t) + 2(Tv)'(t))te^t\,dt = \int_0^1 (Tv)'(t)(1+t)e^t\,dt.$$

Notice $(Tv)'(1) = 0$ and $(Tv)''(t) = -v(t)$. Integrating by parts again yields identity (2.1). □

A consequence of Lemma 2.1 is the following result that is of crucial importance in the proofs of our main results.

Lemma 2.2. *Let $v \in P$. Then*

$$\int_0^1 \bigg((T^{n-1}v)(t) + 2\sum_{i=0}^{n-1}(T^{n-1-i}v)'(t)\bigg)te^t\,dt = \int_0^1 (v(t) + 2v'(t))te^t\,dt.$$

Lemma 2.3. *If $v \in P \cap C^2[0,1]$, $v(0) = v'(1) = 0$, then*

$$\int_0^1 (-v''(t))te^t\,dt = \int_0^1 (v(t) + 2v'(t))te^t\,dt. \tag{2.2}$$

Proof. Integrating by parts and using $v(0) = v'(1) = 0$, we have

$$\int_0^1 (-v''(t))te^t\,dt = \int_0^1 v'(t)(t+1)e^t\,dt = \int_0^1 2v'(t)te^t\,dt + \int_0^1 v'(t)(1-t)e^t\,dt$$

and

$$\int_0^1 v'(t)(1-t)e^t\,dt = \int_0^1 v(t)te^t\,dt,$$

from which (2.2) follows. □

Lemma 2.4. *If $v \in P$, $v(0) = 0$, then*

$$v(1) \le \int_0^1 (v(t) + 2v'(t))te^t\,dt.$$

Proof. Notice $v(0) = 0$. Integrating by parts we have

$$\int_0^1 v(t)te^t\,dt = \int_0^1 v'(t)(1-t)e^t\,dt$$

and thus

$$\int_0^1 (v(t) + 2v'(t))te^t \, dt = \int_0^1 v'(t)(1+t)e^t \, dt \geq \int_0^1 v'(t) \, dt = v(1). \qquad \square$$

Lemma 2.5 (see [15]). *Let E be a real Banach space and P a cone on E. Suppose that $\Omega \subset E$ is a bounded open set and that $T\colon \overline{\Omega} \cap P \to P$ is a completely continuous operator. If there exists $w_0 \in P \setminus \{0\}$ such that*

$$w - Tw \neq \lambda w_0, \quad \text{for all } \lambda \geq 0, \ w \in \partial\Omega \cap P,$$

then $i(T, \Omega \cap P, P) = 0$, where i indicates the fixed point index on P.

Lemma 2.6 (see [15]). *Let E be a real Banach space and P a cone on E. Suppose that $\Omega \subset E$ is a bounded open set with $0 \in \Omega$ and that $T\colon \overline{\Omega} \cap P \to P$ is a completely continuous operator. If*

$$w - \lambda Tw \neq 0, \quad \text{for all } \lambda \in [0,1], \ w \in \partial\Omega \cap P,$$

then $i(T, \Omega \cap P, P) = 1$.

The following is a result that can be obtained by elementary calculus.

Lemma 2.7. *Suppose $h \in P \setminus \{0\}$. Then there exist two positive numbers $b_h > a_h$ such that*

$$a_h w_0(t) \leq \int_0^1 k(t,s)h(s) \, ds \leq b_h w_0(t), \quad \text{for all } t \in [0,1],$$

where

$$w_0(t) := \int_0^1 k(t,s) \, ds = \frac{2t - t^2}{2}. \tag{2.3}$$

3. Main results

For simplicity, we denote by $x := (x_1, \ldots, x_{n+1}) \in \mathbb{R}_+^{n+1}$ and

$$g(x) := x_1 + 2\sum_{i=2}^{n+1} x_i \quad \text{for } x \in \mathbb{R}_+^{n+1}.$$

Now we list our hypotheses on f.

(H2) There exist constants $a > 1$ and $c > 0$ such that $f(t,x) \geq ag(x) - c$ holds for all $x \in \mathbb{R}_+^{n+1}$, $t \in [0,1]$.

(H3) For every $M > 0$, there exists a function $\Phi_M \in C(\mathbb{R}_+, \mathbb{R}_+)$ such that

$$f(t, x_1, \ldots, x_n, y) \leq \Phi_M(y)$$

for all $(x_1, \dots, x_n) \in \underbrace{[0, M] \times \dots \times [0, M]}_{n}$ and $y \geq 0$, and

$$\int_0^\infty \frac{\xi \, d\xi}{\Phi_M(\xi) + \delta} = \infty$$

for all $\delta > 0$.

(H4) There exist constants $b \in (0,1)$ and $r > 0$ such that

$$f(t,x) \leq bg(x)$$

holds for all $x \in \underbrace{[0,r] \times \dots \times [0,r]}_{n+1}$, $t \in [0,1]$.

(H5) There exist constants $\alpha \in (0,1)$ and $c > 0$ such that

$$f(t,x) \leq \alpha g(x) + c$$

holds for all $x \in \mathbb{R}_+^{n+1}$, $t \in [0,1]$.

(H6) There exist constants $\beta > 1$ and $r > 0$ such that

$$f(t,x) \geq \beta g(x)$$

holds for all $x \in \underbrace{[0,r] \times \dots \times [0,r]}_{n+1}$, $t \in [0,1]$.

(H7) f is increasing in x and there is a constant $\omega > 0$ such that

$$\int_0^1 f(s, \underbrace{\omega, \dots, \omega}_{n+1}) \, ds < \omega.$$

Remark 3.1. f is said to be increasing in x if $f(t,x) \leq f(t,y)$ holds for every pair $x, y \in \mathbb{R}_+^{n+1}$ with $x \leq y$, where the partial ordering $\leq$ in $\mathbb{R}^{n+1}$ is understood componentwise.

(H8) $f(t, \lambda x) > \lambda f(t,x)$ for any $\lambda \in (0,1)$, $x \in \underbrace{(0,\infty) \times \dots \times (0,\infty)}_{n+1}$, $t \in [0,1]$.

Theorem 3.2. *If* (H1)–(H4) *hold, then* (1.1) *has at least one positive solution.*

Proof. Let $\mathcal{M}_1 := \{v \in P : v = Av + \lambda\varphi, \ \lambda \geq 0\}$, where $\varphi(t) := te^{-t}$. We are now going to prove that $\mathcal{M}_1$ is bounded. Indeed, if $v_0 \in \mathcal{M}_1$, then $v_0 \in P \cap C^2[0,1]$ and $v_0 = Av_0 + \lambda_0\varphi$ for some $\lambda_0 \in \mathbb{R}_+$, which can be written in the form

$$-v_0''(t) = f(t, (T^{n-1}v_0)(t), (T^{n-1}v_0)'(t), \dots, (Tv_0)'(t), v_0'(t)) + \lambda_0(2-t)e^{-t}.$$

By (H2), we have

$$-v_0''(t) \geq a\bigg((T^{n-1}v_0)(t) + 2\sum_{i=0}^{n-1}(T^{n-i-1}v_0)'(t)\bigg) - c.$$

Multiply by $\psi(t) := te^t$ on both sides of the above and integrate over $[0,1]$ and use Lemmas 2.2 and 2.3 to obtain

$$\int_0^1 (v_0(t) + 2v_0'(t))te^t\,dt \geq a \int_0^1 (v_0(t) + 2v_0'(t))te^t\,dt - c$$

and so

$$\int_0^1 (v_0(t) + 2v_0'(t))te^t\,dt \leq \frac{c}{a-1}, \quad \text{for all } v_0 \in \mathcal{M}_1.$$

Now Lemma 2.4 implies

$$\|v_0\|_0 = v_0(1) \leq \frac{c}{a-1}, \quad \text{for all } v_0 \in \mathcal{M}_1. \tag{3.1}$$

Furthermore, this estimate leads to

$$\|T^{n-1}v_0\|_0 = (T^{n-1}v_0)(1) \leq \frac{c}{a-1}$$

for all $v_0 \in \mathcal{M}_1$ and

$$\|(T^{n-i-1}v_0)'\|_0 = (T^{n-i-1}v_0)'(0) = \int_0^1 (T^{n-i-2}v_0)(t)\,dt \leq \frac{c}{a-1}$$

for all $v_0 \in \mathcal{M}_1$, $i = 0, \dots, n-2$. Let

$$\Pi := \{\mu \geq 0 : \text{there exists some } v \in P \text{ such that } v = Av + \mu\varphi\}.$$

Now (3.1) implies that $\mu_0 := \sup \Pi < +\infty$. Let $M := c/(a-1)$. By (H3), there is a function $\Phi_M \in C(\mathbb{R}_+, \mathbb{R}_+)$ such that

$$f(t, (T^{n-1}v)(t), (T^{n-1}v)'(t), \dots, v(t), v'(t)) \leq \Phi_M(v'(t)), \quad \text{for all } v \in \mathcal{M}_1$$

and hence we obtain

$$\begin{aligned} -v''(t) &= f(t, (T^{n-1}v)(t), (T^{n-1}v)'(t), \dots, v(t), v'(t)) + \mu(2-t)e^{-t} \\ &\leq \Phi_M(v'(t)) + \mu(2-t)e^{-t} \leq \Phi_M(v'(t)) + 2\mu_0, \end{aligned}$$

for all $v \in \mathcal{M}_1$, $\mu \in \Pi$, and

$$\int_0^{v'(0)} \frac{\xi\,d\xi}{\Phi_M(\xi) + 2\mu_0} \leq \int_0^1 v'(t)\,dt = v(1) \leq M, \quad \text{for all } v \in \mathcal{M}_1.$$

By (H3) again, there exists a constant $M_1 > 0$ such that

$$\|v'\|_0 = v'(0) \leq M_1, \quad \text{for all } v \in \mathcal{M}_1.$$

This means that $\mathcal{M}_1$ is bounded. Taking $R > \sup\{||v|| : v \in \mathcal{M}_1\}$, we have

$$v \neq Av + \lambda\varphi, \quad \text{for all } v \in \partial B_R \cap P,\ \lambda \geq 0.$$

Now Lemma 2.5 yields

$$i(A, B_R \cap P, P) = 0. \tag{3.2}$$

Let $\mathcal{M}_2 := \{v \in \overline{B}_r \cap P : v = \lambda Av,\ \lambda \in [0,1]\}$. We are in the position to prove $\mathcal{M}_2 = \{0\}$. Indeed, if $v \in \mathcal{M}_2$, then $v \in \overline{B}_r \cap P \cap C^2[0,1]$, $v(0) = v'(1) = 0$, and there is $\lambda \in [0,1]$ such that

$$\begin{aligned} v(t) &= \lambda (Av)(t) \\ &= \lambda \int_0^1 k(t,s) f(s, (T^{n-1}v)(s), (T^{n-1}v)'(s), \ldots, (Tv)'(s), v'(s))\, ds, \end{aligned}$$

which can be written in the form

$$-v''(t) = \lambda f(t, (T^{n-1}v)(t), (T^{n-1}v)'(t), \ldots, (Tv)'(t), v'(t)).$$

By (H4), we have

$$-v''(t) \le b\bigg((T^{n-1}v)(t) + 2\sum_{i=0}^{n-1} (T^{n-i-1}v)'(t)\bigg).$$

Multiply by $\psi(t) := te^t$ on both sides of the above and integrate over $[0,1]$ and use Lemmas 2.2 and 2.3 to obtain

$$\int_0^1 (v(t) + 2v'(t)) te^t\, dt \le b \int_0^1 (v(t) + 2v'(t)) te^t\, dt,$$

so that $\int_0^1 (v(t)+2v'(t))te^t\, dt = 0$, for all $v \in \mathcal{M}_1$, whence $v(t) \equiv 0$ and $\mathcal{M}_2 = \{0\}$, as required. A consequence of this is $v \ne \lambda Av$, for all $v \in \partial B_r \cap P$, $\lambda \in [0,1]$. Lemma 2.6 yields

$$i(A, B_r \cap P, P) = 1. \tag{3.3}$$

Combining (3.2) and (3.3) we arrive at

$$i(A, (B_R \setminus \overline{B}_r) \cap P, P) = 0 - 1 = -1.$$

Therefore A has at least one fixed point on $(B_R \setminus \overline{B}_r) \cap P$. This implies that (1.1) has at least one positive solution, which completes the proof. $\square$

Theorem 3.3. *If* (H1), (H5) *and* (H6) *are satisfied, then* (1.1) *has at least one positive solution.*

Proof. Let $\mathcal{M}_3 := \{v \in P : v = \lambda Av,\ \lambda \in [0,1]\}$. We shall prove that $\mathcal{M}_3$ is bounded. Indeed, if $v_0 \in \mathcal{M}_3$, then, by definition, $v_0 \in P \cap C^2[0,1]$, $v_0(0) = v_0'(1) = 0$, and there is $\lambda \in [0,1]$ such that

$$\begin{aligned} v_0(t) &= \lambda (Av_0)(t) \\ &= \lambda \int_0^1 k(t,s) f(s, (T^{n-1}v_0)(s), (T^{n-1}v_0)'(s), \ldots, (Tv_0)'(s), v_0'(s))\, ds, \end{aligned}$$

which can be written in the form

$$-v_0''(t) = \lambda f(t, (T^{n-1}v_0))(t), (T^{n-1}v_0)'(t), \ldots, (Tv_0)'(t), v_0'(t)).$$

By (H5), we have

$$-v_0''(t) \le \alpha\Big((T^{n-1}v_0)(t) + 2\sum_{i=0}^{n-1}(T^{n-i-1}v_0)'(t)\Big) + c. \tag{3.4}$$

Multiply by $\psi(t) := te^t$ on both sides of the above and integrate over $[0,1]$ and use Lemmas 2.2 and 2.3 to obtain

$$\int_0^1 (v_0(t) + 2v_0'(t))te^t\,dt \le \alpha\int_0^1 (v_0(t) + 2v_0'(t))te^t\,dt + c,$$

so that $\int_0^1 (v_0(t) + 2v_0'(t))te^t\,dt \le c/(1-\alpha)$. By Lemma 2.4, we obtain

$$\|v_0\|_0 = v_0(1) \le \int_0^1 (v_0(t) + 2v_0'(t))te^t\,dt \le \frac{c}{1-\alpha}, \quad \text{for all } v_0 \in \mathcal{M}_3.$$

Furthermore, this estimate leads to

$$\|T^{n-1}v_0\|_0 = (T^{n-1}v_0)(1) \le \frac{c}{1-\alpha}$$

for all $v_0 \in \mathcal{M}_1$ and

$$\|(T^{n-i-1}v_0)'\|_0 = (T^{n-i-1}v_0)'(0) = \int_0^1 (T^{n-i-2}v_0)(t)\,dt \le \frac{c}{1-\alpha}$$

for all $v_0 \in \mathcal{M}_1$, $i = 0, 1, \cdots, n-2$. Let $c_1 := c/(1-\alpha)$. By (3.4), we have

$$-v_0''(t) \le (2n-1)\alpha c_1 + 2\alpha v_0'(t) + c, \quad \text{for all } v_0 \in \mathcal{M}_3.$$

Noticing $v'(1) = 0$, we obtain

$$v_0'(t) \le \frac{(2n-1)\alpha c_1 + c}{2\alpha}(e^{2\alpha - 2\alpha t} - 1), \quad \text{for all } v_0 \in \mathcal{M}_3,$$

so that

$$\|v_0'\|_0 = v_0'(0) \le \frac{(2n-1)\alpha c_1 + c}{2\alpha}(e^{2\alpha} - 1) \quad \text{for all } v_0 \in \mathcal{M}_3.$$

This proves the boundedness of $\mathcal{M}_3$. Taking $R > \sup\{\|v\| : v \in \mathcal{M}_3\}$, we have

$$v \ne \lambda Av, \quad \text{for all } v \in \partial B_R \cap P,\ \lambda \in [0,1].$$

Lemma 2.6 yields

$$i(A, B_R \cap P, P) = 1. \tag{3.5}$$

Let $\mathcal{M}_4 := \{v \in \overline{B}_r \cap P : v = Av + \lambda\varphi,\ \lambda \ge 0\}$, where $\varphi(t) := te^{-t}$. Next we shall prove $\mathcal{M}_4 \subset \{0\}$. Indeed, if $v \in \mathcal{M}_4$, then $v \in \overline{B}_r \cap P \cap C^2[0,1]$, $v(0) = v'(1) = 0$, and there is $\lambda \ge 0$ such that

$$\begin{aligned} v(t) &= (Av)(t) + \lambda\varphi(t) \\ &= \int_0^1 k(t,s)f(s, (T^{n-1}v)(s), (T^{n-1}v)'(s), \ldots, (Tv)'(s), v'(s))\,ds + \lambda\varphi(t), \end{aligned}$$

which can be written in the form

$$-v''(t) = f(t, (T^{n-1}v)(t), (T^{n-1}v)'(t), \ldots, (Tv)'(t), v'(t)) + \lambda(2-t)e^{-t}.$$

By (H6), we have

$$-v''(t) \geq \beta\bigg((T^{n-1}v)(t) + 2\sum_{i=0}^{n-1}(T^{n-i-1}v)'(t)\bigg).$$

Multiply by $\psi(t) := te^t$ on both sides of the above and integrate over $[0,1]$ and use Lemmas 2.2 and 2.3 to obtain

$$\int_0^1 (v(t) + 2v'(t))te^t\,dt \geq \beta\int_0^1 (v(t) + 2v'(t))te^t\,dt.$$

Consequently $\int_0^1 (v(t) + 2v'(t))te^t\,dt = 0$ and hence $v(t) \equiv 0$. This proves $\mathcal{M}_4 \subset \{0\}$, as required. Consequently, $v \neq Av + \lambda\varphi$, for all $v \in \partial B_r \cap P$, $\lambda \geq 0$. Now Lemma 2.5 yields

$$i(A, B_r \cap P, P) = 0. \tag{3.6}$$

Combining (3.5) and (3.6), we arrive at

$$i(A, (B_R \setminus \overline{B}_r) \cap P, P) = 1 - 0 = 1.$$

Therefore A has at least one fixed point on $(B_R \setminus \overline{B}_r) \cap P$. Hence, (1.1) has at least one positive solution. □

Theorem 3.4. *If* (H1)–(H3), (H6) *and* (H7) *are satisfied, then* (1.1) *has at least two positive solutions.*

Proof. By (H2), (H3), and (H6), we know that (3.2) and (3.6) hold. Note we may choose $R > \omega > r$ in (3.2) and (3.6) (see the proofs of Theorems 3.2 and 3.3). By (H9), we have for all $v \in \partial B_\omega \cap P$,

$$\begin{aligned}
||Av||_0 &= (Av)(1) \\
&= \int_0^1 sf(s, (T^{n-1}v)(s), (T^{n-1}v)'(s), \ldots, (Tv)'(s), v'(s))\,ds \\
&\leq \int_0^1 f(s, (T^{n-1}v)(s), (T^{n-1}v)'(s), \ldots, (Tv)'(s), v'(s))\,ds \\
&\leq \int_0^1 f(s, \omega, \ldots, \omega)\,ds < \omega = \|v\|
\end{aligned}$$

and

$$\begin{aligned}||(Av)'||_0 &= (Av)'(0)\\ &= \int_0^1 f(s,(T^{n-1}v)(s),(T^{n-1}v)'(s),\dots,(Tv)'(s),v'(s))\,ds\\ &\le \int_0^1 f(s,\omega,\dots,\omega)\,ds < \omega = \|v\|.\end{aligned}$$

Thus we obtain $||Av|| < \omega = \|v\|$, for all $v \in \partial B_\omega \cap P$. This implies $v \neq \lambda Av$, for all $u \in \partial B_\omega \cap P$, $\lambda \in [0,1]$. Now Lemma 2.6 yields

$$i(A, B_\omega \cap P, P) = 1.$$

Combining (3.2) and (3.6), we arrive at

$$i(A,(B_R \setminus \overline{B}_\omega)\cap P,P) = 0-1 = -1, \quad i(A,(B_\omega \setminus \overline{B}_r)\cap P,P) = 1-0 = 1.$$

Therefore, A has at least two fixed points, with one on $(B_R \setminus \overline{B}_\omega) \cap P$ and the other on $(B_\omega \setminus \overline{B}_r) \cap P$. Hence (1.1) has at least two positive solutions. □

Theorem 3.5. *If* (H1), (H5), (H6) *and* (H8) *are satisfied, then* (1.1) *has exactly one positive solution.*

Proof. By Theorem 3.3, (1.1) has at least one positive solution. It remains to prove the uniqueness of the positive solutions.

Now suppose that $u_1 \in C^{2n}[0,1]$ and $u_2 \in C^{2n}[0,1]$ are two positive solutions of (1.1). Then $v_1 := (-1)^{n-1}u_1^{(2n-2)} \in C^2[0,1]$ and $v_2 := (-1)^{n-1}u_2^{(2n-2)} \in C^2[0,1]$ are two positive fixed points of A, satisfying

$$\begin{aligned}&v_i(t) > 0, \quad v_i'(t) > 0, \quad \text{for all } t \in (0,1),\\ &v_i(t) = \int_0^1 k(t,s) f(s,(T^{n-1}v_i)(s),(T^{n-1}v_i)'(s),\dots,(Tv_i)'(s),v_i'(s))\,ds\end{aligned}$$

for $i = 1,2$. By Lemma 2.7, there are $a_i > 0$ and $b_i > 0$ such that

$$a_i w_0(t) \le v_i(t) \le b_i w_0(t) \quad (i = 1,2),$$

where $w_0(t)$ is given by (2.3). Thus we have

$$v_2(t) \ge a_2 w_0(t) \ge \frac{a_2}{b_1} v_1(t).$$

Let $\mu_0 := \sup\{\mu > 0 : v_2(t) \ge \mu v_1(t), \text{ for all } t \in [0,1]\}$. It is easy to see that $0 < \mu_0 < \infty$. We claim $\mu_0 \ge 1$. Suppose the contrary $0 < \mu_0 < 1$. Let

$$\begin{aligned}h(t) := &f(t,\mu_0(T^{n-1}v_1)(t),\mu_0(T^{n-1}v_1)'(t),\dots,\mu_0(Tv_1)'(t),\mu_0 v_1'(t))\\ &- \mu_0 f(t,(T^{n-1}v_1)(t),(T^{n-1}v_1)'(t),\dots,(Tv_1)'(t),v_1'(t)).\end{aligned}$$

By (H8), we have $h(t) > 0$, for all $t \in (0,1)$. Lemma 2.7 implies that there exists an $\varepsilon > 0$ such that

$$\int_0^1 k(t,s)h(s)\,ds \geq \varepsilon w_0(t).$$

Therefore,

$$\begin{aligned} v_2(t) &\geq \int_0^1 k(t,s) f(s, \mu_0(T^{n-1}v_1)(s), \mu_0(T^{n-1}v_1)'(s), \dots, \mu_0(Tv_1)'(s), \mu_0 v_1'(s))\,ds \\ &= \int_0^1 k(t,s)h(s)\,ds + \mu_0 v_1(t) \geq \varepsilon w_0(t) + \mu_0 v_1(t) \geq \left(\mu_0 + \frac{\varepsilon}{b_1}\right) v_1(t), \end{aligned}$$

contradicting the definition of μ_0. As a result, we have $\mu_0 \geq 1$. Thus $v_2(t) \geq v_1(t)$. Similarly, we have $v_1(t) \geq v_2(t)$. Therefore, $v_1(t) \equiv v_2(t)$ and hence $u_1(t) \equiv u_2(t)$. This says that (1.1) has exactly one positive solution, which completes the proof. $\square$

4. Some examples

In this section we offer some examples to illustrate our main results.

Example 4.1. Let

$$f(t,x) := \begin{cases} g(x)/2 & \text{if } g(x) \leq 1, \\ 2g(x) - 3/2 & \text{if } g(x) \geq 1. \end{cases}$$

Now (H1)–(H4) are satisfied. By Theorem 3.2, (1.1) has at least one positive solution.

Example 4.2. Let

$$f(t,x) := \begin{cases} 2g(x) & \text{if } g(x) \leq 1, \\ g(x)/2 + 3/2 & \text{if } g(x) \geq 1. \end{cases}$$

Now (H1), (H5) and (H6) are satisfied. By Theorem 3.3, (1.1) has at least one positive solution.

Example 4.3. Let

$$f(t,x) := \left(\sum_{i=1}^{n} a_i x_i\right)^p + a_{n+1} x_{n+1}^q,$$

where $a_i > 0$ $(i = 1, \dots, n+1)$, $p > 1$, $1 < q \leq 2$. Now (H1)–(H4) are satisfied. By Theorem 3.2, (1.1) has at least one positive solution.

Example 4.4. Let

$$f(t,x) := \sum_{i=1}^{n+1} a_i x_i^{p_i} + \sum_{i=1}^{n+1} b_i x_i^{q_i},$$

where $a_i > 0 (i = 1, \dots, n+1)$, $p_i > 1$ $(i = 1, \dots, n)$, $1 < p_{n+1} \leq 2$, $b_i > 0 (i = 1, \dots, n+1)$, $0 < q_i < 1$ $(i = 1, \dots, n+1)$, with $\sum_{i=1}^{n+1}(a_i + b_i) < 1$. Now

(H1)–(H3), (H6) and (H7) are satisfied. By Theorem 3.4, (1.1) has at least two positive solutions.

EXAMPLE 4.5. Let
$$f(t,x) := \Big(\sum_{i=1}^{n+1} a_i x_i\Big)^p,$$
where $a_i > 0$ ($i = 1,\ldots,n+1$), $0 < p < 1$. Now (H1), (H5), (H6) and (H8) are satisfied. By Theorem 3.5, (1.1) has exactly one positive solution.

5. An open question posed by P. W. Eloe and positive symmetric solutions of a Lidstone problem

In [13], P. W. Eloe considered the nonlinear Lidstone boundary value problem more general than (1.2) (with $a(t)$ continuous and nonnegative)
$$\begin{cases} (-1)^n u^{(2n)} = \lambda a(t) f(t, u, -u'', \ldots, (-1)^{n-1}u^{(2n-2)}) & \text{for } 0 < t < 1, \\ u^{(2i)}(0) = u^{(2i)}(1) = 0 & \text{for } i = 0,\ldots,n-1, \end{cases}$$
where $f \in C(\mathbb{R}_+^n, \mathbb{R}_+)$ and $\lambda > 0$ is a real parameter. He posed an open question in this way: "Can the methods employed here apply to a Lidstone BVP with nonlinear dependence on odd order derivatives of the unknown function ?". In this section we shall answer the question partly by considering the simple case $a(t) := 1$. More precisely, we study the existence, multiplicity and uniqueness of positive symmetric solutions for the $2n$-order value boundary value problem
$$\begin{cases} (-1)^n u^{(2n)} = f(u, u', -u''', \ldots, (-1)^{i-1}u^{(2i-1)}, \ldots, (-1)^{n-1}u^{(2n-1)}), \\ u^{(2i)}(-1) = u^{(2i)}(1) = 0, \quad i = 0,\ldots,n-1, \end{cases} \tag{5.1}$$
where f satisfies the following condition:

(H9) $f \in C(\mathbb{R}_+ \times \mathbb{R}_+^n, \mathbb{R}_+)$ and $f(x_1, -x_2, \ldots, -x_{n+1}) = f(x)$ for all $x = (x_1,\ldots,x_{n+1}) \in \mathbb{R}_+^{n+1}$.

Note that the nonlinearity f in (5.1) is a special case of the nonlinearity f in (1.1) when $x \in \mathbb{R}_+^{n+1}$. With this view we apply Theorems 3.2–3.5 and obtain the following results on (5.1).

THEOREM 5.1. *If* (H9) *and* (H2)–(H4) *hold, then* (5.1) *has at least one symmetric positive solution.*

PROOF. By Theorem 3.2, the boundary value problem
$$\begin{cases} (-1)^n u^{(2n)} = f(u, u', -u''', \ldots, (-1)^{i-1}u^{(2i-1)}, \ldots, (-1)^{n-1}u^{(2n-1)}), \\ u^{(2i)}(0) = u^{(2i+1)}(1) = 0, \quad i = 0,\ldots,n-1, \end{cases}$$
has at least one positive solution w. Let
$$u(t) := \begin{cases} w(1-t) & \text{for } 0 \le t \le 1, \\ w(1+t) & \text{for } -1 \le t \le 0. \end{cases}$$

Then $u \in C^{2n}([-1,1],\mathbb{R}_+)$ is a symmetric positive solution of (5.1). $\square$

The following results can be proved analogously:

THEOREM 5.2. *If* (H9)*,* (H5) *and* (H6) *are satisfied, then* (5.1) *has at least one positive solution.*

THEOREM 5.3. *If* (H9)*,* (H2)*,* (H3)*,* (H6) *and* (H7) *are satisfied, then* (5.1) *has at least two symmetric positive solutions.*

THEOREM 5.4. *If* (H9)*,* (H5)*,* (H6) *and* (H8) *are satisfied, then* (5.1) *has exactly one symmetric positive solution.*

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