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ON GLOBAL REGULAR SOLUTIONS TO THE NAVIER–STOKES EQUATIONS IN CYLINDRICAL DOMAINS

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Abstract. We consider the incompressible fluid motion described by the Navier-Stokes equations in a cylindrical domain $\Omega \subset \mathbb{R}^3$ under the slip boundary conditions. First we prove long time existence of regular solutions such that $v \in W_2^{2,1}(\Omega \times (0,T))$, $\nabla p \in L_2(\Omega \times (0,T))$, where v is the velocity of the fluid and p the pressure. To show this we need smallness of $\|v_{,x_3}(0)\|_{L_2(\Omega)}$ and $\|f_{,x_3}\|_{L_2(\Omega\times(0,T))}$, where f is the external force and x_3 is the axis along the cylinder. The above smallness restrictions mean that the considered solution remains close to the two-dimensional solution, which, as is well known, is regular.

Having T sufficiently large and imposing some decay estimates on $\|f(t)\|_{L_2(\Omega)}$ we continue the local solution step by step up to the global one.

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1. Introduction

We consider the initial-boundary value problems to the Navier-Stokes equations

$$
\begin{aligned}
&v_t + v\cdot\nabla v - \operatorname{div}\mathbb{T}(v,p) = f && \text{in } \Omega\times\mathbb{R}_+,\\
&\operatorname{div} v = 0 && \text{in } \Omega\times\mathbb{R}_+,\\
&v\cdot\overline{n} = 0 && \text{on } S\times\mathbb{R}_+,\\
&\nu\overline{n}\cdot\mathbb{D}(v)\cdot\overline{\tau}_\alpha + \gamma v\cdot\overline{\tau}_\alpha = 0,\quad \alpha=1,2, && \text{on } S\times\mathbb{R}_+,\\
&v|_{t=0} = v(0) && \text{in } \Omega,
\end{aligned}
\tag{1.1}
$$

where $\Omega\subset\mathbb{R}^3$ is a cylindrical domain, $S=\partial\Omega$, $v=(v_1(x,t),v_2(x,t),v_3(x,t))\in\mathbb{R}^3$ is the velocity of the fluid, $p=p(x,t)\in\mathbb{R}$ the pressure, $f=(f_1(x,t),f_2(x,t),f_3(x,t))\in\mathbb{R}^3$ the external force field, $x=(x_1,x_2,x_3)$ the global Cartesian system in Ω, $\overline{n}$ is the unit outward vector normal to S, $\overline{\tau}_\alpha$, $\alpha=1,2$, is the tangent vector to S.

By $\mathbb{T}(v,p)$ we denote the stress tensor of the form

$$
\mathbb{T}(v,p)=\nu\mathbb{D}(v)-pI,
\tag{1.2}
$$

where $\nu>0$ is the viscosity coefficient, I is the unit matrix and $\mathbb{D}(v)$ is the dilatation tensor of the form

$$
\mathbb{D}(v)=\{v_{i,x_j}+v_{j,x_i}\}_{i,j=1,2,3}\equiv\nabla v+(\nabla v)^T.
\tag{1.3}
$$

Finally, $\gamma\ge 0$ is the slip coefficient.

By the dot we denote the scalar product in $\mathbb{R}^3$.

We assume that Ω is a cylinder parallel to the x_3-axis with arbitrary cross section. Moreover, $S=S_1\cup S_2$, where S_1 is the part of the boundary parallel to the x_3-axis and S_2 is perpendicular to it. Hence

$$
\begin{aligned}
S_1 &= \{x\in\mathbb{R}^3:\varphi_0(x_1,x_2)=c_0,\ -a<x_3<a\},\\
S_2 &= \{x\in\mathbb{R}^3:\varphi_0(x_1,x_2)<c_0,\ x_3 \text{ is equal either } -a \text{ or } a\},
\end{aligned}
$$

where a, c_0 are given positive numbers and $\varphi_0(x_1,x_2)=c_0$ describes a sufficiently smooth closed curve in the plane $x_3=\text{const}$.

The aim of this paper is to prove the existence of global regular solutions to problem (1.1) without restrictions on the size of norms of the initial velocity and the external force. The problem of wellposedness and regularity of weak solutions has a long history. In 1933 J. Leray (see [13]) proved the existence of global regular two-dimensional solutions in $\mathbb{R}^3$ and in 1959 O. A. Ladyzhenskaya (see [9]) showed the result in a bounded domain. Next many results of global regular solutions were proved under smallness conditions. In [7] H. Fujita and T. Kato assumed smallness of initial velocity in the homogeneous space $\dot{H}^{1/2}$. The result was improved in [20] by F. Weissler to the Lebesgue space L_3. In [3]

M. Cannone, Y. Meyer and F. Planchon proved existence of global unique regular solutions imposing smallness of initial velocity in $\dot{B}^{-1+3/p}_{p,\infty}$, $p < \infty$.

The result allows a construction of global solution with strongly oscilating initial velocity with large norms either in $\dot{H}^{1/2}$ or L_3. Most recently, Koch and Tataru [8] proved the existence of a global unique solutions with sufficiently small initial velocity belonging to the space of vector fields whose components are derivatives of BMO functions.

A very clear and deep historical review can be found in [6].

The existence of global regular two-dimensional solutions (see [9], [13]) implies, by a perturbation argument, the existence of solutions to the 3d Navier–Stokes equations in their neighbourhood in some spaces. To realize the perturbation we are free in choosing basic spaces, domains and boundary conditions. Choosing Besov spaces and periodic boundary conditions, J. Y. Chemin, I. Gallagher and M. Paicu (see [6]) and the first two persons in [5] proved the existence of global regular solutions varying slowly in one direction. Hence, the solutions are in some sense close to two-dimensional solutions.

In [4] a construction of a regular solution is made in two steps. First there is proved the existence of solutions to the Navier–Stokes equations with two independent variables (x_1, x_2) with the initial data being the mean with respect to x_3 of the initial velocity. Next a solution to the Navier–Stokes equations is derived by a perturbation argument applied to the above solutions.

For more references concerning the regularity problem see [6].

In this paper we prove the existence of global regular solutions under smallness of quantities $\|v_{,x_3}(0)\|_{L_2(\Omega)}$ and $\|f_{,x_3}\|_{L_2(0,T;L_{6/5}(\Omega))}$.

The main step of this proof is solvability of the problem for one component of the vorticity, $\chi = v_{2,x_1} - v_{1,x_2}$, which is possible under boundary slip conditions because they imply good boundary conditions for χ (see problem (3.16)).

It seems that our smallness condition (1.5) below is less restrictive than the corresponding one in [6].

The idea of considering the problem for the vorticity is taken from [10] (see also [22], [24]–[27]). But in [22], [25]–[27] stability results for axially symmetric solutions with small swirl were proved.

We should mention that the proof in this paper does not work for non-slip boundary conditions. However, for the space periodic case the proof holds and will be much simpler.

The techniques of papers [4]–[6] need much effords to be applicable for any bounded domain.

The paper is organized in the following way. In Section 2 there are introduced spaces used in this paper. In Section 3 we derived problems for $h = v_{,x_3}$ and χ which help us to show an a priori estimate in Section 4. In Section 5 the existence of solutions to problem (1.1) is proved by the Leray–Schauder fixed point theorem. In Section 6 the existence is extended to any time step by step.

We should mention that long time existence to problem (1.1) was considered in [17] and global existence in [16].

Finally in Appendix some auxiliary results are either proved or formulated.

To prove global existence of solutions to (1.1) we shall examine problem (1.1) step by step in time. Therefore instead of (1.1) we consider the system of problems

$$
\begin{aligned}
&v_t + v\cdot\nabla v - \operatorname{div}\mathbb{T}(v,p) = f && \text{in } \Omega^{(k+1)T} = \Omega\times(kT,(k+1)T),\\
&\operatorname{div} v = 0 && \text{in } \Omega^{(k+1)T},\\
&v\cdot\overline{n} = 0 && \text{on } S^{(k+1)T} = S\times(kT,(k+1)T),\\
&\overline{n}\cdot\mathbb{D}(v)\cdot\overline{\tau}_\alpha + \gamma v\cdot\overline{\tau}_\alpha = 0, &&\\
&\alpha = 1,2, && \text{on } S^{(k+1)T},\\
&v|_{t=KT} = v(kT) && \text{in } \Omega,
\end{aligned}
\tag{1.4}
$$

where $k\in\mathbb{N}_0 = \mathbb{N}\cup\{0\}$ and $v(kT)$ is calculated as a trace of v from the interval $((k-1)T, kT]$.

To formulate existence results for problem (1.4) we introduce the following assumptions

Assumption 1. *Let*

$$
\begin{aligned}
K_2(kT) =& \|f_{,x_3}\|_{L_2(kT,(k+1)T;L_{6/5}(\Omega))} + \|(\operatorname{rot} f)_3\|_{L_2(kT,(k+1)T;L_{6/5}(\Omega))}\\
&+ \|f_3\|_{L_2(S_2^{(k+1)T})} + \|v_{,x_3}(kT)\|_{L_2(\Omega)} + \|(\operatorname{rot} v)_3(kT)\|_{L_2(\Omega)}\\
&+ \|f\|_{L_2(\Omega^{(k+1)T})} + \|v(kT)\|_{H^1(\Omega)} + \varphi(d_1,d_2)(d_1+d_2),\\
K_3(k,T) =& \varphi(d_1,d_2)K_2^2(k,T) + \|f\|_{L_2(\Omega^{(k+1)T})} + \|v(kT)\|_{H^1(\Omega)},\\
K_4(k,T) =& \|f_{,x_3}\|_{L_2(\Omega^{(k+1)T})} + \|v_{,x_3}(kT)\|_{L_2(\Omega)},\\
d_1 =& \|f\|_{L_\infty(\mathbb{R}_+,L_{6/5}(\Omega))} + \|v(0)\|_{L_2(\Omega)},\\
d_2 =& \|f\|_{L_2(kT,(k+1)T;L_{6/5}(\Omega))} + \|v(kT)\|_{L_2(\Omega)}
\end{aligned}
$$

be finite, where φ is an increasing positive function.

Assumption 2. *Let*

$$
d(k,T) = \|f_{,x_3}\|_{L_2(kT,(k+1)T;L_{6/5}(\Omega))} + \|f_3\|_{L_2(S_2^{(k+1)T})} + \|v_{,x_3}(kT)\|_{L_2(\Omega)}
$$

be finite.

Assumption 3. *Assume that $d(k,T)$ is so small that there exists a constant A such that*

$$
c[\varphi(d_1,d_2)A^8 + K_3^4]\exp(cd_2A)\exp(K_2)d(k,T) + K_4 \le A,
\tag{1.5}
$$

where c does not depend on T and k.

Theorem A. *Let the Assumptions* 1–3 *hold. Then there exists a solution to problem* (1.4) *such that*

$$v, v_{,x_3} \in W_2^{2,1}(\Omega^{(k+1)T}), \nabla p, \nabla p_{,x_3} \in L_2(\Omega^{(k+1)T})$$

and

$$\begin{gathered} \|v_{,x_3}\|_{W_2^{2,1}(\Omega^{(k+1)T})} \le A, \\ \|v\|_{W_2^{2,1}(\Omega^{(k+1)T})} + \|\nabla p\|_{L_2(\Omega^{(k+1)T})} \le \varphi(d_1, d_2, A, K_2), \\ \|\nabla p_{,x_3}\|_{L_2(\Omega^{(k+1)T})} \le \varphi(d_1, d_2, A, K_2, K_3, K_4), \end{gathered} \tag{1.6}$$

where φ is an increasing positive function.

Remark A_1. In general the constant $A = A(k,T)$ depends on k, T, because the quantities K_2, K_3, K_4 depend on the time integral norms of f and on

$$\|v(kT)\|_{H^1(\Omega)}, \quad \|v_{,x_3}(kT)\|_{H^1(\Omega)}. \tag{1.7}$$

Hence Theorem A is a local existence theorem and in reality describes existence in the interval $[0,T]$ only, because quantities (1.7) are not yet defined for $k \ge 1$.

Therefore the part of Theorem A for intervals $[kT,(k+1)T]$, $k \ge 1$, is important for the proof of global existence only.

Hence the main step in the proof of global existence consists in obtaining estimations for quantities (1.7) independent of k.

Remark A_2. Since A depends on T by time integral norms of the external force f, T should not be very small for solutions to problem (1.4). It is understandable because smallness restriction on quantity $d(k,T)$ is imposed (see Assumptions 2, 3). However, it is not convenient to have large T because then more restrictions on the external force f must be imposed.

Theorem B. *Let the assumptions of Theorem A hold. Let either Assumption* 6.3 (*see* (6.28)*,* (6.29)) *or Assumption* 6.4 (*see* (6.30)–(6.32)) *hold. Then there exists a global solution to problem* (1.1) *such that $v \in W_2^{2,1}(\Omega \times (kT, (k+1)T))$, $\nabla p \in L_2(\Omega \times (kT,(k+1)T))$ for any $k \in \mathbb{N}_0$.*

2. Notation

By c we denote a generic constant which changes its value from line to line. In general it depends on the constants of imbeddings, regularity of the boundary and so on.

By φ we denote a generic function which is always positive and increasing of its arguments. It also may change its form from line to line.

We denote

$$\Omega^{kT} = \Omega \times ((k-1)T, kT), \qquad \Omega^{T_1,T_2} = \Omega \times (T_1, T_2).$$

We use isotropic and anisotropic Lebesgue spaces

$$\begin{aligned} &L_p(Q), && Q \in \{\Omega^T, S^T, \Omega, S\}, && p \in [1,\infty],\\ &L_q(0,T;L_p(Q)), && Q \in \{\Omega, S\}, && q,p \in [1,\infty]; \end{aligned}$$

Sobolev spaces

$$W_p^{s,s/2}(Q^T), \quad Q \in \{\Omega, S\}, \quad s \in \mathbb{Z}_+ \cup \{0\}, \quad p \in [1,\infty],$$

with the following norm for even s

$$\|u\|_{W_p^{s,s/2}(Q^T)} = \bigg(\sum_{|\alpha|+2a\le s} \int_{Q^T} |D_x^\alpha \partial_t^a u|^p \, dx\, dt\bigg)^{1/p}$$

where $D_x^\alpha = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3}$, $\alpha = (\alpha_1,\alpha_2,\alpha_3)$, $|\alpha| = \alpha_1+\alpha_2+\alpha_3$, $\alpha_i \in \mathbb{Z}_+ \cup \{0\}$, $i = 1,2,3$, and in the case of odd s we have the fractional derivatives. Similarly, we define

$$W_p^s(Q), \quad Q \in \{\Omega, S\}.$$

For $p = 2$ we have

$$H^s(Q) = W_2^s(Q), \quad L_2(Q) = H^0(Q).$$

We define a space natural for weak solutions to the heat and the Stokes equations

$$V_2^k(\Omega^T) = \{u : \|u\|_{V_2^k(\Omega^T)} = \operatorname*{ess\,sup}_{t\le T} \|u(t)\|_{H^k(\Omega)} + \|\nabla u\|_{L_2(0,T;H^k(\Omega))} < \infty\},$$

where $k \in \mathbb{N}_0$.

Next we introduce the Sobolev spaces with mixed norms (see [18], [14]).

We define space $W_{q,r}^{2,1}(\Omega^T)$, $q, r \in [1,\infty]$, as a set of functions with the following norm finite

$$\|u\|_{W_{q,r}^{2,1}(\Omega^T)} = \bigg(\|u_t\|_{L_{q,r}(\Omega^T)}^r + \sum_{|\alpha|\le 2} \|D_x^\alpha u\|_{L_{q,r}(\Omega^T)}^r\bigg)^{1/r},$$

where

$$\|u\|_{L_{q,r}(\Omega^T)} = \bigg(\int_0^T \|u(\cdot,t)\|_{L_q(\Omega)}^r dt\bigg)^{1/r}, \quad q, r \ge 1.$$

Let us consider the Stokes system

$$\begin{aligned} &v_t - \operatorname{div} \mathbb{T}(v,p) = f,\\ &\operatorname{div} v = 0,\\ &v\cdot \overline{n}|_S = 0, \quad \overline{n}\cdot \mathbb{D}(v)\cdot \overline{\tau}_\alpha + \gamma v \cdot \overline{\tau}_\alpha|_S = 0, \quad \alpha = 1,2,\\ &v|_{t=0} = v_0. \end{aligned} \tag{2.1}$$

Similarly as in [18], [14] we prove

Lemma 2.1. *Assume that* $f \in L_{q,r}(\Omega^T)$, $v_0 \in B_{q,r}^{2-2/r}(\Omega)$, $q,r \in (1,\infty)$. *Then there exists a solution to problem* (2.1) *such that* $v \in W_{q,r}^{2,1}(\Omega^T)$, $\nabla p \in L_{q,r}(\Omega^T$ *and the estimate*

$$\|v\|_{W_{q,r}^{2,1}(\Omega^T)} + \|\nabla p\|_{L_{q,r}(\Omega^T)} \le c(\|f\|_{L_{q,r}(\Omega^T)} + \|v_0\|_{B_{q,r}^{2-2/r}(\Omega)}), \tag{2.2}$$

holds.

By $B_{q,r}^l(\Omega)$, $\Omega \subset \mathbb{R}^n$, $q,r \in (1,\infty)$, $l \in \mathbb{R}_+$ we denote the Besov space with the finite norm

$$\|u\|_{B_{q,r}^l(\Omega)} = \|u\|_{W_q^{[l]}(\Omega)} + \|u\|_{\dot{B}_{q,r}^{l-[l]}(\Omega)},$$

where $[l]$ is the integer part of l and

$$\|u\|_{B_{q,r}^\lambda(\Omega)} = \bigg(\sum_{i=1}^n \int_0^\infty \|\Delta_i(h)u\|_{L_q(\Omega)}^r \frac{dh}{h^{1+r\lambda}}\bigg)^{1/r},$$

where $\lambda \in (0,1)$ and

$$\Delta_i(h)u(x) = u(x+e_ih) - u(x)$$

where $e_i = (\delta_{ik})_{k=1,\dots,n}$ and x, $x+e_ih \in \Omega$.

3. Auxiliary results

We start from the weak solutions because in this paper the existence of regularized weak solution will be proved.

Definition 3.1. By a weak solution to problem (1.1) we mean $v \in V_2^0(\Omega^T)$ such that $\operatorname{div} v = 0$, $v \cdot \overline{n}|_S = 0$, and satisfying the integral identity

$$\begin{aligned}\int_{\Omega^T} (-v\cdot\varphi_{,t} + \frac{1}{2}\nu\mathbb{D}(v)\cdot\mathbb{D}(\varphi) + v\cdot\nabla v\cdot\varphi)\,dx\,dt \\ + \gamma\sum_{\alpha=1}^2 \int_{S^T} v\cdot\overline{\tau}_\alpha\varphi\cdot\overline{\tau}_\alpha\,dS\,dt + \int_\Omega v\cdot\varphi|_{t=T}\,dx \\ - \int_\Omega v(0)\cdot\varphi|_{t=0}\,dx = \int_{\Omega^T} f\cdot\varphi\,dx\,dt,\end{aligned} \tag{3.1}$$

which holds for any sufficiently smooth φ such that $\operatorname{div}\varphi = 0$, $\varphi\cdot\overline{n}|_S = 0$.

To prove the existence of weak solutions we need the Korn inequality.

Lemma 3.2 (see [19]). *Assume that* $E_\Omega(v) = \|\mathbb{D}(v)\|_{L_2(\Omega)}^2$, $\operatorname{div} v = 0$ *and* $v\cdot\overline{n}|_S = 0$. *Assume that the cylindrical domain* Ω *is not axially symmetric. Then there exists a constant* c_1, *which depends at most on* Ω *and* S, *such that*

$$\|v\|_{H^1(\Omega)}^2 \le c_1E_\Omega(v). \tag{3.2}$$

PROOF. We have

$$\int_\Omega |\mathbb{D}(v)|^2\,dx = \int_\Omega (v_{i,x_j}+v_{j,x_i})^2\,dx = \int_\Omega (v^2_{i,x_j}+v^2_{j,x_i}+2v_{i,x_j}v_{j,x_i})\,dx,$$

where we used the summation convention and

$$\int_\Omega v_{i,x_j}v_{j,x_i}\,dx = \int_{S_1\cup S_2} n_i v_{i,x_j}v_j\,dS = \int_{S_1\cup S_2} n_{i,x_j}v_iv_j\,dS = \int_{S_1} n_{i,x_j}v_iv_j dS_1.$$

Hence

$$\|\nabla v\|^2_{L_2(\Omega)} \le c\biggl(E_\Omega(v) + \int_{S_1} v^2_{\tau_\alpha}\,dS_1\biggr), \tag{3.3}$$

where $v_{\tau_\alpha} = v\cdot\overline{\tau}_\alpha$, $v^2_{\tau_\alpha} = v^2_{\tau_1}+v^2_{\tau_2}$.

By the Poincaré inequality we have

$$\|v\|^2_{H^1(\Omega)} \le c\biggl(E_\Omega(v) + \int_{S_1} v^2_{\tau_\alpha}\,dS_1\biggr). \tag{3.4}$$

Hence the trace theorem implies

$$\|v\|^2_{H^1(\Omega)} \le c(E_\Omega(v) + \|v\|^2_{L_2(\Omega)}). \tag{3.5}$$

Next we prove the following: there exist positive constants δ and M such that

$$\|v\|^2_{L_2(\Omega)} \le \delta\|\nabla v\|^2_{L_2(\Omega)} + ME_\Omega(v), \tag{3.6}$$

where δ can be chosen sufficiently small.

We prove (3.6) by contradiction. Assume that such M does not exist. Then for any $m\in\mathbb{N}$ there exists $v^m\in H^1(\Omega)$ such that

$$\|v^m\|^2_{L_2(\Omega)} \ge \delta\|\nabla v^m\|^2_{L_2(\Omega)} + mE_\Omega(v^m) \equiv G_m(v^m).$$

Then for $u^m = v^m/\|v^m\|_{L_2(\Omega)}$ we have

$$\|u^m\|_{L_2(\Omega)} = 1,\quad G_m(u^m) = \frac{G_m(v^m)}{\|v^m\|_{L_2(\Omega)}} \le 1.$$

Therefore from the sequence $\{u^m\}$ we can choose a subsequence $\{u^{m_k}\}$ which converges weakly in $H^1(\Omega)$ and strongly in $L_2(\Omega)$ to a limit $u\in H^1(\Omega)$. Moreover, $E_\Omega(u^{m_k}) \le 1/m_k \to 0$. Hence $E_\Omega(u)=0$. Since Ω is not axially symmetric we have $u=0$. This contradicts

$$\|u\|_{L_2(\Omega)} = \lim_{m_k\to\infty}\|u^{m_k}\|_{L_2(\Omega)} = 1.$$

Hence (3.6) holds. From (3.5) and (3.6) we obtain (3.2). □

LEMMA 3.3. *Assume that* $f \in L_\infty(\mathbb{R}_+; L_{6/5}(\Omega))$, $v(0) \in L_2(\Omega)$ *and* Ω *is not axially symmetric. Then*

$$\|v(t)\|_{L_2(\Omega)} \le \frac{1}{\sqrt{2}}\left(\frac{c_1c_2}{\nu}\|f\|_{L_\infty(\mathbb{R}_+;L_{6/5}(\Omega))} + \|v(0)\|_{L_2(\Omega)}\right) \equiv d_1 \tag{3.7}$$

for any $t \in \mathbb{R}_+$, c_2 *is the constant from the imbedding* $H^1(\Omega) \subset L_6(\Omega)$, c_1 *appeared in* (3.2). *Next*

$$\begin{aligned} &\|v\|_{V_2^0(\Omega\times(kT,t))} \\ &\quad \le \frac{1}{\nu_*}\left[\sqrt{\frac{c_1}{\nu}}c_2\|f\|_{L_2(kT,(k+1)T;L_{6/5}(\Omega))} + \|v(kT)\|_{L_2(\Omega)}\right] \equiv d_2(k), \end{aligned} \tag{3.8}$$

for $t \in (kT,(k+1)T]$, $\nu_* = \min\{1, \sqrt{\nu/c_1}\}$.

PROOF. Multiplying $(1.1)_1$ by v, integrating over Ω, using $(1.1)_{2,3,4}$, the Korn inequality and applying the Hölder and the Young inequalities to the term with f yield

$$\frac{1}{2}\frac{d}{dt}\|v\|^2_{L_2(\Omega)} + \frac{\nu}{c_1}\|v\|^2_{H^1(\Omega)} \le \frac{\varepsilon c_2^2}{2}\|v\|^2_{H^1(\Omega)} + \frac{1}{2\varepsilon}\|f\|^2_{L_{6/5}(\Omega)}. \tag{3.9}$$

Setting $\varepsilon = \nu/(c_1c_2^2)$ implies

$$\frac{d}{dt}\|v\|^2_{L_2(\Omega)} + \frac{\nu}{c_1}\|v\|^2_{H^1(\Omega)} \le \frac{c_1c_2^2}{\nu}\|f\|^2_{L_{6/5}(\Omega)} \tag{3.10}$$

Replacing the norm $\|v\|_{H^1(\Omega)}$ by $\|v\|_{L_2(\Omega)}$, multiplying the result by $\exp((\nu/c_1)t)$ and integrating with respect to time we obtain (3.7).

Integrating (3.10) with respect to time from kT to $t \in (kT,(k+1)T]$ we obtain (3.8). □

From the above lemma by an application of the Galerkin method and the considerations from [11, Chapter 6] we have

LEMMA 3.4. *Let the assumptions of Lemma* 3.3 *hold. Then there exists a weak solution to problem* (1.1) *in any interval* $(kT,(k+1)T)$, $k \in \mathbb{N}_0 \equiv \mathbb{N}\cup\{0\}$ *such that*

$$\|v\|_{V_2^0(\Omega\times(kT;(k+1)T)} \le d_2. \tag{3.13}$$

To prove global existence of regular solutions to problem (1.1) we need to obtain an estimate without restrictions on the existence time. We are not able to obtain such an estimate starting directly from problem (1.1). Following [17], [24] we replace problem (1.1) by a sequence of problems. For this purpose we introduce the quantities

$$h = v_{,x_3}, \quad q = p_{,x_3}, \quad g = f_{,x_3}. \tag{3.14}$$

Lemma 3.5 (see [17], [24]). *Assume that* v *is given. Then* (h,q) *is a solution to the problem*

$$
(3.15)\qquad
\begin{array}{ll}
h_{,t}-\operatorname{div}\mathbb{T}(h,q)=-v\cdot\nabla h-h\cdot\nabla v+g & \text{in } \Omega^{(k+1)T},\\
\operatorname{div}h=0 & \text{in } \Omega^{(k+1)T},\\
h\cdot\overline{n}=0,\quad \nu\overline{n}\cdot\mathbb{D}(h)\cdot\overline{\tau}_\alpha+\gamma h\cdot\overline{\tau}_\alpha=0,\quad \alpha=1,2, & \text{on } S_1^{(k+1)T},\\
h_i=0,\quad i=1,2,\quad h_{3,x_3}=0 & \text{on } S_2^{(k+1)T},\\
h|_{t=kT}=h(kT) & \text{in } \Omega,
\end{array}
$$

where $h(kT)$ *is considered as given.*

Lemma 3.6 (see [17], [24]). *Let* $F_3=(\operatorname{rot}f)_3$*,* h,v *be given. Then* $\chi=(\operatorname{rot}v)_3$ *is a solution to the problem*

$$
(3.16)\qquad
\begin{array}{ll}
\chi_{,t}+v\cdot\nabla\chi-h_3\chi+h_2v_{3,x_1}-h_1v_{3,x_2}-\nu\Delta\chi=F_3 & \text{in } \Omega^{(k+1)T},\\
\chi=v_i(n_{i,x_j}\tau_{1j}+\tau_{1i,x_j}n_j) & \\
\qquad+v\cdot\overline{\tau}_1(\tau_{12,x_1}-\tau_{11,x_2})+\dfrac{\gamma}{\nu}v_j\tau_{1j}\equiv\chi_* & \text{on } S_1^{(k+1)T},\\
\chi_{,x_3}=0 & \text{on } S_2^{(k+1)T},\\
\chi|_{t=0}=\chi(kT) & \text{in } \Omega,
\end{array}
$$

where $\chi(kT)$ *is considered as given, and tangent and normal vectors to* S_1 *are defined as follows*

$$
\begin{aligned}
\overline{n}|_{S_1}&=\frac{\nabla\varphi}{|\nabla\varphi|}=\frac{1}{|\nabla\varphi|}(\varphi_{,x_1},\varphi_{,x_2},0),\\
\overline{\tau}|_{S_1}&=\frac{\nabla^\perp\varphi}{|\nabla\varphi|}=\frac{1}{|\nabla\varphi|}(-\varphi_{,x_2},\varphi_{,x_1},0),\quad \overline{\tau}_2|_{S_1}=(0,0,1).
\end{aligned}
$$

4. Estimates

First we obtain estimates for solutions to problem (3.15).

Lemma 4.1. *Assume that*

(a) v *is the weak solution to problem* (1.1)*,*
(b) $h\in L_\infty(kT,(k+1)T;L_3(\Omega))$*,*
(c) $g\in L_2(kT,(k+1)T;L_{6/5}(\Omega))$*,* $f_3\in L_2(S_2\times(kT,(k+1)T))$*,* $h(kT)\in L_2(\Omega)$*,* $k\in\mathbb{N}_0$*.*

Then

$$
(4.1)\quad \nu_*\|h\|^2_{V_2^0(\Omega\times(kT,t))}\le\frac{2c_1c_2^2}{\nu}(d_2^2\|h\|^2_{L_\infty(kT,t;L_3(\Omega))}+\|g\|^2_{L_2(kT,t;L_{6/5}(\Omega))})
+\nu\|f_3\|^2_{L_2(S_2\times(kT,t))}+\|h(kT)\|^2_{L_2(\Omega)},
$$

where $t \in (kT,(k+1)T]$, ν_*, c_2 *are defined in Lemma* 3.3, c_1 *in Lemma* 3.2.

Proof. Multiplying $(3.15)_1$ by h, integrating over Ω, using $(3.15)_{2,3,4}$ and the Korn inequality yields

$$\tag{4.2} \frac{1}{2}\frac{d}{dt}\|h\|^2_{L_2(\Omega)} + \frac{\nu}{c_1}\|h\|^2_{H^1(\Omega)} \le \frac{\varepsilon_1}{2}\|h\|^2_{L_6(\Omega)} + \frac{1}{2\varepsilon_1}\|\nabla v\|^2_{L_2(\Omega)}\|h\|^2_{L_3(\Omega)} + \frac{\varepsilon_2}{2}\|h\|^2_{L_6(\Omega)} + \frac{1}{2\varepsilon_2}\|g\|^2_{L_{6/5}(\Omega)} + \nu\|f_3\|^2_{L_2(S_2)}.$$

Setting $\varepsilon_1 c_2^2 = (1/2)(\nu/c_1)$, $\varepsilon_2 = \varepsilon_1$ in (4.2) we obtain

$$\tag{4.3} \frac{d}{dt}\|h\|^2_{L_2(\Omega)} + \frac{\nu}{2c_1}\|h\|^2_{H^1(\Omega)} \le \frac{2c_1c_2^2}{\nu}(\|\nabla v\|^2_{L_2(\Omega)}\|h\|^2_{L_3(\Omega)} + \|g\|^2_{L_{6/5}(\Omega)}) + \nu\|f_3\|^2_{L_2(S_2)}.$$

Integrating (4.3) from kT to $t \in (kT,(k+1)T]$ and using (3.13) we get (4.1). □

Lemma 4.2. *Let the assumptions* (a) *and* (c) *of Lemma* 4.1 *hold. Let the assumption* (b) *is replaced by* $v \in L_2(kT,(k+1)T;L_3(\Omega))$. *Then*

$$\tag{4.4} \|h(t)\|^2_{L_2(\Omega)} + \int_{kT}^t \|h(t')\|^2_{H^1(\Omega)}dt' \le \exp\left(\frac{4c_1c_2^2}{\nu}\|\nabla v\|_{L_2(kT,(k+1)T;L_3(\Omega))}\right) \cdot \left[\frac{2c_1c_2^2}{\nu}\|g\|^2_{L_2(kT,t;L_{6/5}(\Omega))} + \nu\|f_3\|^2_{L_2(kT,t;L_2(S_2))} + \|h(kT)\|^2_{L_2(\Omega)}\right].$$

Proof. Instead of (4.3) in this case we examine the inequality

$$\tag{4.5} \frac{d}{dt}\|h\|^2_{L_2(\Omega)} + \frac{\nu}{2c_1}\|h\|^2_{H^1(\Omega)} \le \frac{2c_1c_2^2}{\nu}(\|\nabla v\|^2_{L_3(\Omega)}\|h\|^2_{L_2(\Omega)} + \|g\|^2_{L_{6/5}(\Omega)}) + \nu\|f_3\|^2_{L_2(S_2)}.$$

From (4.5) we get

$$\frac{d}{dt}(\|h\|^2_{L_2(\Omega)}e^{-c_3\|\nabla v\|^2_{L_2(kT,t;L_3(\Omega))}}) + \frac{\nu}{2c_1}\|h\|^2_{H^1(\Omega)}e^{-c_3\|\nabla v\|^2_{L_2(kT,t;L_3(\Omega))}} \le (c_3\|g\|^2_{L_{6/5}(\Omega)} + \nu\|f_3\|^2_{L_2(S_2)})e^{-c_3\|\nabla v\|^2_{L_2(kT,t;L_3(\Omega))}},$$

where $c_3 = 2c_1c_2^2/\nu$ and $t \in (kT,(k+1)T]$.

Integrating the above inequality with respect to time yields

$$\|h(t)\|^2_{L_2(\Omega)} + e^{c_3\|\nabla v\|^2_{L_2(kT,t;L_3(\Omega))}}\int_{kT}^t \|h(t')\|^2_{H^1(\Omega)}e^{-c_3\|\nabla v\|^2_{L_2(kT,t';L_3(\Omega))}}\,dt' \le e^{c_3\|\nabla v\|^2_{L_2(kT,t;L_3(\Omega))}}\int_{kT}^t(c_3\|g(t')\|^2_{L_{6/5}(\Omega)} + \nu\|f_3(t')\|^2_{L_2(S_2)}) \cdot e^{-c_3\|\nabla v\|^2_{L_2(kT,t';L_3(\Omega)}}\,dt' + e^{c_3\|\nabla v\|^2_{L_2(kT,t;L_3(\Omega))}}\|h(kT)\|^2_{L_2(\Omega)}.$$

Simplifying, we obtain (4.4). □

Now we examine problem (3.16). Since we need to obtain energy type estimate for solutions to (3.16) we have to make the Dirichlet boundary condition homogeneous. For this purpose we introduce a function $\widetilde{\chi}$ as a solution to the problem

$$
\begin{aligned}
&\widetilde{\chi}_t - \nu\Delta\widetilde{\chi} = 0 && \text{in } \Omega\times(kT,(k+1)T),\\
&\widetilde{\chi} = \chi_* && \text{on } S_1\times(kT,(k+1)T),\\
&\widetilde{\chi}_{,x_3} = 0 && \text{on } S_2\times(kT,(k+1)T),\\
&\widetilde{\chi}|_{t=kT} = 0 && \text{in } \Omega.
\end{aligned}
\tag{4.6}
$$

From [27] we have:

LEMMA 4.3. *For solutions to problem* (4.6) *we have the estimates*

$$
\|\widetilde{\chi}\|_{L_q(kT,(k+1)T;L_p(\Omega))} \le c\|\chi_*\|_{L_q(kT,(k+1)T;L_p(S_1))}
\tag{4.7}
$$

for any $q,p\in[1,\infty]$,

$$
\|\widetilde{\chi}\|_{L_2(kT,(k+1)T;H^1(\Omega))} \le c\|\chi_*\|_{W_2^{1-1/2,1/2-1/4}(S_1^{(k+1)T})}.
$$

Then the new function $\chi' = \chi - \widetilde{\chi}$ *is a solution to the following problem*

$$
\begin{aligned}
&\chi'_{,t} + v\cdot\nabla\chi' - h_3\chi' + h_2v_{3,x_1} - h_1v_{3,x_2}\\
&\qquad - \nu\Delta\chi' = F_3 - v\cdot\nabla\widetilde{\chi} + h_3\widetilde{\chi} && \text{in } \Omega\times(kT,(k+1)T),\\
&\chi' = 0 && \text{on } S_1\times(kT,(k+1)T),\\
&\chi'_{,x_3} = 0 && \text{on } S_2\times(kT,(k+1)T),\\
&\chi'|_{t=kT} = \chi(kT) && \text{in } \Omega.
\end{aligned}
\tag{4.8}
$$

LEMMA 4.4. *Assume that* $h\in L_\infty(kT,(k+1)T;L_3(\Omega))$, $v'\in L_\infty(kT,(k+1)T;H^1(\Omega))\cap L_2(kT,(k+1)T;H^2(\Omega))\cap L_2(\Omega;H^{1/2}(kT,(k+1)T))$, $v'=(v_1,v_2)$, $\chi(kT)\in L_2(\Omega)$, $k\in\mathbb{N}_0$, $F_3\in L_2(kT,(k+1)T;L_{6/5}(\Omega))$. *Then solutions to problem* (3.16) *satisfy the inequality*

$$
\begin{aligned}
\|\chi\|_{V_2^0(\Omega\times(kT,t))} &\le \varphi(c_2,d_2)\sup_t\|h(t)\|_{L_3(\Omega)}\\
&\quad + \varepsilon^{-a}\varphi(c_2,c_4,d_2)\sup_t\|h(t)\|^2_{L_3(\Omega)} + \varepsilon^{-a}\varphi(c_2,c_4,d_1,d_2)\\
&\quad + \frac{8c_2}{\nu}\|F_3\|_{L_2(kT,t;L_{6/5}(\Omega))} + \varepsilon(\|v'\|_{L_\infty(kT,t;H^1(\Omega))}\\
&\quad + \|v'\|_{L_2(kT,t;H^2(\Omega))}) + c_4\|v'\|_{L_2(\Omega;H^{1/2}(kT,t))} + \|\chi(kT)\|_{L_2(\Omega)},
\end{aligned}
\tag{4.9}
$$

where $\varepsilon\in(0,1)$, $a>0$, $t\in(kT,(k+1)T]$, c_4 *is defined below in* (4.14), d_1, d_2, c_2 *in Lemma* 3.3 *and* φ *is a generic function which changes its form from formula to formula.*

Proof. Multiplying $(4.8)_1$ by χ', integrating over Ω, using boundary conditions $(4.8)_{2,3}$ yields

$$(4.10)\quad \frac{1}{2}\frac{d}{dt}\|\chi'\|^2_{L_2(\Omega)}+\nu\|\nabla\chi'\|^2_{L_2(\Omega)}=\int_\Omega h_3{\chi'}^2\,dx$$
$$-\int_\Omega(h_2v_{3,x_1}-h_1v_{3,x_2})\chi'\,dx+\int_\Omega F_3\chi'\,dx-\int_\Omega v\cdot\nabla\widetilde{\chi}\chi'dx+\int_\Omega h_3\widetilde{\chi}\chi'\,dx.$$

The first term on the r.h.s. of (4.10) can be bounded by

$$\bigg|\int_\Omega h_3{\chi'}^2\,dx\bigg|=\bigg|\int_\Omega h_3\chi'(\chi-\widetilde{\chi})\,dx\bigg|\le\bigg|\int_\Omega h_3\chi'\chi\,dx\bigg|+\bigg|\int_\Omega h_3\chi'\widetilde{\chi}\,dx\bigg|$$
$$\le\frac{\varepsilon_1}{4}\|\chi'\|^2_{L_6(\Omega)}+\frac{1}{\varepsilon_1}\|h_3\|^2_{L_3(\Omega)}\|\chi\|^2_{L_2(\Omega)}$$
$$+\frac{\varepsilon_1}{4}\|\chi'\|^2_{L_6(\Omega)}+\frac{1}{\varepsilon_1}\|h_3\|^2_{L_2(\Omega)}\|\widetilde{\chi}\|^2_{L_3(\Omega)}$$

the second by

$$\frac{\varepsilon_2}{2}\|\chi'\|^2_{L_6(\Omega)}+\frac{1}{2\varepsilon_2}\|h\|^2_{L_3(\Omega)}\|v_{3,x'}\|^2_{L_2(\Omega)},$$

and the third by

$$\frac{\varepsilon_3}{2}\|\chi'\|^2_{L_6(\Omega)}+\frac{1}{2\varepsilon_3}\|F_3\|^2_{L_{6/5}(\Omega)}.$$

The fourth term on the r.h.s. of (4.10) we express in the form

$$\int_\Omega v\cdot\nabla\chi'\widetilde{\chi}\,dx$$

and estimate as follows

$$\frac{\varepsilon_4}{2}\|\nabla\chi'\|^2_{L_2(\Omega)}+\frac{1}{2\varepsilon_4}\|v\|^2_{L_6(\Omega)}\|\widetilde{\chi}\|^2_{L_3(\Omega)}.$$

Finally, the last term on the r.h.s. of (4.10) is bounded by

$$\frac{\varepsilon_5}{2}\|\chi'\|^2_{L_6(\Omega)}+\frac{1}{2\varepsilon_5}\|h\|^2_{L_2(\Omega)}\|\widetilde{\chi}\|^2_{L_3(\Omega)}.$$

Using the above estimates in (4.10), setting $\varepsilon_1=\varepsilon_2=\varepsilon_3=\varepsilon_5=\varepsilon$, $\varepsilon_4=\nu/2$, $\varepsilon=\nu/(8c_2)$, where c_2 is introduced in Lemma 3.3, we obtain

$$(4.11)\quad \frac{d}{dt}\|\chi'\|^2_{L_2(\Omega)}+\nu\|\nabla\chi'\|^2_{L_2(\Omega)}\le\frac{8c_2}{\nu}(\|\chi\|^2_{L_2(\Omega)}+\|v_{3,x'}\|^2_{L_2(\Omega)})\|h\|^2_{L_3(\Omega)}$$
$$+\bigg(\frac{2}{\nu}\|v\|^2_{L_6(\Omega)}+\frac{16c_2}{\nu}\|h\|^2_{L_2(\Omega)}\bigg)\|\widetilde{\chi}\|^2_{L_3(\Omega)}+\frac{8c_2}{\nu}\|F_3\|^2_{L_{6/5}(\Omega)}.$$

Integrating (4.11) with respect to time and using the estimate for the weak solutions in Lemma 3.3 we obtain

$$(4.12)\quad \|\chi'\|^2_{L_2(\Omega)}+\nu\|\nabla\chi'\|^2_{L_2(kT,t;L_2(\Omega))}\le\frac{16c_2d_2^2}{\nu}\sup_t\|h(t)\|^2_{L_3(\Omega)}$$
$$+\frac{16c_2+2}{\nu}d_2^2\sup_t\|\widetilde{\chi}\|^2_{L_3(\Omega)}+\frac{8c_2}{\nu}\|F_3\|^2_{L_2(kT,t;L_{6/5}(\Omega))}+\|\chi(kT)\|^2_{L_2(\Omega)}.$$

Since $\chi = \chi' + \widetilde{\chi}$ we have

$$\begin{aligned}(4.13)\quad \|\chi\|^2_{V_2^0(\Omega\times(kT,t))} &\le \frac{16c_2}{\nu} d_2^2 \sup_t \|h(t)\|^2_{L_3(\Omega)} + \frac{16c_2+2}{\nu} d_2^2 \sup_t \|\widetilde{\chi}\|^2_{L_3(\Omega)} \\ &+ \frac{8c_2}{\nu}\|F_3\|^2_{L_2(kT,t;L_{6/5}(\Omega))} + \|\widetilde{\chi}\|^2_{V_2^0(\Omega\times(kT,t))} + \|\chi(kT)\|^2_{L_2(\Omega)},\end{aligned}$$

where $t \in (kT,(k+1)T]$.

In view of Lemma 4.3 and some interpolation inequalities (see [2, Chapter 3, Section 10]) we have

$$\begin{aligned}(4.14)\quad \|\widetilde{\chi}\|_{L_2(kT,t;L_2(\Omega))} &\le c_1'\|v'\|_{L_2(kT,t;L_2(S_1))} \\ &\le \varepsilon\|v'\|_{L_2(kT,t;H^1(\Omega))} + c_2'\varepsilon^{-1}d_2, \\ \|\widetilde{\chi}\|_{L_\infty(kT,t;L_3(\Omega))} &\le c_3'\|v'\|_{L_\infty(kT,t;L_3(S_1))} \\ &\le \varepsilon^{1/6}\|v'\|_{L_\infty(kT,t;H^1(\Omega))} + c_4'\varepsilon^{-5/6}d_1, \\ \|\widetilde{\chi}\|_{L_\infty(kT,t;L_2(\Omega))} &\le c_5'\|v'\|_{L_\infty(kT,t;L_2(S_1))} \\ &\le \varepsilon^{1/2}\|v'\|_{L_\infty(kT,t;H^1(\Omega))} + c_6'\varepsilon^{-1/2}d_1, \\ \|\nabla\widetilde{\chi}\|_{L_2(kT,t;L_2(\Omega))} &\le c_7'\|v'\|_{W_2^{1/2,1/4}(S_1\times(kT,t))} \\ &\le c_8'\|v'\|_{W_2^{1,1/2}(\Omega\times(kT,t))} \\ &= c_8'(\|v'\|_{L_2(kT,t;H^1(\Omega))} + \|v'\|_{L_2(\Omega;H^{1/2}(kT,t))}) \\ &\le \varepsilon^{1/2}\|v'\|_{L_2(kT,t;H^2(\Omega))} \\ &\quad + \varepsilon^{-1/2}c_9'd_2 + c_8'\|v'\|_{L_2(\Omega;H^{1/2}(kT,t))},\end{aligned}$$

where $c_1',\dots,c_9'$ are constants from corresponding imbedding theorems.

Let $c_i' \le c_4$, $i=1,\dots,9$. Using (4.14) and (4.13) we obtain (4.9). □

Let us consider the problem

$$\begin{aligned}(4.15)\quad v_{1,x_2} - v_{2,x_1} &= \chi && \text{in } \Omega', \\ v_{1,x_1} + v_{2,x_2} &= -h_3 && \text{in } \Omega', \\ v'\cdot\overline{n}' &= 0 && \text{on } S_1',\end{aligned}$$

where $\Omega' = \Omega \cap \{\text{plane} : x_3 = \text{const} \in (-a,a)\}$, $S_1' = S_1 \cap \{\text{plane} : x_3 = \text{const} \in (-a,a)\}$ and x_3, t are considered as parameters.

Lemma 4.5. *Let the assumptions of Lemmas* 4.1 *and* 4.4 *be satisfied. Then*

$$\begin{aligned}(4.16)\quad \|v'\|_{V_2^1(\Omega\times(kT,t))} &\le cd_2\|h\|_{L_\infty(kT,t;L_3(\Omega))} + \varphi(d_1,d_2)(d_1+d_2) \\ &+ c\|v'\|_{L_2(\Omega;H^{1/2}(kT,t))} + cK_1(k,T),\end{aligned}$$

where

$$\begin{aligned}(4.17)\quad K_1(k,T) &= \|g\|_{L_2(kT,(k+1)T;L_{6/5}(\Omega))} + \|F_3\|_{L_2(kT,(k+1)T;L_{6/5}(\Omega))} \\ &+ \|f_3\|_{L_2(S_2\times(kT,(k+1)T))} + \|h(kT)\|_{L_2(\Omega)} + \|\chi(kT)\|_{L_2(\Omega)},\end{aligned}$$

and we skip the dependence on c_1, c_2, c_4.

Proof. For solutions to problem (4.15) we have the estimate

$$\int_{-a}^{a} \|v'(x_3)\|^2_{V_2^1(\Omega'\times(kT,t))}\,dx_3 \le c(\|\chi\|^2_{V_2^0(\Omega\times(kT,t))} + \|h\|^2_{V_2^0(\Omega\times(kT,t))}). \tag{4.18}$$

Taking (4.18), estimate (4.1) and using (4.1) and (4.9) with sufficiently small ε we obtain (4.16). □

Let us consider problem (1.4) in the form

$$\begin{aligned} &v_t - \operatorname{div}\mathbb{T}(v,p) = -v'\cdot\nabla v - v_3 h + f && \text{in } \Omega^{(k+1)T},\\ &\operatorname{div} v = 0 && \text{in } \Omega^{(k+1)T},\\ &v\cdot\overline{n} = 0 && \text{on } S^{(k+1)T},\\ &\overline{n}\cdot\mathbb{D}(v)\cdot\overline{\tau}_\alpha + \gamma v\cdot\overline{\tau}_\alpha = 0, \quad \alpha=1,2, && \text{on } S^{(k+1)T},\\ &v|_{t=kT} = v(kT) && \text{in } \Omega. \end{aligned} \tag{4.19}$$

Lemma 4.6. *Let the assumptions of Lemmas* 4.1 *and* 4.4 *hold. Let* $h \in L_{10/3}(\Omega\times(kT,(k+1)T))$, $f \in L_2(\Omega\times(kT,(k+1)T))$, $v(kT)\in H^1(\Omega)$, $k\in\mathbb{N}_0$. *Then for solutions to problem* (4.19) *the following inequality is valid*

$$\begin{aligned} \|v\|_{W_2^{2,1}(\Omega^{kT,t})} + \|\nabla p\|_{L_2(\Omega^{kT,t})} \le{}& \varphi(d_1,d_2)[H(k,T)+K_2(k,T)]^2\\ &+ c(\|f\|_{L_2(\Omega^{kT,t})} + \|v(kT)\|_{H^1(\Omega)}), \end{aligned} \tag{4.20}$$

where c *does not depend on* T, *and* H, K_2 *are defined by* (4.23) *and* (4.24), *respectively. Moreover,* $k\in\mathbb{N}_0$.

Proof. From [23, Lemma 3.7] we have

$$\|v'\|_{L_{10}(\Omega^{kT,t})} \le c_5\|v'\|_{V_2^1(\Omega^{kT,t})},$$

hence

$$\begin{aligned} \|v'\cdot\nabla v\|_{L_{\frac{5}{3}}(\Omega^{kT,t})} &\le \|v'\|_{L_{10}(\Omega^{kT,t})}\|\nabla v\|_{L_2(\Omega^{kT,t})}\\ &\le d_2\|v'\|_{L_{10}(\Omega^{kT,t})} \le c_5 d_2\|v'\|_{V_2^1(\Omega^{kT,t})},\\ \|v_3 h\|_{L_{5/3}(\Omega^{kT,t})} &\le \|v_3\|_{L_{10/3}(\Omega^{kT,t})}\|h\|_{L_{10/3}(\Omega^{kT,t})} \end{aligned}$$

follows.

In view of the above estimates we obtain for solutions to (4.19) the inequality

$$\begin{aligned} \|v\|_{W_{5/3}^{2,1}(\Omega^{kT,t})} + \|\nabla p\|_{L_{5/3}(\Omega^{kT,t})} \le{}& c_6 d_2(c_5\|v'\|_{V_2^1(\Omega^{kT,t})}\\ &+ \|h\|_{L_{10/3}(\Omega^{kT,t})}) + c_6(\|f\|_{L_{5/3}(\Omega^{kT,t})} + \|v(kT)\|_{W_{5/3}^{4/5}(\Omega)}), \end{aligned} \tag{4.21}$$

where $\Omega^{kT,t} = \Omega\times(kT,t)$ and c_6 is the constant appearing in estimation of the nonstationary Stokes system corresponding to (4.19). In view of Lemma A.4 constant c_6 does not depend on T.

Employing (4.16) in the r.h.s. of (4.21) and using the interpolation

$$\|v'\|_{L_2(\Omega\times H^{1/2}(kT,t))} \le \varepsilon\|v'\|_{W^{2,1}_{5/3}(\Omega^{kT,t})} + c(1/\varepsilon)d_2$$

we obtain from (4.21) for sufficiently small ε the inequality

$$\|v\|_{W^{2,1}_{5/3}(\Omega^{kT,t})} + \|\nabla p\|_{L_{5/3}(\Omega^{kT,t})} \le cd_2H(k,T) + cK_2(k,T), \tag{4.22}$$

where c does not depend on T, with

$$\begin{aligned} K_2(k,T) = {}& K_1(k,T) + \varphi(d_1,d_2)(d_1+d_2) \\ &+ \|f\|_{L_{5/3}(\Omega^{kT,t})} + \|v(kT)\|_{W^{4/5}_{5/3}(\Omega)}, \end{aligned} \tag{4.23}$$

where K_1 is defined by (4.17) and

$$H(k,T) = \|h\|_{L_\infty(kT,t;L_3(\Omega))} + \|h\|_{L_{10/3}(\Omega^{kT,t})}. \tag{4.24}$$

From (4.22) and (4.16) we have

$$\|v\|_{W^{2,1}_{5/3}(\Omega^{kT,t})} + \|v'\|_{V^1_2(\Omega^{kT,t})} \le \varphi(d_1,d_2)[H(k,T)+K_2(k,T)]. \tag{4.25}$$

By imbedding theorems we obtain

$$\begin{aligned} \|v'\cdot\nabla v\|_{L_2(\Omega^{kT,t})} &\le \|v'\|_{L_{10}(\Omega^{kT,t})}\|\nabla v\|_{L_{5/2}(\Omega^{kT,t})} \\ &\le c\|v'\|_{V^1_2(\Omega^{kT,t})}\|v\|_{W^{2,1}_{5/3}(\Omega^{kT,t})} \\ &\le c\varphi(d_1,d_2)[H(k,T)+K_2(k,T)]^2 \end{aligned} \tag{4.26}$$

and

$$\begin{aligned} \|v_3h\|_{L_2(\Omega^{kT,t})} &\le \|v_3\|_{L_5(\Omega^{kT,t})}\|h\|_{L_{10/3}(\Omega^{kT,t})} \\ &\le c\|v\|_{W^{2,1}_{5/3}(\Omega^{kT,t})}\|h\|_{L_{10/3}(\Omega^{kT,t})} \\ &\le c\varphi(d_1,d_2)[H(k,T)+K_2(k,T)]H(k,T), \end{aligned} \tag{4.27}$$

where c does not depend on T (see Lemma A.3). In view of (4.26) and (4.27) we derive (4.20). □

Now we consider problem (3.15).

LEMMA 4.7. *Let the assumptions of Lemma* 4.6 *be satisfied. Let* $h(kT)\in H^1(\Omega)$, $g\in L(kT,(k+1)T;L_{6/5}(\Omega))$, $f_3\in L_2(kT,(k+1)T;L_2(S_2))$, $k\in\mathbb{N}_0$. *Then solutions to problem* (3.15) *satisfy the inequality*

$$\begin{aligned} &\|h\|_{W^{2,1}_2(\Omega^{kT,t})} + \|\nabla q\|_{L_2(\Omega^{kT,t})} \\ &\quad\le c[\varphi(d_1,d_2)(H+K_2)^8 + \|f\|^4_{L_2(\Omega^{kT,t})} + \|v(kT)\|^4_{H^1(\Omega)}] \\ &\qquad\cdot\|h\|_{L_2(\Omega^{kT,t})} + c(\|g\|_{L_2(\Omega^{kT,t})} + \|h(kT)\|_{H^1(\Omega)}), \end{aligned} \tag{4.28}$$

where $t \in (kT,(k+1)T]$ *and* c *does not depend on* T.

Proof. For solutions to problem (3.15) we have

$$\|h\|_{W_2^{2,1}(\Omega^{kT,t})} + \|\nabla q\|_{L_2(\Omega^{kT,t})} \le c_7(\|v\cdot\nabla h\|_{L_2(\Omega^{kT,t})} + \|h\cdot\nabla v\|_{L_2(\Omega^{kT,t})} + \|g\|_{L_2(\Omega^{kT,t})} + \|h(kT)\|_{H^1(\Omega)}), \tag{4.29}$$

where c_7 does not depend on T in view of Lemma A.4.

Given $v \in W_2^{2,1}(\Omega^{(k+1)T})$ we have

$$\begin{aligned}\|v\cdot\nabla h\|_{L_2(\Omega^{kT,t})} &\le \|v\|_{L_{10}(\Omega^{kT,t})}\|\nabla h\|_{L_{\frac{5}{2}}(\Omega^{kT,t})}\\ &\le c\|v\|_{W_2^{2,1}(\Omega^{kT,t})}(\varepsilon^{1/4}\|h\|_{W_2^{2,1}(\Omega^{kT,t})} + c\varepsilon^{-3/4}\|h\|_{L_2(\Omega^{kT,t})})\\ &\le \varepsilon_1^{1/4}\|h\|_{W_2^{2,1}(\Omega^{kT,t})} + c\varepsilon_1^{-3/4}\|v\|^4_{W_2^{2,1}(\Omega^{kT,t})}\|h\|_{L_2(\Omega^{kT,t})},\end{aligned} \tag{4.30}$$

where c does not depend on T in view of Lemmas A.3 and A.4.

Similarly, we have

$$\begin{aligned}\|h\cdot\nabla v\|_{L_2(\Omega^{kT,t})} &\le \|\nabla v\|_{L_{10/3}(\Omega^{kT,t})}\|h\|_{L_5(\Omega^{kT,t})}\\ &\le c\|v\|_{W_2^{2,1}(\Omega^{kT,t})}(\varepsilon^{1/4}\|h\|_{W_2^{2,1}(\Omega^{kT,t})} + c\varepsilon^{-3/4}\|h\|_{L_2(\Omega^{kT,t})})\\ &\le \varepsilon_2^{1/4}\|h\|_{W_2^{2,1}(\Omega^{kT,t})} + c\varepsilon_2^{-3/4}\|v\|^4_{W_2^{2,1}(\Omega^{kT,t})}\|h\|_{L_2(\Omega^{kT,t})},\end{aligned} \tag{4.31}$$

where c does not depend on T by Lemmas A.3, A.4.

Using (4.30) and (4.31) in (4.29), assuming that $\varepsilon_1, \varepsilon_2$ are sufficiently small and (4.20) holds we obtain (4.28). □

Let us introduce the notation

$$\begin{aligned}X(kT,t) &= \|h\|_{W_2^{2,1}(\Omega^{kT,t})},\\ d(kT,t) &= \|g\|_{L_2(kT,t;L_{6/5}(\Omega))} + \|f_3\|_{L_2(kT,t;L_2(S_2))} + \|h(kT)\|_{L_2(\Omega)},\\ K_3(kT,t) &= \varphi(d_1,d_2)K_2^2(kT,t) + \|f\|_{L_2(\Omega^{kT,t})} + \|v(kT)\|_{H^1(\Omega)},\\ K_4(kT,t) &= \|g\|_{L_2(\Omega^{kT,t})} + \|h(kT)\|_{H^1(\Omega)}.\end{aligned} \tag{4.32}$$

The above quantities are denoted by $X(kT)$, $d(kT)$, $K_3(kT)$, $K_4(kT)$ for $t = (k+1)T$.

Lemma 4.8. *Assume that the quantities* $d(kT)$, $K_i(kT)$, $i = 2,3,4$, *are finite. Assume that* $d(kT)$ *is sufficiently small. Then there exists a constant* $A(k,T)$ *such that*

$$X(kT,t) \le A(k,T). \tag{4.33}$$

Proof. We shall denote by c constants independent of T. Since

$$\|\nabla v\|_{L_2(kT,(k+1)T;L_3(\Omega))} \le c\|v\|_{W_{5/3}^{2,1}(\Omega^{kT})}$$

we have by Lemma 4.2 and (4.22) that

$$\|h\|_{L_2(\Omega^{kT,t})} \le \exp(cd_2H + K_2)d(kT). \tag{4.34}$$

Using that $H(kT,t) \le cX(kT,t)$ we obtain from (4.28) the inequality

$$\begin{aligned} X(kT,t) \le c[\varphi(d_1,d_2)X^8(kT,t) + K_3^4(kT,t)] \\ \cdot \exp(cd_2X(kT,t) + K_2(kT,t))d(kT,t) + K_4(kT,t). \end{aligned} \tag{4.35}$$

To show that there exists a constant A such that $X(kT,t) \le A(k,T)$ we have to satisfy the inequality

$$c[\varphi(d_1,d_2)A^8 + K_3^4]\exp(cd_2A)\exp(K_2)d(kT) + K_4 \le A. \tag{4.36}$$

Inequality (4.36) holds for $d(kT)$ sufficiently small and A sufficiently large ($A > K_4$). In this case estimate (4.33) holds. □

From [15] we have

REMARK 4.9. To simplify (4.36) we estimate K_2 and K_3 by $\varphi(d_1,d_2)K$ and $\varphi(d_1,d_2)K^2$, respectively, where

$$\begin{aligned} K(kT,t) = \|g\|_{L_2(kT,t;L_{6/5}(\Omega))} + \|F_3\|_{L_2(kT,t;L_{6/5}(\Omega))} \\ + \|f_3\|_{L_2(S_2^{kT,t})} + \|f\|_{L_2(\Omega^{kT,t})} + \|v(kT)\|_{H^1(\Omega)} + d_1 + d_2. \end{aligned}$$

Then (4.36) takes the form

$$\varphi(d_1,d_2)[A^8 + K^8]\exp(cd_2A)\exp(\varphi(d_1,d_2)K)d(kT) + K_4 \le A. \tag{4.37}$$

To show the existence of A satisfying (4.37) we use the method of successive approximations

$$A_{n+1} = \varphi[A_n^8 + K^8]\exp(cd_2A_n)\exp(\varphi K)d + K_4. \tag{4.38}$$

The sequence $\{A_n\}$ converges and

$$\lim_{n\to\infty} A_n = A \le (\eta+1)K_4 \tag{4.39}$$

for any $\eta > 0$, if d is so small that

$$d \le \frac{1}{cd_2\varphi[(\eta+1)^8K_4^8 + K^8]\exp(cd_2(\eta+1)K_4)\exp(\varphi K)}. \tag{4.40}$$

Choosing η large we have A large but then d must be correspondingly small.

5. Existence

To prove the existence of solutions to problem (1.4) we construct the mappings

$$
\begin{aligned}
&v_t - \operatorname{div}\mathbb{T}(v,p) = -\lambda\widetilde{v}\cdot\nabla\widetilde{v} + f && \text{in } \Omega^{(k+1)T},\\
&\operatorname{div} v = 0 && \text{in } \Omega^{(k+1)T},\\
&v\cdot\overline{n} = 0 && \text{on } S^{(k+1)T},\\
&\nu\overline{n}\cdot\mathbb{D}(v)\cdot\overline{\tau}_\alpha + \gamma v\cdot\overline{\tau}_\alpha = 0, \quad \alpha = 1,2, && \text{on } S^{(k+1)T},\\
&v|_{t=kT} = v(kT) && \text{in } \Omega,
\end{aligned}
\tag{5.1}
$$

and

$$
\begin{aligned}
&h_t - \operatorname{div}\mathbb{T}(h,q) = -\lambda(\widetilde{v}\cdot\nabla\widetilde{h} + \widetilde{h}\cdot\nabla\widetilde{v}) + g && \text{in } \Omega^{(k+1)T},\\
&\operatorname{div} h = 0 && \text{in } \Omega^{(k+1)T},\\
&h\cdot\overline{n} = 0, \quad \nu\overline{n}\cdot\mathbb{D}(h)\cdot\overline{\tau}_\alpha + \gamma h\cdot\overline{\tau}_\alpha = 0, \quad \alpha = 1,2, && \text{on } S_1^{(k+1)T},\\
&h_i = 0, \quad i = 1,2, \quad h_{3,x_3} = 0 && \text{on } S_2^{(k+1)T},\\
&h|_{t=kT} = h(kT) && \text{in } \Omega,
\end{aligned}
\tag{5.2}
$$

where $\lambda \in [0,1]$ and $\widetilde{v}$, $\widetilde{h}$ are considered as given functions.

Moreover, we assume that $\widetilde{h} = \widetilde{v}_{,x_3}$, $g = f_{,x_3}$, $h(kT) = v(kT)_{,x_3}$. Differentiating (5.1) with respect to x_3 and subtracting from (5.2) we obtain $h = v_{,x_3}$. Problems (5.1), (5.2) determine the mappings

$$
\Phi_1\colon (\widetilde{v},\lambda) \to (v,p), \qquad \Phi_2\colon (\widetilde{v},\widetilde{h},\lambda) \to (h,q).
$$

Let $\Phi = (\Phi_1,\Phi_2)$. In Section 4 we found a priori estimate for a fixed point of Φ for $\lambda = 1$.

For $\lambda = 0$ we have a unique existence of solutions to problems (5.1) and (5.2).

Let us introduce the space

$$
\mathcal{M}(\Omega^{(k+1)T}) = L_{2r}(kT,(k+1)T; W^2_{6\eta/(3+\eta)}(\Omega)), \quad \eta \ge 2,\ r \ge 2.
$$

We shall find restrictions on r,η such that

$$
\Phi\colon \mathcal{M}(\Omega^{(k+1)T}) \times \mathcal{M}(\Omega^{(k+1)T}) \to \mathcal{M}(\Omega^{(k+1)T}) \times \mathcal{M}(\Omega^{(k+1)T})
$$

is a compact mapping.

Assume that $\widetilde{v} \in L_{2r}(kT,(k+1)T;W^1_{6\eta/(3+\eta)}(\Omega))$. Then

$$\begin{aligned}(5.3)\qquad \|\widetilde{v}\cdot\nabla\widetilde{v}\|_{L_r(kT,(k+1)T;L_\eta(\Omega))} &= \bigg(\int_{kT}^{(k+1)T} dt\|\widetilde{v}\cdot\nabla\widetilde{v}\|^r_{L_\eta(\Omega)}\bigg)^{1/r}\\ &\le \bigg(\int_{kT}^{(k+1)T} dt\|\widetilde{v}\|^r_{L_{6\eta/(3-\eta)}(\Omega)}\|\nabla\widetilde{v}\|^r_{L_{6\eta/(3+\eta)}(\Omega)}\bigg)^{1/r}\\ &\le c\bigg(\int_{kT}^{(k+1)T} dt\|\widetilde{v}\|^{2r}_{W^1_{6\eta/(3+\eta)}(\Omega)}\bigg)^{1/r}\\ &\le c\|\widetilde{v}\|^2_{L_{2r}(kT,(k+1)T;W^1_{6\eta/(3+\eta)}(\Omega))}\end{aligned}$$

In the same way we obtain

$$\begin{aligned}(5.4)\quad &\|\widetilde{v}\cdot\nabla\widetilde{h}\|_{L_r(kT,(k+1)T;L_\eta(\Omega))}+\|\widetilde{h}\cdot\nabla\widetilde{v}\|_{L_r(kT,(k+1)T;L_\eta(\Omega))}\\ &\qquad\le c\|\widetilde{v}\|_{L_{2r}(kT,(k+1)T;W^1_{6\eta/(3+\eta)}(\Omega))}\|\widetilde{h}\|_{L_{2r}(kT,(k+1)T;W^1_{6\eta/(3+\eta)}(\Omega))}.\end{aligned}$$

In view of (5.3) and (5.4) we obtain that solutions to problems (5.1) and (5.2) belong to $W^{2,1}_{\eta,r}(\Omega^{kT})$ (see [18]).

We use the imbeddings (see [2, Chapter 3, Section 10])

$$(5.5)\qquad W^{2,1}_2(\Omega^{kT})\supset W^{2,1}_{\eta,r}(\Omega^{kT})$$

and

$$(5.6)\qquad W^{2,1}_2(\Omega^{kT})\subset L_{2r}(kT,(k+1)T;W^1_{6\eta/(3+\eta)}(\Omega))\equiv\mathcal{M}(\Omega^{kT}),$$

where (5.5) holds for $\eta\ge2$, $r\ge2$ and (5.6) is compact for r,η satisfying the inequality

$$\frac{5}{2}-\frac{3}{6\eta/(3+\eta)}-\frac{2}{2r}<1$$

which takes the form

$$(5.7)\qquad 1<\frac{3}{2\eta}+\frac{1}{r}.$$

Setting $r=\eta=2$ we obtain that $\widetilde{v},\widetilde{h}\in L_4(kT,(k+1)T;W^1_{12/5}(\Omega))$ and then condition (5.7) takes the form

$$(5.8)\qquad 1<\frac{3}{4}+\frac{1}{2}\quad\text{so}\quad\frac{1}{2}<\frac{3}{4}.$$

Hence, we have the compactness of mappings Φ_1 and Φ_2.

To show the continuity of mappings Φ_1 and Φ_2 we consider

$$(5.9)\qquad\begin{aligned}&\begin{cases}v_{st}-\operatorname{div}\mathbb{T}(v_s,p_s)=-\lambda\widetilde{v}_s\cdot\nabla\widetilde{v}_s+f\\ \operatorname{div}v_s=0\end{cases} &&\text{in } \Omega^{(k+1)T},\\ &\overline{n}\cdot v_s=0,\quad \nu\overline{n}\cdot\mathbb{D}(v_s)\cdot\overline{\tau}_\alpha+\gamma v_s\cdot\overline{\tau}_\alpha=0 &&\text{on } S^{(k+1)T},\\ &v_s|_{t=kT}=v(kT) &&\text{in } \Omega,\end{aligned}$$

and

$$
(5.10)\qquad
\begin{cases} h_{st} - \operatorname{div} \mathbb{T}(h_s, q_s) = -\lambda(\widetilde{h}_s \cdot \nabla \widetilde{v}_s + \widetilde{v}_s \cdot \nabla \widetilde{h}_s) + g \\ \operatorname{div} h_s = 0 \end{cases} \quad \text{in } \Omega^{(k+1)T},
$$

$$
\begin{aligned}
&\overline{n} \cdot h_s = 0, \quad \nu\overline{n} \cdot \mathbb{D}(h_s) \cdot \overline{\tau}_\alpha + \gamma h_s \cdot \overline{\tau}_\alpha = 0, \quad \alpha = 1,2 && \text{on } S_1^{(k+1)T},\\
&h_{si} = 0, \quad i = 1,2, \quad h_{s3,x_3} = 0 && \text{on } S_2^{(k+1)T},\\
&h_s|_{t=kT} = h(kT) && \text{in } \Omega,
\end{aligned}
$$

where $s = 1, 2$.

Let

$$
(5.11)\qquad V = v_1 - v_2, \quad H = h_1 - h_2, \quad P = p_1 - p_2, \quad Q = q_1 - q_2.
$$

Then V and H are solutions to the problems

$$
(5.12)\qquad
\begin{aligned}
&V_t - \operatorname{div} \mathbb{T}(V, P) = -\lambda(\widetilde{V} \cdot \nabla \widetilde{v}_1 + \widetilde{v}_2 \cdot \nabla \widetilde{V}),\\
&\operatorname{div} V = 0,\\
&V \cdot \overline{n}|_S = 0, \quad \nu\overline{n} \cdot \mathbb{D}(V) \cdot \overline{\tau}_\alpha + \gamma V \cdot \overline{\tau}_\alpha|_S = 0, \quad \alpha = 1,2,\\
&V|_{t=0} = 0,
\end{aligned}
$$

and

$$
(5.13)\qquad
\begin{aligned}
&H_t - \operatorname{div} \mathbb{T}(H, Q) = -\lambda(\widetilde{H} \cdot \nabla \widetilde{v}_1 + \widetilde{h}_2 \cdot \nabla \widetilde{V} + \widetilde{V} \cdot \nabla \widetilde{h}_1 + \widetilde{v}_2 \cdot \nabla \widetilde{H}),\\
&\operatorname{div} H = 0,\\
&H \cdot \overline{n}|_{S_1} = 0, \quad \nu\overline{n} \cdot \mathbb{D}(H) \cdot \overline{\tau}_\alpha + \gamma H \cdot \overline{\tau}_\alpha|_{S_1} = 0, \quad \alpha = 1,2,\\
&H_i|_{S_2} = 0, \quad i = 1,2, \quad H_{3,x_3}|_{S_2} = 0,\\
&H|_{t=kT} = 0.
\end{aligned}
$$

Assume that $\lambda \neq 0$. Then for solutions of (5.12) we have

$$
(5.14)\qquad
\begin{aligned}
\|V\|_{\mathcal{M}(\Omega^{(k+1)T})} &= \|V\|_{L_{2r}(kT,(k+1)T;W^1_{6\eta/(3+\eta)}(\Omega))}\\
&\le c\,\|V\|_{W_2^{2,1}(\Omega^{(k+1)T})} \le c\|V\|_{W^{2,1}_{\eta,r}(\Omega^{(k+1)T})}\\
&\le c \sum_{s=1}^{2} \|\widetilde{v}_s\|_{L_{2r}(kT;(k+1)T;W^1_{6\eta/(3+\eta)}(\Omega))}\\
&\quad \cdot \|\widetilde{V}\|_{L_{2r}(kT;(k+1)T;W^1_{6\eta/(3+\eta)}(\Omega))}\\
&\le c(A)\|\widetilde{V}\|_{\mathcal{M}(\Omega^{(k+1)T})},
\end{aligned}
$$

where $r \ge 2$, $\eta \ge 2$ and satisfy either (5.7) or (5.8).

Similarly we have

$$
(5.15)\qquad \|H\|_{\mathcal{M}(\Omega^{(k+1)T})} \le c(A)(\|\widetilde{V}\|_{\mathcal{M}(\Omega^{(k+1)T})} + \|\widetilde{H}\|_{\mathcal{M}(\Omega^{(k+1)T})}),
$$

where A is the constant from Lemma 4.8.

Inequalities (5.14) and (5.15) imply the continuity of mapping Φ. Continuity with respect to λ is evident. Hence by the Leray-Schauder fixed point theorem we have the existence of solutions to problem (1.1) such that $v \in W_2^{2,1}(\Omega^{(k+1)T})$, $\nabla p \in L_2(\Omega^{(k+1)T})$. This concludes the proof of Theorem A.

6. Global existence

To prove the global existence of solutions to problem (1.1) we have to show that the constant A appearing in Lemma 4.8 does not depend on k. For this purpose we have to show that, for any $k \in \mathbb{N}_0$,

$$\|v((k+1)T)\|_{H^1(\Omega)} \le \|v(kT)\|_{H^1(\Omega)}, \tag{6.1}$$

$$\|h((k+1)T)\|_{H^1(\Omega)} \le \|h(kT)\|_{H^1(\Omega)}. \tag{6.2}$$

To show (6.1) we need

LEMMA 6.1. *Assume that there exists a local solution to problem* (1.4) *in the interval* $[kT,(k+1)T]$. *Then there exist constants* c_1', c_3', c_4' *independent of* T *such that*

$$\begin{aligned}\|v((k+1)T)\|^2_{H^1(\Omega)} \le{}& c_3' e^{-c_1'(k+1)T+c_4'\varphi(A(k,T))} \int_{kT}^{(k+1)T} \|f(t)\|^2_{L_2(\Omega)} e^{c_1' t}\,dt \\ &+ c_3' e^{-c_1' T + c_4'\varphi(A(k,T))} \|v(kT)\|^2_{H^1(\Omega)}.\end{aligned} \tag{6.3}$$

Assume that

$$\|f(t)\|_{L_2(\Omega)} \le \|f(kT)\|_{L_2(\Omega)} e^{-\delta(t-kT)}, \tag{6.4}$$

$t \in (kT,(k+1)T]$, $\delta > 0$. *Then* (6.3) *implies*

$$\begin{aligned}\|v((k+1)T)\|^2_{H^1(\Omega)} \le{}& c_3' e^{-2\delta T + c_4'\varphi(A(k,T))} \|f(kT)\|^2_{L_2(\Omega)} \\ &+ c_3' e^{-c_1' T + c_4'\varphi(A(k,T))} \|v(kT)\|^2_{H^1(\Omega)}.\end{aligned} \tag{6.5}$$

PROOF. To prove the lemma we use problem (1.4) in the form

$$\begin{aligned}&v_t - \operatorname{div} \mathbb{T}(v,p) = -v\cdot\nabla v + f,\\ &\operatorname{div} v = 0,\\ &v\cdot\overline{n}|_S = 0, \quad \overline{n}\cdot\mathbb{T}(v,p)\cdot\overline{\tau}_\alpha + \gamma v\cdot\overline{\tau}_\alpha|_S = 0, \quad v|_{t=kT} = v(kT).\end{aligned} \tag{6.6}$$

Multiplying (6.6) by $\operatorname{div}\mathbb{T}(v,p)$ and integrating the result over Ω yields

$$\begin{aligned}&\int_\Omega v_t\cdot \operatorname{div}\mathbb{T}(v,p)\,dx - \int_\Omega |\operatorname{div}\mathbb{T}(v,p)|^2\,dx \\ &\quad = -\int_\Omega v\cdot\nabla v\cdot\operatorname{div}\mathbb{T}(v,p)dx + \int_\Omega f\cdot\operatorname{div}\mathbb{T}(v,p)\,dx.\end{aligned} \tag{6.7}$$

Integrating by parts the first integral leads to

$$\begin{aligned}(6.8)\qquad \int_\Omega v_{i,t}T_{ij}(v,p)_{,x_j}\,dx &= \int_\Omega (v_{i,t}T_{ij}(v,p))_{,x_j}\,dx - \int_\Omega v_{i,x_jt}T_{ij}(v,p)\,dx\\ &= \int_S v_{i,t}n_jT_{ij}(v,p)\,dS - \int_\Omega v_{i,x_jt}D_{ij}(v)\,dx\\ &= \int_S (v_{\tau_\alpha t}\tau_{\alpha i} + v_{nt}n_i)n_jT_{ij}(v,p)\,dS - \frac{1}{2}\int_\Omega D_{ij}(v_t)D_{ij}(v)\,dx\\ &= -\gamma\int_{S_1} v_{\tau_\alpha t}v_{\tau_\alpha}\,dS_1 - \frac{1}{4}\frac{d}{dt}\int_\Omega |\mathbb{D}(v)|^2\,dx,\end{aligned}$$

where we used that $v_{\tau_\alpha} = v\cdot\overline{\tau}_\alpha$, $v_n = v\cdot\overline{n}$, $v = v_{\tau_\alpha}\overline{\tau}_\alpha + v_n\overline{n}$ in a neighbourhood of S,

$$\mathbb{D}(v) = \{D_{ij}(v)\}_{i,j=1,2,3},\quad \mathbb{T}(v,p) = \{T_{ij}(v,p)\}_{i,j=1,2,3}$$

and the summation convention over the repeated indices is assumed: $i,j=1,2,3$, $\alpha = 1,2$.

Using (6.8) in (6.7), and applying the Hölder and the Young inequalities to the integrals on the r.h.s. of (6.7), we obtain

$$\begin{aligned}(6.9)\qquad \frac{d}{dt}\bigg(\frac{1}{4}\int_\Omega |\mathbb{D}(v)|^2\,dx + \gamma\int_{S_1}|v\cdot\overline{\tau}_\alpha|^2\,dS_1\bigg) + \frac{1}{2}\int_\Omega |\operatorname{div}\mathbb{T}(v,p)|^2\,dx\\ \le \int_\Omega |v\cdot\nabla v|^2 + \int_\Omega f^2\,dx.\end{aligned}$$

Using Lemma A.5 and the Korn inequality in Lemma 3.2 we obtain

$$\begin{aligned}(6.10)\qquad &\frac{d}{dt}\bigg(\frac{1}{4}\int_\Omega |\mathbb{D}(v)|^2\,dx + \gamma\int_{S_1}|v\cdot\overline{\tau}_\alpha|^2\,dS_1\bigg)\\ &+ c_1'\bigg(\frac{1}{4}\int_\Omega |\mathbb{D}(v)|^2\,dx + \gamma\int_{S_1}|v\cdot\overline{\tau}_\alpha|^2\,dS_1\bigg)\\ &\le c_2'\|v\|^2_{L_\infty(\Omega)}\bigg(\frac{1}{4}\int_\Omega |\mathbb{D}(v)|^2\,dx + \gamma\int_{S_1}|v\cdot\overline{\tau}_\alpha|^2\,dS_1\bigg) + c_2'\|f\|^2_{L_2(\Omega)},\end{aligned}$$

where c_1', c_2' do not depend on t.

Let

$$(6.11)\qquad X(t) = \frac{1}{4}\|\mathbb{D}(v)\|^2_{L_2(\Omega)} + \gamma\|v\cdot\overline{\tau}_\alpha\|^2_{L_2(S_1)}.$$

Then from (6.10) we have

$$(6.12)\quad \frac{d}{dt}\big(X(t)e^{c_1't - c_2'\int_{kT}^t \|v(t')\|^2_{L_\infty(\Omega)}\,dt'}\big) \le c_2'\|f\|^2_{L_2(\Omega)}e^{c_1't - c_2'\int_{kT}^t \|v(t')\|^2_{L_\infty(\Omega)}\,dt'},$$

where $t\in(kT,(k+1)T]$. Integrating (6.12) with respect to time yields

$$\begin{aligned}(6.13)\quad X(t) \le e^{-c_1't + \int_{kT}^t \alpha\,dt'}c_2'\int_{kT}^t \|f\|^2_{L_2(\Omega)}e^{c_1't' - \int_{kT}^{t'}\alpha\,dt''}\,dt'\\ + e^{-c_1'(t-kT) + \int_{kT}^t \alpha dt'}X(kT),\end{aligned}$$

where $\alpha = c_2'\|v(t)\|^2_{L_\infty(\Omega)}$.

First we consider the case

$$\|f(t)\|^2_{L_2(\Omega)} \le \|f(kT)\|^2_{L_2(\Omega)} e^{-2\delta(t-kT)} \tag{6.14}$$

Then (6.13) implies

$$X(t) \le e^{2\delta kT - c_1't + \int_{kT}^t \alpha dt'} \|f(kT)\|^2_{L_2(\Omega)} \int_{kT}^t e^{(c_1'-2\delta)t'}\,dt' + e^{-c_1'(t-kT)+\int_{kT}^t \alpha dt'} X(kT). \tag{6.15}$$

Hence

$$X(t) \le e^{-2\delta(t-kT)+\int_{kT}^t \alpha(t')\,dt'} \frac{1}{c_1'-2\delta} \|f(kT)\|^2_{L_2(\Omega)} + e^{-c_1'(t-kT)+\int_{kT}^t \alpha\,dt'} X(kT). \tag{6.16}$$

Setting $t=(k+1)T$ and using the Korn inequality we obtain from (6.16) the relation

$$\|v((k+1)T)\|^2_{H^1(\Omega)} \le c_3' e^{-2\delta T+\int_{kT}^{(k+1)T} \|v(t)\|^2_{L_\infty(\Omega)}\,dt} \frac{1}{c_1'-2\delta} \|f(kT)\|^2_{L_2(\Omega)} + c_3' e^{-c_1'T+\int_{kT}^{(k+1)T} \|v(t)\|^2_{L_\infty(\Omega)}\,dt} \|v(kT)\|^2_{H^1(\Omega)}. \tag{6.17}$$

In view of Theorem A and the equivalence of $X(t)$ and $\|v(t)\|_{H^1(\Omega)}$ we obtain (6.5).

Assume that (6.14) does not hold. Then (6.13) takes the form

$$X(t) \le c_2' e^{-c_1 t+\int_{kT}^t \alpha dt'} \int_{kT}^t \|f(t')\|^2_{L_2(\Omega)} e^{c_1't'}\,dt' + e^{-c_1'(t-kT)+\int_{kT}^t \alpha\,dt'} X(kT). \tag{6.18}$$

Setting $t=(k+1)T$, using (6.11) and Theorem A we obtain (6.3). □

To show (6.2) we need

LEMMA 6.2. *Assume that $f_3|_{S_2}=0$. Assume that there exists a local solution to problem* (1.4) *in the interval* $[kT,(k+1)T]$. *Then there exist constants c_1', c_2', c_3' such that*

$$\|h((k+1)T)\|^2_{H^1(\Omega)} \le c_3' e^{-c_1'(k+1)T+c_4'\varphi(A(kT))} \cdot \int_{kT}^{(k+1)T} \|g(t)\|^2_{L_2(\Omega)} e^{c_1't}\,dt + c_3' e^{-c_1'T+c_4'\varphi(A(kT))} \|h(kT)\|^2_{H^1(\Omega)}. \tag{6.19}$$

Assume that, for $t\in(kT,(k+1)T]$, $\delta>0$,

$$\|g(t)\|_{L_2(\Omega)} \le \|g(kT)\|_{L_2(\Omega)} e^{-\delta(t-kT)}. \tag{6.20}$$

Then

$$\|h((k+1)T)\|_{H^1(\Omega)}^2 \le c_3' e^{-2\delta T + c_4'\varphi(A(k,T))}\|g(kT)\|_{L_2(\Omega)}^2 + c_3' e^{-c_1' T + c_4'\varphi(A(k,T))}\|h(kT)\|_{H^1(\Omega)}^2. \tag{6.21}$$

Proof. Let us consider problem (3.15). Multiplying $(3.15)_1$ by $\operatorname{div}\mathbb{T}(h,q)$ and integrating over Ω yields

$$\int_\Omega h_t\cdot\operatorname{div}\mathbb{T}(h,q)dx - \int_\Omega |\operatorname{div}\mathbb{T}(h,q)|^2\,dx = -\int_\Omega (v\cdot\nabla h + h\cdot\nabla v)\operatorname{div}\mathbb{T}(h,q)\,dx + \int_\Omega f\cdot\operatorname{div}\mathbb{T}(h,q)\,dx. \tag{6.22}$$

Repeating the considerations from (6.8) we have

$$\int_\Omega h_t\cdot\operatorname{div}\mathbb{T}(h,q)\,dx = -\gamma\int_{S_1} h_{\tau_\alpha,t}h_{\tau_\alpha}\,dS_1 - \frac{1}{4}\frac{d}{dt}\int_\Omega|\mathbb{D}(h)|^2\,dx + \int_{S_2} h_{3,t}T_{33}(h,q)\,dS_2,$$

where the last term equals

$$\int_{S_2} h_{3,t}(2h_{3,x_3}+q)\,dS_2 = \int_{S_2} h_{3,t}q\,dS_2 = \int_{S_2} h_{3,t}f_3\,dS_2.$$

We do not know how to cope with this term. Therefore, we assume that

$$f_3|_{S_2} = 0. \tag{6.23}$$

Then (6.22) implies

$$\frac{d}{dt}\left[\frac{1}{4}\int_\Omega|\mathbb{D}(h)|^2\,dx + \gamma\int_{S_1}|h\cdot\overline{\tau}_\alpha|^2\,dS_1\right] + \|\operatorname{div}\mathbb{T}(h,q)\|_{L_2(\Omega)}^2 \le (\|v\|_{L_\infty(\Omega)}\|\nabla h\|_{L_2(\Omega)} + \|\nabla v\|_{L_3(\Omega)}\|h\|_{L_6(\Omega)})\|\operatorname{div}\mathbb{T}(h,q)\|_{L_2(\Omega)} + \|g\|_{L_2(\Omega)}\|\operatorname{div}\mathbb{T}(h,q)\|_{L_2(\Omega)}. \tag{6.24}$$

From (6.24) we have

$$\frac{d}{dt}\left[\frac{1}{4}\int_\Omega|\mathbb{D}(h)|^2\,dx + \gamma\int_{S_1}|h\cdot\overline{\tau}_\alpha|^2\,dS_1\right] + \frac{1}{2}\int_\Omega|\operatorname{div}\mathbb{T}(h,q)|^2\,dx \le c(\|v\|_{L_\infty(\Omega)}^2 + \|\nabla v\|_{L_3(\Omega)}^2)\|h\|_{H^1(\Omega)}^2 + \|g\|_{L_2(\Omega)}^2. \tag{6.25}$$

Using Lemma A.5 and the Korn inequality yields

$$\frac{d}{dt}X + c_1'X \le c_2'(\|v\|_{L_\infty(\Omega)}^2 + \|\nabla v\|_{L_3(\Omega)}^2)X + c_2'\|g\|_{L_2(\Omega)}^2, \tag{6.26}$$

where

$$X = \frac{1}{4}\|\mathbb{D}(h)\|_{L_2(\Omega)}^2 + \gamma\|h_{\tau_\alpha}\|_{L_2(S_1)}^2, \tag{6.27}$$

and c_1', c_2' do not depend on t.

Using that

$$\int_{kT}^{(k+1)T} (\|v(t)\|^2_{L_\infty(\Omega)} + \|\nabla v(t)\|^2_{L_3(\Omega)})\,dt \le \varphi(A(k,T)),$$

and repeating the considerations from the proof of Lemma 6.1 we conclude the proof of Lemma 6.2. □

To prove global existence of solution to problem (1.1) we have to show inequalities (6.1) and (6.2) for any $k \in \mathbb{N}_0$. For this purpose we use Lemmas 6.1 and 6.2. Let us introduce the assumptions

ASSUMPTION 6.3. *Assume that*

$$\begin{aligned}(6.28)\quad & c_3' e^{-c_1'(k+1)T + c_4'\varphi(A(T))} \int_{kT}^{(k+1)T} \|f(t)\|^2_{L_2(\Omega)} e^{c_1' t}\,dt \\ &\quad + e^{-c_1'T + c_4'\varphi(A(T))} \|v(0)\|^2_{H^1(\Omega)} \le \|v(0)\|^2_{H^1(\Omega)},\end{aligned}$$

and

$$\begin{aligned}(6.29)\quad & c_3' e^{-c_1'(k+1)T + c_4'\varphi(A(T))} \int_{kT}^{(k+1)T} \|g(t)\|^2_{L_2(\Omega)} e^{c_1' t}\,dt \\ &\quad + e^{-c_1'T + c_4'\varphi(A(T))} \|h(0)\|^2_{H^1(\Omega)} \le \|h(0)\|^2_{H^1(\Omega)}.\end{aligned}$$

ASSUMPTIONS 6.4. *Assume that*

$$(6.30)\qquad \begin{aligned}\|f(t)\|_{L_2(\Omega)} &\le \|f(kT)\|_{L_2(\Omega)} e^{-\delta(t-kT)},\\ \|g(t)\|_{L_2(\Omega)} &\le \|g(kT)\|_{L_2(\Omega)} e^{-\delta(t-kT)},\end{aligned}$$

for $\delta > 0$, $t \in (kT, (k+1)T]$, $k \in \mathbb{N}_0$. Assume also

$$\begin{aligned}(6.31)\quad & c_3' e^{-2\delta T + c_4'\varphi(A(T))} \|f(kT)\|^2_{L_2(\Omega)} + c_3' e^{-c_1'T + c_4'\varphi(A(T))} \|v(0)\|^2_{H^1(\Omega)} \\ &\qquad \le \|v(0)\|^2_{H^1(\Omega)},\end{aligned}$$

$$\begin{aligned}(6.32)\quad & c_3' e^{-2\delta T + c_4'\varphi(A(T))} \|g(kT)\|^2_{L_2(\Omega)} + c_3' e^{-c_1'T + c_4'\varphi(A(T))} \|h(0)\|^2_{H^1(\Omega)} \\ &\qquad \le \|h(0)\|^2_{H^1(\Omega)},\end{aligned}$$

for any $k \in \mathbb{N}_0$.

PROOF OF THEOREM B. Take $k = 0$. Then $A(0,T) = A(T)$ and in view of Assumptions 6.3 and 6.4 we obtain

$$(6.33)\qquad \begin{aligned}\|v(T)\|_{H^1(\Omega)} &\le \|v(0)\|_{H^1(\Omega)},\\ \|h(t)\|_{H^1(\Omega)} &\le \|h(0)\|_{H^1(\Omega)}.\end{aligned}$$

Take $k = 1$. Then in view of (6.33) we can repeat the proof of Theorem A in the interval $[T, 2T]$ and we obtain that $A(1, T) = A(T)$. Then Assumptions 6.3 and 6.4 imply

$$\begin{aligned}\|v(2T)\|_{H^1(\Omega)} &\le \|v(T)\|_{H^1(\Omega)} \le \|v(0)\|_{H^1(\Omega)},\\ \|h(2T)\|_{H^1(\Omega)} &\le \|h(T)\|_{H^1(\Omega)} \le \|h(0)\|_{H^1(\Omega)}.\end{aligned} \tag{6.34}$$

Hence, repeating the above considerations we prove Theorem B. □

A. Appendix

Let us consider the problem

$$\begin{aligned} &u_t - \Delta u = 0 && \text{in } \Omega^T,\\ &u|_S = \varphi && \text{on } S^T,\\ &u|_{t=0} = 0 && \text{in } \Omega,\end{aligned} \tag{A.1}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain.

From [26] we have for solutions to problem (A.1) the estimate

$$\|u\|_{L_q(0,T;L_p(\Omega))} \le a_1 \|\varphi\|_{L_q(0,T;L_p(S))}, \quad p, q \in [1, \infty], \tag{A.2}$$

holds, where a_1 is a constant.

Lemma A.1. *The constant a_1 does not depend on T.*

Proof. From [26] we have

$$\|u\|_{L_q(0,T;L_p(\Omega))} \le c \int_0^T \frac{1}{\tau^{3/2}} \bigg(\int_0^d x_n^p e^{-px_n^2/(4\tau)}\, dx_n \bigg)^{1/p} d\tau \cdot \|\varphi\|_{L_q(0,T;L_p(\Omega))}, \tag{A.3}$$

where $d \ge \operatorname{diam}\Omega$ and c does not depend on T.

We express the above integral in the form

$$\int_0^1 d\tau \frac{1}{\tau^{3/2}} \bigg(\int_0^d x_n^p e^{-px_n^2/(4\tau)}\, dx_n \bigg)^{1/p} + \int_1^T d\tau 1/\tau^{3/2} \bigg(\int_0^d x_n^p e^{-px_n^2/(4\tau)}\, dx_n \bigg)^{1/p} \equiv I_1 + I_2,$$

where

$$I_1 \le \int_0^1 d\tau \frac{1}{\tau^{3/2}} \bigg(\int_0^\infty x_n^p e^{-px_n^2/(4\tau)}\, dx_n \bigg)^{1/p} \equiv I_1'.$$

Changing variables $y_n = x_n/\sqrt{\tau}$, $dx_n = \sqrt{\tau} dy_n$ we get

$$I_1' = \int_0^1 d\tau \tau^{1/2p+1/2-3/2} \bigg(\int_0^\infty y_n^p e^{-py_n^2}\, dy_n \bigg)^{1/p} \le c(p) \int_0^1 d\tau \tau^{1/2p-1} \le c(p).$$

Next

$$I_2 \le \int_1^T d\tau \frac{1}{\tau^{3/2}} \bigg(\int_0^d d^p \, dx_n \bigg)^{1/p} = d^{1+1/p} \int_1^T \frac{d\tau}{\tau^{3/2}} \le 2d^{1+1/p} \bigg(1 - \frac{1}{\sqrt{T}} \bigg) \le 2d^{1+1/p},$$

which concludes the proof. □

LEMMA A.2. *The constant a_2 in the imbedding*

$$\|u\|_{L_{10}(\Omega^T)} \le a_2 \|u\|_{V_2^1(\Omega^T)} \tag{A.4}$$

does not depend on T.

PROOF. First we show that the constant a' from the imbedding

$$\|u\|_{L_q(0,T;L_p(\Omega))} \le a' \|u\|_{V_2^0(\Omega^T)} \tag{A.5}$$

does not depend on T.

We follow the considerations from [12, Chapter 2, Section 3]. To apply the interpolation inequality (3.1) from [12, Chapter 2, Section 3] we have to extend u outside of Ω in such a way that the extended function $\widetilde{u}$ vanishes outside a compact set. Then the interpolation inequality holds without a lower order term on the r.h.s. Hence, we extend u by the Hestenes–Whitney method in such a way that $\widetilde{u}|_\Omega = u$, $\operatorname{supp} \widetilde{u} = \widetilde{\Omega}$ and $\widetilde{\Omega}$ is a compact set. Moreover, $\|\widetilde{u}\|_{H^1(\widetilde{\Omega})} \le c_1' \|u\|_{H^1(\Omega)}$, $\|\widetilde{u}\|_{L_2(\widetilde{\Omega})} \le c_2' \|u\|_{L_2(\Omega)}$, where the constants c_1', c_2' depend on Ω. For $\widetilde{u}$ we have the interpolation

$$\|\widetilde{u}\|_{L_p(\widetilde{\Omega})} \le c_3' \|\nabla \widetilde{u}\|_{L_2(\widetilde{\Omega})}^{\alpha} \|\widetilde{u}\|_{L_2(\widetilde{\Omega})}^{1-\alpha}, \quad \alpha = \frac{3}{3} - \frac{3}{p} \le 1,$$

where c_3' depends on $\widetilde{\Omega}$.

Next, we have

$$\|\widetilde{u}\|_{L_q(0,T;L_p(\Omega))} \le c_3' \bigg(\int_0^T \|\widetilde{u}_x\|_{L_2(\widetilde{\Omega})}^{\alpha q} \, dx \bigg)^{1/q} \sup_t \|\widetilde{u}\|_{L_2(\widetilde{\Omega})}^{1-\alpha}.$$

Setting $\alpha q = 2$ and applying the Young inequality yields

$$\|\widetilde{u}\|_{L_q(0,T;L_p(\widetilde{\Omega}))} \le c_4' \|\widetilde{u}\|_{V_2^0(\widetilde{\Omega}^T)},$$

where c_4' depends on $\widetilde{\Omega}$ only.

In view of the definition of the extension we obtain

$$\|u\|_{L_q(0,T;L_p(\Omega))} \le \|\widetilde{u}\|_{L_q(0,T;L_p(\widetilde{\Omega}))} \le c_4' \|\widetilde{u}\|_{V_2^0(\widetilde{\Omega}^T)} \le c_5' \|u\|_{V_2^0(\Omega^T)} \tag{A.6}$$

with constant c_5' independent of T.

Let $u \in V_2^1(\Omega^T)$. By (A.5) we have

$$\|u\|_{L_q(0,T;W_p^1(\Omega))} \le c_5' \|u\|_{V_2^1(\Omega^T)}. \tag{A.7}$$

To show that $u \in L_q(0,T;L_\sigma(\Omega))$ we have to satisfy the relations

$$\frac{3}{p}+\frac{2}{q}=\frac{3}{2},\quad \frac{3}{p}-\frac{3}{\sigma}\le 1 \quad \text{so} \quad p=\frac{3\sigma}{3+\sigma}$$

Hence

$$\frac{3+\sigma}{\sigma}+\frac{2}{q}=\frac{3}{2}$$

Choosing $q=\sigma$ we obtain $\sigma=10$, so the lemma is proved. □

From [1, Chapter 5, 5.14] we have

Lemma A.3. *Imbeddings*

$$\|\nabla^\alpha u\|_{L_p(\Omega^T)} \le a_3\|u\|_{W^{2,1}_\sigma(\Omega^T)}, \tag{A.8}$$

$\sigma<p$, $\alpha=0,1$, hold with constant a_3 independent of T.

Let us consider the problem

$$\begin{aligned} &v_t-\operatorname{div}\mathbb{T}(v,p)=f && \text{in } \Omega^T,\\ &\operatorname{div} v=0 && \text{in } \Omega^T,\\ &v\cdot\overline{n}=0 && \text{on } S^T,\\ &\overline{n}\cdot\mathbb{D}(v)\cdot\overline{\tau}_\alpha+\gamma v\cdot\overline{\tau}_\alpha=0,\quad \alpha=1,2, && \text{on } S^T,\\ &v|_{t=0}=v_0 && \text{in } \Omega. \end{aligned} \tag{A.9}$$

Lemma A.4. *Given $f\in L_r(\Omega^T)$, $v_0\in W_r^{2-2/r}(\Omega)$ there exists a solution to problem* (A.9) *such that $v\in W_r^{2,1}(\Omega^T)$, $\nabla p\in L_r(\Omega^T)$ and*

$$\|v\|_{W_r^{2,1}(\Omega^T)}+\|\nabla p\|_{L_r(\Omega^T)}\le a_4(\|f\|_{L_r(\Omega^T)}+\|v_0\|_{W_r^{2-2/r}(\Omega)}), \tag{A.10}$$

where a_4 does not depend on T.

Proof. We use [21, Chapter 3, Theorem 3.1.1]. □

Let us consider the elliptic problem

$$\begin{aligned} &-\operatorname{div}\mathbb{T}(v,p)=f && \text{in } \Omega,\\ &\operatorname{div} v=0 && \text{in } \Omega,\\ &v\cdot\overline{n}=0 && \text{on } S,\\ &\overline{n}\cdot\mathbb{T}(v,p)\cdot\overline{\tau}_\alpha+\gamma v\cdot\overline{\tau}_\alpha=0,\quad \alpha=1,2, && \text{on } S. \end{aligned} \tag{A.11}$$

Lemma A.5. *Let $f\in L_2(\Omega)$, $S\in C^2$. Then there exists a solution to problem* (A.11) *such that $v\in H^2(\Omega)$ and $\nabla p\in L_2(\Omega)$ and the following estimate*

$$\|v\|_{H^2(\Omega)}+\|\nabla p\|_{L_2(\Omega)}\le c\|f\|_{L_2(\Omega)} \tag{A.12}$$

holds. Moreover, we have

$$\|D(v)\|^2_{L_2(\Omega)}+\gamma\sum_{\alpha=1}^2\|v\cdot\overline{\tau}_\alpha\|^2_{L_2(S)}\le c\|f\|^2_{L_2(\Omega)}. \tag{A.13}$$

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