Topological Methods in Nonlinear Analysis
Journal of the Juliusz Schauder Center
Volume 37, 2011, 1–35

LOCALIZED SINGULARITIES AND CONLEY INDEX

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Abstract. We establish some abstract convergence and Conley index continuation principles for families of singularly perturbed semilinear parabolic equations and apply them to reaction-diffusion equations with nonlinear boundary conditions and localized large diffusion. This extends and refines previous results of [9] and [1].

1. Introduction

Let Ω be a bounded smooth domain in $\mathbb{R}^N$ and $\Omega_{0,i}$, $i \in [1\,.\,.\,m]$ be smooth domains whose closures are pairwise disjoint and included in Ω. Let $\Omega_0 = \bigcup_{i=1}^m \Omega_{0,i}$, $\Gamma = \partial\Omega$ and $\Gamma_{0,i} = \partial\Omega_{0,i}$, $i \in [1\,.\,.\,m]$. Set $\Omega_1 = \Omega \backslash \mathrm{Cl}\,\Omega_0$. For each $\varepsilon > 0$, consider the following parabolic problem

$$(\mathrm{E}_\varepsilon) \qquad \begin{cases} u_t - \mathrm{Div}(d_\varepsilon(x)\nabla u) + (\lambda + V_\varepsilon(x))u = \varphi_\varepsilon(x,u), & t > 0,\ x \in \Omega, \\ d_\varepsilon(x)\partial_\nu u + b_\varepsilon(x)u = \psi_\varepsilon(x,u), & t > 0,\ x \in \Gamma. \end{cases}$$

Here, $\lambda \in \mathbb{R}$ and ν is the exterior normal vector field on $\partial\Omega$. Moreover, $d_\varepsilon \geq m > 0$, V_ε and b_ε, resp. φ_ε and ψ_ε, are given functions on Ω, resp. $\Omega \times \mathbb{R}$ satisfying some regularity assumptions. We assume that, for $\varepsilon \to 0$, $V_\varepsilon \to V_0$, $b_\varepsilon \to b_0$, $\varphi_\varepsilon \to \varphi_0$, $\psi_\varepsilon \to \psi_0$ and $d_\varepsilon|_{\Omega_1} \to d_0$ (in some sense) while $d_\varepsilon|\Omega_0 \to \infty$.

2010 *Mathematics Subject Classification.* Primary 37B30, 35B25; Secondary 35B40.
Key words and phrases. Conley index, homology index braids, localized large diffusion, singular perturbations.
The research of the first author was partially supported by grants CNPq 304590/2006-1 and FAPESP 08/53440-0.

Under some general conditions on the functions involved equation (E_ε) can be written abstractly as a semilinear problem

$$(\mathrm{E}'_\varepsilon) \qquad \dot{u} + A_\varepsilon u = f_\varepsilon(u)$$

generating a local semiflow π_ε on $H^1(\Omega)$.

Consider the limit equation

$$(\mathrm{E}_0) \quad \begin{cases} u_t - \mathrm{Div}(d_0(x)\nabla u) + (\lambda + V_0(x))u = \varphi_0(x,u), & t>0,\ x\in\Omega_1, \\ d_0(x)\partial_\nu u + b_0(x)u = \psi_0(x,u), & t>0,\ x\in\Gamma, \\ \gamma_{0,i}(u) = u_{\Omega_{0,i}}, & \text{on } \Gamma_{0,i},\ i\in[1\,..\,m], \\ \dot{u}_{\Omega_{0,i}} + |\Omega_{0,i}|^{-1}\displaystyle\int_{\Gamma_{0,i}} d_0(x)\partial_{\nu_{0,i}} u\,d\sigma & \\ \quad +(\lambda+\widehat{c}_i)u_{\Omega_{0,i}} = |\Omega_{0,i}|^{-1}\displaystyle\int_{\Omega_{0,i}} \varphi_0(x,u_{\Omega_{0,i}})\,dx, & t>0,\ i\in[1\,..\,m], \end{cases}$$

where $\gamma_{0,i}$ is the trace operator on $\Gamma_{0,i}$, $\nu_{0,i}$ is the interior normal vector field on $\partial\Omega_{0,i}$, $u_{\Omega_{0,i}}$ is the value of u on $\Omega_{0,i}$ and $\widehat{c}_i := |\Omega_{0,i}|^{-1}\int_{\Omega_{0,i}} V_0\,dx$.

Equation (E_0) can similarly be written abstractly as a semilinear problem

$$(\mathrm{E}'_0) \qquad \dot{u} + A_0 u = f_0(u)$$

generating a local semiflow π_0 on a closed subspace $H^1_{\Omega_0}(\Omega)$ of $H^1(\Omega)$. In the paper [9] the spectral convergence of the family $(A_\varepsilon)_{\varepsilon>0}$ to A_0 for $\varepsilon\to 0$ is proved while the authors of [1] establish existence and upper semicontinuity results for global attractors of π_ε, $\varepsilon \ge 0$, under additional dissipativity conditions on the nonlinearities.

In this paper, we extend and refine these results. In particular, we prove that, as $\varepsilon \to 0$, the semiflows π_ε converge in a singular sense to the semiflow π_0 and we establish a singular compactness result for the family π_ε, $\varepsilon\ge 0$. As a consequence of these results, we obtain singular Conley index and homology index braid continuation principles for this family of semiflows.

In this paper we proceed as in [2] and keep the presentation of our results at an abstract level. In fact we only assume certain spectral convergence properties and compactness assumptions on a family of linear operators $(A_\varepsilon)_{\varepsilon\ge 0}$ (see conditions (Spec) and (Comp) in Section 4). We also make an abstract convergence hypothesis (condition (Conv) in section 5) on a family of nonlinear operators $(f_\varepsilon)_{\varepsilon\ge 0}$.

Our abstract approach permits applications to some other singular systems of reaction-diffusion equations. This will be treated in a subsequent publication.

2. Main results

In this section we will introduce some notation and state the main results of this paper.

Let N and m be a positive integers and $\widetilde{\varepsilon}_0$ be a positive real number. Let Ω be a bounded smooth domain in $\mathbb{R}^N$ and $\Omega_{0,i}$, $i \in [1\,.\,.\,m]$ be smooth domains whose closures are pairwise disjoint and are included in Ω. Let $\Omega_0 = \bigcup_{i=1}^m \Omega_{0,i}$, $\Gamma = \partial\Omega$ and $\Gamma_{0,i} = \partial\Omega_{0,i}$, $i \in [1\,.\,.\,m]$. Set $\Omega_1 = \Omega \setminus \operatorname{Cl}\Omega_0$.

For each $\varepsilon \in [0, \widetilde{\varepsilon}_0]$, let $d_\varepsilon \colon \Omega \to \mathbb{R}$ be a smooth function such that

$$0 < m_0 \leq d_\varepsilon(x) \leq M_\varepsilon \quad \text{for all } x \in \Omega,$$

where m_0 and M_ε, $\varepsilon \in [0, \widetilde{\varepsilon}_0]$, are positive constants, and

$$d_\varepsilon(x) \to d_0(x) \quad \text{as } \varepsilon \to 0 \text{ uniformly for } x \in \Omega_1$$

and

$$d_\varepsilon(x) \to \infty \quad \text{as } \varepsilon \to 0 \text{ uniformly on compact subsets of } \Omega_0.$$

Let $V_\varepsilon \in L^{q_0}(\Omega)$ and $b_\varepsilon \in L^{q_1}(\Gamma)$ be such that

$$|V_\varepsilon - V_0|_{L^{q_0}(\Omega)} \to 0 \quad \text{and} \quad |b_\varepsilon - b_0|_{L^{q_1}(\Gamma)} \to 0 \quad \text{as } \varepsilon \to 0,$$

where q_0 and q_1 are constants such that

$$q_0 \begin{cases} > 1 & \text{for } N = 1; \\ > 1 & \text{for } N = 2; \\ > N/2 & \text{for } N \geq 3 \end{cases} \qquad \text{and} \qquad q_1 \begin{cases} > 1 & \text{for } N = 1; \\ > 1 & \text{for } N = 2; \\ > N-1 & \text{for } N \geq 3. \end{cases}$$

Now let $L^2_{\Omega_0}(\Omega)$ be the set of all functions in $L^2(\Omega)$ which are (almost everywhere) constant on each $\Omega_{0,i}$, $i \in [1\,.\,.\,m]$. Set $H^1_{\Omega_0}(\Omega) = H^1(\Omega) \cap L^2_{\Omega_0}(\Omega)$. It follows that $H^1_{\Omega_0}(\Omega)$ (resp. $L^2_{\Omega_0}(\Omega)$) is a closed subspace of $H^1(\Omega)$ (resp. $L^2(\Omega)$).

Let $\gamma \colon H^1(\Omega) \to H^{1/2}(\Gamma)$ be the trace operator. For $\lambda \in \mathbb{R}$ and $\varepsilon \in \,]0, \widetilde{\varepsilon}_0]$ define the bilinear form $\zeta_\varepsilon \colon H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ by

$$\zeta_\varepsilon(u, v) = \int_\Omega d_\varepsilon \nabla u \nabla v \, dx + \int_\Omega (\lambda + V_\varepsilon) uv \, dx + \int_\Gamma b_\varepsilon \gamma(u)\gamma(v) \, d\sigma, \quad u,\, v \in H^1(\Omega).$$

Here, dx is the N-Lebesgue measure and $d\sigma$ is the surface measure on Γ.

It follows from results in [9] that ζ_ε is defined and continuous on $H^1(\Omega) \times H^1(\Omega)$. Furthermore, define the bilinear form $\zeta_0 \colon H^1_{\Omega_0}(\Omega) \times H^1_{\Omega_0}(\Omega) \to \mathbb{R}$ by

$$\zeta_0(u, v) = \int_{\Omega_1} d_0 \nabla u \nabla v \, dx + \int_\Omega (\lambda + V_0) uv \, dx + \int_\Gamma b_0 \gamma(u)\gamma(v) \, d\sigma, \quad u, v \in H^1_{\Omega_0}(\Omega).$$

Results in [9] imply that ζ_0 is defined and continuous on $H^1_{\Omega_0}(\Omega) \times H^1_{\Omega_0}(\Omega)$. Moreover, there are an $\varepsilon_0 \in \,]0, \widetilde{\varepsilon}_0]$, a $\widetilde{\mu} \in \,]0, \infty[$ and a $\widetilde{\lambda} \in \mathbb{R}$ such that for all $\lambda \geq \widetilde{\lambda}$

$$\begin{aligned} \zeta_\varepsilon(u, u) &\geq \widetilde{\mu} |u|^2_{H^1(\Omega)}, \quad && \varepsilon \in \,]0, \varepsilon_0]\,,\, u \in H^1(\Omega) \\ \zeta_0(u, u) &\geq \widetilde{\mu} |u|^2_{H^1_{\Omega_0}(\Omega)}, \quad && u \in H^1_{\Omega_0}(\Omega). \end{aligned}$$

For $\varepsilon \in]0,\varepsilon_0]$ the pair $(\zeta_\varepsilon, \langle\,\cdot\,,\,\cdot\,\rangle_{L^2(\Omega)})$ defines an operator $A_\varepsilon\colon D(A_\varepsilon) \to H^\varepsilon := L^2(\Omega)$. Specifically, let $D(A_\varepsilon)$ be the set of all $u \in H^1(\Omega)$ such that there is a $w = w_u \in L^2(\Omega)$ with the property that

$$\zeta_\varepsilon(u,v) = \langle w, v\rangle_{L^2(\Omega)}$$

for all $v \in H^1(\Omega)$. Then w_u is uniquely determined by u, the set $D(A_\varepsilon)$ is a dense linear subspace both of $H^1(\Omega)$ and of $L^2(\Omega)$, and the map

$$A_\varepsilon\colon D(A_\varepsilon) \to L^2(\Omega), \quad u \mapsto w_u$$

is a linear positive self-adjoint operator in $(L^2, \langle\,\cdot\,,\,\cdot\,\rangle_{L^2(\Omega)})$ with A_ε^{-1} compact.

Analogously, the pair $(\zeta_0, \langle\,\cdot\,,\,\cdot\,\rangle_{L^2_{\Omega_0}(\Omega)})$ defines a linear positive self-adjoint operator A_0 in $H^0 := L^2_{\Omega_0}(\Omega)$ with A_0^{-1} compact.

It is proved in [9] that, for $\varepsilon \in]0,\varepsilon_0]$, $D(A_\varepsilon)$ is the set of all $u \in H^1(\Omega)$ such that $-\operatorname{Div}(d_\varepsilon \nabla u) + V_\varepsilon u \in L^2(\Omega)$ and $d_\varepsilon \partial_\nu u + b_\varepsilon u = 0$ in Γ. Here, ν is the exterior normal vector field on $\partial\Omega$ and $d_\varepsilon \partial_\nu u$ is the conormal derivative of u in some generalized sense. The linear operator A_ε is then given by

$$A_\varepsilon u = -\operatorname{Div}(d_\varepsilon \nabla u) + (\lambda + V_\varepsilon)u$$

for $u \in D(A_\varepsilon)$.

Moreover, $D(A_0)$ is the set of all $u \in H^1_{\Omega_0}(\Omega)$ such that $-\operatorname{Div}(d_0\nabla u) + V_0 u \in L^2(\Omega_1)$ with $d_0\partial_\nu u + b_0 u = 0$ in Γ. The linear operator A_0 is then given by

$$\begin{aligned} A_0 u = {}&(-\operatorname{Div}(d_0\nabla u) + (\lambda + V_0)u)\chi_{\Omega_1} \\ &+ \sum_{i=1}^m \biggl(|\Omega_{0,i}|^{-1} \int_{\Gamma_{0,i}} d_0 \partial_{\nu_{0,i}} u \, d\sigma + (\lambda + \widehat{c}_i) u_{\Omega_{0,i}} \biggr) \chi_{\Omega_{0,i}}, \quad u \in D(A_0). \end{aligned}$$

Here, $\nu_{0,i}$ is the interior normal vector field on $\partial\Omega_{0,i}$, $u_{\Omega_{0,i}}$ is the constant value of u on $\Omega_{0,i}$, $\widehat{c}_i := |\Omega_{0,i}|^{-1} \int_{\Omega_{0,i}} V_0 \, dx$, $i \in [1\,..\,m]$, and χ_B denotes the characteristic function of a given set B.

For $\varepsilon \in [0,\varepsilon_0]$ the operator A_ε is sectorial so it defines a family of fractional power operators $A_\varepsilon^\beta\colon D(A_\varepsilon^\beta) \to H^\varepsilon$, $\beta \in [0,\infty[$ and we write, for $\alpha \in [0,\infty[$, $H^\varepsilon_\alpha := D(A_\varepsilon^{\alpha/2})$. In particular, $H^\varepsilon_0 = H^\varepsilon$. In a canonical way, H^ε_α is a Hilbert space and we set $H^\varepsilon_{-\alpha}$ to be the dual of H^ε_α.

Let

$$2^*_\Omega = \begin{cases} \dfrac{2N}{N-2} & \text{if } N \ge 3; \\ \text{an arbitrary } p^* \in]0,\infty[& \text{if } N = 2; \\ \infty & \text{if } N = 1 \end{cases}$$

and

$$2^*_\Gamma = \begin{cases} \dfrac{2(N-1)}{N-2} & \text{if } N \ge 3; \\ \text{an arbitrary } p^{**} \in]0,\infty[& \text{if } N = 2; \\ \infty & \text{if } N = 1. \end{cases}$$

Now assume the following

HYPOTHESIS 2.1. *For* $\varepsilon \in [0, \varepsilon_0]$, $\varphi_\varepsilon \colon \Omega \times \mathbb{R} \to \mathbb{R}$ *and* $\psi_\varepsilon \colon \Gamma \times \mathbb{R} \to \mathbb{R}$, $(x, s) \mapsto \varphi_\varepsilon(x, s)$, $(x, s) \mapsto \psi_\varepsilon(x, s)$, *are functions such that*

(a) *there is a null set* N_Ω *in* Ω *with* $\varphi_\varepsilon(x, \cdot\,) \in C^1(\mathbb{R}, \mathbb{R})$ *for all* $x \in \Omega \setminus N_\Omega$*;*
(b) *there is a null set* N_Γ *in* Γ *(rel. to the surface measure on* Γ*) with* $\psi_\varepsilon(x, \cdot\,) \in C^1(\mathbb{R}, \mathbb{R})$ *for all* $x \in \Gamma \setminus N_\Gamma$*;*
(c) *for all* $s \in \mathbb{R}$, $\varphi_\varepsilon(\,\cdot\,, s)$ *and* $\partial_s \varphi_\varepsilon(\,\cdot\,, s)$ *is measurable on* Ω*;*
(d) *for all* $s \in \mathbb{R}$, $\psi_\varepsilon(\,\cdot\,, s)$ *and* $\partial_s \psi_\varepsilon(\,\cdot\,, s)$ *is measurable on* Γ.

Moreover, $q_2 \in \,](1 - (1/2^*_\Omega))^{-1}, 2^*_\Omega[$, $q_3 \in \,](1 - (1/2^*_\Gamma))^{-1}, 2^*_\Gamma[$ *and*

$$r_2 = \frac{2^*_\Omega q_2}{2^*_\Omega - q_2}, \quad r_3 = \frac{2^*_\Gamma q_3}{2^*_\Gamma - q_3}, \quad \beta_2 = \frac{2^*_\Omega}{q_2} - 1, \quad \beta_3 = \frac{2^*_\Gamma}{q_3} - 1.$$

There is a constant $\widetilde{C} \in \,]0, \infty[$ *and functions* $a_2 \in L^{r_2}(\Omega)$, $b_2 \in L^{q_2}(\Omega)$, $a_3 \in L^{r_3}(\Gamma)$, $b_3 \in L^{q_3}(\Gamma)$ *such that, for all* $\varepsilon \in [0, \varepsilon_0]$,

$$\begin{aligned} |\partial_s \varphi_\varepsilon(x, s)| &\le \widetilde{C}(a_2(x) + |s|^{\beta_2}), && \text{for } (x, s) \in (\Omega \setminus N_\Omega) \times \mathbb{R}, \\ |\varphi_\varepsilon(x, 0)| &\le b_2(x), && \text{for } x \in \Omega \setminus N_\Omega, \\ |\partial_s \psi_\varepsilon(x, s)| &\le \widetilde{C}(a_3(x) + |s|^{\beta_3}), && \text{for } (x, s) \in (\Gamma \setminus N_\Gamma) \times \mathbb{R}, \\ |\psi_\varepsilon(x, 0)| &\le b_3(x), && \text{for } x \in \Gamma \setminus N_\Gamma. \end{aligned}$$

Finally, as $\varepsilon \to 0^+$,

$$\begin{aligned} |\varphi_\varepsilon(x, s) - \varphi_0(x, s)| &\to 0, \quad \text{for } (x, s) \in (\Omega \setminus N_\Omega) \times \mathbb{R} \\ |\psi_\varepsilon(x, s) - \psi_0(x, s)| &\to 0, \quad \text{for } (x, s) \in (\Gamma \setminus N_\Gamma) \times \mathbb{R}. \end{aligned}$$

Under Hypothesis 2.1, there is an $\alpha \in \,]1/2, 1[$ such that whenever $\varepsilon \in [0, \varepsilon_0]$, $u \in H^\varepsilon_1$ and $h \in H^\varepsilon_\alpha$, the functions $x \mapsto \varphi_\varepsilon(x, u(x)) \cdot h(x)$ and $x \mapsto \psi_\varepsilon(x, \gamma(u)(x)) \cdot \gamma(h)(x)$ are integrable on Ω and Γ, respectively. (Cf. Section 8 below.) Defining

$$f_\varepsilon(u)(h) = \int_\Omega \varphi_\varepsilon(x, u(x)) \cdot h(x)\, dx + \int_\Gamma \psi_\varepsilon(x, \gamma(u)(x)) \cdot \gamma(h)(x)\, d\sigma$$

we obtain a locally Lipschitzian map $f_\varepsilon \colon H^\varepsilon_1 \to H^\varepsilon_{-\alpha}$.

The linear isometry $A_\varepsilon \colon D(A_\varepsilon) = H^\varepsilon_2 \to H^\varepsilon_0$ can be extended to a unique linear isometry $\widetilde{A}_\varepsilon \colon D(\widetilde{A}_\varepsilon) = H^\varepsilon_{2-\alpha} \to H^\varepsilon_{-\alpha}$. Moreover, $\widetilde{A}_\varepsilon$ is a positive selfadjoint operator on $H^\varepsilon_{-\alpha}$.

Therefore, we may consider the abstract parabolic equation

$$\dot{u} = \widetilde{A}_\varepsilon u + f_\varepsilon(u) \tag{2.1}$$

on H^ε_1. This equation generates a local semiflow π_ε on H^ε_1.

For each $\varepsilon \in]0, \varepsilon_0]$, equation (2.1) is an abstract formulation of the following parabolic partial differential equation with localized large diffusion and nonlinear boundary conditions:

$$\begin{cases} u_t - \operatorname{Div}(d_\varepsilon(x)\nabla u) + (\lambda + V_\varepsilon(x))u = \varphi_\varepsilon(x,u), & t > 0,\ x \in \Omega, \\ d_\varepsilon(x)\partial_\nu u + b_\varepsilon(x)u = \psi_\varepsilon(x,u), & t > 0,\ x \in \partial\Omega. \end{cases}$$

For $\varepsilon = 0$, (2.1) is an abstract formulation of the following boundary value problem:

$$\begin{cases} u_t - \operatorname{Div}(d_0(x)\nabla u) + (\lambda + V_0(x))u = \varphi_0(x,u), & t > 0,\ x \in \Omega_1, \\ d_0(x)\partial_\nu u + b_0(x)u = \psi_0(x,u), & t > 0,\ x \in \Gamma, \\ \gamma_{0,i}(u) = u_{\Omega_{0,i}}, & \text{on } \Gamma_{0,i},\ i \in [1\,..\,m]\,, \\ \dot{u}_{\Omega_{0,i}} + |\Omega_{0,i}|^{-1} \displaystyle\int_{\Gamma_{0,i}} d_0(x)\partial_{\nu_{0,i}} u \, d\sigma + (\lambda + \widehat{c}_i)u_{\Omega_{0,i}} & \\ \qquad = |\Omega_{0,i}|^{-1} \displaystyle\int_{\Omega_{0,i}} \varphi_0(x, u_{\Omega_{0,i}})\, dx, & t > 0,\ i \in [1\,..\,m]\,. \end{cases}$$

Here, for $i \in [1\,..\,m]$, $\gamma_{0,i}$ is the trace operator on $\Gamma_{0,i}$.

It was proved in [9] that, as $\varepsilon \to 0$, the spectrum of A_ε converges to the spectrum of A_0. Using this one can obtain results on convergence of e^{-tA_ε} to e^{-tA_0}. Now by using the variation-of-constants formula one suspects that, in some sense, some families of solutions of π_ε converge to solutions of π_0. This was proved in [1] for full bounded solutions under some additional dissipativeness conditions both on the linear and on the nonlinear problem, cf. [1, conditions S, D_ε and D_0]. This latter result also implies existence of global attractors of both π_ε and π_0 and their upper semicontinuity at $\varepsilon = 0$.

In this paper we extend and refine these results. More specifically, working first in an abstract setting, we establish in Sections 4, 5, 6 and 7 various singular convergence, compactness and Conley index continuation results for abstract families of equations of type (2.1). These abstract results imply the following main theorems of this paper:

Theorem 2.2. *Let $(\varepsilon_n)_n$ be a sequence in $]0, \varepsilon_0]$ with $\varepsilon_n \to 0$ and $(t_n)_n$ be a sequence in $[0, \infty[$ with $t_n \to t_0$, for some $t_0 \in [0, \infty[$. Let $u_0 \in H_1^0$ and $(u_n)_n$ be a sequence with $u_n \in H_1^{\varepsilon_n}$ for every $n \in \mathbb{N}$ and*

$$|u_n - u_0|_{H_1^{\varepsilon_n}} \to 0 \quad \text{as } n \to \infty.$$

Assume $u_0\pi_0 t_0$ is defined. Then there exists an $n_0 \in \mathbb{N}$ such that $u_n \pi_{\varepsilon_n} t_n$ is defined for all $n \geq n_0$ and

$$|u_n \pi_{\varepsilon_n} t_n - u_0 \pi_0 t_0|_{H_1^{\varepsilon_n}} \to 0 \quad \text{as } n \to \infty.$$

Theorem 2.3. *Suppose* $\kappa \in]0,\infty[$, $(\varepsilon_n)_n$ *is a sequence in* $]0,\varepsilon_0]$ *with* $\varepsilon_n \to 0$, $(t_n)_n$ *is a sequence in* $[0,\infty[$ *with* $t_n \geq \kappa$ *for every* $n \in \mathbb{N}$ *and* $(u_n)_n$ *is a sequence with* $u_n \in H_1^{\varepsilon_n}$ *for every* $n \in \mathbb{N}$. *Assume that there exists a constant* $C \in]0,\infty[$ *such that* $u_n\pi_{\varepsilon_n}t_n$ *is defined and*

$$|u_n\pi_{\varepsilon_n}s|_{H_1^{\varepsilon_n}} \leq C \quad \textit{for all } n \in \mathbb{N} \textit{ and for all } s \in [0,t_n].$$

Then there exist a $v \in H_1^0$ *and a sequence* $(n_k)_k$ *in* $\mathbb{N}$ *with* $n_k \to \infty$ *as* $k \to \infty$ *such that*

$$|u_{n_k}\pi_{\varepsilon_{n_k}}t_{n_k} - v|_{H_1^{\varepsilon_{n_k}}} \to 0 \quad \textit{as } k \to \infty.$$

Theorem 2.4. *For each* $\varepsilon \in]0,\varepsilon_0]$, *let* I_ε *be the identity map on* H_1^ε *and* $Q_\varepsilon\colon H_1^\varepsilon \to H_1^\varepsilon$ *be the* H_1^ε*-orthogonal projection of* H_1^ε *onto* H_1^0. *Let* N *be a closed and bounded isolating neighbourhood of an invariant set* K_0 *relative to* π_0. *For* $\varepsilon \in]0,\varepsilon_0]$ *and for every* $\eta \in]0,\infty[$ *set*

$$N_{\varepsilon,\eta} := \{u \in H_1^\varepsilon \mid Q_\varepsilon u \in N \textit{ and } |(I_\varepsilon - Q_\varepsilon)u|_{H_1^\varepsilon} \leq \eta\}$$

and $K_{\varepsilon,\eta} := \operatorname{Inv}_{\pi_\varepsilon}(N_{\varepsilon,\eta})$ *i.e.* $K_{\varepsilon,\eta}$ *is the largest* π_ε*-invariant set in* $N_{\varepsilon,\eta}$. *Then for every* $\eta \in]0,\infty[$ *there exists an* $\varepsilon^{\mathrm{c}} = \varepsilon^{\mathrm{c}}(\eta) \in]0,\varepsilon_0]$ *such that for every* $\varepsilon \in]0,\varepsilon^{\mathrm{c}}]$ *the set* $N_{\varepsilon,\eta}$ *is a strongly admissible isolating neighbourhood of* $K_{\varepsilon,\eta}$ *relative to* π_ε *and*

$$h(\pi_\varepsilon, K_{\varepsilon,\eta}) = h(\pi_0, K_0).$$

Furthermore, for every $\eta > 0$, *the family* $(K_{\varepsilon,\eta})_{\varepsilon\in[0,\varepsilon^{\mathrm{c}}(\eta)]}$ *of invariant sets, where* $K_{0,\eta} = K_0$, *is upper semicontinuous at* $\varepsilon = 0$ *with respect to the family* $|\cdot|_{H_1^\varepsilon}$ *of norms i.e.*

$$\lim_{\varepsilon\to 0^+} \sup_{w\in K_{\varepsilon,\eta}} \inf_{u\in K_0} |w-u|_{H_1^\varepsilon} = 0.$$

The family $(K_{\varepsilon,\eta})_{\varepsilon\in]0,\varepsilon^{\mathrm{c}}(\eta)]}$ *is asymptotically independent of* η *i.e. whenever* η_1 *and* $\eta_2 \in]0,\infty[$ *then there is an* $\varepsilon' \in]0,\min(\varepsilon^{\mathrm{c}}(\eta_1),\varepsilon^{\mathrm{c}}(\eta_2))]$ *such that* $K_{\varepsilon,\eta_1} = K_{\varepsilon,\eta_2}$ *for* $\varepsilon \in]0,\varepsilon']$.

Theorem 2.5. *Assume the hypotheses of Theorem* 2.4 *and for every* $\eta \in]0,\infty[$ *let* $\varepsilon^{\mathrm{c}}(\eta) \in]0,\varepsilon_0]$ *be as in that theorem.*

Let $(P,\prec)$ *be a finite poset and* $(M_{p,0})_{p\in P}$ *be a* $\prec$*-ordered Morse decomposition of* K_0 *relative to* π_0. *For each* $p \in P$, *let* $V_p \subset N$ *be closed in* X_0 *and such that* $M_{p,0} = \operatorname{Inv}_{\pi_0}(V_p) \subset \operatorname{Int}_{H_1^0}(V_p)$. (*Such sets* V_p, $p \in P$, *exist.*) *For* $\varepsilon \in]0,\varepsilon_0]$, *for every* $\eta \in]0,\infty[$ *and* $p \in P$ *set* $M_{p,\varepsilon,\eta} := \operatorname{Inv}_{\pi_\varepsilon}(V_{p,\varepsilon,\eta})$, *where*

$$V_{p,\varepsilon,\eta} := \{u \in H_1^\varepsilon \mid Q_\varepsilon u \in V_p \textit{ and } |(I_\varepsilon - Q_\varepsilon)u|_{H_1^\varepsilon} \leq \eta\}.$$

Then for every $\eta \in]0,\infty[$ *there is an* $\widetilde{\varepsilon} = \widetilde{\varepsilon}(\eta) \in]0,\varepsilon^{\mathrm{c}}(\eta)]$ *such that for every* $\varepsilon \in]0,\widetilde{\varepsilon}]$ *and* $p \in P$, $M_{p,\varepsilon,\eta} \subset \operatorname{Int}_{H_1^\varepsilon}(V_{p,\varepsilon,\eta})$ *and the family* $(M_{p,\varepsilon,\eta})_{p\in P}$ *is a* $\prec$*-ordered Morse decomposition of* $K_{\varepsilon,\eta}$ *relative to* π_ε *and the* (*co*)*homology* *index braids of* $(\pi_0, K_0, (M_{p,0})_{p\in P})$ *and* $(\pi_\varepsilon, K_{\varepsilon,\eta}, (M_{p,\varepsilon,\eta})_{p\in P}))$, $\varepsilon \in]0, \widetilde{\varepsilon}]$, *are isomorphic and so they determine the same collection of* C*-connection matrices.*

Again, for each $p \in P$, *the family* $(M_{p,\varepsilon,\eta})_{\varepsilon\in[0,\widetilde{\varepsilon}(\eta)]}$, *where* $M_{p,0,\eta} = M_{p,0}$ *is upper semicontinuous at* $\varepsilon = 0$ *with respect to the family* $|\cdot|_{H_1^\varepsilon}$ *of norms and the family* $(M_{p,\varepsilon,\eta})_{\varepsilon\in]0,\widetilde{\varepsilon}(\eta)]}$ *is asymptotically independent of* η.

The above theorems are proved in Section 8.

3. Preliminaries

Suppose H is an infinite dimensional linear space which is complete with respect to the scalar product $\langle\,\cdot\,,\,\cdot\,\rangle_H$ and let $A: D(A) \subset H \to H$ be a (densely defined) positive self-adjoint operator on $(H, \langle\,\cdot\,,\,\cdot\,\rangle_H)$ with $A^{-1}: H \to H$ compact. Let $(\lambda_j)_j$ be *the repeated sequence of eigenvalues of* A, i.e. the uniquely determined nondecreasing sequence $(\lambda_j)_j$ containing exactly the eigenvalues of A and such that the number of occurrences of every eigenvalue of A in this sequence is equal to its multiplicity. Let $(w_j)_j$ be an H-orthonormal sequence of eigenvectors of A corresponding to $(\lambda_j)_j$. For $\alpha \in [0, \infty[$, let $H_\alpha = H_\alpha(A) = D(A^{\alpha/2})$. In particular,

$$H_0 = H.$$

Note that H_α is a Hilbert space under the scalar product

$$\langle u, v\rangle_{H_\alpha} = \langle A^{\alpha/2}u, A^{\alpha/2}v\rangle_H, \quad u, v \in H_\alpha.$$

For every $j \in \mathbb{N}$, $w_j \in H_\alpha$ and the sequence $(\lambda_j^{-\alpha/2} w_j)_j$ is H_α-orthonormal and H_α-complete. If $u \in H_\alpha$ we have

$$\left| u - \sum_{j=1}^{k} \langle u, w_j\rangle_H w_j \right|_{H_\alpha} \to 0 \quad \text{as } k \to \infty \tag{3.1}$$

and so

$$|u|_{H_\alpha}^2 = \sum_{j=1}^{\infty} \lambda_j^\alpha |\langle u, w_j\rangle_H|^2. \tag{3.2}$$

If $\alpha \in]0, \infty[$, let $H_{-\alpha} = H'_\alpha$ be the dual of H_α. It follows that $H_{-\alpha}$ is a Hilbert space under the dual scalar product

$$\langle u, v\rangle_{H_{-\alpha}} = \langle F_\alpha^{-1}v, F_\alpha^{-1}u\rangle_{H_\alpha}, \quad u, v \in H_{-\alpha},$$

where $F_\alpha: H_\alpha \to H_{-\alpha}$, $u \mapsto \langle\,\cdot\,, u\rangle_{H_\alpha}$, is the Fréchet–Riesz isomorphism.

Define the map $\psi_\alpha: H = H_0 \to H_{-\alpha}$ by $\psi_\alpha(u) = y$, where $y: H_\alpha \to \mathbb{K}$ is defined by

$$y(v) = \langle v, u\rangle_H, \quad v \in H_\alpha.$$

ψ_α is an injection so that we can (and will) identify elements $u \in H$ with $\psi_\alpha(u) \in H_{-\alpha}$. We thus consider H as a linear subspace of $H_{-\alpha}$. With this identification, the sequence $(\lambda_j^{\alpha/2} w_j)_j$ is $H_{-\alpha}$-orthonormal and $H_{-\alpha}$-complete. If $u \in H_{-\alpha}$ then

$$\Big| u - \sum_{j=1}^{k} u(w_j) w_j \Big|_{H_{-\alpha}} \to 0 \quad \text{as } k \to \infty \tag{3.3}$$

and so

$$|u|^2_{H_{-\alpha}} = \sum_{j=1}^{\infty} \lambda_j^{-\alpha} |u(w_j)|^2. \tag{3.4}$$

For $\alpha \in \,]0, \infty[$ there is a unique continuous extension $\widetilde{A}^{-1} = \widetilde{A}_\alpha^{-1} \colon H_{-\alpha} \to H_{2-\alpha}$ of $A^{-1} \colon H \to H_2$. The map $\widetilde{A}^{-1}$ is a bijective linear isometry. Let $\widetilde{A} \colon H_{2-\alpha} \to H_{-\alpha}$ be the inverse of $\widetilde{A}^{-1}$. Then $\widetilde{A}$ is a positive densely defined self-adjoint operator on $H_{-\alpha}$. Moreover, for $\beta \in [0, \infty]$ the β-fractional power space $H_\beta(\widetilde{A})$ of $\widetilde{A}$ is isomorphic (as a Hilbert space) to $H_{\beta-\alpha} = H_{\beta-\alpha}(A)$.

The linear semigroup $e^{-t\widetilde{A}} \colon H_{-\alpha} \to H_{-\alpha}$, $t \in [0, \infty[$, is an extension of the semigroup $e^{-tA} \colon H \to H$, $t \in [0, \infty[$. Since, for every $j \in \mathbb{N}$ and $t \in [0, \infty[$,

$$e^{-tA} w_j = e^{-t\widetilde{A}} w_j = e^{-t\lambda_j} w_j$$

we conclude that, for every $u \in H$, every $\beta \in [0, \infty[$ and every $t \in \,]0, \infty[$

$$\Big| e^{-tA} u - \sum_{j=1}^{k} e^{-t\lambda_j} \langle u, w_j \rangle_H w_j \Big|_{H_\beta} \to 0 \quad \text{as } k \to \infty. \tag{3.5}$$

We also conclude that, for every $u \in H_{-\alpha}$, every $\beta \in [0, \infty[$ and every $t \in \,]0, \infty[$

$$\Big| e^{-t\widetilde{A}} u - \sum_{j=1}^{k} e^{-t\lambda_j} u(w_j) w_j \Big|_{H_\beta} \to 0 \quad \text{as } k \to \infty. \tag{3.6}$$

4. Singular convergence of linear semiflows

In this section we introduce two abstract hypotheses, conditions (Spec) and (Comp), and we show that condition (Spec) enables us to prove some singular convergence results for linear semiflows.

First we introduce the following spectral convergence definition for a family of Hilbert spaces and linear operators.

Definition 4.1. Given $\varepsilon_0 > 0$ we say that the family $(H^\varepsilon, \langle \cdot\,, \cdot \rangle_{H^\varepsilon}, A_\varepsilon)_{\varepsilon \in [0, \varepsilon_0]}$ *satisfies condition* (Spec) if the following properties are satisfied:

(a) for every $\varepsilon \in [0, \varepsilon_0]$, $(H^\varepsilon, \langle \cdot\,, \cdot \rangle_{H^\varepsilon})$ is an infinite dimensional Hilbert space and $A_\varepsilon \colon D(A_\varepsilon) \subset H^\varepsilon \to H^\varepsilon$ is a densely defined positive self-adjoint operator on the space $(H^\varepsilon, \langle \cdot\,, \cdot \rangle_{H^\varepsilon})$ with ${A_\varepsilon}^{-1} \colon H^\varepsilon \to H^\varepsilon$ compact. For $\alpha \in \mathbb{R}$ write $H^\varepsilon_\alpha := H_\alpha(A_\varepsilon)$. In particular, $H^\varepsilon_0 = H^\varepsilon$;

(b) for each $\varepsilon \in]0,\varepsilon_0]$, H^0 is a linear subspace of H^ε and H_1^0 is a linear subspace of H_1^ε;

(c) there exists a constant $C \in]1,\infty[$ such that

$$|u|_{H_1^\varepsilon} \leq C|u|_{H_1^0} \quad \text{and} \quad |u|_{H_1^0} \leq C|u|_{H_1^\varepsilon}$$

for all $u \in H_1^0$ and all $\varepsilon \in]0,\varepsilon_0]$;

(d) for every $\varepsilon \in]0,\varepsilon_0]$ let $(\lambda_{\varepsilon,j})_j$ be the repeated sequence of eigenvalues of A_ε and $(w_{\varepsilon,j})_j$ be a corresponding H^ε-orthonormal sequence of eigenfunctions. Furthermore, let $(\lambda_{0,j})_j$ be the repeated sequence of eigenvalues of A_0.
Whenever $(\varepsilon_n)_n$ is a sequence in $]0,\varepsilon_0]$ with $\varepsilon_n \to 0$ then
(d1) $\lambda_{\varepsilon_n,j} \to \lambda_{0,j}$ as $n \to \infty$, for all $j \in \mathbb{N}$.
Moreover, there is a sequence $(n_k)_k$ in $\mathbb{N}$ with $n_k \to \infty$ as $k \to \infty$ and there is an H^0-orthonormal sequence of eigenfunctions $(w_{0,j})_j$ of A_0 corresponding to $(\lambda_{0,j})_j$ such that
(d2) $|w_{\varepsilon_{n_k},j} - w_{0,j}|_{H_1^{\varepsilon_{n_k}}} \to 0$ as $k \to \infty$, for all $j \in \mathbb{N}$;
(d3) $\langle u, w_{\varepsilon_{n_k},j}\rangle_{H^{\varepsilon_{n_k}}} \to \langle u, w_{0,j}\rangle_{H^0}$ as $k \to \infty$, for all $u \in H^0$ and all $j \in \mathbb{N}$.

We also require the following definition.

Definition 4.2. Let the family $(H^\varepsilon, \langle\,\cdot\,,\,\cdot\,\rangle_{H^\varepsilon}, A_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ satisfy condition (Spec). We say that $(H^\varepsilon, \langle\,\cdot\,,\,\cdot\,\rangle_{H^\varepsilon}, A_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ *satisfies condition* (Comp) if whenever $(\varepsilon_n)_n$ is a sequence in $]0,\varepsilon_0]$ with $\varepsilon_n \to 0$ and $(\xi_n)_n$ is a sequence with $\xi_n \in H_1^{\varepsilon_n}$ for every $n \in \mathbb{N}$ and

$$\sup_{n\in\mathbb{N}} |\xi_n|_{H_1^{\varepsilon_n}} < \infty,$$

then there exist a $v \in H_1^0$ and a sequence $(n_k)_k$ in $\mathbb{N}$ with $n_k \to \infty$ as $k \to \infty$ such that

$$|\xi_{n_k} - v|_{H^{\varepsilon_{n_k}}} \to 0 \quad \text{as } k \to \infty.$$

Now we will show that condition (Spec) allows us to obtain two singular convergence theorems for linear semiflows. We start with following preliminary result.

Proposition 4.3. *If* $(H^\varepsilon, \langle\,\cdot\,,\,\cdot\,\rangle_{H^\varepsilon}, A_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ *satisfy condition* (Spec)*, then for every* $\varepsilon \in]0,\varepsilon_0]$*, the subspace* H_1^0 *is closed in* $(H_1^\varepsilon, |\cdot|_{H_1^\varepsilon})$.

Proof. Let $\varepsilon \in]0,\varepsilon_0]$ and suppose $(u_n)_n$ is a sequence in H_1^0 with $|u_n - u|_{H_1^\varepsilon} \to 0$ as $n \to \infty$ for some $u \in H_1^\varepsilon$. Part (c) of condition (Spec) implies that

$$|u_n - u_m|_{H_1^0} \leq C|u_n - u_m|_{H_1^\varepsilon},$$

so $(u_n)_n$ is a Cauchy sequence in the Banach space $(H_1^0, |\cdot|_{H_1^0})$. Therefore $(u_n)_n$ converges in H_1^0 to some v in H_1^0. But part (c) of condition (Spec) implies that

$$|u_n - v|_{H_1^\varepsilon} \le C|u_n - v|_{H_1^0}.$$

Hence $u = v$ and thus $u \in H_1^0$. This proves the proposition. □

REMARK 4.4. Note that, for α, $t \in]0,\infty[$ and $\lambda \in [0,\infty[$

$$\lambda^\alpha e^{-\lambda t} \le C(\alpha)t^{-\alpha} \quad \text{with } C(\alpha) = (\alpha/e)^\alpha.$$

Let $(H^\varepsilon, \langle\cdot,\cdot\rangle_{H^\varepsilon}, A_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ satisfy condition (Spec). Let $\alpha \in [0,\infty[$, $\varepsilon \in [0,\varepsilon_0]$ and $r \in]0,\infty[$. Using the above estimate, we obtain for every $u \in H_{-\alpha}^\varepsilon$

$$\begin{aligned}
|e^{-\widetilde{A}_\varepsilon r}u|_{H_1^\varepsilon}^2 &= \sum_{j=1}^\infty \lambda_{\varepsilon,j}^{\alpha+1}(e^{-\lambda_{\varepsilon,j}r})^2\lambda_{\varepsilon,j}^{-\alpha}|u(w_{\varepsilon,j})|^2 \\
&= \sum_{j=1}^\infty ((\lambda_{\varepsilon,j})^{(\alpha+1)/2}e^{-\lambda_{\varepsilon,j}r})^2\lambda_{\varepsilon,j}^{-\alpha}|u(w_{\varepsilon,j})|^2 \\
&\le (C((\alpha+1)/2)^2 r^{-(\alpha+1)})|u|_{H_{-\alpha}^\varepsilon}^2.
\end{aligned}$$

Consequently, we obtain for every $u \in H_{-\alpha}^\varepsilon$

$$|e^{-\widetilde{A}_\varepsilon r}u|_{H_1^\varepsilon} \le C_0 r^{-(\alpha+1)/2}|u|_{H_{-\alpha}^\varepsilon}, \tag{4.1}$$

where $C_0 = C((\alpha+1)/2)$.

We shall need these estimates in the results to follow.

We now prove our first result on the convergence of the linear semiflows.

THEOREM 4.5. *Let $(H^\varepsilon, \langle\cdot,\cdot\rangle_{H^\varepsilon}, A_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ satisfy condition* (Spec). *Suppose $(\varepsilon_n)_n$ is a sequence in $]0,\varepsilon_0]$ with $\varepsilon_n \to 0$. Let $u_0 \in H_1^0$ and $(u_n)_n$ be a sequence such that, for every $n \in \mathbb{N}$, $u_n \in H_1^{\varepsilon_n}$ and*

$$|u_n - u_0|_{H_1^{\varepsilon_n}} \to 0 \quad \text{as } n \to \infty.$$

Then

$$\sup_{t\in[0,\infty[} |e^{-tA_{\varepsilon_n}}u_n - e^{-tA_0}u_0|_{H_1^{\varepsilon_n}} \to 0 \quad \text{as } n \to \infty.$$

PROOF. Since $\lambda_{\varepsilon,j} > 0$ for all $\varepsilon \in]0,\varepsilon_0]$ and for all $j \in \mathbb{N}$, we have

$$|e^{-tA_\varepsilon}v|_{H_1^\varepsilon}^2 = \sum_{j=1}^\infty (e^{-t\lambda_{\varepsilon,j}})^2\lambda_{\varepsilon,j}|\langle v, w_{\varepsilon,j}\rangle_{H^\varepsilon}|^2 \le \sum_{j=1}^\infty \lambda_{\varepsilon,j}|\langle v, w_{\varepsilon,j}\rangle_{H^\varepsilon}|^2 = |v|_{H_1^\varepsilon}^2,$$

for all $v \in H_1^\varepsilon$, $\varepsilon \in]0,\varepsilon_0]$ and $t \in [0,\infty[$. Thus we obtain, for all $n \in \mathbb{N}$ and all $t \in [0,\infty[$,

$$\begin{aligned}
|e^{-tA_{\varepsilon_n}}u_n - e^{-tA_0}u_0|_{H_1^{\varepsilon_n}} &\le |e^{-tA_{\varepsilon_n}}(u_n - u_0)|_{H_1^{\varepsilon_n}} + |e^{-tA_{\varepsilon_n}}u_0 - e^{-tA_0}u_0|_{H_1^{\varepsilon_n}} \\
&\le |u_n - u_0|_{H_1^{\varepsilon_n}} + |e^{-tA_{\varepsilon_n}}u_0 - e^{-tA_0}u_0|_{H_1^{\varepsilon_n}}.
\end{aligned}$$

Therefore we only have to prove that

$$\sup_{t\in[0,\infty[} |e^{-tA_{\varepsilon_n}}u_0 - e^{-tA_0}u_0|_{H_1^{\varepsilon_n}} \to 0 \quad \text{as } n \to \infty. \tag{4.2}$$

Suppose (4.2) is not true. Then there are a $\delta_0 > 0$ and a sequence $(n_k)_k$ in $\mathbb{N}$ with $n_k \to \infty$ as $k \to \infty$ such that

$$\sup_{t\in[0,\infty[} |e^{-tA_{\varepsilon_{n_k}}}u_0 - e^{-tA_0}u_0|_{H_1^{\varepsilon_{n_k}}} \geq \delta_0 \quad \text{for all } k \in \mathbb{N}. \tag{4.3}$$

Taking a further subsequence, if necessary, and using condition (Spec) we may also assume that there exists an H^0-orthonormal sequence of eigenfunctions $(w_{0,j})_j$ corresponding to $(\lambda_{0,j})_j$ such that, for all $j \in \mathbb{N}$ and $u \in H^0$,

$$|w_{\varepsilon_{n_k},j} - w_{0,j}|_{H_1^{\varepsilon_{n_k}}} \to 0 \quad \text{and} \quad \langle u, w_{\varepsilon_{n_k},j}\rangle_{H^{\varepsilon_{n_k}}} \to \langle u, w_{0,j}\rangle_{H^0} \tag{4.4}$$

as $k \to \infty$. For each $k \in \mathbb{N}$ and $j \in \mathbb{N}$, let $P_{k,j}\colon H^{\varepsilon_{n_k}} \to H^{\varepsilon_{n_k}}$ be the $H^{\varepsilon_{n_k}}$-orthogonal projection of $H^{\varepsilon_{n_k}}$ onto the span of $\{ w_{\varepsilon_{n_k},1}, \ldots, w_{\varepsilon_{n_k},j-1} \}$ and let $P_{0,j}\colon H^0 \to H^0$ be the H^0-orthogonal projection of H^0 onto the span of $\{ w_{0,1}, \ldots, w_{0,j-1} \}$.

Let $t \in [0,\infty[$ be arbitrary. Then for each $j \in \mathbb{N}$ and each $k \in \mathbb{N}$ we have

$$\begin{aligned} |e^{-tA_{\varepsilon_{n_k}}}u_0 - e^{-tA_0}u_0|_{H_1^{\varepsilon_{n_k}}} &\leq |P_{k,j}e^{-tA_{\varepsilon_{n_k}}}u_0 - P_{0,j}e^{-tA_0}u_0|_{H_1^{\varepsilon_{n_k}}} \\ &\quad + |(I - P_{k,j})e^{-tA_{\varepsilon_{n_k}}}u_0|_{H_1^{\varepsilon_{n_k}}} + |(I - P_{0,j})e^{-tA_0}u_0|_{H_1^{\varepsilon_{n_k}}}. \end{aligned}$$

Notice that for each $j \in \mathbb{N}$,

$$|P_{k,j}u_0 - P_{0,j}u_0|_{H_1^{\varepsilon_{n_k}}} \to 0 \quad \text{as } k \to \infty. \tag{4.5}$$

Indeed, for each $k \in \mathbb{N}$ we have

$$\begin{aligned} |P_{k,j}u_0 - P_{0,j}u_0|_{H_1^{\varepsilon_{n_k}}} &= \bigg| \sum_{i=1}^{j-1} \langle u_0, w_{\varepsilon_{n_k},i}\rangle_{H^{\varepsilon_{n_k}}} w_{\varepsilon_{n_k},i} - \sum_{i=1}^{j-1} \langle u_0, w_{0,i}\rangle_{H^0} w_{0,i} \bigg|_{H_1^{\varepsilon_{n_k}}} \\ &\leq \sum_{i=1}^{j-1} |\langle u_0, w_{\varepsilon_{n_k},i}\rangle_{H^{\varepsilon_{n_k}}}| \, |w_{\varepsilon_{n_k},i} - w_{0,i}|_{H_1^{\varepsilon_{n_k}}} \\ &\quad + \sum_{i=1}^{j-1} |\langle u_0, w_{\varepsilon_{n_k},i}\rangle_{H^{\varepsilon_{n_k}}} - \langle u_0, w_{0,i}\rangle_{H^0}| \, |w_{0,i}|_{H_1^{\varepsilon_{n_k}}}. \end{aligned}$$

Condition (Spec) and (4.4) now imply (4.5).

Let $\delta > 0$ be arbitrary. By (3.1), $|(I - P_{0,j})u_0|_{H_1^0} \to 0$ as $j \to \infty$, so there is a $j_0 \in \mathbb{N}$ such that

$$|(I - P_{0,j_0})u_0|_{H_1^0} < \delta.$$

Since $|(I - P_{k,j})u_0 - (I - P_{0,j})u_0|_{H_1^{\varepsilon_{n_k}}} = |P_{k,j}u_0 - P_{0,j}u_0|_{H_1^{\varepsilon_{n_k}}}$ for all $j \in \mathbb{N}$ and for all $k \in \mathbb{N}$, it follows from (4.5) that there is a $k_0 \in \mathbb{N}$ such that

$$|(I - P_{k,j_0})u_0 - (I - P_{0,j_0})u_0|_{H_1^{\varepsilon_{n_k}}} < \delta \quad \text{for all } k \geq k_0. \tag{4.6}$$

Hence for all $k \geq k_0$,

$$
\begin{aligned}
(4.7)\ |(I - P_{k,j_0})e^{-tA_{\varepsilon_{n_k}}} u_0|_{H_1^{\varepsilon_{n_k}}} &= |e^{-tA_{\varepsilon_{n_k}}}(I - P_{k,j_0})u_0|_{H_1^{\varepsilon_{n_k}}} \\
&\leq |(I - P_{k,j_0})u_0|_{H_1^{\varepsilon_{n_k}}} \leq \delta + |(I - P_{0,j_0})u_0|_{H_1^{\varepsilon_{n_k}}} \\
&\leq \delta + C|(I - P_{0,j_0})u_0|_{H_1^0} \leq (1 + C)\delta.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
(4.8) \qquad |(I - P_{0,j_0})e^{-tA_0}u_0|_{H_1^{\varepsilon_{n_k}}} &\leq C|(I - P_{0,j_0})e^{-tA_0}u_0|_{H_1^0} \\
&= C|e^{-tA_0}(I - P_{0,j_0})u_0|_{H_1^0} \\
&\leq C|(I - P_{0,j_0})u_0|_{H_1^0} \leq C\delta.
\end{aligned}
$$

We further have

$$
\begin{aligned}
&|P_{k,j_0}e^{-tA_{\varepsilon_{n_k}}} u_0 - P_{0,j_0}e^{-tA_0}u_0|_{H_1^{\varepsilon_{n_k}}} \\
&\quad \leq \sum_{i=1}^{j_0-1} |e^{-t\lambda_{\varepsilon_{n_k},i}} \langle u_0, w_{\varepsilon_{n_k},i}\rangle_{H^{\varepsilon_{n_k}}} w_{\varepsilon_{n_k},i} - e^{-t\lambda_{0,i}} \langle u_0, w_{0,i}\rangle_{H^0} w_{0,i}|_{H_1^{\varepsilon_{n_k}}} \\
&\quad \leq \sum_{i=1}^{j_0-1} |e^{-t\lambda_{\varepsilon_{n_k},i}} \langle u_0, w_{\varepsilon_{n_k},i}\rangle_{H^{\varepsilon_{n_k}}} (w_{\varepsilon_{n_k},i} - w_{0,i})|_{H_1^{\varepsilon_{n_k}}} \\
&\qquad + \sum_{i=1}^{j_0-1} |e^{-t\lambda_{\varepsilon_{n_k},i}} \langle u_0, w_{\varepsilon_{n_k},i}\rangle_{H^{\varepsilon_{n_k}}} w_{0,i} - e^{-t\lambda_{0,i}} \langle u_0, w_{0,i}\rangle_{H^0} w_{0,i}|_{H_1^{\varepsilon_{n_k}}} \\
&\quad \leq \sum_{i=1}^{j_0-1} |\langle u_0, w_{\varepsilon_{n_k},i}\rangle_{H^{\varepsilon_{n_k}}}|\, |w_{\varepsilon_{n_k},i} - w_{0,i}|_{H_1^{\varepsilon_{n_k}}} \\
&\qquad + C \sum_{i=1}^{j_0-1} |e^{-t\lambda_{\varepsilon_{n_k},i}} \langle u_0, w_{\varepsilon_{n_k},i}\rangle_{H^{\varepsilon_{n_k}}} - e^{-t\lambda_{0,i}} \langle u_0, w_{0,i}\rangle_{H^0}|\, |w_{0,i}|_{H_1^0}.
\end{aligned}
$$

Since, for every $i \in \mathbb{N}$,

$$
\sup_{t \in [0,\infty[} |e^{-t\lambda_{\varepsilon_{n_k},i}} - e^{-t\lambda_{0,i}}| \to 0 \quad \text{as } k \to \infty,
$$

it follows that

$$
(4.9) \qquad \sup_{t \in [0,\infty[} |P_{k,j_0}e^{-tA_{\varepsilon_{n_k}}} u_0 - P_{0,j_0}e^{-tA_0}u_0|_{H_1^{\varepsilon_{n_k}}} \to 0 \quad \text{as } k \to \infty.
$$

Since $\delta > 0$ is arbitrary, (4.7), (4.8) and (4.9) imply that

$$
(4.10) \qquad \sup_{t \in [0,\infty[} |e^{-tA_{\varepsilon_{n_k}}} u_0 - e^{-tA_0}u_0|_{H_1^{\varepsilon_{n_k}}} \to 0 \quad \text{as } k \to \infty,
$$

but this contradicts (4.3). The proof is complete. □

We also require a second, more technical, theorem on the convergence of the linear semiflows.

Theorem 4.6. *Let $(H^\varepsilon, \langle\,\cdot\,,\,\cdot\,\rangle_{H^\varepsilon}, A_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ satisfy condition* (Spec). *Suppose $(\varepsilon_n)_n$ is a sequence in $]0,\varepsilon_0]$ with $\varepsilon_n \to 0$. Let $\alpha \in [0,\infty[$, $u_0 \in H^0_{-\alpha}$ be arbitrary and let $(u_n)_n$ and $(v_n)_n$ be sequences such that u_n and $v_n \in H^{\varepsilon_n}_{-\alpha}$ for $n \in \mathbb{N}$. Suppose that*

(a) $|u_n - v_n|_{H^{\varepsilon_n}_{-\alpha}} \to 0$ *as $n \to \infty$.*
(b) *For all $j \in \mathbb{N}$, $v_n(w_{\varepsilon_n,j}) \to u_0(w_{0,j})$ as $n \to \infty$.*
(c) $\sup_{n\in\mathbb{N}} |v_n|_{H^{\varepsilon_n}_{-\alpha}} < \infty$.

For every $\varepsilon \in [0,\varepsilon_0]$, let $\widetilde{A}_\varepsilon = \widetilde{A}_{\varepsilon,-\alpha}\colon H^\varepsilon_{2-\alpha} \to H^\varepsilon_{-\alpha}$ be the extension of A_ε to $H^\varepsilon_{-\alpha}$. Then, for every $\beta \in]0,\infty[$,

$$\sup_{t\in[\beta,\infty[} |e^{-t\widetilde{A}_{\varepsilon_n}} u_n - e^{-t\widetilde{A}_0} u_0|_{H^{\varepsilon_n}_1} \to 0 \quad \textit{as } n \to \infty.$$

Proof. Fix $\beta \in]0,\infty[$. Suppose the theorem is not true. Then there are a $\delta_0 > 0$ and a sequence $(n_k)_k$ in $\mathbb{N}$ with $n_k \to \infty$ as $k \to \infty$ such that

$$\sup_{t\in[\beta,\infty[} |e^{-t\widetilde{A}_{\varepsilon_{n_k}}} u_{n_k} - e^{-t\widetilde{A}_0} u_0|_{H^{\varepsilon_{n_k}}_1} \geq \delta_0 \quad \text{for all } k \in \mathbb{N}. \tag{4.11}$$

Taking a further subsequence, if necessary, and using condition (Spec) we may also assume that there exists an H^0-orthonormal sequence of eigenfunctions $(w_{0,j})_j$ corresponding to $(\lambda_{0,j})_j$ such that, for all $j \in \mathbb{N}$ and $u \in H^0$,

$$|w_{\varepsilon_{n_k},j} - w_{0,j}|_{H^{\varepsilon_{n_k}}_1} \to 0 \quad \text{and} \quad \langle u, w_{\varepsilon_{n_k},j}\rangle_{H^{\varepsilon_{n_k}}} \to \langle u, w_{0,j}\rangle_{H^0} \tag{4.12}$$

as $k \to \infty$. Let $\delta > 0$ be arbitrary. By Remark 4.4 there is an $s_0 = s_0(\delta,\beta) > 0$ such that $s^{(\alpha+1)/2} e^{-st} < \delta$ for $s \geq s_0$ and $t \geq \beta$. Since $\lambda_{0,j} \to \infty$ as $j \to \infty$, there is a $j_0 = j_0(\delta,\beta) \in \mathbb{N}$ such that $\lambda_{0,j_0} > s_0$ for all $j \geq j_0$. Thus there is an $k_0 = k_0(\delta,\beta) \in \mathbb{N}$ such that $\lambda_{\varepsilon_{n_k},j_0} > s_0$ for $k \geq k_0$. Therefore we obtain

$$\lambda_{\varepsilon_{n_k},j} \geq s_0(\delta,\beta) \quad \text{for } k \geq k_0(\delta,\beta) \text{ and } j \geq j_0(\delta,\beta). \tag{4.13}$$

Formula (3.6) implies that, for all $\varepsilon \in [0,\varepsilon_0]$, all $t \in]0,\infty[$ and all $u \in H^\varepsilon_{-\alpha}$,

$$\Big| e^{-t\widetilde{A}_\varepsilon} u - \sum_{j=1}^{k} e^{-t\lambda_{\varepsilon,j}} u(w_{\varepsilon,j}) w_{\varepsilon,j} \Big|_{H^\varepsilon_1} \to 0 \quad \text{as } k \to \infty. \tag{4.14}$$

Let $t \geq \beta$ be arbitrary. Then

$$\begin{aligned} &|e^{-t\widetilde{A}_{\varepsilon_{n_k}}} u_{n_k} - e^{-t\widetilde{A}_0} u_0|_{H^{\varepsilon_{n_k}}_1} \\ &\quad \leq \sum_{j=1}^{j_0-1} |e^{-t\lambda_{\varepsilon_{n_k},j}} u_{n_k}(w_{\varepsilon_{n_k},j}) w_{\varepsilon_{n_k},j} - e^{-t\lambda_{0,j}} u_0(w_{0,j}) w_{0,j}|_{H^{\varepsilon_{n_k}}_1} \end{aligned} \tag{4.15}$$

$$+\left|e^{-t\widetilde{A}_{\varepsilon_{n_k}}}u_{n_k}-\sum_{j=1}^{j_0-1}e^{-t\lambda_{\varepsilon_{n_k},j}}u_{n_k}(w_{\varepsilon_{n_k},j})w_{\varepsilon_{n_k},j}\right|_{H_1^{\varepsilon_{n_k}}}$$
$$+\left|e^{-t\widetilde{A}_0}u_0-\sum_{j=1}^{j_0-1}e^{-t\lambda_{0,j}}u_0(w_{0,j})w_{0,j}\right|_{H_1^{\varepsilon_{n_k}}}.$$

Now (4.13) implies that

$$\begin{aligned}(4.16)\qquad &\left|e^{-t\widetilde{A}_{\varepsilon_{n_k}}}u_{n_k}-\sum_{j=1}^{j_0-1}e^{-t\lambda_{\varepsilon_{n_k},j}}u_{n_k}(w_{\varepsilon_{n_k},j})w_{\varepsilon_{n_k},j}\right|^2_{H_1^{\varepsilon_{n_k}}}\\ &=\sum_{j=j_0}^{\infty}(\lambda_{\varepsilon_{n_k},j}^{(\alpha+1)/2}e^{-t\lambda_{\varepsilon_{n_k},j}})^2\lambda_{\varepsilon_{n_k},j}^{-\alpha}|u_{n_k}(w_{\varepsilon_{n_k},j})|^2\\ &\le\delta^2\sum_{j=j_0}^{\infty}\lambda_{\varepsilon_{n_k},j}^{-\alpha}|u_{n_k}(w_{\varepsilon_{n_k},j})|^2\le\delta^2|u_{n_k}|^2_{H_{-\alpha}^{\varepsilon_{n_k}}}\le\delta^2\widetilde{C}^2,\end{aligned}$$

where $\widetilde{C}:=\sup_{k\in\mathbb{N}}|u_{n_k}|^2_{H_{-\alpha}^{\varepsilon_{n_k}}}$. Note that $\widetilde{C}<\infty$ by our assumptions (a) and (c). Analogously,

$$\begin{aligned}(4.17)\qquad &\left|e^{-t\widetilde{A}_0}u_0-\sum_{j=1}^{j_0-1}e^{-t\lambda_{0,j}}u_0(w_{0,j})w_{0,j}\right|^2_{H_1^{\varepsilon_{n_k}}}\\ &\le C^2\left|e^{-t\widetilde{A}_0}u_0-\sum_{j=1}^{j_0-1}e^{-t\lambda_{0,j}}u_0(w_{0,j})w_{0,j}\right|^2_{H_1^0}\\ &=C^2\sum_{j=j_0}^{\infty}(\lambda_{0,j}^{(\alpha+1)/2}e^{-t\lambda_{0,j}})^2\lambda_{0,j}^{-\alpha}|u_0(w_{0,j})|^2\\ &\le C^2\delta^2\sum_{j=j_0}^{\infty}\lambda_{0,j}^{-\alpha}|u_0(w_{0,j})|^2\le C^2\delta^2|u_0|^2_{H_{-\alpha}^0}.\end{aligned}$$

Let $j\in[1\,..\,j_0-1]$ be arbitrary. Then

$$\begin{aligned}(4.18)\qquad &|e^{-t\lambda_{\varepsilon_{n_k},j}}u_{n_k}(w_{\varepsilon_{n_k},j})w_{\varepsilon_{n_k},j}-e^{-t\lambda_{0,j}}u_0(w_{0,j})w_{0,j}|_{H_1^{\varepsilon_{n_k}}}\\ &\le|e^{-t\lambda_{\varepsilon_{n_k},j}}(u_{n_k}-v_{n_k})(w_{\varepsilon_{n_k},j})w_{\varepsilon_{n_k},j}|_{H_1^{\varepsilon_{n_k}}}\\ &\quad+|e^{-t\lambda_{\varepsilon_{n_k},j}}v_{n_k}(w_{\varepsilon_{n_k},j})(w_{\varepsilon_{n_k},j}-w_{0,j})|_{H_1^{\varepsilon_{n_k}}}\\ &\quad+|e^{-t\lambda_{\varepsilon_{n_k},j}}(v_{n_k}(w_{\varepsilon_{n_k},j})-u_0(w_{0,j}))w_{0,j}|_{H_1^{\varepsilon_{n_k}}}\\ &\quad+|(e^{-t\lambda_{\varepsilon_{n_k},j}}-e^{-t\lambda_{0,j}})u_0(w_{0,j})w_{0,j}|_{H_1^{\varepsilon_{n_k}}}\\ &\le|u_{n_k}-v_{\varepsilon_{n_k},j}|_{H_{-\alpha}^{\varepsilon_{n_k}}}|w_{\varepsilon_{n_k},j}|_{H_{\alpha}^{\varepsilon_{n_k}}}|w_{\varepsilon_{n_k},j}|_{H_1^{\varepsilon_{n_k}}}\\ &\quad+|v_{n_k}|_{H_{-\alpha}^{\varepsilon_{n_k}}}|w_{\varepsilon_{n_k},j}|_{H_{\alpha}^{\varepsilon_{n_k}}}\cdot|w_{\varepsilon_{n_k},j}-w_{0,j}|_{H_1^{\varepsilon_{n_k}}}\end{aligned}$$

$$+ |v_{n_k}(w_{\varepsilon_{n_k},j}) - u_0(w_{0,j})| \cdot |w_{0,j}|_{H_1^{\varepsilon_{n_k}}}$$
$$+ |e^{-t\lambda_{\varepsilon_{n_k},j}} - e^{-t\lambda_{0,j}}| \cdot |u_0(w_{0,j})| \cdot |w_{0,j}|_{H_1^{\varepsilon_{n_k}}}.$$

Note that, for every $\gamma \in [0,\infty[$, $|w_{\varepsilon_{n_k},j}|_{H_\gamma^{\varepsilon_{n_k}}} = \lambda_{\varepsilon_{n_k},j}^{\gamma/2}$. Moreover, $|w_{0,j}|_{H_1^{\varepsilon_{n_k}}} \leq C|w_{0,j}|_{H_1^0}$ and

$$\sup_{t\in[\beta,\infty[} |e^{-t\lambda_{\varepsilon_{n_k},j}} - e^{-t\lambda_{0,j}}| \to 0 \quad \text{as } k \to \infty.$$

Hence, our assumptions and (4.18) show that

$$\sup_{t\in[\beta,\infty[} |e^{-t\lambda_{\varepsilon_{n_k},j}} u_{n_k}(w_{\varepsilon_{n_k},j}) w_{\varepsilon_{n_k},j} - e^{-t\lambda_{0,j}} u_0(w_{0,j}) w_{0,j}|_{H_1^{\varepsilon_{n_k}}} \to 0 \tag{4.19}$$

as $k \to \infty$. Thus formulas (4.15)–(4.17), (4.19) and the fact that $\delta > 0$ is arbitrary imply that

$$\sup_{t\in[\beta,\infty[} |e^{-t\widetilde{A}_{\varepsilon_{n_k}}} u_{n_k} - e^{-t\widetilde{A}_0} u_0|_{H_1^{\varepsilon_{n_k}}} \to 0 \quad \text{as } k \to \infty$$

which contradicts (4.11). The theorem is proved. □

Corollary 4.7. *Let* $(H^\varepsilon, \langle\cdot\,,\,\cdot\rangle_{H^\varepsilon}, A_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ *satisfy condition* (Spec). *Suppose* $(\varepsilon_n)_n$ *is a sequence in* $]0,\varepsilon_0]$ *with* $\varepsilon_n \to 0$. *Let* $u_0 \in H^0$ *be arbitrary and let* $(u_n)_n$ *be a sequence such that* $u_n \in H^{\varepsilon_n}$ *for* $n \in \mathbb{N}$. *Suppose that*

$$|u_n - u_0|_{H^{\varepsilon_n}} \to 0 \quad \text{as } n \to \infty.$$

Then, for every $\beta \in]0,\infty[$,

$$\sup_{t\in[\beta,\infty[} |e^{-tA_{\varepsilon_n}} u_n - e^{-tA_0} u_0|_{H_1^{\varepsilon_n}} \to 0 \quad \text{as } n \to \infty.$$

Proof. Use Theorem 4.6 with $\alpha = 0$ and $v_n = u_0$ for all $n \in \mathbb{N}$. □

5. Singular convergence of nonlinear semiflows

We now introduce a natural condition on a family of nonlinearities, condition (Conv), and we show that conditions (Spec) and (Conv) imply a general singular convergence theorem for semiflows.

Definition 5.1. Let $\varepsilon_0 > 0$ be arbitrary and $(H^\varepsilon, \langle\cdot\,,\,\cdot\rangle_{H^\varepsilon}, A_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ be a family satisfying condition (Spec). Let $\alpha \in [0,1[$ be given and for every $\varepsilon \in [0,\varepsilon_0]$ let $\widetilde{A}_\varepsilon = \widetilde{A}_{\varepsilon,-\alpha}: H_{2-\alpha}^\varepsilon \to H_{-\alpha}^\varepsilon$ be the extension of A_ε to $H_{-\alpha}^\varepsilon$. We say that the family $(f_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ of maps satisfies condition (Conv) if the following properties are satisfied:

(a) $f_\varepsilon: H_1^\varepsilon \to H_{-\alpha}^\varepsilon$ for every $\varepsilon \in [0,\varepsilon_0]$.
(b) $\lim_{\varepsilon\to 0^+} |e^{-t\widetilde{A}_\varepsilon} f_\varepsilon(u) - e^{-t\widetilde{A}_0} f_0(u)|_{H_1^\varepsilon} = 0$ for every $u \in H_1^0$ and every $t \in]0,\infty[$.

(c) For every $M \in [0,\infty[$ there is an $L = L_M \in [0,\infty[$ such that

$$|f_\varepsilon(u) - f_\varepsilon(v)|_{H^\varepsilon_{-\alpha}} \le L|u-v|_{H^\varepsilon_1}$$

for all $\varepsilon \in [0,\varepsilon_0]$ and $u, v \in H^\varepsilon_1$ satisfying $|u|_{H^\varepsilon_1}, |v|_{H^\varepsilon_1} \le M$.

(d) For every $u \in H^0_1$ there is an $\varepsilon'_0 \in]0,\varepsilon_0]$ such that

$$\sup_{\varepsilon\in[0,\varepsilon'_0]} |f_\varepsilon(u)|_{H^\varepsilon_{-\alpha}} < \infty.$$

The next result shows that the above condition (b) is valid uniformly for t bounded away from zero.

PROPOSITION 5.2. *Assume condition* (Conv) *and let $\beta \in]0,\infty]$ be arbitrary. Then, for every $u \in H^0_1$,*

$$\lim_{\varepsilon\to 0^+} \sup_{t\in[\beta,\infty[} |e^{-t\widetilde{A}_\varepsilon} f_\varepsilon(u) - e^{-t\widetilde{A}_0} f_0(u)|_{H^\varepsilon_1} = 0$$

PROOF. Let $v = e^{-\beta\widetilde{A}_0} f_0(u) \in H^0_1$. For every $t \in [\beta,\infty[$ we have

$$\begin{aligned}
&|e^{-t\widetilde{A}_\varepsilon} f_\varepsilon(u) - e^{-t\widetilde{A}_0} f_0(u)|_{H^\varepsilon_1} \\
&\le |e^{-(t-\beta)\widetilde{A}_\varepsilon}(e^{-\beta\widetilde{A}_\varepsilon} f_\varepsilon(u) - e^{-\beta\widetilde{A}_0} f_0(u))|_{H^\varepsilon_1} + |e^{-(t-\beta)\widetilde{A}_\varepsilon} v - e^{-(t-\beta)\widetilde{A}_0} v|_{H^\varepsilon_1} \\
&\le |e^{-\beta\widetilde{A}_\varepsilon} f_\varepsilon(u) - e^{-\beta\widetilde{A}_0} f_0(u)|_{H^\varepsilon_1} + |e^{-(t-\beta)\widetilde{A}_\varepsilon} v - e^{-(t-\beta)\widetilde{A}_0} v|_{H^\varepsilon_1}
\end{aligned}$$

Since, by Theorem 4.5

$$\lim_{\varepsilon\to 0} \sup_{s\in[0,\infty[} |e^{-s\widetilde{A}_\varepsilon} v - e^{-s\widetilde{A}_0} v|_{H^\varepsilon_1} = 0,$$

the assertion follows from condition (Conv) part (b) (with $t = \beta$). $\square$

For the rest of the paper, if $(H^\varepsilon, \langle\cdot\,,\cdot\rangle_{H^\varepsilon}, A_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ satisfies condition (Spec) and $(f_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ satisfies condition (Conv) then we will write, for every $\varepsilon \in [0,\varepsilon_0]$, $\pi_\varepsilon := \pi_{A_\varepsilon, f_\varepsilon}$ to denote the local semiflow on H^ε_1 generated by the abstract parabolic equation

$$\dot{u} = -\widetilde{A}_\varepsilon u + f_\varepsilon(u). \tag{5.1}$$

To prove the theorems of this section we will need the following auxiliary result.

LEMMA 5.3. *Suppose that $(H^\varepsilon, \langle\cdot\,,\cdot\rangle_{H^\varepsilon}, A_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ satisfies condition* (Spec) *and $(f_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ satisfies condition* (Conv). *For every $\overline{u} \in H^0_1$ there exist a $\delta > 0$ and a $\tau > 0$ such that for every $a_0 \in H^0_1$ with $|a_0 - \overline{u}|_{H^0_1} \le \delta$, $a_0\pi_0 s$, $s \in [0,\tau]$, is defined and whenever $(\varepsilon_n)_n$ is a sequence in $]0,\varepsilon_0]$ with $\varepsilon_n \to 0$ and $(a_n)_n$ is a sequence with $a_n \in H^{\varepsilon_n}_1$ for every $n \in \mathbb{N}$ and*

$$|a_n - a_0|_{H^{\varepsilon_n}_1} \to 0 \quad \text{as } n \to \infty,$$

then there is an $n_0 \in \mathbb{N}$ *such that* $a_n \pi_{\varepsilon_n} s$, $s \in [0,\tau]$, *is defined for* $n \geq n_0$. *Moreover there exists an* $M' \in [0,\infty[$ *such that* $|a_n \pi_{\varepsilon_n} s|_{H_1^{\varepsilon_n}} \leq M'$ *for all* $n \geq n_0$ *and for all* $s \in [0,\tau]$.

Proof. Let $\overline{u} \in H_1^0$ be arbitrary and $C_1 \in]0,\infty[$ be such that $|\overline{u}|_{H_1^0} \leq C_1$. Let C be as in part (c) of condition (Spec), set

$$M' := 3C_1 + 3CC_1 \tag{5.2}$$

and let $L := L_{M'}$ be as in Definition 5.1 with M replaced by M'.

Part (d) of Definition 5.1 implies that there is an $\varepsilon_0' \in]0,\varepsilon_0]$ such that

$$C_2 = \sup_{\varepsilon \in [0,\varepsilon_0']} |f_\varepsilon(\overline{u})|_{H^\varepsilon} < \infty.$$

Now choose τ and $\delta \in]0,\infty[$ such that

$$2(1-\alpha)^{-1} C_0 L \tau^{(1-\alpha)/2} \leq 1/2, \tag{5.3}$$

$$2(1-\alpha)^{-1} C_0 \tau^{(1-\alpha)/2} (2LC_1 + C_2) \leq C_1/4, \tag{5.4}$$

$$2C\delta \leq C_1/4 \tag{5.5}$$

and

$$C|e^{-tA_0}\overline{u} - \overline{u}|_{H_1^0} \leq C_1/4 \quad \text{for } t \in [0,\tau], \tag{5.6}$$

where the constant C_0 is as in Remark 4.4.

For every $\varepsilon \in [0,\varepsilon_0']$ and $a \in H_1^\varepsilon$ with

$$|a - \overline{u}|_{H_1^\varepsilon} \leq C_1, \tag{5.7}$$

define

$$\begin{aligned} S_{\varepsilon,a} := \{\, u \mid u\colon [0,\tau] \to H_1^\varepsilon &\text{ is continuous} \\ &\text{and } |u(t) - a|_{H_1^\varepsilon} \leq C_1 \text{ for all } t \in [0,\tau]\}. \end{aligned}$$

For $u \in S_{\varepsilon,a}$ define the map $T_{\varepsilon,a}(u)\colon [0,\tau] \to H_1^\varepsilon$ by

$$\begin{aligned} T_{\varepsilon,a}(u)(t) :&= e^{-t\widetilde{A}_\varepsilon} a + \int_0^t e^{-(t-s)\widetilde{A}_\varepsilon} f_\varepsilon(u(s))\,ds \\ &= e^{-tA_\varepsilon} a + \int_0^t e^{-(t-s)\widetilde{A}_\varepsilon} f_\varepsilon(u(s))\,ds. \end{aligned}$$

The map $T_{\varepsilon,a}(u)$ is continuous. Moreover, whenever $u \in S_{\varepsilon,a}$, then, for all $t \in [0,\tau]$,

$$|u(t)|_{H_1^\varepsilon} \leq C_1 + |a|_{H_1^\varepsilon} \leq C_1 + C_1 + |\overline{u}|_{H_1^\varepsilon} \leq 2C_1 + C_1 C \leq M',$$

where the last inequality follows from (5.2). Thus for all $u, v \in S_{\varepsilon,a}$ arbitrary and for all $t \in [0,\tau]$, we have, by (4.1),

$$
\begin{aligned}
(5.8)\qquad & |T_{\varepsilon,a}(u)(t) - T_{\varepsilon,a}(v)(t)|_{H_1^\varepsilon} \\
&= \left| \int_0^t e^{-(t-s)\widetilde{A}_\varepsilon} (f_\varepsilon(u(s)) - f_\varepsilon(v(s)))\, ds \right|_{H_1^\varepsilon} \\
&\le C_0 \int_0^t (t-s)^{-(\alpha+1)/2} |f_\varepsilon(u(s)) - f_\varepsilon(v(s))|_{H_{-\alpha}^\varepsilon}\, ds \\
&\le C_0 L \int_0^t (t-s)^{-(\alpha+1)/2}\, ds \sup_{s\in[0,\tau]} |u(s) - v(s)|_{H_1^\varepsilon} \\
&= 2(1-\alpha)^{-1} C_0 L \tau^{(1-\alpha)/2} \sup_{s\in[0,\tau]} |u(s) - v(s)|_{H_1^\varepsilon} \\
&\le 1/2 \sup_{s\in[0,\tau]} |u(s) - v(s)|_{H_1^\varepsilon}.
\end{aligned}
$$

The last inequality follows from (5.3). Moreover, for all $u \in S_{\varepsilon,a}$ and $t \in [0,\tau]$,

$$
|T_{\varepsilon,a}(u)(t) - a|_{H_1^\varepsilon} \le |e^{-tA_\varepsilon} a - a|_{H_1^\varepsilon} + \left| \int_0^t e^{-(t-s)\widetilde{A}_\varepsilon} f_\varepsilon(u(s))\, ds \right|_{H_1^\varepsilon}.
$$

Since for $\varepsilon \in [0, \varepsilon_0']$ and $s \in [0,\tau]$ we have

$$
\begin{aligned}
|f_\varepsilon(u(s))|_{H_{-\alpha}^\varepsilon} &\le |f_\varepsilon(u(s)) - f_\varepsilon(a)|_{H_{-\alpha}^\varepsilon} + |f_\varepsilon(a)|_{H_{-\alpha}^\varepsilon} \\
&\le L|u(s) - a|_{H_1^\varepsilon} + |f_\varepsilon(a) - f_\varepsilon(\overline{u})|_{H_{-\alpha}^\varepsilon} + |f_\varepsilon(\overline{u})|_{H_{-\alpha}^\varepsilon} \\
&\le LC_1 + LC_1 + C_2 = 2LC_1 + C_2,
\end{aligned}
$$

we obtain, by (4.1),

$$
\begin{aligned}
\left| \int_0^t e^{-(t-s)\widetilde{A}_\varepsilon} f_\varepsilon(u(s))\, ds \right|_{H_1^\varepsilon} &\le C_0 \int_0^t (t-s)^{-(\alpha+1)/2} |f_\varepsilon(u(s))|_{H_{-\alpha}^\varepsilon}\, ds \\
&\le 2(1-\alpha)^{-1} C_0 \tau^{(1-\alpha)/2} (2LC_1 + C_2) \le C_1/4.
\end{aligned}
$$

In the previous computation we used the fact that $|a|_{H_1^\varepsilon} \le C_1 + |\overline{u}|_{H_1^\varepsilon} \le C_1 + CC_1 \le M'$ and $|\overline{u}|_{H_1^\varepsilon} \le M'$. If $\widetilde{a}_0 \in H_1^0$ satisfies $|\widetilde{a}_0 - \overline{u}|_{H_1^0} \le \delta$, then

$$
\begin{aligned}
|e^{-tA_\varepsilon} a - a|_{H_1^\varepsilon} \le & |e^{-tA_\varepsilon} a - e^{-tA_0}\widetilde{a}_0|_{H_1^\varepsilon} + |e^{-tA_0}\widetilde{a}_0 - e^{-tA_0}\overline{u}|_{H_1^\varepsilon} \\
& + |e^{-tA_0}\overline{u} - \overline{u}|_{H_1^\varepsilon} + |\overline{u} - \widetilde{a}_0|_{H_1^\varepsilon} + |a - \widetilde{a}_0|_{H_1^\varepsilon} \\
\le & |e^{-tA_\varepsilon} a - e^{-tA_0}\widetilde{a}_0|_{H_1^\varepsilon} + C|e^{-tA_0}\widetilde{a}_0 - e^{-tA_0}\overline{u}|_{H_1^0} \\
& + C|e^{-tA_0}\overline{u} - \overline{u}|_{H_1^0} + C|\overline{u} - \widetilde{a}_0|_{H_1^0} + |a - \widetilde{a}_0|_{H_1^\varepsilon} \\
\le & |e^{-tA_\varepsilon} a - e^{-tA_0}\widetilde{a}_0|_{H_1^\varepsilon} + |a - \widetilde{a}_0|_{H_1^\varepsilon} + C|e^{-tA_0}\overline{u} - \overline{u}|_{H_1^0} + 2C\delta \\
\le & |e^{-tA_\varepsilon} a - e^{-tA_0}\widetilde{a}_0|_{H_1^\varepsilon} + |a - \widetilde{a}_0|_{H_1^\varepsilon} + C_1/2,
\end{aligned}
$$

where the last inequality follows from (5.5) and (5.6).

Thus putting things together, we obtain for all $\varepsilon \in [0, \varepsilon_0']$, all $a \in H_1^\varepsilon$ satisfying (5.7) and all $\widetilde{a}_0 \in H_1^0$ with $|\widetilde{a}_0 - \overline{u}|_{H_1^0} \leq \delta$

$$|T_{\varepsilon,a}(u)(t) - a|_{H_1^\varepsilon} \leq 3C_1/4 + |e^{-tA_\varepsilon} a - e^{-tA_0}\widetilde{a}_0|_{H_1^\varepsilon} + |a - \widetilde{a}_0|_{H_1^\varepsilon}, \tag{5.9}$$

$u \in S_{\varepsilon,a}$, $t \in [0, \tau]$.

In particular, for all $a \in H_1^0$ satisfying $|a - \overline{u}|_{H_1^0} \leq \delta$ and all $u \in S_{0,a}$ we have

$$|T_{0,a}(u)(t) - a|_{H_1^0} \leq 3C_1/4 \leq C_1 \quad \text{for all } t \in [0, \tau].$$

Hence we conclude that $T_{0,a}(S_{0,a}) \subset S_{0,a}$ and so, by Banach Fixed Point Theorem, there is a unique fixed point of $T_{0,a}$ in $S_{0,a}$. In particular $a\ \pi_0 s$ is defined for all $s \in [0, \tau]$.

Now let $(\varepsilon_n)_n$ be a sequence in $]0, \varepsilon_0]$ with $\varepsilon_n \to 0$. Suppose $a_0 \in H_1^0$ satisfies $|a_0 - \overline{u}|_{H_1^0} \leq \delta$ and $(a_n)_n$ is a sequence with $a_n \in H_1^{\varepsilon_n}$ for every $n \in \mathbb{N}$ and $|a_n - a_0|_{H_1^{\varepsilon_n}} \to 0$. By what we just proved, it follows that $a_0 \pi_0 s$ is defined for all $s \in [0, \tau]$.

Theorem 4.5 implies that there is an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$

$$\sup_{t \in [0,\tau]} |e^{-tA_{\varepsilon_n}} a_n - e^{-tA_0} a_0|_{H_1^{\varepsilon_n}} + |a_n - a_0|_{H_1^{\varepsilon_n}} \leq C_1/4. \tag{5.10}$$

Thus, it follows from (5.9) that $T_{\varepsilon_n,a_n}(S_{\varepsilon_n,a_n}) \subset S_{\varepsilon_n,a_n}$ for all $n \geq n_0$. Hence T_{ε_n,a_n} has a fixed point in S_{ε_n,a_n}. In particular $a_n \pi_{\varepsilon_n} s$ is defined for all $s \in [0, \tau]$ and for all $n \geq n_0$.

Moreover, (5.9) and (5.10) imply that, for $s \in [0, \tau]$ and $n \geq n_0$,

$$|a_n \pi_{\varepsilon_n} s|_{H_1^{\varepsilon_n}} \leq |a_n \pi_{\varepsilon_n} s - a_n|_{H_1^{\varepsilon_n}} + |a_n - \overline{u}|_{H_1^{\varepsilon_n}} + |\overline{u}|_{H_1^{\varepsilon_n}} \leq 2C_1 + CC_1 \leq M'.$$

The lemma is proved. □

We can now state our first singular convergence result for semiflows.

Theorem 5.4. *Suppose that the family* $(H^\varepsilon, \langle\cdot\,,\cdot\rangle_{H^\varepsilon}, A_\varepsilon)_{\varepsilon \in [0,\varepsilon_0]}$ *satisfies condition* (Spec), *and the family* $(f_\varepsilon)_{\varepsilon \in [0,\varepsilon_0]}$ *satisfies condition* (Conv). *Let* $(\varepsilon_n)_n$ *be a sequence in* $]0, \varepsilon_0]$ *with* $\varepsilon_n \to 0$. *Let* $u_0 \in H_1^0$ *and* $(u_n)_n$ *be a sequence with* $u_n \in H_1^{\varepsilon_n}$ *for every* $n \in \mathbb{N}$ *and*

$$|u_n - u_0|_{H^{\varepsilon_n}} \to 0 \quad \textit{as } n \to \infty.$$

Let $b \in]0, \infty[$ *and suppose that* $u_n \pi_{\varepsilon_n} t$ *and* $u\pi_0 t$ *are defined for all* $n \in \mathbb{N}$ *and* $t \in [0, b]$. *Moreover suppose there exists an* $M' \in [0, \infty[$ *such that* $|u_n \pi_{\varepsilon_n} s|_{H_1^{\varepsilon_n}} \leq M'$ *for all* $n \in \mathbb{N}$ *and for all* $s \in [0, b]$. *Then for every* $t \in]0, b]$ *and every sequence* $(t_n)_n$ *in* $]0, b]$ *converging to* t

$$|u_n \pi_{\varepsilon_n} t_n - u_0 \pi_0 t_n|_{H_1^{\varepsilon_n}} \to 0 \quad \textit{as } n \to \infty.$$

PROOF. For every $t \in [0,b]$ we have, by the variation-of-constants formula,

$$\begin{aligned} u_n\pi_{\varepsilon_n}t - u_0\pi_0 t = & e^{-tA_{\varepsilon_n}}u_n - e^{-tA_0}u_0 \\ & + \int_0^t e^{-(t-s)\widetilde{A}_{\varepsilon_n}}(f_{\varepsilon_n}(u_n\pi_{\varepsilon_n}s) - f_{\varepsilon_n}(u_0\pi_0 s))\,ds \\ & + \int_0^t (e^{-(t-s)\widetilde{A}_{\varepsilon_n}}f_{\varepsilon_n}(u_0\pi_0 s) - e^{-(t-s)\widetilde{A}_0}f_0(u_0\pi_0 s))\,ds. \end{aligned}$$

Define the function $g_n\colon [0,b]\times[0,b]\to\mathbb{R}$ as follows: If $0<s<t$ then set

$$g_n(t,s) = |e^{-(t-s)\widetilde{A}_{\varepsilon_n}}f_{\varepsilon_n}(u_0\pi_0 s) - e^{-(t-s)\widetilde{A}_0}f_0(u_0\pi_0 s)|_{H_1^{\varepsilon_n}}$$

and set $g_n(t,s)=0$ otherwise. The function g_n restricted to the set of (s,t) with $0<s<t$ is continuous. Thus g_n is measurable on $[0,b]\times[0,b]$. By Fubini's theorem the function

$$c_n(t) := \int_0^b g_n(t,s)\,ds = \int_0^t g_n(t,s)\,ds$$

is almost everywhere defined and measurable on $[0,b]$. Set

$$a_n(t) := |e^{-tA_{\varepsilon_n}}u_n - e^{-tA_0}u_0|_{H_1^{\varepsilon_n}} + c_n(t) \quad \text{for } t\in\,]0,b].$$

It follows that a_n is measurable on $[0,b]$. Using (4.1) we obtain for $0<s<t$

$$\begin{aligned} |g_n(t,s)| &\le C_2C_0(t-s)^{-(\alpha+1)/2} + CC_2C_0(t-s)^{-(\alpha+1)/2} \\ &=: C_3(t-s)^{-(\alpha+1)/2} \end{aligned} \tag{5.11}$$

and so for $t\in\,]0,b]$

$$a_n(t) \le M' + C|u_0|_{H_1^0} + 2C_3(1-\alpha)^{-1}b^{(1-\alpha)/2} =: C_4,$$

where

$$C_2 := \max\left\{\sup_{s\in[0,b]}\sup_{n\in\mathbb{N}}|f_{\varepsilon_n}(u_0\pi_0 s)|_{H_{-\alpha}^{\varepsilon_n}},\ \sup_{s\in[0,b]}|f_0(u_0\pi_0 s)|_{H_{-\alpha}^0}\right\}.$$

Notice that condition (Conv) implies that $C_2<\infty$. Now let $t\in\,]0,b]$ be arbitrary and $(t_n)_n$ be any sequence in $]0,b]$ converging to 0. If $0<s<t$ then $0<s<t_n$ for all n large enough and so, by Proposition 5.2 $g_n(t_n,s)\to 0$ as $n\to\infty$. If $0<t<s$, then $0<t_n<s$ for all n large enough and so $g_n(t_n,s)=0$ for such n. Again $g_n(t_n,s)\to 0$ as $n\to\infty$. Thus (5.11) and the dominated convergence theorem imply that

$$c_n(t_n)\to 0 \quad \text{as } n\to\infty.$$

Thus, using Corollary 4.7 we obtain

$$a_n(t_n)\to 0, \quad \text{as } n\to\infty. \tag{5.12}$$

Notice that $\widetilde{M} := \sup_{s\in[0,b]} |u_0\pi_0 s|_{H_1^0} < \infty$. Hence $|u_0\pi_0 s|_{H_1^{\varepsilon_n}} \leq C\widetilde{M}$ for all $s \in [0,b]$. Set $M'' := \max\{M', C\widetilde{M}\}$ and let $L := L_{M''}$ be as in Definition 5.1 with M replaced by M''. We have, for all $r \in \,]0,b]$

$$\begin{aligned}
|u_n\pi_{\varepsilon_n} r - u_0\pi_0 r|_{H_1^{\varepsilon_n}} &\leq |e^{-rA_{\varepsilon_n}}u_n - e^{-rA_0}u_0|_{H_1^{\varepsilon_n}} \\
&\quad + \int_0^r |e^{-(r-s)\widetilde{A}_{\varepsilon_n}}(f_{\varepsilon_n}(u_n\pi_{\varepsilon_n}s) - f_{\varepsilon_n}(u_0\pi_0 s))|_{H_1^{\varepsilon_n}}\,ds \\
&\quad + \int_0^r |(e^{-(r-s)\widetilde{A}_{\varepsilon_n}}f_{\varepsilon_n}(u_0\pi_0 s) - e^{-(r-s)\widetilde{A}_0}f_0(u_0\pi_0 s))|_{H_1^{\varepsilon_n}}\,ds \\
&\leq a_n(r) + C_0\int_0^r (r-s)^{-(\alpha+1)/2}|f_{\varepsilon_n}(u_n\pi_{\varepsilon_n}s) - f_{\varepsilon_n}(u_0\pi_0 s))|_{H_{-\alpha}^{\varepsilon_n}}\,ds \\
&\leq a_n(r) + C_0 L\int_0^r (r-s)^{-(\alpha+1)/2}|u_n\pi_{\varepsilon_n}s - u_0\pi_0 s|_{H_1^{\varepsilon_n}}\,ds.
\end{aligned}$$

An application of Henry's Inequality [7, Lemma 7.1.1] implies that

$$|u_n\pi_{\varepsilon_n} r - u_0\pi_0 r|_{H_1^{\varepsilon_n}} \leq a_n(r) + \int_0^r \rho(r-s)a_n(s)\,ds \quad \text{for } r \in \,]0,b],$$

where

$$\rho(x) := \sum_{n=1}^{\infty} \frac{(C_0 L\Gamma(\beta))^n}{\Gamma(n\beta)} x^{n\beta-1}$$

with $\beta := (1-\alpha)/2$.

The function $\rho\colon]0,\infty[\, \to \,]0,\infty[$ is well defined and continuous on $]0,\infty[$ and it satisfies the estimate

$$\rho(x) \leq C_5 x^{-(\alpha+1)/2} + C_5 \quad \text{for } x \in \,]0,b].$$

Let t and $(t_n)_n$ be as above. Fix a $\delta_0 \in \,]0,t[$ and let $\delta \in \,]0,\delta_0/2[$ be arbitrary. There is an $n_0 = n_0(\delta) \in \mathbb{N}$ such that $|t_n - t| < \delta$ for $n \geq n_0$. Therefore for all such $n \in \mathbb{N}$ and all $s \in [0, t-2\delta]$ it follows that $t_n - s > \delta$ so $\rho(t_n - s) \leq C_5\delta^{-(\alpha+1)/2} + C_5$. Thus

$$\rho(t_n - s)a_n(s) \leq C_6 \quad \text{for } s \in \,]0, t-2\delta].$$

Therefore (5.12) (with $t_n \equiv s$) and the dominated convergence theorem show that

$$\int_0^{t-2\delta} \rho(t_n - s)a_n(s)\,ds \to 0 \quad \text{as } n \to \infty.$$

On the other hand,

$$\int_{t-2\delta}^{t_n} \rho(t_n - s)a_n(s)\,ds \leq C_7(\delta^{(1-\alpha)/2} + \delta).$$

Since $\delta \in \,]0,\delta_0/2[$ is arbitrary, it follows that

$$\int_0^{t_n} \rho(t_n - s)a_n(s)\,ds \to 0 \quad \text{as } n \to \infty.$$

Consequently,

$$|u_n\pi_{\varepsilon_n}t_n - u_0\pi_0 t_n|_{H_1^{\varepsilon_n}} \to 0 \quad \text{as } n\to\infty.$$

The theorem is proved. □

Our second convergence result reads as follows:

THEOREM 5.5. *Suppose that the family* $(H^\varepsilon, \langle\,\cdot\,,\,\cdot\,\rangle_{H^\varepsilon}, A_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ *satisfies condition* (Spec) *and the family* $(f_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ *satisfies condition* (Conv). *Let* $(\varepsilon_n)_n$ *be a sequence in* $]0,\varepsilon_0]$ *with* $\varepsilon_n\to 0$ *and let* $(t_n)_n$ *be a sequence in* $[0,\infty[$ *with* $t_n\to 0$. *Let* $u_0\in H_1^0$ *and* $(u_n)_n$ *be a sequence with* $u_n\in H_1^{\varepsilon_n}$ *for every* $n\in\mathbb{N}$ *and*

$$|u_n - u_0|_{H_1^{\varepsilon_n}} \to 0 \quad \textit{as } n\to\infty.$$

Then there exists an $n_0\in\mathbb{N}$ *such that* $u_0\pi_0 t_n$ *and* $u_n\pi_{\varepsilon_n}t_n$ *are defined for all* $n\ge n_0$ *and*

$$|u_n\pi_{\varepsilon_n}t_n - u_0\pi_0 t_n|_{H_1^{\varepsilon_n}} \to 0 \quad \textit{as } n\to\infty.$$

PROOF. Set $\overline{u} := u_0$ in Lemma 5.3 and let $\tau>0$ be as in that lemma. It follows from Lemma 5.3 that $u_0\pi_0 s$, $s\in[0,\tau]$, is defined and there is an $n_0\in\mathbb{N}$ such that $u_n\pi_{\varepsilon_n}s$, $s\in[0,\tau]$, is defined for $n\ge n_0$. Moreover there exists a $M'\ge 0$ such that $|u_n\pi_{\varepsilon_n}s|_{H_1^{\varepsilon_n}}\le M'$ for all $n\ge n_0$ and for all $s\in[0,\tau]$.

Since $t_n\to 0$ as $n\to\infty$, we may assume that $t_n\in[0,\tau]$ for all $n\in\mathbb{N}$. For every $t\in[0,\tau]$ we have

$$u_n\pi_{\varepsilon_n}t - u_0 = e^{-tA_{\varepsilon_n}}u_n - u_0 + \int_0^t e^{-(t-s)\widetilde{A}_{\varepsilon_n}} f_{\varepsilon_n}(u_n\pi_{\varepsilon_n}s)\,ds.$$

Notice that $\widetilde{M} := \sup_{s\in[0,b]}|u_0\pi_0 s|_{H_1^0} < \infty$. Hence $|u_0\pi_0 s|_{H_1^{\varepsilon_n}}\le C\widetilde{M}$ for all $s\in[0,\tau]$. Set $M'' := \max\{M', C\widetilde{M}\}$ and let $L := L_{M''}$ be as in Definition 5.1 with M replaced by M''. It follows that for all $n\ge n_0$ and for every $s\in[0,\tau]$

$$\begin{aligned}|f_{\varepsilon_n}(u_n\pi_{\varepsilon_n}s)|_{H_{-\alpha}^{\varepsilon_n}} &\le |f_{\varepsilon_n}(u_n\pi_{\varepsilon_n}s) - f_{\varepsilon_n}(u_0)|_{H_{-\alpha}^{\varepsilon_n}} + |f_{\varepsilon_n}(u_0)|_{H_{-\alpha}^{\varepsilon_n}}\\ &\le L|u_n\pi_{\varepsilon_n}s - u_0|_{H_1^{\varepsilon_n}} + |f_{\varepsilon_n}(u_0)|_{H_{-\alpha}^{\varepsilon_n}}\\ &\le L(M' + C|u_0|_{H_1^0}) + |f_{\varepsilon_n}(u_0)|_{H_{-\alpha}^{\varepsilon_n}}.\end{aligned}$$

Part (d) of condition (Conv) now implies

$$|f_{\varepsilon_n}(u_n\pi_{\varepsilon_n}s)|_{H_{-\alpha}^{\varepsilon_n}}\le\widetilde{C},\quad \text{for all } s\in[0,\tau] \text{ and for all } n\ge n_0,$$

for some positive constant $\widetilde{C}$. Therefore for all $n\ge n_0$

$$\begin{aligned}|u_n\pi_{\varepsilon_n}t_n - u_0\pi_0 t_n|_{H_1^{\varepsilon_n}} &\le |u_n\pi_{\varepsilon_n}t_n - u_0|_{H_1^{\varepsilon_n}} + C|u_0\pi_0 t_n - u_0|_{H_1^0}\\ \le& |e^{-t_nA_{\varepsilon_n}}u_n - e^{-t_nA_0}u_0|_{H_1^{\varepsilon_n}} + C|e^{-t_nA_0}u_0 - u_0|_{H_1^0}\\ &+ C_0\widetilde{C}\int_0^{t_n}(t_n-s)^{-(\alpha+1)/2}\,ds + C|u_0\pi_0 t_n - u_0|_{H_1^0}.\end{aligned}$$

Since $|e^{-t_nA_0}u_0 - u_0|_{H_1^0} \to 0$ and $|u_0\pi_0 t_n - u_0|_{H_1^0} \to 0$ as $n \to \infty$, an application of Theorem 4.5 completes the proof. □

Theorem 5.4 and Theorem 5.5 imply the following corollary.

Corollary 5.6. *Let* $(H^\varepsilon, \langle\,\cdot\,,\,\cdot\,\rangle_{H^\varepsilon}, A_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ *satisfy condition* (Spec) *and* $(f_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ *satisfy condition* (Conv). *Then for every* $\overline{u} \in H_1^0$ *there exist a* $\delta > 0$ *and a* $\tau > 0$ *such that for every* $a_0 \in H_1^0$ *with* $|a_0 - \overline{u}|_{H_1^0} \le \delta$, $a_0\pi_0 s$, $s \in [0,\tau]$, *is defined and whenever* $(\varepsilon_n)_n$ *is a sequence in* $]0,\varepsilon_0]$ *with* $\varepsilon_n \to 0$ *and* $(a_n)_n$ *is a sequence with* $a_n \in H_1^{\varepsilon_n}$ *for every* $n \in \mathbb{N}$ *and*

$$|a_n - a_0|_{H_1^{\varepsilon_n}} \to 0 \quad \textit{as } n \to \infty,$$

then there is an $n_0 \in \mathbb{N}$ *such that* $a_n\pi_{\varepsilon_n}s$, $s \in [0,\tau]$, *is defined for* $n \ge n_0$ *and*

$$\sup_{s\in[0,\tau]} |a_n\pi_{\varepsilon_n}s - a_0\pi_0 s|_{H_1^{\varepsilon_n}} \to 0 \quad \textit{as } n \to \infty.$$

Proof. Lemma 5.3 implies that for every $\overline{u} \in H_1^0$ there exist a $\delta > 0$ and a $\tau > 0$ such that for every $a_0 \in H_1^0$ with $|a_0 - \overline{u}|_{H_1^0} \le \delta$, $a_0\pi_0 s$, $s \in [0,\tau]$, is defined and whenever $(\varepsilon_n)_n$ is a sequence in $]0,\varepsilon_0]$ with $\varepsilon_n \to 0$ and $(a_n)_n$ is a sequence with $a_n \in H_1^{\varepsilon_n}$ for every $n \in \mathbb{N}$ and

$$|a_n - a_0|_{H_1^{\varepsilon_n}} \to 0 \quad \text{as } n \to \infty,$$

then there is an $n_0 \in \mathbb{N}$ such that $a_n\pi_{\varepsilon_n}s$, $s \in [0,\tau]$, is defined for $n \ge n_0$. Moreover there exists a $M' \ge 0$ such that $|a_n\pi_{\varepsilon_n}s|_{H_1^{\varepsilon_n}} \le M'$ for all $n \ge n_0$ and for all $s \in [0,\tau]$.

To complete the proof of the corollary we need to show that

$$\sup_{s\in[0,\tau]} |a_n\pi_{\varepsilon_n}s - a_0\pi_0 s|_{H_1^{\varepsilon_n}} \to 0 \quad \text{as } n \to \infty.$$

Suppose this is not true. Then there are a $\delta_0 > 0$ and a sequence $(n_k)_k$ in $\mathbb{N}$ with $n_k \to \infty$ as $k \to \infty$ such that

$$\sup_{s\in[0,\tau]} |a_{n_k}\pi_{\varepsilon_{n_k}}s - a_0\pi_0 s|_{H_1^{\varepsilon_{n_k}}} \ge \delta_0 \quad \text{for all } k \in \mathbb{N}.$$

Thus for each $k \in \mathbb{N}$ there exists an $s_k \in [0,\tau]$ such that

$$|a_{n_k}\pi_{\varepsilon_{n_k}}s_k - a_0\pi_0 s_k|_{H_1^{\varepsilon_{n_k}}} \ge \delta_0. \tag{5.13}$$

Without loss of generality we can assume that there is an $s_0 \in [0,\tau]$ such that $s_k \to s_0$ as $k \to \infty$. If $s_0 = 0$, it follows from Theorem 5.5 that $|a_{n_k}\pi_{\varepsilon_{n_k}}s_k - a_0\pi_0 s_k|_{H_1^{\varepsilon_{n_k}}} \to 0$ as $k \to \infty$ which contradicts (5.13). If $s_0 > 0$, then Theorem 5.4 implies that $|a_{n_k}\pi_{\varepsilon_{n_k}}s_k - a_0\pi_0 s_k|_{H_1^{\varepsilon_{n_k}}} \to 0$ as $k \to \infty$ which again contradicts (5.13). □

We conclude this section proving our main convergence result for semiflows.

Theorem 5.7. *Suppose that the family* $(H^\varepsilon, \langle\,\cdot\,,\,\cdot\,\rangle_{H^\varepsilon}, A_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ *satisfies condition* (Spec) *and the family* $(f_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ *satisfies condition* (Conv). *Let* $(\varepsilon_n)_n$ *be a sequence in* $]0,\varepsilon_0]$ *with* $\varepsilon_n \to 0$ *and let* $(t_n)_n$ *be a sequence in* $[0,\infty[$ *with* $t_n \to t_0$*, for some* $t_0 \in [0,\infty[$*. Let* $u_0 \in H^0_1$ *and* $(u_n)_n$ *be a sequence with* $u_n \in H^{\varepsilon_n}_1$ *for every* $n \in \mathbb{N}$ *and*

$$|u_n - u_0|_{H^{\varepsilon_n}_1} \to 0 \quad \text{as } n \to \infty.$$

Assume $u_0\pi_0 t_0$ *is defined. Then there exists an* $n_0 \in \mathbb{N}$ *such that* $u_n\pi_{\varepsilon_n}t_n$ *is defined for all* $n \geq n_0$ *and*

$$|u_n\pi_{\varepsilon_n}t_n - u_0\pi_0 t_0|_{H^{\varepsilon_n}_1} \to 0 \quad \text{as } n \to \infty.$$

Proof. Since $u_0\pi_0 t_0$ is defined, there is a $b > t_0$, $b \in]0,\infty[$, such that $u_0\pi_0 t$ is defined for all $t \in [0,b[$. Define

$$I := \{t \in [0,b[\mid \text{there exists an } n_0 \in \mathbb{N} \text{ such that } u_n\pi_{\varepsilon_n}t \text{ is defined for } n \geq n_0 \\ \text{and } \sup_{s\in[0,t]} |u_n\pi_{\varepsilon_n}s - u_0\pi_0 s|_{H^{\varepsilon_n}_1} \to 0 \text{ as } n \to \infty\}.$$

It is clear that $0 \in I$. Furthermore if $0 \leq t' < t$ and $t \in I$, then $t' \in I$. Let

$$\bar{t} := \sup I.$$

It follows that $\bar{t} \leq b$ and so $[0,\bar{t}[\subset I$. An application of Corollary 5.6 with $\bar{u} := u_0$ shows that $\bar{t} > 0$. We claim that $\bar{t} = b$. Suppose, on the contrary, that $\bar{t} < b$. It follows that $\bar{u} := u_0\pi_0\bar{t}$ is defined. Let $\delta > 0$ and $\tau > 0$ be as in Corollary 5.6 with respect to this choice of $\bar{u}$.

Choose $t \in \mathbb{R}$ with $0 < t < \bar{t} < t + \tau$ and $|u_0\pi_0 t - u_0\pi_0\bar{t}|_{H^0_1} < \delta$. We have that $t \in I$ so there exists an $n_0 \in \mathbb{N}$ such that $u_n\pi_{\varepsilon_n}t$ is defined for all $n \geq n_0$ and

$$\sup_{s\in[0,t]} |u_n\pi_{\varepsilon_n}s - u_0\pi_0 s|_{H^{\varepsilon_n}_1} \to 0 \quad \text{as } n \to \infty. \tag{5.14}$$

Set $\widetilde{u}_n := u_n\pi_{\varepsilon_n}t$ and $\widetilde{u} := u_0\pi_0 t$. Applying Corollary 5.6 with a_n replaced by $\widetilde{u}_n$ and a_0 replaced by $\widetilde{u}$ we thus have that $\widetilde{u}\pi_0\tau$ is defined and we obtain the existence of an $n_1 \geq n_0$ such that $\widetilde{u}_n\pi_{\varepsilon_n}\tau$ is defined for all $n \geq n_1$ and

$$\sup_{s\in[0,\tau]} |\widetilde{u}_n\pi_{\varepsilon_n}s - \widetilde{u}\pi_0 s|_{H^{\varepsilon_n}_1} \to 0 \quad \text{as } n \to \infty. \tag{5.15}$$

Formulas (5.14) and (5.15) imply that $u_0\pi_0(t+\tau)$ is defined, $u_n\pi_{\varepsilon_n}(t+\tau)$ is also defined for all $n \geq n_1$ and

$$\sup_{s\in[0,t+\tau]} |u_n\pi_{\varepsilon_n}s - u_0\pi_0 s|_{H^{\varepsilon_n}_1} \to 0 \quad \text{as } n \to \infty.$$

Thus $t + \tau \in I$, but $t + \tau > \bar{t}$, a contradiction, which proves that $\bar{t} = b$.

Since $t_0 \in [0, b[$, it follows that there is a $t \in [0, b[$ with $t_0 < t$ and $t_n < t$ for all n large enough. In particular $u_0\pi_0 t_n$ and $u_n\pi_{\varepsilon_n} t_n$ are defined for all n large enough and

$$|u_n\pi_{\varepsilon_n} t_n - u_0\pi_0 t_n|_{H_1^{\varepsilon_n}} \to 0 \quad \text{as } n \to \infty.$$

Since

$$|u_0\pi_0 t_n - u_0\pi_0 t_0|_{H_1^{\varepsilon_n}} \le C|u_0\pi_0 t_n - u_0\pi_0 t_0|_{H_1^0}$$

and $|u_0\pi_0 t_n - u_0\pi_0 t_0|_{H_1^0} \to 0$ as $n \to \infty$, the theorem follows. □

6. Singular compactness

In this section we shall prove that under the abstract compactness hypothesis, condition (Comp), on the family $(H^\varepsilon, \langle\cdot\,,\cdot\rangle_{H^\varepsilon}, A_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ the corresponding family of semiflows satisfies a singular compactness property. This property, together with the singular convergence result obtained in the previous section, is crucial for establishing the singular continuation principle for Conley index and for (co)homology index braid.

We start with the following result.

Theorem 6.1. *Suppose that the family $(H^\varepsilon, \langle\cdot\,,\cdot\rangle_{H^\varepsilon}, A_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ satisfies condition* (Spec) *and the family $(f_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ satisfies condition* (Conv). *Let $\varepsilon \in [0, \varepsilon_0]$ be arbitrary. Then every closed and bounded set in H_1^ε is strongly π_ε-admissible.*

Proof. Let N be a closed and bounded set in H_1^ε and let $M > 0$ such that for all $u \in N$ we have $|u|_{H_1^\varepsilon} \le M$. Let $L := L_M$ be as in Definition 5.1. Let $u_0 \in N$. Hence for all $u \in N$

$$\begin{aligned}|f_\varepsilon(u)|_{H_{-\alpha}^\varepsilon} &\le |f_\varepsilon(u) - f_\varepsilon(u_0)|_{H_{-\alpha}^\varepsilon} + |f_\varepsilon(u_0)|_{H_{-\alpha}^\varepsilon}\\ &\le L|u - u_0|_{H_1^\varepsilon} + |f_\varepsilon(u_0)|_{H_{-\alpha}^\varepsilon} \le 2ML + |f_\varepsilon(u_0)|_{H_{-\alpha}^\varepsilon}.\end{aligned}$$

Now the result follows from [11, Theorem III 4.4]. □

We can now state the following singular compactness theorem.

Theorem 6.2. *Suppose that the family $(H^\varepsilon, \langle\cdot\,,\cdot\rangle_{H^\varepsilon}, A_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ satisfies conditions* (Spec) *and* (Comp) *and the family $(f_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ satisfies* (Conv). *Suppose $\kappa \in]0, \infty[$, $(\varepsilon_n)_n$ is a sequence in $]0, \varepsilon_0]$ with $\varepsilon_n \to 0$, $(t_n)_n$ is a sequence in $[0, \infty[$ with $t_n \ge \kappa$ for every $n \in \mathbb{N}$ and $(u_n)_n$ is a sequence with $u_n \in H_1^{\varepsilon_n}$ for every $n \in \mathbb{N}$. Assume that there exists a constant $C'' \in]0, \infty[$ such that $u_n\pi_{\varepsilon_n} t_n$ is defined and*

$$|u_n\pi_{\varepsilon_n} s|_{H_1^{\varepsilon_n}} \le C'' \quad \text{for all } n \in \mathbb{N} \text{ and for all } s \in [0, t_n].$$

Then there exist a $v \in H_1^0$ and a sequence $(n_k)_k$ in $\mathbb{N}$ with $n_k \to \infty$ as $k \to \infty$ such that

$$|u_{n_k}\pi_{\varepsilon_{n_k}} t_{n_k} - v|_{H_1^{\varepsilon_{n_k}}} \to 0 \quad \text{as } k \to \infty.$$

Proof. Set $\xi_n = u_n \pi_{\varepsilon_n}(t_n - \kappa)$ for $n \in \mathbb{N}$. Hence

$$|\xi_n|_{H_1^{\varepsilon_n}} \le C'' \quad \text{for all } n \in \mathbb{N}.$$

Condition (Comp) implies that there exist a $\widetilde{v} \in H_1^0$ and a sequence $(n_k)_k$ in $\mathbb{N}$ with $n_k \to \infty$ as $k \to \infty$ such that

$$|\xi_{n_k} - \widetilde{v}|_{H^{\varepsilon_{n_k}}} \to 0 \quad \text{as } k \to \infty.$$

We claim that $\widetilde{v}\pi_0\kappa$ is defined. Suppose our claim is not true. Let $\beta \in [0, \infty[$ be such that $CC'' < \beta$ and $|\widetilde{v}|_{H_1^0} < \beta$. By Theorem 6.1 with $\varepsilon = 0$ and $N = \{u \in H_1^0 \mid |u|_{H_1^0} \le \beta\}$, there exists a $t_0 \in]0, \kappa[$ such that $\widetilde{v}\pi_0 t_0$ is defined and $|\widetilde{v}\pi_0 t_0|_{H_1^0} > \beta$. Condition (Spec) implies that

$$|\widetilde{v}\pi_0 t_0|_{H_1^\varepsilon} \ge \beta / C > C'', \quad \varepsilon \in [0, \varepsilon_0]. \tag{6.1}$$

It follows that there exists a $k_0 \in \mathbb{N}$ such that $\widetilde{v}\pi_0 s$ and $\xi_{n_k}\pi_{\varepsilon_{n_k}} s$ are defined for all $s \in [0, t_0]$, for all $k \ge k_0$ and

$$|\xi_{n_k}\pi_{\varepsilon_{n_k}} s|_{H_1^{\varepsilon_{n_k}}} \le C'' \quad \text{for all } s \in [0, t_0] \text{ and for all } k \ge k_0.$$

Hence Theorem 5.4 implies that

$$|\xi_{n_k}\pi_{\varepsilon_{n_k}} t_0 - \widetilde{v}\pi_0 t_0|_{H_1^{\varepsilon_{n_k}}} \to 0 \quad \text{as } k \to \infty.$$

This together with formula (6.1) implies that

$$|\xi_{n_k}\pi_{\varepsilon_{n_k}} t_0|_{H_1^{\varepsilon_{n_k}}} > C'' \quad \text{for all } k \in \mathbb{N} \text{ large enough}$$

which is a contradiction. Thus $\widetilde{v}\pi_0\kappa$ is defined. Another application of Theorem 5.4 shows that

$$|\xi_{n_k}\pi_{\varepsilon_{n_k}} \kappa - \widetilde{v}\pi_0\kappa|_{H_1^{\varepsilon_{n_k}}} \to 0 \quad \text{as } k \to \infty.$$

Set $v := \widetilde{v}\pi_0\kappa \in H_1^0$. Since $\xi_{n_k}\pi_{\varepsilon_{n_k}}\kappa = u_{n_k}\pi_{\varepsilon_{n_k}} t_{n_k}$ for all $k \in \mathbb{N}$, the proof is complete. □

Recall that Proposition 4.3 implies that for every $\varepsilon \in]0, \varepsilon_0]$, the set H_1^0 is a closed subspace of H_1^ε. For each $\varepsilon \in]0, \varepsilon_0]$, let $Q_\varepsilon \colon H_1^\varepsilon \to H_1^\varepsilon$ be the H_1^ε-orthogonal projection of H_1^ε onto H_1^0.

Theorem 6.2 easily implies the following corollary:

Corollary 6.3. *Let* $(H^\varepsilon, \langle\cdot\,,\cdot\rangle_{H^\varepsilon}, A_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ *satisfy conditions* (Spec) *and* (Comp) *and suppose the family of maps* $(f_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ *satisfies condition* (Conv). *Suppose* $N \subset H_1^0$ *is a closed and bounded set and* $(\varepsilon_n)_n$ *is a sequence in* $]0, \varepsilon_0]$ *with* $\varepsilon_n \to 0$. *Let* $\eta > 0$ *be arbitrary and define*

$$N_n = N_{n,\eta} := \{u \in H_1^{\varepsilon_n} \mid Q_{\varepsilon_n} u \in N \text{ and } |(I - Q_{\varepsilon_n})u|_{H_1^{\varepsilon_n}} \le \eta\}.$$

Suppose $(t_n)_n$ *is a sequence in* $[0,\infty[$ *with* $t_n\to\infty$ *and* $(u_n)_n$ *is a sequence with* $u_n\in H_1^{\varepsilon_n}$ *for every* $n\in\mathbb{N}$ *and*

$$u_n\pi_{\varepsilon_n}[0,t_n]\subset N_n \quad \textit{for all } n\in\mathbb{N}.$$

Then there exist a $v\in N$ *and a sequence* $(n_k)_k$ *in* $\mathbb{N}$ *with* $n_k\to\infty$ *as* $k\to\infty$ *such that*

$$|u_{n_k}\pi_{\varepsilon_{n_k}}t_{n_k}-v|_{H_1^{\varepsilon_{n_k}}}\to 0 \quad \textit{as } k\to\infty.$$

7. Singular continuation principle for the Conley index and for (co)homology index braids

In this section, under the conditions (Spec), (Conv) and (Comp), we obtain a singular continuation principle for the Conley index and for (co)homology index braids for the class of abstract parabolic equations described in (5.1).

This section is not self-contained, in particular, we use some results established in the papers [3], [4], [5]. We will also assume that the reader is familiar with the Conley index theory for semiflows on (not necessarily locally compact) metric spaces, as expounded in [10] or [11].

Let $(H^\varepsilon,\langle\,\cdot\,,\,\cdot\,\rangle_{H^\varepsilon},A_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ satisfy conditions (Spec) and (Comp) and suppose the family of maps $(f_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ satisfies condition (Conv).

Set $X_0:=H_1^0$. For every $\varepsilon\in\,]0,\varepsilon_0]$, define $Y_\varepsilon:=(I-Q_\varepsilon)H_1^\varepsilon$ and endow Y_ε with the norm $|\cdot|_{H_1^\varepsilon}$ restricted to Y_ε. Define on $Z_\varepsilon=X_0\times Y_\varepsilon$ the following norm:

$$||(u,v)||_\varepsilon:=\max\{|u|_{H_1^0},|v|_{H_1^\varepsilon}\} \quad \text{for } (u,v)\in Z_\varepsilon.$$

We will denote by Γ_ε the metric on Z_ε induced by the norm $||\cdot||_\varepsilon$. For each $\varepsilon\in\,]0,\varepsilon_0]$, define $\theta_\varepsilon:=0$.

Let $\Psi_\varepsilon\colon H_1^\varepsilon\to Z_\varepsilon$ be the linear map defined by

$$\Psi_\varepsilon(w):=(Q_\varepsilon w,(I-Q_\varepsilon)w) \quad \text{for } w\in H_1^\varepsilon.$$

It follows that Ψ_ε is a bijective linear map and its inverse map is given by

$$\Psi_\varepsilon{}^{-1}(u,v)=u+v \quad \text{for } (u,v)\in Z_\varepsilon.$$

Moreover both Ψ_ε and $\Psi_\varepsilon{}^{-1}$ are continuous maps. This fact is a consequence of the following inequalities:

$$\|\Psi_\varepsilon(w)\|_\varepsilon\le C|w|_{H_1^\varepsilon} \quad \text{for } w\in H_1^\varepsilon, \tag{7.1}$$

$$|\Psi_\varepsilon{}^{-1}(u,v)|_{H_1^\varepsilon}\le(1+C^2)^{1/2}||(u,v)||_\varepsilon \quad \text{for } (u,v)\in Z_\varepsilon, \tag{7.2}$$

where the constant $C\in\,]1,\infty[$ was defined in hypothesis (Spec).

Given $(u,v)\in Z_\varepsilon$ and $t\in[0,\infty[$ define

$$(u,v)\widetilde{\pi}_\varepsilon t:=\Psi_\varepsilon(\Psi_\varepsilon{}^{-1}(u,v)\pi_\varepsilon t)$$

whenever $\Psi_\varepsilon{}^{-1}(u,v)\pi_\varepsilon t$ is defined. It follows that $\widetilde{\pi}_\varepsilon$ is a local semiflow on Z_ε, the *conjugate to* π_ε *via* Ψ_ε. Theorem 5.7 and inequalities (7.1) and (7.2) immediately imply the following

Corollary 7.1. *Under the above hypotheses the family* $(\widetilde{\pi}_\varepsilon)_{\varepsilon\in]0,\varepsilon_0]}$ *converges singularly to* π_0. □

Theorem 6.1, Corollary 6.3 and inequalities (7.1) and (7.2) imply the following:

Corollary 7.2. *Let* N *be a closed and bounded subset of* X_0. *Then for every* $\eta > 0$ *the set* N *is singularly strongly admissible with respect to* η *and the family* $(\widetilde{\pi}_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$, *where* $\widetilde{\pi}_0 = \pi_0$.

We can now prove the following Conley index continuation principle for singular families of abstract parabolic equations:

Theorem 7.3. *Let* N *be a closed and bounded isolating neighbourhood of an invariant set* K_0 *relative to* π_0. *For* $\varepsilon \in]0,\varepsilon_0]$ *and for every* $\eta \in]0,\infty[$ *set*

$$N_{\varepsilon,\eta} := \{u \in H_1^\varepsilon \mid Q_\varepsilon u \in N \text{ and } |(I-Q_\varepsilon)u|_{H_1^\varepsilon} \le \eta\}$$

and $K_{\varepsilon,\eta} := \mathrm{Inv}_{\pi_\varepsilon}(N_{\varepsilon,\eta})$ *i.e.* $K_{\varepsilon,\eta}$ *is the largest* π_ε*-invariant set in* $N_{\varepsilon,\eta}$. *Then for every* $\eta \in]0,\infty[$ *there exists an* $\varepsilon^{\mathrm{c}} = \varepsilon^{\mathrm{c}}(\eta) \in]0,\varepsilon_0]$ *such that for every* $\varepsilon \in]0,\varepsilon^{\mathrm{c}}]$ *the set* $N_{\varepsilon,\eta}$ *is a strongly admissible isolating neighbourhood of* $K_{\varepsilon,\eta}$ *relative to* π_ε *and*

$$h(\pi_\varepsilon, K_{\varepsilon,\eta}) = h(\pi_0, K_0).$$

Furthermore, for every $\eta > 0$, *the family* $(K_{\varepsilon,\eta})_{\varepsilon\in[0,\varepsilon^{\mathrm{c}}(\eta)]}$ *of invariant sets, where* $K_{0,\eta} = K_0$, *is upper semicontinuous at* $\varepsilon = 0$ *with respect to the family* $|\cdot|_{H_1^\varepsilon}$ *of norms i.e.*

$$\lim_{\varepsilon\to 0^+} \sup_{w\in K_{\varepsilon,\eta}} \inf_{u\in K_0} |w-u|_{H_1^\varepsilon} = 0.$$

Proof. The isomorphism Ψ_ε conjugates the local semiflow π_ε to the local semiflow $\widetilde{\pi}_\varepsilon$. Thus whenever S is a strongly admissible isolating neighbourhood with respect to π_ε, then $\Psi_\varepsilon(S)$ is a strongly admissible isolating neighbourhood with respect to $\widetilde{\pi}_\varepsilon$ and

$$h(\pi_\varepsilon, S) = h(\widetilde{\pi}_\varepsilon, \Psi_\varepsilon(S)).$$

Corollaries 7.1 and 7.2 imply that the family of semiflows $(\widetilde{\pi}_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ and the set N satisfy the hypotheses of [3, Theorem 4.1]. Notice also that any closed ball in Y_ε is contractible. Hence [3, Theorem 4.1] and [3, Corollary 4.11] completes the proof. □

Remark 7.4. The family $(K_{\varepsilon,\eta})_{\varepsilon\in]0,\varepsilon^{\mathrm{c}}(\eta)]}$ is asymptotically independent of η i.e. whenever η_1 and $\eta_2 \in]0,\infty[$ then there is an $\varepsilon' \in]0, \min(\varepsilon^{\mathrm{c}}(\eta_1), \varepsilon^{\mathrm{c}}(\eta_2))]$ such that $K_{\varepsilon,\eta_1} = K_{\varepsilon,\eta_2}$ for $\varepsilon \in]0,\varepsilon']$.

We also prove the following (co)homology index continuation principle:

Theorem 7.5. *Assume the hypotheses of Theorem 7.3 and for every* $\eta \in]0,\infty[$ *let* $\varepsilon^{\mathrm{c}}(\eta) \in]0,\varepsilon_0]$ *be as in that theorem. Let* $(P,\prec)$ *be a finite poset. Let* $(M_{p,0})_{p\in P}$ *be a* $\prec$*-ordered Morse decomposition of* K_0 *relative to* π_0*. For each* $p \in P$*, let* $V_p \subset N$ *be closed in* X_0 *and such that* $M_{p,0} = \mathrm{Inv}_{\pi_0}(V_p) \subset \mathrm{Int}_{H_1^0}(V_p)$*. (Such sets* V_p*,* $p \in P$*, exist.) For* $\varepsilon \in]0,\varepsilon_0]$*, for every* $\eta \in]0,\infty[$ *and* $p \in P$ *set* $M_{p,\varepsilon,\eta} := \mathrm{Inv}_{\pi_\varepsilon}(V_{p,\varepsilon,\eta})$*, where*

$$V_{p,\varepsilon,\eta} := \{u \in H_1^\varepsilon \mid Q_\varepsilon u \in V_p \text{ and } |(I-Q_\varepsilon)u|_{H_1^\varepsilon} \leq \eta\}.$$

Then for every $\eta \in]0,\infty[$ *there is an* $\widetilde{\varepsilon} = \widetilde{\varepsilon}(\eta) \in]0,\varepsilon^{\mathrm{c}}(\eta)]$ *such that for every* $\varepsilon \in]0,\widetilde{\varepsilon}]$ *and* $p \in P$*,* $M_{p,\varepsilon,\eta} \subset \mathrm{Int}_{H_1^\varepsilon}(V_{p,\varepsilon,\eta})$ *and the family* $(M_{p,\varepsilon,\eta})_{p\in P}$ *is a* $\prec$*-ordered Morse decomposition of* $K_{\varepsilon,\eta}$ *relative to* π_ε *and the (co)homology index braids of* $(\pi_0, K_0, (M_{p,0})_{p\in P})$ *and* $(\pi_\varepsilon, K_{\varepsilon,\eta}, (M_{p,\varepsilon,\eta})_{p\in P}))$*,* $\varepsilon \in]0,\widetilde{\varepsilon}]$*, are isomorphic and so they determine the same collection of* C*-connection matrices.*

Proof. Since the isomorphism Ψ_ε conjugates the local semiflow π_ε to the local semiflow $\widetilde{\pi}_\varepsilon$, using [5, Proposition 2.7], it follows that whenever S is a strongly admissible isolating neighbourhood with respect to π_ε and $(M_p)_{p\in P}$ is a $\prec$-ordered Morse decomposition of S relative to π_ε, then $\Psi_\varepsilon(S)$ is a strongly admissible isolating neighbourhood with respect to $\widetilde{\pi}_\varepsilon$ and $(\Psi_\varepsilon(M_p))_{p\in P}$ is a $\prec$-ordered Morse decomposition of S relative to $\widetilde{\pi}_\varepsilon$ and the (co)homology index braids of $(\pi_\varepsilon, S, (M_p)_{p\in P})$ and $(\widetilde{\pi}_\varepsilon, \Psi_\varepsilon(S), (\Psi_\varepsilon(M_p))_{p\in P}))$, $\varepsilon \in]0,\varepsilon_0]$, are isomorphic.

Corollaries 7.1 and 7.2 imply that the family of semiflows $(\widetilde{\pi}_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ and the set N satisfy the hypotheses of [4, Theorem 3.10]. Since any closed ball in Y_ε is contractible, an application of [4, Theorem 3.10] completes the proof. □

Remark 7.6. Again, for each $p \in P$, the family $(M_{p,\varepsilon,\eta})_{\varepsilon\in[0,\widetilde{\varepsilon}(\eta)]}$, where $M_{p,0,\eta} = M_{p,0}$ is upper semicontinuous at $\varepsilon = 0$ with respect to the family $|\cdot|_{H_1^\varepsilon}$ of norms and the family $(M_{p,\varepsilon,\eta})_{\varepsilon\in]0,\widetilde{\varepsilon}(\eta)]}$ is asymptotically independent of η.

8. Application to parabolic problems with localized large diffusion

Now let $\varepsilon_0 \in]0,\infty[$ and the operators A_ε, $\varepsilon \in [0,\varepsilon_0]$, be as in section 2.

For $\varepsilon \in]0,\varepsilon_0[$ set $H^\varepsilon = L^2(\Omega)$ and $\langle\cdot,\cdot\rangle_{H^\varepsilon} = \langle\cdot,\cdot\rangle_{L^2(\Omega)}$. Moreover, write $H^0 = L^2_{\Omega_0}(\Omega)$ and $\langle\cdot,\cdot\rangle_{H^0} = \langle\cdot,\cdot\rangle_{L^2(\Omega)}$. Notice that $H_0^\varepsilon = H^\varepsilon$ for all $\varepsilon \in [0,\varepsilon_0]$.

For $\varepsilon \in [0,\varepsilon_0]$ and $\alpha \in \mathbb{R}$ write $H_\alpha^\varepsilon = H_\alpha(A_\varepsilon)$. Then, if $\varepsilon \in]0,\varepsilon_0]$, it follows that $H_1^\varepsilon = H_1(A_\varepsilon) = H^1(\Omega)$ and $\langle\cdot,\cdot\rangle_{H_1^\varepsilon} = \zeta_\varepsilon(\cdot,\cdot)$. Furthermore, $H_1^0 = H^1_{\Omega_0}(\Omega)$ and $\langle\cdot,\cdot\rangle_{H_1^0} = \zeta_0(\cdot,\cdot)$.

In particular, if $u \in H_1^0$, then, for all $\varepsilon \in]0,\varepsilon_0]$,

$$|u|_{H_1^\varepsilon} = |u|_{H_1^0}.$$

It is now easy to conclude that parts (a), (b) and (c) of Condition (Spec) are satisfied. Moreover, the following result holds:

Proposition 8.1. *The above family* $(H^\varepsilon, \langle\,\cdot\,,\,\cdot\,\rangle_{H^\varepsilon}, A_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ *satisfies conditions* (Spec) *and* (Comp).

Proof. [9, Theorem 5.1] implies that part (d) of Condition (Spec) also holds. Condition (Comp) follows from [9, Theorem 4.4]. □

By interpolation theory (cf. [12]) for every $\theta \in [0,1]$ and every $\varepsilon \in [0,\varepsilon_0]$ there is a continuous imbedding from H^ε_θ to $H^\theta(\Omega)$ with imbedding constant $C_{1,\theta} \in \,]0,\infty[$ independent of $\varepsilon \in [0,\varepsilon_0]$. Furthermore, there is a continuous imbedding from $H^\theta(\Omega)$ into $L^{p_{\theta,\Omega}}(\Omega)$ with imbedding constant $C_{2,\theta} \in \,]0,\infty[$. Here,

$$p_{\theta,\Omega} = \left(\theta\frac{1}{2^*_\Omega} + (1-\theta)\frac{1}{2}\right)^{-1}.$$

Moreover, for every $\rho \in [0,1]$ there is a continuous imbedding from $H^{\rho/2}(\Gamma)$ into $L^{p_{\rho,\Gamma}}(\Gamma)$ with imbedding constant $C_{3,\rho} \in \,]0,\infty[$. Here,

$$p_{\rho,\Gamma} = \left(\rho\frac{1}{2^*_\Gamma} + (1-\rho)\frac{1}{2}\right)^{-1}.$$

Finally, by [8], for every $\theta \in \,]1/2,1]$ there is a bounded linear trace operator $\gamma = \gamma_\theta \colon H^\theta(\Omega) \to H^{\theta-(1/2)}(\Gamma)$ with a bound $C_{4,\theta} \in \,]0,\infty[$. Now the continuity of the functions $\theta \mapsto p_{\theta,\Omega}$ and $\theta \mapsto p_{2\theta-1,\Gamma}$ at $\theta = 1$ implies the following result.

Lemma 8.2. *Let* $q_2 \in \,](1-(1/2^*_\Omega))^{-1},\infty[$ *and* $q_3 \in \,](1-(1/2^*_\Gamma))^{-1},\infty[$ *be arbitrary. Then there is a* $\theta \in \,]1/2,1[$ *such that*

$$p_2 = \frac{q_2}{q_2-1} < p_{\theta,\Omega} \quad \textit{and} \quad p_3 = \frac{q_3}{q_3-1} < p_{2\theta-1,\Gamma}.$$

Set $\alpha = \theta$ *and let* $C_5 \in \,]0,\infty[$ *(resp.* $C_6 \in \,]0,\infty[$*) be a bound of the imbedding* $L^{p_{\alpha,\Omega}}(\Omega) \to L^{p_2}(\Omega)$ *(resp.* $L^{p_{2\alpha-1,\Gamma}}(\Gamma) \to L^{p_3}(\Gamma)$*). Then, whenever* $\Phi \in L^{q_2}(\Omega)$, $\Psi \in L^{q_3}(\Gamma)$, $\varepsilon \in [0,\varepsilon_0]$ *and* $h \in H^\varepsilon_\alpha$*, then* $\Phi \cdot h \in L^1(\Omega)$, $\Psi \cdot \gamma(h) \in L^1(\Gamma)$,

$$\int_\Omega |\Phi \cdot h|\,dx \le C_{1,\alpha}C_{2,\alpha}C_5|\Phi|_{L^{q_2}(\Omega)}|h|_{H^\varepsilon_\alpha},$$

and

$$\int_\Gamma |\Psi \cdot \gamma(h)|\,d\sigma \le C_{1,\alpha}C_{4,\alpha}C_{3,2\alpha-1}C_6|\Psi|_{L^{q_3}(\Gamma)}|h|_{H^\varepsilon_\alpha}.$$

In particular, there is a unique $f_\varepsilon \in H^\varepsilon_{-\alpha}$ *such that*

$$f_\varepsilon(h) = \int_\Omega \Phi \cdot h\,dx + \int_\Gamma \Psi \cdot \gamma(h)\,d\sigma, \quad h \in H^\varepsilon_\alpha.$$

Moreover,

$$|f_\varepsilon|_{H^\varepsilon_{-\alpha}} \le C_{7,\alpha}(|\Phi|_{L^{q_2}(\Omega)} + |\Psi|_{L^{q_3}(\Gamma)})$$

where $C_{7,\alpha} = \max(C_{1,\alpha}C_{2,\alpha}C_5, C_{1,\alpha}C_{4,\alpha}C_{3,2\alpha-1}C_6)$.

The next two theorems describe how to obtain a family of maps that satisfies hypothesis (Conv).

Theorem 8.3. *For* $\varepsilon \in [0,\varepsilon_0]$, *let* $\Phi_\varepsilon\colon H^1(\Omega) \to L^{q_2}(\Omega)$ *and* $\Psi_\varepsilon\colon H^{1/2}(\Gamma) \to L^{q_3}(\Gamma)$ *be maps satisfying the following assumptions:*

(a) *For all* $M \in [0,\infty[$ *there is an* $L = L_M \in [0,\infty[$ *such that*
 (a1) *for all* $\varepsilon \in [0,\varepsilon_0]$ *and all* u, $v \in H^1(\Omega)$ *with* $|u|_{H^1(\Omega)}$, $|v|_{H^1(\Omega)} \le M$,

$$|\Phi_\varepsilon(u) - \Phi_\varepsilon(v)|_{L^{q_2}(\Omega)} \le L|u-v|_{H^1(\Omega)}$$

 (a2) *for all* $\varepsilon \in [0,\varepsilon_0]$ *and all* u, $v \in H^{1/2}(\Gamma)$ *with* $|u|_{H^{1/2}(\Gamma)}$, $|v|_{H^{1/2}(\Gamma)} \le M$,

$$|\Psi_\varepsilon(u) - \Psi_\varepsilon(v)|_{L^{q_3}(\Gamma)} \le L|u-v|_{H^{1/2}(\Gamma).}$$

(b) *For every* $u \in H^1_{\Omega_0}(\Omega)$,

$$|\Phi_\varepsilon(u) - \Phi_0(u)|_{L^{q_2}(\Omega)} \to 0 \quad \textit{as } \varepsilon \to 0^+.$$

(c) *For every* $u \in H^{1/2}(\Gamma)$,

$$|\Psi_\varepsilon(u) - \Psi_0(u)|_{L^{q_3}(\Gamma)} \to 0 \quad \textit{as } \varepsilon \to 0^+.$$

Let $\alpha \in \,]1/2,1[$ *be as in Lemma* 8.2. *For* $\varepsilon \in [0,\varepsilon_0]$ *and* $u \in H^\varepsilon_1$ *define, for* $h \in H^\varepsilon_\alpha$,

$$f_\varepsilon(u)(h) = \int_\Omega \Phi_\varepsilon(u)\cdot h\,dx + \int_\Gamma \Psi_\varepsilon(\gamma(u))\cdot\gamma(h)\,d\sigma.$$

Then $f_\varepsilon(u) \in H^\varepsilon_{-\alpha}$ *and the family* $(f_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ *of maps satisfies condition* (Conv).

Remark 8.4. By the definition of 2^*_Ω and 2^*_Γ we may, for $N = 1$, 2, take q_2 and q_3 arbitrary in $]1,\infty[$.

Proof of Theorem 8.3. Lemma 8.2 implies that the family $(f_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ satisfies (a) of condition (Conv). Let $M \in [0,\infty[$ be arbitrary and $L = L_M$ be as in assumption (a). If $\varepsilon \in [0,\varepsilon_0]$ and u, $v \in H^\varepsilon_1$ with $|u|_{H^\varepsilon_1}$, $|v|_{H^\varepsilon_1} \le M/C_{1,1}$ then u, $v \in H^1(\Omega)$ with $|u|_{H^1(\Omega)}$, $|v|_{H^1(\Omega)} \le M$ so

$$\begin{aligned}|f_\varepsilon(u) - f_\varepsilon(v)|_{H^\varepsilon_{-\alpha}} &\le C_{7,\alpha}(|\Phi_\varepsilon(u) - \Phi_\varepsilon(v)|_{L^{q_2}(\Omega)} + |\Psi_\varepsilon(\gamma(u)) - \Psi_\varepsilon(\gamma(v))|_{L^{q_3}(\Gamma)}) \\ &\le C_{7,\alpha}(L + LC_{4,1})|u-v|_{H^1(\Omega)} \le C_{7,\alpha}(L+LC_{4,1})C_{1,1}|u-v|_{H^\varepsilon_1}.\end{aligned}$$

This together with assumption (a) implies part (c) of condition (Conv). If $u \in H^0_1$ then

$$|f_\varepsilon(u)|_{H^\varepsilon_{-\alpha}} \le C_{7,\alpha}(|\Phi_\varepsilon(u)|_{L^{q_2}(\Omega)} + |\Psi_\varepsilon(\gamma(u))|_{L^{q_3}(\Gamma)}).$$

This together with assumptions (b) and (c) easily implies part (d) of condition (Conv).

Now let $w \in H_1^0$ be arbitrary and $(\varepsilon_n)_n$ be a sequence in $]0, \varepsilon_0]$ with $\varepsilon_n \to 0$. Let $t \in]0, \infty[$ be arbitrary. We will show that

$$\lim_{n\to\infty} |e^{-t\widetilde{A}_{\varepsilon_n}} f_{\varepsilon_n}(w) - e^{-t\widetilde{A}_0} f_0(w)|_{H_1^{\varepsilon_n}} = 0, \tag{8.1}$$

proving (b) of condition (Conv).

It follows from Proposition 8.1 that the families $(H^\varepsilon, \langle\,\cdot\,,\,\cdot\,\rangle_{H^\varepsilon})_{\varepsilon\in[0,\varepsilon_0]}$ and $(A_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ satisfy condition (Spec). For $n \in \mathbb{N}$ set $u_n = f_{\varepsilon_n}(w)$ and define $v_n \in H_{-\alpha}^{\varepsilon_n}$ by

$$v_n(h) = \int_\Omega \Phi_0(w) \cdot h\,dx + \int_\Gamma \Psi_0(\gamma(w)) \cdot \gamma(h)\,d\sigma, \quad h \in H_\alpha^{\varepsilon_n}.$$

Finally, set $u = f_0(w)$. Then

$$|u_n - v_n|_{H_{-\alpha}^{\varepsilon_n}} \le C_{7,\alpha}(|\Phi_{\varepsilon_n}(w) - \Phi_0(w)|_{L^{q_2}(\Omega)} + |\Psi_{\varepsilon_n}(\gamma(w)) - \Psi_0(\gamma(w))|_{L^{q_3}(\Gamma)})$$

Notice that the right hand side of this estimate goes to zero as $n \to \infty$. Thus assumption (a) of Theorem 4.6 is satisfied.

Let $C_8 \in]0, \infty[$ be a bound for the imbedding $H^1(\Omega) \to H^\alpha(\Omega)$. Then, for every $j \in \mathbb{N}$,

$$\begin{aligned} |v_n(w_{\varepsilon_n,j}) - u(w_{0,j})| &\le |\Phi_0(w)|_{L^{q_2}(\Omega)} |w_{\varepsilon_n,j} - w_{0,j}|_{L^{p_2}(\Omega)} \\ &\quad + |\Psi_0(\gamma(w))|_{L^{q_3}(\Gamma)} |\gamma(w_{\varepsilon_n,j} - w_{0,j})|_{L^{p_3}(\Gamma)} \le \widetilde{C} |w_{\varepsilon_n,j} - w_{0,j}|_{H_1^{\varepsilon_n}}, \end{aligned}$$

where $\widetilde{C} := C_5 C_{2,\alpha} C_8 C_{1,1} |\Phi_0(w)|_{L^{q_2}(\Omega)} + C_6 C_{3,2\alpha-1} C_{4,\alpha} C_8 C_{1,1} |\Psi_0(\gamma(w))|_{L^{q_3}(\Gamma)}$. Hence $|v_n(w_{\varepsilon_n,j}) - u(w_{0,j})| \to 0$ as $n \to \infty$. Thus assumption (b) of Theorem 4.6 is satisfied.

Now, for all $n \in \mathbb{N}$,

$$|v_n|_{H_{-\alpha}^{\varepsilon_n}} \le C_{7,\alpha}(|\Phi_0(w)|_{L^{p_2}(\Omega)} + |\Psi_0(\gamma(w))|_{L^{p_3}(\Gamma)}).$$

Thus assumption (c) of Theorem 4.6 is satisfied. Now (8.1) follows from Theorem 4.6. □

Theorem 8.4. *Assume Hypothesis* 2.1. *For* $\varepsilon \in [0, \varepsilon_0]$ *and* $u \in H^1(\Omega)$ (*resp.* $u \in H^{1/2}(\Gamma)$) *define* $\Phi_\varepsilon(u)(x) = \varphi_\varepsilon(x, u(x))$ (*resp.* $\Psi_\varepsilon(u)(x) = \psi_\varepsilon(x, u(x))$) *for* $x \in \Omega$ (*resp.* $x \in \Gamma$). *Then* $\Phi_\varepsilon \colon H^1(\Omega) \to L^{q_2}(\Omega)$ *and* $\Psi_\varepsilon \colon H^{1/2}(\Gamma) \to L^{q_3}(\Gamma)$ *are defined and satisfy the assumptions of Theorem* 8.3.

Proof. Use results and arguments in [6, Chapter 2]. □

Finally we obtain the following

Corollary 8.5. *For* $\varepsilon \in [0, \varepsilon_0]$ *let* $\varphi_\varepsilon: \Omega \times \mathbb{R} \to \mathbb{R}$ *and* $\psi_\varepsilon: \Gamma \times \mathbb{R} \to \mathbb{R}$, $(x, s) \mapsto \varphi_\varepsilon(x, s)$, $(x, s) \mapsto \psi_\varepsilon(x, s)$, *be functions as in Theorem* 8.4. *For* $\varepsilon \in [0, \varepsilon_0]$ *and* $u \in H^1(\Omega)$ *(resp.* $u \in H^{1/2}(\Gamma)$*) define* $\Phi_\varepsilon(u)(x) = \varphi_\varepsilon(x, u(x))$ *(resp.* $\Psi_\varepsilon(u)(x) = \psi_\varepsilon(x, u(x))$*) for* $x \in \Omega$ *(resp.* $x \in \Gamma$*) and let* $\alpha \in \,]1/2, 1]$ *be as in Lemma* 8.2. *For* $\varepsilon \in [0, \varepsilon_0]$ *and* $u \in H_1^\varepsilon$ *define, for* $h \in H_\alpha^\varepsilon$,

$$f_\varepsilon(u)(h) = \int_\Omega \Phi_\varepsilon(u) \cdot h \, dx + \int_\Gamma \Psi_\varepsilon(\gamma(u)) \cdot \gamma(h) \, d\sigma.$$

Then $f_\varepsilon(u) \in H_{-\alpha}^\varepsilon$ *and the family* $(f_\varepsilon)_{\varepsilon \in [0, \varepsilon_0]}$ *of maps satisfies condition* (Conv).

Proof. This follows from Theorems 8.4 and 8.3. □

We can now prove the results stated in Section 2.

Proof of Theorem 2.2. The theorem follows from Proposition 8.1, Corollary 8.5 and Theorem 5.7. □

Proof of Theorem 2.3. The theorem follows from Proposition 8.1, Corollary 8.5 and Corollary 6.3. □

Proof of Theorem 2.4. Proposition 8.1, Corollary 8.5, Theorem 7.3 and Remark 7.4 imply the theorem. □

Proof of Theorem 2.5. Proposition 8.1, Corollary 8.5, Theorem 7.5 and Remark 7.6 imply the theorem. □

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Manuscript received October 18, 2010

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