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STRUCTURE OF THE FIXED-POINT SET OF MAPPINGS WITH LIPSCHITZIAN ITERATES

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Dedicated to Kazimierz Goebel on the occasion of his 70th birthday

Abstract. We prove, by asymptotic center techniques and some inequalities in Banach spaces, that if E is p-uniformly convex Banach space, C is a nonempty bounded closed convex subset of E, and $T: C \to C$ has lipschitzian iterates (with some restrictions), then the set of fixed-points is not only connected but even a retract of C. The results presented in this paper improve and extend some results in [6], [8].

1 Introduction

We consider Banach spaces E over the real field only. Our notation and terminology are standard. Let C be a nonempty bounded closed convex subset of E. We say that a mapping $T: C \to C$ is nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\| \quad \text{for every } x, y \in C.$$

The result of Bruck [1] asserts that if a nonexpansive mapping $T: C \to C$ has a fixed point in every nonempty closed convex subset of C which is invariant under T and if C is convex and weakly compact, then $\operatorname{Fix} T = \{x \in C : Tx = x\}$, the set of fixed points, is nonexpansive retract of C (that is, there exists a nonexpansive mapping $R: C \to \operatorname{Fix} T$ such that $R_{|\operatorname{Fix} T} = I$). A few years

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ago, the Bruck result was extended by Domínguez Benavides and Lorenzo Ramírez [3] to the case of asymptotically nonexpansive mappings if the space E was sufficiently regular.

On the other hand it is known that the set of fixed points of a k-lipschitzian mapping can be very irregular for any $k > 1$.

Example 1.1 ([10]). Let F be a nonempty closed subset of C of a Banach space. Fix $z \in F$, $0 < \varepsilon < 1$ and put

$$Tx = x + \varepsilon \cdot \operatorname{dist}(x, F) \cdot (z - x), x \in C.$$

It is not difficult to see that $\operatorname{Fix} T = F$ and the Lipschitz constant of T tends to 1 if $\varepsilon \downarrow 0$.

For more information on the structure of fixed-point sets see [2], [5].

In 1973, K. Goebel and W. A. Kirk [4] introduced the class of uniformly k-lipschitzian mappings for $k \geq 1$ which is a natural generalization of the nonexpansive mappings. Recall that a mapping $T: C \to C$ is *uniformly k-lipschitzian*, $k \geq 0$, if

$$\|T^n x - T^n y\| \leq k\|x - y\| \quad \text{for every } x, y \in C \text{ and } n \in \mathbb{N}.$$

Goebel and Kirk stated a relationship between the existence of a fixed point for uniformly lipschitzian mappings and the Clarkson modulus of convexity. Recall that the modulus of convexity δ_E is the function $\delta_E: [0, 2] \to [0, 1]$ defined by

$$\delta_E(\varepsilon) = \inf\left\{1 - \frac{1}{2}\|x + y\| : \|x\| \leq 1,\ \|y\| \leq 1,\ \|x - y\| \geq \varepsilon\right\},$$

and uniform convexity means $\delta_E(\varepsilon) > 0$ for $\varepsilon > 0$. For example, a Hilbert space is uniformly convex. This fact is a direct consequence of parallelogram identity.

Theorem 1.2 ([4]). *Let E be a uniformly convex Banach space with modulus of convexity δ_E and let C be a nonempty bounded closed convex subset of E. Suppose $T: C \to C$ is uniformly k-lipschitzian and $k < \gamma$, where $\gamma > 1$ is the unique solution of the equation*

$$\gamma\left(1 - \delta_E\left(\frac{1}{\gamma}\right)\right) = 1.$$

Then T has a fixed point in C. (Note that in a Hilbert space, $k < \gamma = \sqrt{5}/2$, in L^p-spaces, $2 < p < \infty$, $k < (1 + 2^{-p})^{1/p}$).

E. Sędłak and A. Wiśnicki [10] proved: *under the assumptions of Theorem* 1.2, $\operatorname{Fix} T$ *is not only connected but even a retract of C.* Recently the present author extended this result [7], [8].

In this paper, via one of number of inequalities in Banach spaces, we show new theorems on the structure fixed point sets for mappings with lipschitzian iterates. We improve some results from the papers [6]–[8].

2. Preliminaries

K. Goebel and W. A. Kirk's result was extended in various directions by Lifshitz (1975), Casini and Maluta (1985), Tan and Xu (1993), Domínguez Banavides and Xu (1995), Domínguez Benavides (1998), see [5]. In present paper we continue this study.

Let $(E, \|\cdot\|)$ be a Banach space, C be a nonempty subset of E and $T\colon C \to C$ a lipschitzian mapping. We denote by $\|T^n\|$ the Lipschitz norm of T^n, $n = 1, 2, \ldots$, i.e.

$$\|T^n\| = \sup\left\{\frac{\|T^n x - T^n y\|}{\|x-y\|} : x, y \in C,\ x \neq y\right\}.$$

In 1988 the present author and M. Krüppel [9] constructed a uniformly lipschitzian mapping for which

$$1 \le \liminf_{n\to\infty} \|T^n\| < \limsup_{n\to\infty} \|T^n\|,$$

see Example 5.3.

Let $p > 1$ and denote by λ a number in $[0,1]$ and by $W_p(\lambda)$ the function $\lambda \cdot (1-\lambda)^p + \lambda^p \cdot (1-\lambda)$. The functional $\|\cdot\|^p$ is said to be uniformly convex [12] on the Banach space E if

there exists a positive constant c_p such that for all $\lambda \in [0,1]$ and $x, y \in E$ the following inequality holds

$$\|\lambda x + (1-\lambda) y\|^p \le \lambda \|x\|^p + (1-\lambda)\|y\|^p - c_p \cdot W_p(\lambda) \cdot \|x-y\|^p.$$

H. K. Xu [11] proved that the functional $\|\cdot\|^p$ is uniformly convex in the whole Banach space E if and only if E is p-uniformly convex, i.e. there exists a constant $c > 0$ such that the modulus of convexity $\delta_E(\varepsilon) \ge c \cdot \varepsilon^p$ for $0 \le \varepsilon \le 2$. We note that a Hilbert space is 2-uniformly convex (indeed, $\delta_H(\varepsilon) = 1 - \sqrt{1-(\varepsilon/2)^2} \ge \varepsilon^2/8$) and an L^p-space ($1 < p < \infty$) is $\max\{p, 2\}$-uniformly convex.

An infinite real matrix $A = [a_{n,k}]_{n,k\ge 1}$ is called *strongly ergodic* if

(1) $a_{n,k} \ge 0$, for all n, k,
(2) $\lim_{n\to\infty} a_{n,k} = 0$, forall k,
(3) $\sum_{k=1}^{\infty} a_{n,k} = 1$, for all n,
(4) $\lim_{n\to\infty} \sum_{k=1}^{\infty} |a_{n,k+1} - a_{n,k}| = 0$.

In the paper [6] the present author proved the following generalization of theorems for uniformly lipschitzian mappings:

Theorem 2.1. *Let $p > 1$ and let E be a p-uniformly convex Banach space, C a nonempty bounded closed convex subset of E, and $A = [a_{n,k}]_{n,k\geq 1}$ a strongly ergodic matrix. If $T: C \to C$ is a mapping such that*

$$\liminf_{n\to\infty} \inf_{m=0,1,\dots} \sum_{k=1}^{\infty} a_{n,k} \|T^{k+m}\|^p < 1 + c_p,$$

then T has a fixed point in C.

This result generalizes, among others Lifshitz's Theorem [5] (in case of a Hilbert space), by show that the above mentioned theorem admits certain perturbations in the behavior of the norm of successive iterations in infinite sets; see [6, Example 1]. From this example it follows that the class of mappings with lipschitzian iterates is significantly greater than the class of uniformly lipschitzian mappings.

3. Asymptotic centers

Now we prove a generalization of [8, Lemma 2.2]. Let C be a nonempty bounded closed convex subset of a p-uniformly convex Banach space E and let $T: C \to C$ be a mapping such that $\|T^k\| \geq 1$ for all $k = 1, 2, \dots$, and

$$\limsup_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} \|T^k\|^p < 1 + c_p$$

for some constant $c_p > 0$.

For $x, y \in C$ we use

$$r(y, \{T^k x\}) = \limsup_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} \|y - T^k x\|^p \quad \text{and} \quad r(C, \{T^k x\}) = \inf_{y\in C} r(y, \{T^k x\})$$

to denote the asymptotic radius of $\{T^k x\}$ at y and the asymptotic radius of $\{T^k x\}$ in C, respectively. By Lemma 1 from the paper [6] in p-uniformly convex Banach spaces the asymptotic center of $\{T^k x\}$ in C:

$$A(C, \{T^k x\}) = \{y \in C : r(y, \{T^k x\}) = r(C, \{T^k x\})\}$$

is a singleton.

Let $A: C \to C$ denote a mapping which associates with a given $x \in C$ a unique $z \in A(C, \{T^k x\})$, that is, $z = Ax$. Then we have the following lemma:

Lemma 3.1. *Let E be a p-uniformly convex Banach space and let C be a nonempty bounded closed convex subset of E. Then the mapping $A: C \to C$ is continuous.*

Proof. Without loss of generality we assume that $0 \in C$. On the contrary, suppose that there exists $x_0 \in C$ and $\varepsilon_0 > 0$ such that for all $\eta > 0$ there exists

$x_1 \in C$ such that $\|x_1 - x_0\| < \eta$ and $\|z_1 - z_0\| \geq \varepsilon_0$, where $\{z_0\} = A(C, \{T^k x_0\})$, $\{z_1\} = A(C, \{T^k x_1\})$.

Fix $\eta > 0$ and take $x_1 \in C$ such that

$$\|x_1 - x_0\| < \eta \quad \text{and} \quad \|z_1 - z_0\| \geq \varepsilon_0.$$

Let $R_0 = r(C, \{T^k x_0\})$, $R_1 = r(C, \{T^k x_1\})$ and

$$R = \limsup_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} \|z_1 - T^k x_0\|^p.$$

Notice that

$$R_0 < R. \tag{3.1}$$

Choose $\varepsilon > 0$. Then

$$\begin{cases} \|z_1 - T^k x_0\| < \sqrt[p]{R+\varepsilon}, \\ \|z_0 - T^k x_0\| < \sqrt[p]{R_0+\varepsilon} < \sqrt[p]{R+\varepsilon}, \\ \|z_0 - z_1\| \geq \varepsilon_0, \end{cases} \tag{3.2}$$

for all but finitely many k.

If, for example, $\|z_1 - T^k x_0\| \geq \sqrt[p]{R+\varepsilon}$ for all k, then

$$\|z_1 - T^k x_0\|^p \geq R + \varepsilon.$$

Multiplying both sides of this inequality (for fixed k) by suitable element of the matrix A and summing up such obtained inequalities for $k \geq 1$, we have for $n = 1, 2, \ldots$:

$$\sum_{k=1}^{\infty} a_{n,k} \|z_1 - T^k x_0\|^p \geq (R+\varepsilon) \cdot \sum_{k=1}^{\infty} a_{n,k} = R + \varepsilon.$$

Taking the limit superior as $n \to \infty$ on each side we get

$$R = \limsup_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} \|z_1 - T^k x_0\|^p \geq R + \varepsilon > R,$$

which is contradiction.

It follows by (3.2) and the properties of δ_E that

$$\left\| T^k x_0 - \frac{z_1 + z_0}{2} \right\| \leq \left(1 - \delta_E\left(\frac{\varepsilon_0}{\sqrt[p]{R+\varepsilon}}\right)\right) \sqrt[p]{R+\varepsilon},$$

and

$$\left\| T^k x_0 - \frac{z_1 + z_0}{2} \right\|^p \leq \left(1 - \delta_E\left(\frac{\varepsilon_0}{\sqrt[p]{R+\varepsilon}}\right)\right)^p (R+\varepsilon).$$

Multiplying both sides of this inequality (for fixed k) by suitable element of the matrix A and summing up such obtained inequalities for $k \geq 1$, we have for $n = 1, 2, \dots$:

$$\begin{aligned}\sum_{k=1}^{\infty} a_{n,k} \left\| \frac{z_1 + z_0}{2} - T^k x_0 \right\|^p &\leq \left(1 - \delta_E\left(\frac{\varepsilon_0}{\sqrt[p]{R+\varepsilon}}\right)\right)^p (R+\varepsilon) \cdot \sum_{k-1}^{\infty} a_{n,k} \\ &= \left(1 - \delta_E\left(\frac{\varepsilon_0}{\sqrt[p]{R+\varepsilon}}\right)\right)^p (R+\varepsilon).\end{aligned}$$

Taking the limit superior as $n \to \infty$ on each side we get

$$R_0 < \limsup_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} \left\| \frac{z_1 + z_0}{2} - T^k x_0 \right\|^p \leq \left(1 - \delta_E\left(\frac{\varepsilon_0}{\sqrt[p]{R+\varepsilon}}\right)\right)^p (R+\varepsilon).$$

Moreover (for fixed k), from triangle inequality we have

$$\|T^k x_0 - z_1\|^p \leq (\|T^k x_1 - z_1\| + \|T^k x_0 - T^k x_1\|)^p.$$

Let $a = \|T^k x_1 - z_1\|$, $b = \|T^k x_0 - T^k x_1\|$. By Mean Value Theorem (Lagrange): $(a+b)^p = a^p + b \cdot p \cdot \xi^{p-1}$ for some number $\xi \in (a, a+b)$. Thus, $\xi < a+b \leq 2 \cdot \operatorname{diam} C$, because $0 \in C$, and

$$\begin{aligned}\|T^k x_0 - z_1\|^p &\leq \|T^k x_1 - z_1\|^p + \|T^k x_0 - T^k x_1\| \cdot p \cdot (2 \cdot \operatorname{diam} C)^{p-1} \\ &\leq \|T^k x_1 - z_1\|^p + \|T^k\| \cdot \|x_0 - x_1\| \cdot p \cdot (2 \cdot \operatorname{diam} C)^{p-1}.\end{aligned}$$

Multiplying both sides of this inequality (for fixed k) by a suitable element of the matrix A and summing up such obtained inequalities for $k \geq 1$, we have for $n = 1, 2, \dots$:

$$\sum_{k=1}^{\infty} a_{n,k} \|T^k x_0 - z_1\|^p \leq \sum_{k=1}^{\infty} a_{n,k} \|T^k x_1 - z_1\|^p + p \cdot (2 \cdot \operatorname{diam} C)^{p-1} \cdot \|x_0 - x_1\| \cdot \sum_{k=1}^{\infty} a_{n,k}.$$

Taking the limit superior as $n \to \infty$ on each side we get

$$\begin{aligned}R &= \limsup_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} \|T^k x_0 - z_1\|^p \\ &\leq \limsup_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} \|T^k x_1 - z_1\|^p \\ &\quad + p \cdot (2 \cdot \operatorname{diam} C)^{p-1} \cdot \|x_0 - x_1\| \cdot \limsup_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} \|T^k\| \\ &\leq R_1 + p \cdot (2 \cdot \operatorname{diam} C)^{p-1} \cdot \|x_0 - x_1\| \cdot \limsup_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} \|T^k\|^p \\ &\leq R_1 + \varepsilon + p \cdot (2 \cdot \operatorname{diam} C)^{p-1} \cdot \eta \cdot (1 + c_p).\end{aligned} \tag{3.3}$$

Similarly,

$$(3.4)\quad R_1 < \limsup_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} \|T^k x_1 - z_0\|^p \le R_0 + \varepsilon + p\cdot(2\cdot\operatorname{diam} C)^{p-1}\cdot\eta\cdot(1+c_p).$$

From (3.3) and (3.4), we have

$$(3.5)\qquad \begin{aligned} R &\le R_1 + \varepsilon + p\cdot(2\cdot\operatorname{diam} C)^{p-1}\cdot\eta\cdot(1+c_p)\\ &< R_0 + 2\cdot\varepsilon + 2^p\cdot p\cdot(\operatorname{diam} C)^{p-1}\cdot\eta\cdot(1+c_p). \end{aligned}$$

If $R_0 = 0$, then from (3.5) it follows $R = 0$. This contradicts (3.1). If $R_0 > 0$, then combining (3.5) with (3.3) and applying the monotonicity of δ_E, we obtain

$$\begin{aligned} R_0 < {} & \left(1 - \delta_E\left(\frac{\varepsilon_0}{\sqrt[p]{R_0 + 3\varepsilon + 2^p\cdot p\cdot(\operatorname{diam} C)^{p-1}\cdot\eta\cdot(1+c_p)}}\right)\right)^p\\ & \cdot(R_0 + 3\varepsilon + 2^p\cdot p\cdot(\operatorname{diam} C)^{p-1}\cdot\eta\cdot(1+c_p)). \end{aligned}$$

Letting $\eta, \varepsilon \downarrow 0$ and using the continuity of δ_E, we conclude that

$$1 \le \left(1 - \delta_E\left(\frac{\varepsilon_0}{\sqrt[p]{R_0}}\right)\right)^p < 1.$$

This contradiction proves the continuity of the mapping A. □

4. Main result

In this section we study the structure of fixed point sets for mappings with lipschitzian iterates. The following theorem shows that the fixed-point set in Theorem 2.1 (Theorem 2 of [6]) is actually a retract of C.

THEOREM 4.1. *Let $p > 1$ and let E be a p-uniformly convex Banach space, C a nonempty bounded closed convex subset of E, and $A = [a_{n,k}]_{n,k\ge1}$ a strongly ergodic matrix. If $T\colon C \to C$ is a mapping such that*

$$g = \liminf_{n\to\infty} \inf_{m=0,1,\ldots} \sum_{k=1}^{\infty} a_{n,k} \|T^{k+m}\|^p < 1 + c_p,$$

then T has a fixed point in C and $\operatorname{Fix} T$ *is a retract of C.*

PROOF. We may assume that $\|T^k\| \ge 1$ for all $k = 1, 2, \ldots$, otherwise the well known Banach Contraction Principle guarantees the existence a fixed point of T which is a singleton. Without loss of generality we assume that $0 \in C$.

Let $\{n_i\}$ and $\{m_i\}$ be sequences of natural numbers such that

$$g = \lim_{i\to\infty} \sum_{k=1}^{\infty} a_{n_i,k} \|T^{k+m_i}\|^p < 1 + c_p.$$

From Theorem 2.1, $\operatorname{Fix} T \neq \emptyset$. For any $x \in C$ we can inductively define a sequence $\{z_j\}$ in the following manner: z_1 is the unique point in C that minimizes the functional

$$\overline{r}(y, \{T^k x\}) = \limsup_{i\to\infty} \sum_{k=1}^{\infty} a_{n_i,k} \|y - T^{k+m_i} x\|^p$$

over $y \in C$ and z_{j+1} is the unique point in C that minimizes the functional

$$\overline{r}(y, \{T^k z_j\}) = \limsup_{i\to\infty} \sum_{k=1}^{\infty} a_{n_i,k} \|y - T^{k+m_i} z_j\|^p$$

over $y \in C$, that is, $z_j = A^j x$, $j = 1, 2, \ldots$ Analogically as in the proof of Theorem 2.1 (see [6, Theorem 2]),

$$\begin{aligned}\overline{r}(z_j, \{T^k z_j\}) &= \limsup_{i\to\infty} \sum_{k=1}^{\infty} a_{n_i,k} \|z_j - T^{k+m_i} z_j\|^p \\ &\leq B^j \cdot \limsup_{i\to\infty} \sum_{k=1}^{\infty} a_{n_i,k} \|x - T^{k+m_i} x\|^p = B^j \cdot \overline{r}(x, \{T^k x\})\end{aligned} \tag{4.1}$$

for $j = 1, 2, \ldots$, where

$$B = \frac{1}{c_p}\left(\lim_{i\to\infty} \sum_{k=1}^{\infty} a_{n_i,k} \|T^{k+m_i}\|^p - 1\right) < 1.$$

For a fixed $N \in \mathbb{N}$, from Jensen's inequality we have

$$\|z_{j+1} - z_j\|^p = \|A^{j+1}x - A^j x\|^p \leq 2^{p-1}(\|A^{j+1}x - T^N A^j x\|^p + \|T^N A^j x - A^j x\|^p).$$

Multiplying this inequality for $N = k + m_i$ by suitable element $a_{n_i,k}$, summing up these inequalities for $k = 1, 2, \ldots$, and next taking the limit superior on each side as $i \to \infty$, we obtain

$$\begin{aligned}\|A^{j+1}x - A^j x\|^p &\leq 2^{p-1}\left(\limsup_{i\to\infty} \sum_{k=1}^{\infty} a_{n_i,k} \|A^{j+1}x - T^{k+m_i} A^j x\|^p \right. \\ &\qquad \left. + \limsup_{i\to\infty} \sum_{k=1}^{\infty} a_{n_i,k} \|T^{k+m_i} A^j x - A^j x\|^p\right) \\ &\leq 2^{p-1}\left(\overline{r}(A^{j+1}x, \{T^k A^j x\}) + \overline{r}(A^j x, \{T^k A^j x\})\right) \\ &\leq 2^p \cdot \overline{r}(A^j x, \{T^k A^j x\}) \\ &\leq 2^p \cdot B^j \cdot \overline{r}(x, \{T^k x\}) \leq 2^p \cdot B^j \cdot (\operatorname{diam} C)^p\end{aligned}$$

for $j = 1, 2, \ldots$, $x \in C$. Thus

$$\|A^{j+1}x - A^j x\| \leq 2 \cdot \operatorname{diam} C \cdot (B^{1/p})^j = 2 \cdot \operatorname{diam} C \cdot D^j$$

where $D = B^{1/p} < 1$ and

$$\sup_{x\in C} \|A^s x - A^j x\| \le \frac{D^j}{1-D} \cdot 2 \cdot \operatorname{diam} C \to 0 \quad \text{if } s, j \to \infty,$$

which implies that sequence $\{A^j x\}$ converges uniformly to a function

$$Rx = \lim_{j\to\infty} A^j x, \quad x \in C.$$

It follows from Lemma 3.1 that $R: C \to C$ is continuous.

Moreover, for fixed $N \in \mathbb{N}$, we have

$$\begin{aligned}
&\|Rx - T^N Rx\|^p \\
&\qquad \le (\|Rx - A^j x\| + \|A^j x - T^N A^j x\| + \|T^N A^j x - T^N Rx\|)^p \\
&\qquad \le 3^{p-1}(\|Rx - A^j x\|^p + \|A^j x - T^N A^j x\|^p + \|T^N\|^p \cdot \|A^j x - Rx\|^p) \\
&\qquad \le 3^{p-1}(1 + \|T^N\|^p) \cdot \|Rx - A^j x\|^p + 3^{p-1} \cdot \|A^j x - T^N A^j x\|^p.
\end{aligned}$$

Multiplying this inequality for $N = k + m_i$ by suitable element $a_{n_i,k}$, summing up these inequalities for $k = 1, 2, \dots$, we obtain

$$\begin{aligned}
\sum_{k=1}^{\infty} a_{n_i,k} \|Rx - T^{k+m_i} Rx\|^p &\le 3^{p-1} \cdot \left(1 + \sum_{k=1}^{\infty} a_{n_i,k} \|T^{k+m_i}\|^p\right) \cdot \|Rx - A^j x\|^p \\
&\quad + 3^{p-1} \cdot \sum_{k=1}^{\infty} a_{n_i,k} \|A^j x - T^{k+m_i} Rx\|^p
\end{aligned}$$

for $i = 1, 2, \dots$ Taking the limit superior on each side as $i \to \infty$, we get

$$\begin{aligned}
\overline{r}(Rx, \{T^k Rx\}) &= \limsup_{i\to\infty} \sum_{k=1}^{\infty} a_{n_i,k} \|Rx - T^{k+m_i} Rx\|^p \\
&\le 3^{p-1} \cdot \left(1 + \lim_{i\to\infty} \sum_{k=1}^{\infty} a_{n_i,k} \|T^{k+m_i}\|^p\right) \cdot \|Rx - A^j x\|^p \\
&\quad + 3^{p-1} \cdot \limsup_{i\to\infty} \sum_{k=1}^{\infty} a_{n_i,k} \|A^j x - T^{k+m_i} A^j x\|^p \\
&\overset{(4.1)}{\le} 3^{p-1} \cdot (2 + c_p) \cdot \|Rx - A^j x\|^p + 3^{p-1} \cdot B^j \cdot (\operatorname{diam} C)^p \to 0
\end{aligned}$$

if $j \to \infty$. Thus $\overline{r}(Rx, \{T^k Rx\}) = 0$. This implies that $Rx = TRx$. Indeed, for any $\varepsilon > 0$ there exists natural numbers $n, n+1$ such that

$$\|T^n Rx - Rx\| < \varepsilon \quad \text{and} \quad \|T^{n+1} Rx - Rx\| < \varepsilon.$$

Otherwise, we have for any n and m,

$$\sum_{k=1}^{\infty} a_{n,k} \|Rx - T^{k+m} Rx\|^p \ge \frac{1}{2} \varepsilon^p$$

and therefore

$$\begin{aligned}\overline{r}(Rx, \{T^k Rx\}) &= \limsup_{i\to\infty} \sum_{k=1}^{\infty} a_{n_i,k} \|Rx - T^{k+m_i} Rx\|^p \\ &\geq \liminf_{i\to\infty} \sum_{k=1}^{\infty} a_{n_i,k} \|Rx - T^{k+m_i} Rx\|^p \geq \frac{1}{2}\varepsilon^p.\end{aligned}$$

Thus, for every natural numbers s, there exists a natural number n_s such that

$$\|Rx - T^{n_s} Rx\| < \frac{1}{s} \quad \text{and} \quad \|Rx - T^{n_s+1} Rx\| < \frac{1}{s},$$

i.e.

$$T^{n_s} Rx \to Rx \quad \text{and} \quad T^{n_s+1} Rx \to Rx \quad \text{as } s \to \infty.$$

Since T is continuous,

$$TRx = T\Big(\lim_{s\to\infty} T^{n_s} Rx\Big) = \lim_{s\to\infty} T^{n_s+1} Rx = Rx,$$

and the proof is complete. □

5. Some applications

Now we give applications of some well known inequalities in Hilbert spaces, see [11].

Lemma 5.1. *Let H be a Hilbert space. Then*

$$\|\lambda x + (1-\lambda) y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2$$

for all $x, y \in H$, $0 \leq \lambda \leq 1$ ($c_2 = 1$).

Let $A = [a_{n,k}]_{n,k\geq 1}$ be the Cesaro matrix, that is, for $n = 1, 2, \ldots$,

$$a_{n,k} = \begin{cases} \dfrac{1}{n} & \text{for } k = 1, \ldots, n, \\ 0 & \text{for } k \geq n+1. \end{cases}$$

Hence the following result follows from Theorem 4.1:

Corollary 5.2 ([8, Theorem 4.1]). *Let H be a Hilbert space, C a nonempty bounded closed convex subset of E. If $T\colon C \to C$ is a mapping such that*

$$g = \liminf_{n\to\infty} \inf_{m=0,1,\ldots} \frac{1}{n} \sum_{k=1}^{n} \|T^{k+m}\|^2 < 2,$$

then T has a fixed point in C and $\operatorname{Fix} T$ is a retract of C.

Corollary 5.2 improves Lifshitz's Theorem [5], [7, Corollary 9] (in case of a Hilbert space). To illustrate this let us consider the following example.

Example 5.3. Let l^2 be a Hilbert space, $B = \{x \in l^2 : \|x\| \le 1\}$ and define $T\colon B \to B$ such that

$$\|T^n\| = \begin{cases} 1.4 & \text{for } 1\;0^k < n \le 9 \cdot 10^k, \\ 1.96 & \text{for } 9 \cdot 10^k < n \le 10^{k+1}, \end{cases}$$

$n = 1, 2, \ldots,\ k = 0, 1, \ldots$ The method of construction of this mapping is described in [9]. For this mapping

$$\limsup_{n\to\infty} \|T^n\| = 1.96 > \sqrt{2}$$

and

$$\begin{aligned} \liminf_{n\to\infty} \inf_{m=0,1,\ldots} \frac{1}{n} \sum_{j=1}^{n} \|T^{j+m}\|^2 & \\ \le \liminf_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \|T^j\|^2 &= \lim_{n\to\infty} \frac{1}{9\cdot 10^n} \sum_{j=1}^{9\cdot 10^n} \|T^j\|^2 \\ = \lim_{n\to\infty} \frac{1}{9\cdot 10^n} \bigg[&(1.4)^2 \cdot (9 + 8(10 + 10^2 + \ldots + 10^n)) \\ &+ (1.96)^2 \cdot (1 + 10 + 10^2 + \ldots + 10^{n-1}) \bigg] \\ = \frac{160.6416}{81} \approx 1.98 < 2, & \end{aligned}$$

which means that the assumptions of Corollary 5.2 are satisfied.

When E is particularly an L^p-space $(1 < p < \infty)$, we have the following inequalities, see [11] and the references given there.

Lemma 5.4. *Suppose E is an L^p-space.*

(a) *If $1 < p \le 2$, then*

$$\|\lambda x + (1-\lambda)y\|^2 \le \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - (p-1)\cdot\lambda\cdot(1-\lambda)\cdot\|x-y\|^2$$

for all $x, y \in E$ and $0 \le \lambda \le 1$ $(c_p = p - 1)$.

(b) *If $2 < p < \infty$, then*

$$\|\lambda x + (1-\lambda)y\|^p \le \lambda\|x\|^p + (1-\lambda)\|y\|^p - c_p \cdot W_p(\lambda)\cdot\|x-y\|^p$$

for all $x, y \in E$, $0 \le \lambda \leqslant 1$, where $W_p(\lambda) = \lambda(1-\lambda)^p + \lambda^p(1-\lambda)$ and

$$c_p = \frac{1 + t_p^{p-1}}{(1+t_p)^{p-1}} = (p-1)(1+t_p)^{2-p}$$

with t_p being the unique solution of the equation

$$(p-2)t^{p-1} + (p-1)t^{p-2} - 1 = 0,\ \ 0 < t < 1.$$

All constant appeared in the above inequalities are best possible (*for example, in* L^3*-space*, $c_{L^3} = 2 - \sqrt{2}$, *in* L^4*-space*, $c_{L^4} = 1/3$).

Thus for the Cesaro matrix we have the following corollaries:

Corollary 5.5. *Let* C *be a nonempty bounded closed convex subset of* L^p $(1 < p \leq 2)$. *If* $T: C \to C$ *is a mapping such that*

$$g = \liminf_{n\to\infty} \inf_{m=0,1,\ldots} \frac{1}{n} \sum_{k=1}^{n} \|T^{k+m}\|^2 < p,$$

then T *has a fixed point in* C *and* $\operatorname{Fix} T$ *is a retract of* C.

Corollary 5.6. *Let* C *be a nonempty bounded closed convex subset of* L^p $(2 < p \leq \infty)$. *If* $T: C \to C$ *is a mapping such that*

$$g = \liminf_{n\to\infty} \inf_{m=0,1,\ldots} \frac{1}{n} \sum_{k=1}^{n} \|T^{k+m}\|^p < 1 + c_p,$$

then T *has a fixed point in* C *and* $\operatorname{Fix} T$ *is a retract of* C.

More consequences (among others, for Hardy and Sobolev spaces) are analogous to those presented in [6].

Remark 5.7. Note that Theorem 1.2 was significantly generalized by Lifshitz (1975), Casini and Maluta (1985), Tan and Xu (1993), Domínguez Benavides and Xu (1995), Domínguez Benavides(1998) and the present author (2009) (see [5], [7]) but it is not very clear whether our statements are also valid in all these cases.

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