Topological Methods in Nonlinear Analysis
Journal of the Juliusz Schauder Center
Volume 36, 2010, 285–310

EXISTENCE AND MULTIPLICITY OF NONTRIVIAL SOLUTIONS FOR SEMILINEAR ELLIPTIC DIRICHLET PROBLEMS ACROSS RESONANCE

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Abstract. With the linear growth of the nonlinearity and a new compactness condition involving the asymptotic behavior of its potential at infinity, we establish the existence and multiplicity results of nontrivial solutions for semilinear elliptic Dirichlet problems. The nonlinearity may cross multiple eigenvalues.

1. Introduction

This paper concerns the existence and multiplicity of nontrivial solutions for the following semilinear elliptic Dirichlet boundary value problem

$$\begin{aligned} -\Delta u &= f(x,u) && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n (n \geq 1)$ is an open bounded domain with smooth boundary $\partial\Omega$ and $f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$.

2010 *Mathematics Subject Classification.* Primary 35J60; Secondary 35J20, 47J30.
Key words and phrases. Semilinear elliptic problems, compactness condition, across resonance, Mountain pass theorem, Leray–Schauder degree.
This work was supported by NSFC Grants No. 10531050, 10225107, SRFDP Grant No. 20040183030, the 985 program of Jilin University and the Science Research Foundation for Excellent Young Teachers of College of Mathematics at Jilin University.

Define the functional $J\colon H_0^1(\Omega) \to \mathbb{R}$ by

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega F(x,u)\,dx,$$

where $F(x,u) = \int_0^u f(x,s)\,ds$. Clearly, $J \in C^1(H_0^1(\Omega),\mathbb{R})$ (see [31]) and

$$(\nabla J(u), z) = \int_\Omega (\nabla u \nabla z - f(x,u)z)\,dx, \quad \text{for all } u,z \in H_0^1(\Omega).$$

Thus, u is a weak solution to (1.1) if and only if u is a critical point of J. Denote by $0 < \lambda_1 \leq \ldots \leq \lambda_k \leq \ldots$ the eigenvalues of $(-\Delta, H_0^1(\Omega))$. In order to find nontrivial critical points of the functional J, one usually uses the well known Mountain Pass Theorem presented by Ambrosetti and Rabinowitz in [2], where they introduced the superquadraticity condition, that is, for some $\theta > 2$ and $M > 0$,

$$(\mathrm{AR})_\theta \qquad 0 < \theta F(x,s) \leq f(x,s)s, \quad \text{for all } |s| \geq M, \text{ uniformly for a.e. } x \in \Omega,$$

to ensure that the functional J satisfies the (PS) condition. Clearly, if the $(\mathrm{AR})_\theta$ condition holds, we have $\lim_{|s|\to\infty} F(x,s)/s^2 = \infty$, i.e. $f(x,s)$ is superlinear with respect to s at infinity. However, for some physical problems (see [32] and [33]), the nonlinear term $f(x,s)$ is asymptotically linear with respect to s at infinity and $(\mathrm{AR})_\theta$ is not satisfied. In 1994, D. G. Costa and C. A. Magalhães [12] proposed the nonquadraticity conditions, i.e.

$$(\mathrm{F}_1)_q \qquad \limsup_{|s|\to\infty} \frac{F(x,s)}{|s|^q} \leq b < \infty \quad \text{uniformly for a.e. } x \in \Omega,$$

$$(\mathrm{F}_2^+)_\mu \qquad \liminf_{|s|\to\infty} \frac{sf(x,s) - 2F(x,s)}{|s|^\mu} \geq a > 0 \quad \text{uniformly for a.e. } x \in \Omega,$$

$$(\mathrm{F}_2^-)_\mu \qquad \limsup_{|s|\to\infty} \frac{sf(x,s) - 2F(x,s)}{|s|^\mu} \leq -a < 0 \quad \text{uniformly for a.e. } x \in \Omega,$$

where $\mu > n(q-2)/2$, to ensure the compactness of J. Here the assumptions $(\mathrm{F}_2^\pm)_\mu$ allows the nonlinearity $f(x,s)$ to be superlinear or asymptotically linear. Note that $(\mathrm{F}_2^\pm)_\mu$ implies that

$$(\mathrm{F}_2)_\pm \qquad \lim_{|s|\to\infty} [sf(x,s) - 2F(x,s)] = \pm\infty \quad \text{uniformly for a.e. } x \in \Omega,$$

which is a weaker compactness condition and ensures that J satisfies a weak version of the (PS) condition, namely the Cerami condition. Recently, H. S. Zhou (see [36]) obtained some existence and multiplicity results of solutions of (1.1) by assuming that

$$\lim_{|s|\to\infty} \frac{f(x,s)}{s} = l \quad \text{uniformly for a.e. } x \in \Omega, \tag{1.2}$$

where $l \in [\lambda_1, \infty)$ is a constant. Some relate results can be seen in [26], [34] and [37]. There the fact that the limit of the ratio $f(x,s)/s$ exists at infinity is very important in their proofs to prove the compactness of the functional J.

In this paper, we obtain some existence and multiplicity results of nontrivial solutions for the problem (1.1) under a new compactness condition. Motivated by some ideas in [19], we prove that if f is of linear growth at infinity and the ratio $2F(x,s)/s^2$ satisfies

$$\lambda_k \preceq \alpha_\infty(x) \le \liminf_{|s|\to\infty} \frac{2F(x,s)}{s^2} \le \limsup_{|s|\to\infty} \frac{2F(x,s)}{s^2} \le \beta_\infty(x) \preceq \lambda_{k+1}, \tag{1.3}$$

where k is a positive integer, $\alpha_\infty, \beta_\infty \in L^\infty(\Omega)$ and $a(x) \preceq b(x)$ denotes that $a(x) \le b(x)$ for almost every $x \in \Omega$ with $a(x) < b(x)$ holding on some subset of Ω with positive measure, then the functional J satisfies the (PS) condition. If imposing some additional behavior of the ratio $2F(x,s)/s^2$ at zero and infinity, we shall show that some nontrivial solutions of problem (1.1) are obtained. Note that, under our conditions, the asymptotic behavior of the ratio $f(x,s)/s$ at zero or at infinity can cross several eigenvalues and the nonlinearity f satisfying (1.3) may not satisfy any of the compactness conditions previously mentioned. Note that if the asymptotic behavior of the ratio $f(x,s)/s$ at infinity stays between two consecutive eigenvalues λ_k and λ_{k+1}, which is usually called nonresonance or double resonance, the (PS) condition was obtained in [28] and [15], respectively. However, if the asymptotic behavior of the ratio $f(x,s)/s$ at infinity stays between any interval $[A,B]$ with $A \in (0,\lambda_m), B \in (\lambda_k,\infty)$ $(m \le k)$, i.e. the ratio $f(x,s)/s$ at infinity interacts with multiple eigenvalues of $-\Delta$ on $H_0^1(\Omega)$, neither the Landesman–Lazer type condition (see [21]) nor the Ahmad–Lazer–Paul condition (see [1]) holds and hence our results can not be covered by the previous results. In fact, if we take f to be

$$f(x,s) = \frac{\lambda_k + \lambda_{k+1}}{2} s + C_0 s \sin s \quad \text{for } x \in \Omega,\ |s| \ge M > 0,$$

where $C_0 > 0$ is sufficiently large, then we can obtain that, for $|s| \ge M$,

$$F(x,s) = \frac{\lambda_k + \lambda_{k+1}}{2} \frac{s^2}{2} + C_0 \sin s - C_0 s \cos s$$

and then

$$\lim_{|s|\to\infty} \frac{2F(x,s)}{s^2} = \frac{\lambda_k + \lambda_{k+1}}{2},$$

which implies that (1.3) is satisfied. In addition, it is easy to see that none of the compactness conditions $(\mathrm{AR})_\theta$, $(\mathrm{F}_2)_\pm$, (1.2), the Landesman–Lazer type condition and the Ahmad–Lazer–Paul condition is satisfied. Furthermore, we have

$$f(x,s)s - 2F(x,s) = C_0[s^2 \sin s - 2\sin s - 2s\cos s],$$

which implies that

$$\begin{aligned}\limsup_{|s|\to\infty}[f(x,s)s-2F(x,s)] &= +\infty,\\ \liminf_{|s|\to\infty}[f(x,s)s-2F(x,s)] &= -\infty.\end{aligned}$$

Hence the condition $(\mathrm{F}_2)_\pm$ doesn't hold and so the nonquadraticity conditions $(\mathrm{F}_1)_q - (\mathrm{F}_2^\pm)_\mu$ are not satisfied. On the other hand, note that

$$\frac{f(x,s)}{s} = \frac{\lambda_k+\lambda_{k+1}}{2} + C_0 \sin s$$

with C_0 large enough for $|s| \geq M$, it follows that the range of the ratio $f(x,s)/s$ may cross multiple eigenvalues of $(-\Delta, H_0^1(\Omega))$. Our methods are based on the variational methods and the Leray–Schauder degree theory.

The paper is organized as follows. Section 2 contains the statements of our main results. Some preliminary lemmas are obtained in Section 3. In Section 4, we get the proof of Theorem 2.1. In Section 5, proofs of Theorem 2.3, 2.5 and 2.7 are given.

For convenience, we introduce some denotations. The space $H_0^1(\Omega)$ denoted by H is provided with the inner product

$$\langle u, v\rangle = \int_\Omega \nabla u \cdot \nabla v \, dx$$

and associated norm $\|\cdot\|$; $L^p(\Omega)(1 < p \leq \infty)$ is the usual Sobolev space with inner product and norm denoted by $\langle\,\cdot\,,\,\cdot\,\rangle_p$ and $\|\cdot\|_p$, respectively; $W^{k,p}(k \geq 0, p \geq 1)$ is the Sobolev space with norm

$$\|u\|_{k,p} = \bigg(\sum_{|\alpha|\leq k}\int_\Omega |D^\alpha u(x)|^p \, dx\bigg)^{1/p};$$

$C^k(\overline{\Omega})(k \in \mathbb{Z}^+)$ denotes the space of all k-times continuously differentiable functions defined on $\overline{\Omega}$ with norm

$$\|u\|_{C^k} = \sum_{i=0}^k \|D^k u\|_\infty.$$

$C_1, C_2, \ldots$ denote (possibly different) positive constants.

2. Main results

We make the following assumptions:

(H1) $f \in C(\overline{\Omega}\times\mathbb{R},\mathbb{R})$, $|f(x,s)| \leq C_1(1+|s|)$ for almost every $x \in \Omega$ and all $|s| \geq M > 0$.

(H2) $f(x,0) = 0$ for almost every $x \in \Omega$.

(H3) $\lambda_k \preceq \alpha_\infty(x) \le \liminf_{|s|\to\infty} \dfrac{2F(x,s)}{s^2} \le \limsup_{|s|\to\infty} \dfrac{2F(x,s)}{s^2} \le \beta_\infty(x) \preceq \lambda_{k+1}$, $k \ge 2$ is a positive integer.

(H4) $2F(x,s) \le \eta_1 s^2$ for almost every $x \in \Omega$, $|s| \le \delta_0$, where $\delta_0 > 0$, $\eta_1 \in (\lambda_{m-1}, \lambda_m)$ ($\lambda_0 = 0$), $m \ge 1$ is a positive integer.

(H5) $2F(x,s) \ge \lambda_{r-1} s^2$ for almost every $x \in \Omega$, $|s| \le \delta_1$, where $\delta_1 > 0$, $r \ge 1$ is a positive integer.

Our main results are as follows.

Theorem 2.1. *Assume that* (H1)–(H4) *with* $m = 1$ *hold. Then problem* (1.1) *admits at least two nontrivial solutions, one of which is positive, another one is negative. Furthermore, we have the following results.*

(a) *If k is even, then problem* (1.1) *possesses at least three nontrivial solutions, one of which is positive, another one is negative.*

(b) *If all the critical points of the functional J are nondegenerate, then problem* (1.1) *admits at least four nontrivial solutions.*

(c) *If $f \in C^1(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ and there exists $\mu \in L^\infty(\Omega)$ such that*

$$f'(x,s) \le \mu(x) \preceq \lambda_{k+1} \tag{2.1}$$

for almost every $x \in \Omega$ and for all $s \in \mathbb{R}$, then problem (1.1) *admits at least four nontrivial solutions.*

Remark 2.2. For $f(x,u) = f(u)$, H. Amann and E. Zehnder [4] proved that there exists at least one nontrivial solution whenever the interval $(f'(0), f'(\infty)) \cup (f'(\infty), f'(0))$ contains at least one eigenvalue, where $f'(0) = \lim_{|s|\to 0} f(s)/s$ and $f'(\infty) = \lim_{|s|\to\infty} f(s)/s$. In [11], A. Castro and A. C. Lazer obtained that under the conditions of H. Amann and E. Zehnder and the additional condition $f'(s) < \lambda_{k+1}$ for all $s \in \mathbb{R}$, problem (1.1) has at least three solutions. In 1994, A. Castro and J. Cossio [10] extended this result and proved that if there exists $k \ge 2$ such that

$$f'(0) < \lambda_1 < \lambda_k < f'(\infty) < \lambda_{k+1} \tag{2.2}$$

and $f'(s) \le \gamma < \lambda_{k+1}$ for all $s \in \mathbb{R}$, then (1.1) has at least four nontrivial solutions. In [16], J. Cossio and C. Vélez utilized mountain pass arguments and Leray–Schauder degree to prove the existence of at least three nontrivial solutions of problem (1.1) when k is an even positive integer. J. Cossio and S. Herrón [9] applied the Morse index arguments of the type Lazer–Solimini to show that there exist at least three nontrivial solutions if (2.2) hold and all the critical points of the functional J are nondegenerate. We should also mention the work of T. Bartsch, K. C. Chang and Z.-Q. Wang [6] who applied the Morse theory to obtain four nontrivial solutions of problem (1.1) under the assumption (2.2) and $f'(s) > f(s)/s$, for all $s > 0$. When double resonance occurs at infinity, V. O. V. De Paiva [18] proved the existence of at least three nontrivial solutions by computing critical groups under the conditions

$$f'(x,0) < \lambda_1 < \lambda_k \preceq \liminf_{|s|\to\infty} \frac{f(x,s)}{s} \leq \limsup_{|s|\to\infty} \frac{f(x,s)}{s} \preceq \lambda_{k+1},$$

where $k \geq 2$. Moreover, if $f(x,s)/s$ is strictly increasing on $s \leq 0$ and strictly decreasing with respect on $s \leq 0$, he obtained the existence of at least four nontrivial solutions. Recently, S. J. Li and Z. T. Zhang [25] also dealt with the case that resonance occur at infinity:

$$f'(0) < \lambda_1 < f_\infty = \lim_{|s|\to\infty} \frac{f(s)}{s} = \lambda_k,$$

whose result still depends on the global bound of the derivative of the nonlinearity. Some related results can also be found in [24] and [27]. Involving the ratio $2F(x,s)/s^2$, J. Mawhin, J. R. Ward and M. Willem [29] obtained the solvability of (1.1) by assuming the ratio $2F(x,s)/s^2$ to stay below the first eigenvalue λ_1. In [12], D. G. Costa and C. A. Magalhães obtained a nontrivial solution by assuming that f is of subcritical growth and satisfies the nonquadraticity condition $(\mathrm{F}_1)_q - (\mathrm{F}_2^{\pm})_\mu$ with the following condition

$$\limsup_{|s|\to 0} \frac{2F(x,s)}{s^2} < \lambda_1 < \lambda_k < \liminf_{|s|\to\infty} \frac{2F(x,s)}{s^2}$$

holds uniformly for almost every $x \in \Omega$. In our results, we obtain multiple nontrivial solutions by requiring that f is of linear growth and using the asymptotic behavior of $2F(x,s)/s^2$ at infinity and zero. Thus, as shown in Section 1, the ratio $f(x,s)/s$ may interact with multiple eigenvalues of $-\Delta$. In addition, here the nonquadraticity condition $(\mathrm{F}_1)_q$–$(\mathrm{F}_2^{\pm})_\mu$ is not required.

Theorem 2.3. *Assume that* $f \in C^1(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ *and* (H1), (H3)–(H5) *hold with* $m = r < k$. *If there exists* $\mu \in L^\infty(\Omega)$ *such that* (2.1) *holds, then problem* (1.1) *admits at least two nontrivial solutions.*

Remark 2.4. In [35], W. M. Zou and J. Q. Liu obtained at least two nontrivial solutions of (1.1) by assuming (H3)–(H5) with $m = r < k$ and the following conditions:

(a) $\lambda_k \leq \liminf_{|s|\to\infty} f(x,s)/s$ uniformly for almost every $x \in \Omega$ and there exists $\alpha \in C(\overline{\Omega})$ such that $f'(x,s) \leq \alpha(x) \preceq \lambda_{k+1}$ for almost every $x \in \Omega$ and $s \in \mathbb{R}$.

(b) There exist $\overline{\mu} \in (0,2)$ and $\overline{\beta} \in C(\overline{\Omega})$ such that

$$\limsup_{|s|\to\infty} \frac{sf(x,s) - 2F(x,s)}{|s|^{\overline{\mu}}} \leq \overline{\beta}(x) \preceq 0 \quad \text{uniformly for a.e. } x \in \Omega.$$

Comparing with their result, we don't require the nonquadraticity condition (b) which ensures J satisfies the (PS) condition. In addition, the condition $\lambda_k \leq \liminf_{|s|\to\infty} f(x,s)/s$ is replaced by the assumption that f is of linear growth and $\lambda_k \preceq \alpha_\infty(x) \leq \liminf_{|s|\to\infty} 2F(x,s)/s^2$ and some crossing at infinity may occur.

Theorem 2.5. *Assume that* (H1), (H3), (H4) *hold with* $m = k$. *If the following condition holds:*

$$2F(x,s) \geq \lambda_{k-1}s^2 \quad \text{for a.e. } x \in \Omega \text{ and for all } s \in \mathbb{R}, \tag{2.3}$$

then problem (1.1) *admits at least one nontrivial solution* u *with* $J(u) > 0$.

Remark 2.6. D. G. Costa and C. A. Magalhães [13] obtained at least one nontrivial solution by assuming (H4) with $m = k$, (2.3) and $\lambda_k < \nu_0 \leq \liminf_{|s|\to\infty} 2F(x,s)/s^2$ to ensure the link geometry of J and the nonquadraticity condition to ensure the compactness of J. Theorem 2.3 requires (H1), (H3) but the nonquadraticity condition may be not satisfied.

Theorem 2.7. *Assume that* (H1) *and* (H3) *hold. If there exist constants* $\eta_2 > \lambda_{k+1}$ *and* $\delta_2 > 0$ *such that*

$$2F(x,s) \geq \eta_2 s^2 \quad \text{for a.e. } x \in \Omega,\ |s| \leq \delta_2 \tag{2.4}$$

and

$$2F(x,s) \leq \lambda_{k+2}s^2 \quad \text{for a.e. } x \in \Omega \text{ and for all } s \in \mathbb{R}, \tag{2.5}$$

then problem (1.1) *admits at least one nontrivial solution* u *with* $J(u) < 0$.

Remark 2.8. Since the work of H. Amann and E. Zehnder [4], some existence and multiplicity results were obtained by many authors (see [14], [17]) under the assumption $\lambda_k < f'(\infty) < \lambda_{k+1} < f'(0) < \lambda_{k+2}$. Here, we replace the ratio $f(x,s)/s$ by the ratio $2F(x,s)/s^2$ and some crossing of eigenvalues is allowed.

3. Preliminary lemmas

Let ϕ_i $(i = 1, 2, \ldots)$ denote the eigenfunction of the operator $-\Delta$ on $H_0^1(\Omega)$ corresponding to the eigenvalue λ_i. E_i denotes the eigenspace corresponding to λ_i $(i \geq 1)$ and $N_i = E_1 \oplus \ldots \oplus E_i$. Then $H = N_i \oplus N_i^\perp$. In addition, we indicate by P_i the orthogonal projection in $L^2(\Omega)$ onto E_i. We have the following results:

Lemma 3.1. *Assume that* (H1) *and* (H3) *hold. Then the functional* J *satisfies the* (PS) *condition.*

Proof. Assume that $\{u_n\}_{n\in\mathbb{N}} \subset H$ be a (PS) sequence, i.e. for some $M > 0$,

$$|J(u_n)| \leq M, \quad J'(u_n) \to 0 \quad \text{as } n \to \infty. \tag{3.1}$$

It suffices to prove that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in H. Then a standard argument shows that $\{u_n\}_{n\in\mathbb{N}}$ has a convergent subsequence, which implies that J satisfies the (PS) condition.

Since $\{u_n\}$ is a (PS) sequence, there exists a sequence $\{\varepsilon_n\}$ with $\varepsilon_n \to 0$ in $H_0^{-1}(\Omega)$ such that

$$-\Delta u_n = f(x,u_n) + \varepsilon_n \quad \text{in } H_0^{-1}(\Omega). \tag{3.2}$$

By f is of linear growth at infinity it follows that $u_n \in W^{2,q}(\Omega)$ for some $q > n$ and by the regularity theory we can see that

$$\|u_n\|_{2,q} \le C_1\|u_n\|_\infty + C_2$$

holds for some $C_1, C_2 > 0$. Then by the Sobolev embedding $W^{2,q}(\Omega) \hookrightarrow C^1(\Omega)$ it follows that there exist constants $C_3, C_4 > 0$ such that

$$\|\nabla u_n\|_\infty \le C_3\|u_n\|_\infty + C_4. \tag{3.3}$$

Thus it suffices to prove that there exists some $C_5 > 0$ such that

$$\|u_n\|_\infty \le C_5. \tag{3.4}$$

Assume by the contrary that there exists a sequence $\{u_n\}$ satisfying (3.1) such that

$$\|u_n\|_\infty \to \infty \quad \text{as } n \to \infty. \tag{3.5}$$

Set $z_n = u_n/\|u_n\|_\infty$. By the linear growth of f and the regularity theory, $\{z_n\}$ remains bounded in $W^{2,q}(\Omega)(q > n)$ and so, passing to a subsequence if possible, we have

$$z_n \rightharpoonup z_0 \quad \text{weakly in } W^{2,q}(\Omega), \tag{3.6}$$

$$z_n \to z_0 \quad \text{strongly in } C^1(\overline{\Omega}). \tag{3.7}$$

Clearly, $\|z_0\|_\infty = 1$. On the other hand, for a further subsequence, $f(u_n)/\|u_n\|_\infty$ converges in $L^\infty(\Omega)$ with respect to the weak* topology and the limit function can be written as $m(x)z_0(x)$, where $|m(x)| \le C_6$, for almost every $x \in \Omega$ and some $C_6 > 0$. It follows that z_0 satisfies

$$\begin{aligned} -\Delta z_0 &= m(x)z_0 && \text{in } \Omega, \\ z_0 &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{3.8}$$

Moreover, we can obtain, up to a subsequence,

$$\int_\Omega \left| \frac{f(x,u_n)}{\|u_n\|_\infty} - m(x)z_0(x) \right| dx \to 0. \tag{3.9}$$

Indeed, the fact that $f(x,u_n)/\|u_n\|_\infty$ converges to $m(x)z_0(x)$ in $L^\infty(\Omega)$ with respect to the weak* topology means that

$$\int_\Omega \left[\frac{f(x,u_n)}{\|u_n\|_\infty} - m(x)z_0(x)\right]\phi\,dx \to 0 \quad \text{for all } \phi \in L^1(\Omega).$$

Then by the L^p theory we can see $f(x,u_n)/\|u_n\|_\infty$ converges weakly to $m(x)z_0(x)$ in $L^p(\Omega)$ with $1 \le p < \infty$.

By the linear theory, we know that, for every $h \in L^p(\Omega)(p>1)$, the problem

$$\begin{aligned} -\Delta w &= h(x) \quad \text{in } \Omega, \\ w &= 0 \qquad \text{on } \partial\Omega, \end{aligned}$$

has a unique classical solution $w \in W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$. Furthermore, we have $\|w\|_{2,p} \le C\|h\|_p$. Denote K as the operator mapping h onto w. It is easy to see that K is invertible. Note that, by (H1), there exist some positive integer m and $M>0$ such that

$$|f(x,u)| \le \lambda_m|u|, \quad \text{for all } |u| \ge M.$$

Then, for every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|f(x,u)\cdot u| \ge \frac{|f(x,u)|^2}{\lambda_m} - \varepsilon u^2 - C_\varepsilon,$$

which implies that, if u is a solution of (1.1), then

$$\begin{aligned} \int_\Omega |f(x,u)\cdot u|\,dx &\ge \int_\Omega \frac{|f(x,u)|^2}{\lambda_m}\,dx - \varepsilon\int_\Omega u^2\,dx - C_\varepsilon \text{meas}(\Omega) \\ &= \sum_{j=1}^m \frac{1}{\lambda_m}\|P_jf\|_2^2 + \frac{1}{\lambda_m}\|(I-\overline{P})f\|_2^2 - \varepsilon\|u\|_2^2 - C_\varepsilon\text{meas}(\Omega), \end{aligned} \tag{3.10}$$

where $\overline{P} = P_1 + \ldots + P_m$.

On the other hand, we have

$$\begin{aligned} \int_\Omega |f(x,u)\cdot u|\,dx &= \int_\Omega |f(x,u)\cdot Kf(x,u)|\,dx \\ &\le \sum_{j=1}^m \int_\Omega |P_jf\cdot KP_jf|\,dx + \int_\Omega |(I-\overline{P})f\cdot K(I-\overline{P})f|\,dx \\ &\le \sum_{j=1}^m \frac{1}{\lambda_j}\int_\Omega |P_jf|^2\,dx + \frac{1}{\lambda_{m+1}}\int_\Omega |(I-\overline{P})f|^2\,dx \\ &= \sum_{j=1}^m \frac{1}{\lambda_j}\|P_jf\|_2^2 + \frac{1}{\lambda_{m+1}}\|(I-\overline{P})f\|_2^2. \end{aligned} \tag{3.11}$$

Together (3.10) and (3.11), it follows that

$$\left(\frac{1}{\lambda_m} - \frac{1}{\lambda_{m+1}}\right)\|(I-\overline{P})f\|_2^2 \le \sum_{j=1}^m \left(\frac{1}{\lambda_j} - \frac{1}{\lambda_m}\right)\|P_jf\|_2^2 + \varepsilon\|u\|_2^2 + C_\varepsilon\text{meas}(\Omega).$$

Hence, there exists constant $M_1 > 0$ such that

$$\|f\|_2 \le M_1 \|\overline{P} f\|_2. \tag{3.12}$$

Denote $w_n = z_n - z_0$. Then w_n satisfies

$$\begin{aligned} -\Delta w_n &= \frac{f(x,u_n)}{\|u_n\|_\infty} - m(x) z_0(x) && \text{in } \Omega, \\ w_n &= 0 && \text{on } \partial\Omega, \end{aligned}$$

which combing with (3.12) implies that

$$\|z_n - z_0\|_{2,2} = \|w_n\|_{2,2} \le M_1 \bigg\| \overline{P} \bigg[\frac{f(x,u_n)}{u_n(x)} z_n(x) - m(x) z_0(x) \bigg] \bigg\|_2 .$$

By the fact that $f(x,u_n)/u_n(x) z_n(x)$ converges weakly to $m(x) z_0(x)$ in $L^2(\Omega)$ and $\overline{P}$ is of finite dimensional, it follows that

$$z_n \to z_0 \quad \text{strongly in } W^{2,2}(\Omega),$$

which implies that $f(x,u_n)/\|u_n(x)\|_\infty$ converges to some function $\varrho(x)$ in $L^2(\Omega)$, and for a subsequence,

$$\frac{f(x,u_n)}{\|u_n(x)\|_\infty} \to \varrho(x) \quad \text{uniformly for a.e. } x \in \Omega.$$

Thus it is easily seen that $\varrho(x) = m(x) z_0(x)$ and (3.9) follows.

Claim 1. *For any point $x_0 \in \Omega$, there exists $x^* \in \partial\Omega$, with the segment $[x_0, x^*] \subset \overline{\Omega}$, such that, passing to a subsequence if possible,*

$$\frac{\int_0^1 \overline{f}(x, u_n(x_0 + t(x^* - x_0)))\, dt}{\|u_n\|_\infty} \to 0 \quad \textit{uniformly for a.e. } x \in \Omega, \tag{3.13}$$

where $\overline{f}(x,s) \doteq f(x,s) - m(x)s$.

Proof. Fix $x_0 \in \Omega$. By (3.6), (3.7), (3.9), it follows that

$$\begin{aligned} \int_\Omega \bigg| \frac{\overline{f}(x,u_n)}{\|u_n\|_\infty} \bigg| \, dx \le \int_\Omega \bigg| \frac{f(x,u_n)}{\|u_n\|_\infty} - m(x) z_0 \bigg| \, dx \\ + \int_\Omega |m(x) z_n - m(x) z_0| \, dx \to 0. \end{aligned} \tag{3.14}$$

Take

$$\alpha_n(x) = \frac{\overline{f}(x,u_n)}{\|u_n\|_\infty} \quad \text{in } \Omega, \qquad \alpha_n(x) = 0 \quad \text{in } \overline{B}_{R_0} \setminus \Omega,$$

where $B_{R_0} \doteq \{x \in \mathbb{R}^n : \|x - x_0\|_{\mathbb{R}^n} \le R_0\}$. Then by (3.14) we can see that

$$\int_{\overline{B}_0} |\alpha_n(x)| \, dx \to 0. \tag{3.15}$$

Now we introduce spherical coordinates in $\mathbb{R}^n$. Without loss of generality, we suppose that x_0 is the origin of $\mathbb{R}^n$. We denote $x \in \overline{B}_{R_0}$ as $x = (x_1, \ldots, x_n)$. Take

$x_1 = r\cos\theta_1, x_2 = r\sin\theta_1\cos\theta_2, \dots, x_{n-1} = r\sin\theta_1\dots\sin\theta_{n-2}\cos\theta_{n-1}, x_n = r\sin\theta_1\dots\sin\theta_{n-2}\sin\theta_{n-1}$, where $r \in [0,R_0]$, $\theta_i \in [0,\pi]$, $i = 1,\dots,n-2$, $\phi \in [0,2\pi]$. From (3.15) it follows that

$$\int_{[0,\pi]^{n-2}\times[0,2\pi]} \bigg(\int_0^{R_0} |\alpha_n(x(r,\theta_1,\dots,\theta_{n-2},\phi))| \cdot \frac{D(x_1,\dots,x_n)}{D(r,\theta_1,\dots,\theta_{n-2},\phi)}\,dr \bigg)\,d\theta_1\dots\,d\theta_{n-2}\,d\phi \to 0,$$

where

$$\frac{D(x_1,\dots,x_n)}{D(r,\theta_1,\dots,\theta_{n-2},\phi)} = r^{n-1}(\sin\theta_1)^{n-2}(\sin\theta_2)^{n-3}\dots(\sin\theta_{n-2}).$$

Then, passing to a subsequence if possible, for almost every $(\theta_1,\dots,\theta_{n-2},\phi)$,

$$\int_0^{R_0} |\alpha_n(x(r,\theta_1,\dots,\theta_{n-2},\phi))|r^{n-1}\,dr \to 0.$$

For a further subsequence, we obtain that, for almost every $(\theta_1,\dots,\theta_{n-2},\phi)$,

$$\alpha_n(x(r,\theta_1,\dots,\theta_{n-2},\phi)) \to 0,$$

for almost every $r \in [0,R_0]$. Then, for almost every $\overline{x} \in \partial B_{R_0}$, there exists a subsequence of $\{\alpha_n\}$, we still denote it by $\{\alpha_n\}$, such that

$$\alpha_n(x_0 + t(\overline{x} - x_0)) \to 0,$$

for almost every $t \in [0,1]$. Since the functions α_n are uniformly bounded almost every in $\overline{B}$, we obtain by the Lebesgue dominated convergence theorem that

$$\int_0^1 \alpha_n(x_0 + t(\overline{x} - x_0))\,dt \to 0.$$

Hence, taking $x^* \in [x_0,\overline{x}] \cap \partial\Omega$ such that

$$|x^* - x_0| = \min\{|x - x_0| : x \in [x_0,\overline{x}] \cap \partial\Omega\},$$

we can see that (3.13) holds. □

We now distinguish three cases:

(i) $z_0 \ge 0$ in Ω,
(ii) $z_0 \le 0$ in Ω,
(iii) z_0 changes sign in Ω.

It will be shown in the following that each case leads to a contradiction. For convenience we denote $s_n = \max u_n$ and $t_n = \min u_n$.

Case (i). In this case we have $s_n = \|u_n\|_\infty \to +\infty$ and $\{t_n\}$ are bounded. By (1.3), we get

$$\liminf_{n\to\infty} \frac{2F(x,s_n)}{s_n^2} = \eta(x) \tag{3.16}$$

with

$$\lambda_k \preceq \eta(x) \preceq \lambda_{k+1} \quad \text{for a.e. } x \in \Omega. \tag{3.17}$$

Define $\overline{F}(x,s) = \int_0^s \overline{f}(x,t)\,dt$. Then by (3.16) it follows that

$$\liminf_{n\to\infty} \frac{2\overline{F}(x,s_n)}{s_n^2} = \limsup_{n\to\infty} \left[\frac{2F(x,s_n)}{s_n^2} - m(x)\right] = \eta(x) - m(x). \tag{3.18}$$

By (3.18), there exists some subset Ω_0 of Ω with positive measure such that

$$\eta(x) - m(x) \neq 0 \quad \text{for a.e. } x \in \Omega_0. \tag{3.19}$$

Indeed, if not, we have $m(x) = \eta(x)$ for a.e. $x \in \Omega$. However, this together with (3.17) implies that $z_0 \equiv 0$, which is contrary to that $\|z_0\|_\infty = 1$.

For all n, let $x_n \in \overline{\Omega}$ be such that $u_n(x_n) = \max_{x\in\overline{\Omega}} u_n(x)$. Passing to a subsequence if possible, we can suppose that $x_n \to x_0 \in \Omega$, and we have $v(x_0) = \max_{x\in\overline{\Omega}} v(x)$. Let $x^* \in \partial\Omega$ be a point provided by Claim 1. Set

$$\gamma(t) = x^* + t(x_0 - x^*), \qquad \gamma_n(t) = x_0 + t(x_n - x_0),$$

where $t \in [0,1]$. Then by the linear growth of f, the boundedness of Ω and (3.3) we have

$$\begin{aligned}
|\overline{F}(x,s_n)| &= |\overline{F}(x,s_n) - \overline{F}(x,0)| \\
&= |\overline{F}(x,s_n) - \overline{F}(x,u_n(x_0)) + \overline{F}(x,u_n(x_0)) - \overline{F}(x,0)| \\
&= \bigg| \int_0^1 \overline{f}(x,u_n(\gamma_n(t)))(u_n(x_n) - u_n(x_0)) \\
&\qquad + \int_0^1 \overline{f}(x,u_n(\gamma(t)))(u_n(x_0) - u_n(x^*)) \bigg| \\
&= \bigg| \int_0^1 \overline{f}(x,u_n(\gamma_n(t)))(\nabla u_n \cdot (x_n - x_0))\,dt \\
&\qquad + \int_0^1 \overline{f}(x,u_n(\gamma(t)))(\nabla u_n \cdot (x_0 - x^*))\,dt \bigg| \\
&\leq \int_0^1 \frac{|\overline{f}(x,u_n(\gamma_n(t)))|}{\|u_n\|_\infty}\,dt \|u_n\|_\infty \|\nabla u_n\| |x_n - x_0| \\
&\qquad + \bigg| \int_0^1 \frac{\overline{f}(x,u_n(\gamma(t)))}{\|u_n\|_\infty}\,dt \|u_n\|_\infty \|\nabla u_n\| \operatorname{diam}(\Omega) \bigg| \\
&\leq C_7 \bigg(|x_n - x_0| + \bigg| \int_0^1 \frac{\overline{f}(x,u_n(\gamma(t)))}{\|u_n\|_\infty}\,dt \bigg| \bigg) s_n^2,
\end{aligned}$$

for some $C_7 > 0$. Thus, by $x_n \to x_0$ and Claim 1 it follows that

$$\left| \frac{\overline{F}(x,s_n)}{s_n^2} \right| \to 0 \quad \text{uniformly for a.e. } x \in \Omega,$$

which leads to a contradiction with (3.18) and (3.19). Hence, (3.5) can't hold and so the conclusion follows.

Case (ii). In this case $t_n = -\|u_n\|_\infty \to -\infty$ and $\{s_n\}$ are bounded. Then we can get a contradiction using an argument similar to the proof of Case (i). We omit it here.

Case (iii). We first prove that there exists a constant $\overline{n} \in \mathbb{Z}^+$ and $0 < \kappa_1 < 1 < \kappa_2$ such that

$$\kappa_1 \le \frac{\max u_n}{-\min u_n} \le \kappa_2 \tag{3.20}$$

for all $n \ge \overline{n}$. Indeed, if not, we assume by contradiction, that there exists a subsequence of $\{u_n\}$, we still denote it as $\{u_n\}$ with $\max u_n \to \infty$ and $\min u_n \to -\infty$, such that

$$\text{either} \quad \frac{\max u_n}{-\min u_n} \to 0 \quad \text{or} \quad \frac{\max u_n}{-\min u_n} \to \infty.$$

By z_0 changes sign in Ω and the fact that $\|z_0\|_\infty = 1$ it follows that

$$\frac{\max(u_n/\|u_n\|_\infty)}{-\min(u_n/\|u_n\|_\infty)} \to \frac{\max z_0}{-\min z_0} = \overline{\eta} \in (0,\infty).$$

A contradiction. Thus (3.20) holds. Hence we have

$$s_n \to \infty \quad \text{and} \quad t_n \to -\infty. \tag{3.21}$$

The other parts can be treated as in Case (i).

In a word, we obtained a contradiction to (3.5). Thus by (3.3) it follows that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in H and standard arguments imply that J satisfies the (PS) condition.

The proof of Lemma 3.1 is complete. □

LEMMA 3.2. *Assume that f satisfies* (H1) *and* (H4). *Then there exist $\rho_0 > 0$ and $r_0 > 0$ such that*

$$J(u) \ge \rho_0 \quad \textit{for } \|u\| = r_0,\ u \in N_{m-1}^\perp.$$

PROOF. By (H1), there exists $l > \lambda_m + 1$ large enough such that

$$2F(x,s) \le ls^2 \quad \text{for } |s| \ge \delta_0,\ \text{a.e. } x \in \Omega.$$

It is easily seen that

$$2F(x,s) \le 2ls^2 - l\delta_0^2 \quad \text{for } |s| \ge \delta_0,\ \text{a.e. } x \in \Omega. \tag{3.22}$$

Denote $N_{m-1}^\perp = V_1 \oplus W_1$, where

$$V_1 = \operatorname{span}\{\phi_m,\dots,\phi_{\gamma-1}\}, \qquad W_1 = \operatorname{span}\{\phi_\gamma,\dots\}.$$

Here γ is large enough such that $\lambda_\gamma > 2\eta_1^2/(\lambda_m - \eta_1) + 5l + 2\eta_1$. For $u = N_{m-1}^\perp$, write $u = v + w$, where $v \in V_1$ and $w \in W_1$. Since V_1 is of finite dimensional, there exists a constant $C_{\gamma-1}$ such that

$$\|u\|_{L^\infty} \leq C_{\gamma-1}\|u\| \quad \text{for all } u \in V_1. \tag{3.23}$$

Let

$$\vartheta = \frac{\lambda_m + \eta_1}{4}v^2 + \frac{1}{4}\lambda_\gamma w^2 - F(x, v + w).$$

If $|v + w| \leq \delta_0$, then by (2.4) and the choose of γ it follows that

$$\begin{aligned}\vartheta &\geq \frac{\lambda_m + \eta_1}{4}v^2 + \frac{1}{4}\lambda_\gamma w^2 - \frac{1}{2}\eta_1 v^2 - \frac{1}{2}\eta_1 w^2 - \eta_1|v||w| \\ &\geq \frac{1}{4}(\lambda_m - \eta_1)v^2 + \frac{1}{4}(\lambda_\gamma - 2\eta_1)w^2 - \eta_1|v||w| \\ &\geq \left[\frac{1}{2}(\lambda_m - \eta_1)^{1/2}(\lambda_\gamma - 2\eta_1)^{1/2} - \eta_1\right]|v||w| \geq 0.\end{aligned} \tag{3.24}$$

If $|v + w| > \delta_0$, then by (3.22) and the choose of γ we get

$$\begin{aligned}\vartheta &\geq \frac{\lambda_m + \eta_1}{4}v^2 + \frac{1}{4}\lambda_\gamma w^2 - lv^2 - lw^2 - lvw + \frac{1}{2}l\delta_0^2 \\ &\geq \frac{1}{4}(\lambda_\gamma - 4l)w^2 + lv^2 + \left(\frac{\lambda_m + \eta_1}{4} - 2l\right)v^2 - l|v||w| + \frac{1}{2}l\delta_0^2 \\ &\geq [(\lambda_\gamma - 4l)^{1/2}l^{1/2} - l]|v||w| - \left(2l - \frac{\lambda_m + \eta_1}{4}\right)v^2 + \frac{1}{2}l\delta_0^2 \\ &\geq -\left(2l - \frac{\lambda_m + \eta_1}{4}\right)v^2 + \frac{1}{2}l\delta_0^2.\end{aligned} \tag{3.25}$$

Let

$$\Omega_1 \doteq \{x \in \Omega : |v + w| \leq \delta_0\}, \qquad \Omega_2 \doteq \{x \in \Omega : |v + w| \geq \delta_0\}$$

and

$$\xi_0 \doteq \frac{3}{4(C_{\gamma-1})^2(8l - \lambda_m - \eta_1)}\left(1 - \frac{\eta_1}{\lambda_m}\right).$$

Then by (3.24)–(3.25) we have

$$\begin{aligned}\int_\Omega \vartheta\, dx &= \int_{\Omega_1} \vartheta\, dx + \int_{\Omega_2} \vartheta\, dx \\ &\geq \int_{\Omega_2}\left[-\left(2l - \frac{\lambda_m + \eta_1}{4}\right)v^2 + \frac{1}{2}l\delta_0^2\right]dx.\end{aligned} \tag{3.26}$$

Hence

$$\begin{aligned}J(u) &= J(v + w) = \frac{1}{2}\|v\|^2 + \frac{1}{2}\|w\|^2 - \int_\Omega F(x, v + w)\, dx \\ &\geq \frac{1}{4}\|v\|^2 + \frac{1}{4}\|w\|^2 + \frac{1}{4}\lambda_m\|v\|_2^2 + \frac{1}{4}\lambda_\gamma\|w\|_2^2 - \int_\Omega F(x, v + w)\, dx \\ &\geq \frac{1}{4}\left(1 - \frac{\eta_1}{\lambda_m}\right)\|v\|^2 + \frac{1}{4}\|w\|^2 + \int_\Omega \vartheta\, dx.\end{aligned}$$

If $\operatorname{meas}\Omega_2 \geq \xi_0$, then by (3.26) and the choose of l it follows that

$$
\begin{aligned}
J(u) &\geq \left[\frac{1}{4}\left(1-\frac{\eta_1}{\lambda_m}\right)-\left(2l-\frac{\lambda_m+\eta_1}{4}\right)\right]\|v\|^2+\frac{1}{4}\|w\|^2+\frac{1}{2}l\delta_0^2\xi_0 \\
&\geq \left[\frac{1}{4}\left(1-\frac{\eta_1}{\lambda_m}\right)-\left(2l-\frac{\lambda_m+\eta_1}{4}\right)\right]\|u\|^2+\frac{1}{2}l\delta_0^2\xi_0 \\
&\geq -\delta_1\|u\|^2+\frac{1}{2}l\delta_0^2\xi_0,
\end{aligned}
\tag{3.27}
$$

where $\delta_1 = (8l-\lambda_m-\eta_1-1)/4 > 0$.

If $\operatorname{meas}\Omega_2 < \xi_0$, then by (3.23), (3.26) and the definition of ξ_0 we get

$$
\begin{aligned}
J(u) &\geq \frac{1}{4}\left(1-\frac{\eta_1}{\lambda_m}\right)\|v\|^2+\frac{1}{4}\|w\|^2 \\
&\quad -\left(2l-\frac{\lambda_m+\eta_1}{4}\right)C_{\gamma-1}^2\|v\|^2\operatorname{meas}\Omega_2 \\
&\geq \frac{1}{4}\left(1-\frac{\eta_1}{\lambda_m}\right)\|u\|^2-\frac{3}{16}\left(1-\frac{\eta_1}{\lambda_m}\right)\|v\|^2 \\
&\geq \frac{1}{16}\left(1-\frac{\eta_1}{\lambda_m}\right)\|u\|^2.
\end{aligned}
\tag{3.28}
$$

By (3.27)–(3.28) we may find $\rho_0, r_0 > 0$ such that $J(u) \geq \rho_0$ for $u \in N_{m-1}^{\perp}$ with $\|u\| = r_0$. □

Lemma3.3. *Assume that* (H1) *and* (H3) *hold. Then*

(a) *For any $v \in N_k$, $J(v+w) \to \infty$ as $w \in N_k^{\perp}$ and $\|w\| \to \infty$;*
(b) *$J(v) \to -\infty$ as $v \in N_k$ and $\|v\| \to \infty$.*

Proof. We just prove (a), for (b) see Proposition 2(d) in [15]. Define

$$\zeta(w) \doteq \int_\Omega [|\nabla w|^2 - \beta_\infty(x)w^2]\,dx, \quad w \in N_k^{\perp}.$$

Clearly, by $\int_\Omega |\nabla w|^2\,dx \geq \lambda_{k+1}\|w\|_2^2$ for all $w \in N_k^{\perp}$, we have

$$\zeta(w) \geq \int_\Omega [\lambda_{k+1} - \mu(x)]w^2\,dx \geq 0, \quad \text{for all } w \in N_k^{\perp}. \tag{3.29}$$

We claim that there exist $\mu_2, M > 0$ such that

$$\zeta(w) \geq \mu_2\|w\|^2 \quad \text{for } w \in N_k^{\perp} \text{ with } \|w\| \geq M. \tag{3.30}$$

Indeed, we assume, by contradiction, that (3.30) doesn't hold. Then there exists a sequence $\{w_n\} \subset W$ with $\|w_n\| \to \infty$ such that $\beta(w_n)/\|w_n\|^2 \to 0$ as $n \to \infty$. Let $z_n = w_n/\|w_n\|$. Then $\|z_n\| = 1$. Passing, if necessary, to a subsequence we assume that $z_n \rightharpoonup z_0 \in W$ weakly and $z_n \to z_0$ in $C(\overline{\Omega})$ with $\|z_0\| = 1$. Clearly, z_0 is not identical to 0 and hence by $z_n, z_0 \in W$ it follows that

$$0 \leq \beta(z_0) \leq \liminf \beta(z_n) = 0, \tag{3.31}$$

which implies that

$$0 = \beta(z_0) \geq \int_\Omega [|\nabla z|^2 - \lambda_{k+1} z_0^2]\, dx \geq 0.$$

Thus z_0 is an eigenfunction corresponding to λ_{k+1}. Note that (2.1), (3.29) and (3.31) imply that $z_0 = 0$ on the set $\Omega_1 \doteq \{x \in \Omega : \mu(x) < \lambda_{k+1}\}$, by the unique continuation principle we get $z_0 \equiv 0$. A contradiction. Thus (3.30) holds.

Let $0 < \varepsilon < \mu_2 \lambda_{k+1}$. Owing to (H3) there exists $M_\varepsilon \in L^1(\Omega)$ such that

$$F(x,s) \leq \frac{1}{2}(\beta_\infty(x) + \varepsilon)s^2 + M_\varepsilon(x) \quad \text{for a.e. } x \in \Omega \text{ and all } s \in \mathbb{R}.$$

Hence, for $u = v + w \in H = N_k \oplus N_k^\perp$, we have

$$\begin{aligned}
J(u) &= \frac{1}{2}\|v+w\|^2 - \int_\Omega [F(x, v+w)]\, dx \\
&\geq \frac{1}{2}(\|v\|^2 + \|w\|^2) - \frac{1}{2}\int_\Omega (\beta_\infty(x) + \varepsilon)(v^2 + w^2)\, dx - \int_\Omega M_\varepsilon(x)\, dx \\
&= \frac{1}{2}\zeta(w) - \frac{1}{2}\varepsilon \int_\Omega w^2\, dx + \frac{1}{2}\|v\|^2 - \frac{1}{2}\int_\Omega (\beta_\infty(x) + \varepsilon) v^2\, dx - \|M_\varepsilon\|_1 \\
&\geq \frac{1}{2}\left(\mu_2 - \frac{\varepsilon}{\lambda_{i+1}}\right)\|w\|^2 + \frac{1}{2}\|v\|^2 - \frac{1}{2}\int_\Omega (\beta_\infty(x) + \varepsilon) v^2\, dx - \|M_\varepsilon\|_1.
\end{aligned}$$

Obviously, $(\mu_2 - \varepsilon/\lambda_{i+1})/2 > 0$. Thus it follows that for given $v \in N_k$ we have $J(v + w) \to \infty$ as $w \in N_k^\perp$ and $\|w\| \to \infty$. $\square$

4. Proof of Theorem 2.1

Proof of The First Part of Theorem 2.1. Consider the following truncated problem

$$\begin{aligned}
-\Delta u &= f^+(x,u) && \text{in } \Omega, \\
u &= 0 && \text{on } \partial\Omega,
\end{aligned}$$

where $f^+(x,s) = f(x,s)$, if $s \geq 0$, $f^+(x,s) = 0$, if $s < 0$. Define the functional $J^+\colon H \to \mathbb{R}$ by

$$J^+(u) = \int_\Omega \left[\frac{1}{2}|\nabla u|^2 - F^+(x,u)\right] dx,$$

where $F^+(x,u) = \int_0^u f^+(x,s)\, ds$. By (H1), $J^+ \in C^1(H, \mathbb{R})$ (see [31]). Similar arguments as in the proof of Lemma 3.1 imply that J^+ satisfies the (PS) condition. In the following we shall show that the functional J^+ has a mountain pass geometry. Precisely, we shall prove that J^+ satisfies:

(a) There exists $r, \delta > 0$ such that $J^+(u) \geq \delta$ for all $u \in H_0^1(\Omega)$ with $\|u\| = r$.

(b) $J^+(t\phi_1) \to -\infty$ as $t \to \infty$.

(a) By Lemma 3.2 with $m = 1$, this is an immediate consequence of condition (H4) with $m = 1$.

(b) By (H1) and (H3), there exists $M_2 \in \mathbb{R}$ and $A_0 \in (\lambda_k, \lambda_{k+1})$ such that

$$2F^+(x,s) \geq A_0|s|^2 + M_2 \quad \text{for } s \geq 0.$$

Then we have

$$\begin{aligned} J^+(t\phi_1) &= \int_\Omega \frac{t^2}{2}|\nabla\phi_1|^2\,dx - \int_\Omega F^+(x,t\phi_1)\,dx \\ &\leq \frac{t^2}{2}\lambda_1 \int_\Omega |\phi_1|^2\,dx - \frac{t^2}{2}A_0 \int_\Omega |\phi_1|^2\,dx - M_2|\Omega| \\ &= \frac{t^2}{2}[\lambda_1 - A_0] \int_\Omega |\phi_1|^2\,dx - M_2|\Omega|. \end{aligned}$$

By $\lambda_1 - A_0 < \lambda_1 - \lambda_k < 0$ it follows that $J^+(t\phi_1) \to -\infty$ as $t \to \infty$. Therefore, by the Mountain Pass Theorem, J^+ has a critical point u^+ with $J^+(u^+) > 0$, which implies that the functional J has a nontrivial critical point $u^+ \geq 0$. By the Maximum Principle it follows that $u^+ > 0$ in Ω. Similar arguments show that J admits a nontrivial critical point $u^- < 0$. □

Lemma 4.1. *If P is a bounded region containing the positive solution u^+ of* (1.1) *and no other critical point of J then*

$$\deg(\nabla J, P, 0) = -1. \tag{4.2}$$

If N is a bounded region containing the negative solution u^- of (1.1) *and no other critical point of J then*

$$\deg(\nabla J, N, 0) = -1. \tag{4.3}$$

Proof. Since u^+ and u^- are all of mountain pass type, the conclusion follows immediately by a result of H. Hofer [20]. □

Lemma 4.2. *If* 0 *is an isolated critical point, then there exists ρ_1 small such that* $\deg(\nabla J, B_\rho, 0) = 1$, *for all* $0 < \rho \leq \rho_1$, *where* $B_\rho = \{x \in \mathbb{R}^n : |x| < \rho\}$.

Proof. Similar arguments as in the proof of the first part of Theorem 2.1 imply that 0 is a strict local minimizer of J. Then the conclusion is obtained by Corollary 2 of H. Amann [3]. □

Lemma 4.3. *Assume that* (H1) *and* (H3) *hold. If u is a solution of* (1.1), *then there exists constant $C > 0$ such that $\|u\|_{C^1} \leq C$.*

Proof. Since f is of linear growth at infinity, any solution $u \in H_0^1(\Omega)$ of (1.1) belongs to $W^{2,q}(\Omega)$ for some $q > n$ and we obtain by the regularity theory that $\|u\|_{2,q} \leq C\|u\|_\infty + C$. Then by the Sobolev embedding $W_0^{2,q} \hookrightarrow C^1(\overline{\Omega})$ it follows that there exists constant $C_8 > 0$ such that, for any solution of (1.1),

$\|\nabla u\|_\infty \leq C_8\|u\|_\infty + C_8$. Thus it suffices to prove that any solution u of (1.1) is bounded in $L^\infty(\Omega)$. This can be treated as in the proof of Lemma 3.1. We omit it here. □

Denote $\mu_0 = (\lambda_k + \lambda_{k+1})/2$. Then the following problem

$$-\Delta u = \mu_0 u, \quad \text{in } \Omega, \qquad u = 0, \quad \text{on } \partial\Omega$$

has the only weak solution 0. Define

$$\Psi(u) = \frac{1}{2}\bigg[\|u\|^2 - \int_\Omega \mu_0 u^2\,dx\bigg], \quad \text{for all } u \in H.$$

Then 0 is the only critical point of Ψ. By the Riesz representation theorem there is a continuous map N on H such that, for each $u \in H$,

$$(N(u), v) = \int_\Omega \mu_0 uv\,dx, \quad \text{for all } v \in H.$$

By the Sobolev embedding theorem it follows that N is compact. In view of

$$(\nabla\Psi(u), v) = (u, v) - (N(u), v), \quad \text{for all } u, v \in H,$$

we have $\nabla\Psi = I - N$. Denote $B_R = \{u \in H : \|u\| \leq R,\ R \in \mathbb{R}^+\}$. We have the following result.

Lemma 4.4. *There exists $R_0 > 0$ such that* $\deg(\nabla\Psi, B_R, 0) = (-1)^k$ *for* $R \geq R_0$.

Proof. Since

$$\deg(\nabla\Psi, B_R, 0) = \deg(\mathrm{id} - N, B_0, 0) = \mathrm{index}_{\mathrm{LS}}(\mathrm{id} - N, 0) = (-1)^\beta,$$

where

$$\beta = \sum_{\lambda_j > 1,\, \lambda_j \in \sigma(N)} \beta_j, \quad \beta_j = \dim \bigcup_{i=1}^{\infty} \ker(\lambda_j \cdot \mathrm{id} - N)^i,$$

it suffices to compute $\mathrm{index}_{\mathrm{LS}}(\mathrm{id} - N, 0)$. If $Nu = \lambda u$ for some $\lambda \in \mathbb{R}$ and $u \neq 0$, then we have

$$(Nu, v) = \int_\Omega \rho uv\,dx = \int_\Omega \lambda\nabla u \cdot \nabla v\,dx, \quad \text{for all } u, v \in H,$$

which implies that $\lambda = \mu_0/\lambda_i$ for some $i \in \mathbb{Z}^+$. By $\lambda_k < \mu_0 < \lambda_{k+1}$, we obtain that $\lambda > 1$ holds for all $i \leq k$, which implies that $\deg(\nabla\Phi, B_R, 0) = (-1)^k$ for all $R \geq R_0$. □

Applying above lemma, we can obtain the following result.

LEMMA 4.5. *Assume that* (H1)–(H3) *hold. Then there exists* $R_1 > 0$ *such that*

$$\deg(\nabla J, B_R, 0) = (-1)^k \quad \textit{for } R \geq R_1.$$

PROOF. Consider the following auxiliary problem:

$$-\Delta u = \lambda f(x,u) + (1-\lambda)\mu_0 u \equiv f_\lambda(x,u), \quad \text{for } x \in \Omega,\ \lambda \in [0,1],\ u|_{\partial\Omega} = 0.$$

Define

$$J_\lambda(u) = \int_\Omega \left[\frac{1}{2}|\nabla u|^2\,dx - F_\lambda(x,u)\right] dx,$$

where $F_\lambda(x,u) = \int_0^u f_\lambda(x,s)\,ds$. By (H1)–(H3) and the definition of μ_0, we can see from Lemma 4.3 that there exists $R_1 > 0$ large enough such that,

$$\nabla J_\lambda|_{\partial B_R}(u) \neq 0, \quad \text{for all } R \geq R_1,$$

uniformly for $\lambda \in [0,1]$. Then by the homotopy invariance of the Leray–Schauder degree and Lemma 4.4, we obtain that

$$\deg(\nabla J, B_R, 0) = \deg(\nabla J_1, B_R, 0) = \deg(\nabla J_0, B_R, 0) = \deg(\nabla \Psi, B_R, 0) = (-1)^k,$$

for all $R \geq R_1$. $\square$

PROOF OF THEOREM 2.1(a). Suppose that the set of all critical points of the functional J is finite. The assumptions (H1)–(H3) and Lemma 4.3 imply that the solutions of (1.1) are bounded, so there exists $C > 0$ large enough such that if u is a critical point of J then $\|u\| \leq C$. Take $R \geq \max\{R_1, C\}$. Then from Lemma 4.5 and the fact that k is even it follows that

$$\deg(\nabla J, B_R, 0) = 1. \tag{4.6}$$

By Lemmas 4.1 and 4.2, there exist $\rho \in (0, \rho_1)$ and $r_1 > 0$ small sufficiently such that

$$B_\rho \cap B_{r_1}(u^+) = \emptyset, \quad B_\rho \cap B_{r_1}(u^-) = \emptyset, \quad B_{r_1}(u^+) \cap B_{r_1}(u^-) = \emptyset,$$
$$\overline{B}_\rho, \overline{B}_{r_1}(u^+), \overline{B}_{r_1}(u^-) \subset B_R$$

with

$$\deg(\nabla J, B_\rho, 0) = 1, \tag{4.7}$$

$$\deg(\nabla J, B_{r_1}(u^+), 0) = -1, \quad \deg(\nabla J, B_{r_1}(u^-), 0) = -1. \tag{4.8}$$

By the additivity and excision properties of the Leray-Schauder degree, we have

$$\begin{aligned}
&\deg(\nabla J, B_R \setminus \overline{(B_\rho \cup B_{r_1}(u^+) \cup B_{r_1}(u^-))}, 0) \\
&\quad = \deg(\nabla J, B_R, 0) - \deg(\nabla J, B_\rho, 0) - \deg(\nabla J, B_{r_1}(u^+), 0) \\
&\qquad - \deg(\nabla J, B_{r_1}(u^-), 0) - \deg(\nabla J, B_R \setminus \overline{(B_\rho \cup B_{r_1}(u^+) \cup B_{r_1}(u^-))}, 0).
\end{aligned}$$

From (4.6)–(4.8) and the excision property of the Leray–Schauder degree we obtain

$$\deg(\nabla I, B_R \setminus \overline{(B_\rho \cup B_{r_1}(u^+) \cup B_{r_1}(u^-))}, 0) = 2.$$

Hence, by the existence property of the Leray–Schauder degree we can see that there exists $u^* \in B_R \setminus \overline{(B_\rho \cup B_{r_1}(u^+) \cup B_{r_1}(u^-))}$ such that $\nabla J(u^*) = 0$, which together with u^+, u^- gives the existence of at least three nontrivial solutions of problem (1.1). This completes the proof. □

Proof of Theorem 2.1(b). Firstly we shall show that the functional J satisfies the hypotheses of the Saddle Point Theorem. In fact, since f is continuous and satisfies (H1)–(H3) it follows that $J \in C^1(H, \mathbb{R})$. Furthermore, by Lemma 3.1, J satisfies the (PS) condition. Consequently, it suffices to show that the functional J satisfies the saddle point geometry. But this can be obtained by a similar argument as in [9]. Hence, using Lemma 1.1 of [24], since J has finite critical points which are all nondegenerate, there exists a critical u^* with Morse index equal to $\dim V = k + 1 \geq 2$. Moreover, in view of that 0 is a local minimum of J and $J(0) = 0$, it follows that u^* is nontrivial. On the other hand, since J has a nontrivial positive solution u^+ and a nontrivial negative solution u^- which are all nondegenerate and of mountain pass type, by [22] it follows that u^+ and u^- are all of Morse index less that or equal to 1. Then

$$u^* \neq u^+ \quad \text{and} \quad u^* \neq u^-.$$

Now, similar as the arguments in the proof of Theorem 2.1(a), we can take $\rho \in (0, \rho_1)$ and $r_2, \tau > 0$ small sufficiently such that

$$\begin{gathered} B_\rho \cap B_{r_2}(u^+) = \emptyset, \qquad B_\rho \cap B_{r_2}(u^-) = \emptyset, \qquad B_\rho \cap B_\tau(u^*) = \emptyset, \\ B_{r_2}(u^+) \cap B_\tau(u^*) = \emptyset, \quad B_{r_2}(u^-) \cap B_\tau(u^*) = \emptyset, \quad B_{r_2}(u^+) \cap B_{r_2}(u^-) = \emptyset, \\ \overline{B}_\rho, \overline{B}_{r_2}(u^+), \overline{B}_{r_2}(u^-), \overline{B}_\tau(u^*) \subset B_R \end{gathered}$$

with

$$\deg(\nabla J, B_\rho, 0) = 1, \tag{4.9}$$

$$\deg(\nabla J, B_{r_2}(u^+), 0) = -1, \quad \deg(\nabla J, B_{r_2}(u^-), 0) = -1. \tag{4.10}$$

By the additivity and excision properties of the Leray–Schauder degree, we get

$$\begin{aligned} \deg(\nabla J, B_R, 0) = \deg(\nabla J, B_\tau(u^*), 0) &+ \deg(\nabla J, B_\rho, 0) \\ &+ \deg(\nabla J, B_{r_2}(u^+), 0) + \deg(\nabla J, B_{r_2}(u^-), 0) \\ &+ \deg(\nabla J, B_R \setminus \overline{(B_\tau(u^*) \cup B_\rho \cup B_{r_2}(u^+) \cup B_{r_2}(u^-))}, 0). \end{aligned}$$

Since all the critical points of I are nondegenerate, we have

$$|\deg(\nabla J, B_\tau(x^*), 0)| = 1. \tag{4.11}$$

Then, from (4.9)–(4.11) and the excision property of the Leray–Schauder degree, we obtain

$$\deg(\nabla J, B_R \setminus (\overline{B_\tau(u^*) \cup B_\rho \cup B_{r_2}(u^+) \cup B_{r_2}(u^-)}), 0) \neq 0.$$

Hence by the existence property of the Leray–Schauder degree it follows that there exists $u_4 \in B_R \setminus (\overline{B_\tau(u^*) \cup B_\rho \cup B_{r_2}(u^+) \cup B_{r_2}(u^-)})$ such that $\nabla J(u_4) = 0$. Thus problem (1.1) has at least four nontrivial solutions: u^+, u^-, u^*, u_4. □

Before proving Theorem 2.1(c), we recall a global version of the Lyapunov–Schmidt method.

LEMMA 4.6 ([10]). *Let H be a real separable Hilbert space. Let V and W be closed subspaces of H such that $H = V \oplus W$. Assume that $J \in C^1(H, \mathbb{R})$. If there are $\mu_1 > 0$ and $\tau_1 > 1$ such that*

$$(\nabla J(v + w_1) - \nabla J(v + w_2), w_1 - w_2) \geq \mu_1 \|w_1 - w_2\|^{\tau_1}$$

for all $v \in V$, $w_1, w_2 \in W$, then we have:

(a) *There exists $\psi \in C(V, W)$ such that*

$$J(v + \psi(v)) = \min_{w \in W} J(v + w).$$

Moreover, $\psi(x)$ is the unique member of W such that

$$(\nabla J(v + \psi(v), w) = 0 \quad \text{for all } w \in W.$$

(b) *If we define $\overline{J}(v) = J(v + \psi(v))$, then $\overline{J} \in C^1(V, \mathbb{R})$ and*

$$(\nabla \overline{J}(v), v_1) = (\nabla J(v + \psi(v), v_1) \quad \text{for all } v, v_1 \in V.$$

(c) *An element $v \in V$ is a critical point of $\overline{J}$ if and only if $v + \psi(v)$ is a critical point of J.*

(d) *Let $\dim X < \infty$ and P be the projection onto V across W. Let $S \subset V$ and $D \subset H$ be open bounded regions such that*

$$\{v + \psi(v) \mid v \in S\} = D \cap \{v + \psi(v) \mid v \in V\}.$$

If $\nabla \overline{J}(v) \neq 0$ for $v \in \partial S$, then

$$\deg(\nabla \overline{J}, S, 0) = \deg(\nabla J, D, 0).$$

(e) *If $x_0 = v_0 + w_0$ is a critical point of mountain pass type of J, then v_0 is a critical point of mountain pass type of $\overline{J}$.*

PROOF OF THEOREM 2.1(c). Denote $V = N_k$ and $W = N_k^\perp$. Define the functional $\beta(w): W \to \mathbb{R}$ by

$$\beta(w) = \int_\Omega [|\nabla w|^2 - \mu(x) w^2] \, dx, \quad \text{for all } w \in W.$$

As in the proof of Lemma 2.3, we can see that there exist $\delta, M > 0$ such that

$$\beta(w) \geq \delta\|w\|^2, \quad \|w\| \geq M. \tag{4.12}$$

By the mean value theorem it follows that

$$\begin{aligned}(\nabla J(v+w) &- \nabla J(v+w_1), w-w_1) \\ &= \int_\Omega [|\nabla w - \nabla w_1|^2 - f'(x,\xi(x))(w-w_1)^2]\,dx \\ &\geq \int_0^{2\pi} [|\nabla w - \nabla w_1|^2 - \mu(x)(w-w_1)^2]\,dx \\ &\geq \bigg(1 - \frac{\delta}{\lambda_{k+1}}\bigg)\|w-w_1\|^2.\end{aligned}$$

Then by Lemma 4.6 there exists $\psi: V \to W$ such that

$$J(v+\psi(v)) = \min_{w\in W} J(v+w).$$

Moreover, $\psi(v)$ is the unique element of W such that

$$(\nabla J(v+\psi(v)), w) = 0 \quad \text{for all } w \in W.$$

Define $\overline{I}: V \to \mathbb{R}$ by

$$\overline{I}(v) = J(v+\psi(v)).$$

Then $\overline{I}$ is of class C^1, and

$$(\nabla\overline{I}(v), v_1) = (\nabla J(v+\psi(v))(u), u_1) \quad \text{for all } v, v_1 \in V.$$

By (H3), it is easily seen that

$$J(v) \to -\infty \quad \text{as } \|v\| \to \infty \text{ and } v \in V.$$

Then in view of $\overline{I} \leq J(v)$, we can obtain that

$$\overline{I}(v) \to -\infty \quad \text{as } \|v\| \to \infty \text{ and } v \in V.$$

Since V is of finite dimension, there exists $v_0 \in V$ such that

$$\overline{I}(v_0) = \max_{v\in V} J(v+\psi(v)). \tag{4.13}$$

Then $u_0 = v_0 + \psi(v_0)$ is a critical point of J, i.e. $\nabla J(u_0) = 0$. Suppose that v_0 is an isolated critical point of $\overline{I}$, so u_0 is an isolated critical point of J. By (4.13), v_0 is a strictly local maximum of the functional $\overline{I}$. Then there exists $\widehat{v}$ in some neighborhood S_0 of v_0 such that $\overline{I}(\widehat{v}) < \overline{I}(v_0)$, i.e.

$$J(\widehat{v} + \psi(\widehat{v})) < J(v_0 + \psi(v_0)),$$

which means that u_0 can't be local minimum of the functional J. Thus u_0 is nontrivial. On the other hand, if we denote $u^+ = v^+ + \psi(v^+)$, then by Lemma 4.6 we can see that v^+ is a critical point of mountain pass type of $\overline{I}$, which implies $u_0 \neq u^+$. Similarly, $u_0 \neq u^-$. Furthermore, denoting $B_\sigma(u_0) = \{u \in H \mid \|u - u_0\| \leq \sigma\}$, by (4.13) there exists $\sigma_0 > 0$ small such that

$$\deg(\nabla \overline{I}, B_\sigma(u_0)\mid_V, 0) = (-1)^k, \quad \text{for all } 0 < \sigma \leq \sigma_0.$$

Hence from Lemma 4.6 it follows that

$$\deg(\nabla J, B_\sigma(u_0), 0) = (-1)^k, \quad \text{for all } 0 < \sigma \leq \sigma_0.$$

Now similar arguments as in the proof of Theorem 2.1(b) implies that there exists at least a nontrivial critical point u_4 of J that different from u_0, u^+, u^-.□

5. Proof of Theorems 2.3, 2.5 and 2.7

Proof of Theorem 2.3. As in the proof of Theorem 2.1(c), there exists $\psi: N_k \to N_k^\perp$ such that $J(v + \psi(v)) = \min_{w \in N_k^\perp} J(v + w)$. Moreover, the functional $\overline{I}: N_k \to \mathbb{R}$ defined by $\overline{I}(v) = J(v + \psi(v))$ is of class C^1 and an element $v \in N_k$ is a critical point $\overline{I}$ if and only if $u = v + \psi(v)$ is a critical point of J. By Lemma 3.3(b), we can obtain that $\overline{I}(v) \to -\infty$ as $\|v\| \to \infty$ and $v \in N_k$. Thus $-\overline{I}$ is bounded below on N_k. Denote $H^1 = E_1 \oplus \ldots \oplus E_{r-1}$ and $H^2 = E_r \oplus \ldots \oplus E_{k-1}$. Since $\dim H^1 < \infty$, there exists $C_9 > 0$ such that

$$\|u\|_\infty \leq C_9 \|u\|, \quad \text{for all } u \in H^1.$$

Then by (2.5), there exists $\widetilde{\delta}_1 > 0$ such that, for $u \in H^1$ with $\|u\| \leq \widetilde{\delta}_1$,

$$-\overline{I}(u) \geq -J(u) \geq -\frac{1}{2}\|u\|^2 + \frac{1}{2}\int_\Omega \lambda_{r-1} u^2\, dx \geq 0. \tag{5.1}$$

On the other hand, by Lemma 3.2 and the fact that $\psi: H^1 \oplus H^2 \to N_{k-1}^\perp$ is of C^1, denoting $z = u + \psi(u)$, there exists $\widetilde{\delta}_2 > 0$ such that, for $u \in H^2$ with $\|u\| \leq \widetilde{\delta}_2$,

$$-\overline{I}(u) = -J(u + \psi(u)) \leq -\frac{1}{2}\|z\|^2 + \int_\Omega F(x, z)\, dx \geq 0. \tag{5.2}$$

Take $\delta^* = \min\{\widetilde{\delta}_1, \widetilde{\delta}_2\}$. Then (5.1) and (5.2) imply that $-\overline{I}$ has the geometry of local linking at 0 on $H^1 \oplus H^2$. Since J satisfies the (PS) condition, the functional $-\overline{I}$ also satisfies the (PS) condition (see [5]). Now, by the Local Linking Theorem(see [8], [23]) it follows that $-\overline{I}$ and hence J has at least two nontrivial critical points. □

Proof of Theorem 2.5. By Lemma 3.1, the functional J satisfies the (PS) condition. By (2.3) we get

$$J(u) \leq \int_\Omega \frac{1}{2}[|\nabla u|^2\, dx - \lambda_{k-1} u^2] dx \leq 0, \quad \text{for all } u \in N_{k-1}.$$

In view of Lemma 3.3(b), there exists R sufficiently large such that $J(u) < 0$ for $u \in N_k$ with $\|u\| = R$. From (H1), (H3) and (H4) with $m = k$ it follows that there exist $r_0 > 0$ such that $J(u) \geq \rho_0$ for $\|u\| = r_0$, $u \in N_{k-1}^{\perp}$. Hence by the Benci–Rabinowitz Linking Theorem (see [7]) we can see that the functional J admits a nontrivial critical point $u \in H$ with $J(u) \geq \rho_0 > 0$. □

To prove Theorem 2.7, we need the following abstract result.

Lemma 5.1 ([30]). *Let $X = X_1 \oplus X_2$ be a Banach space with $0 < k = \dim X_1 < \infty$. Suppose that $J \in C^1(X, \mathbb{R})$ satisfies*

- (I_1) *there exists $\rho > 0$ such that $\sup_{S_\rho^1} J < 0$, where $S_\rho^1 = \{u \in X_1 : \|u\| = \rho\}$,*
- (I_2) *$J \geq 0$ on X_2,*
- (I_3) *there exists a nonzero vector $e \in X_1$ such that J is bounded below on the half-space $\{se + u_2 : s \geq 0,\ u_2 \in X_2\}$.*

In addition, assume that J satisfies the (PS) *condition and has only isolated critical values with each critical value corresponding to a finite number of critical points. Then J has two different critical points u_1, u_2 with $J(u_1) < 0 \leq J(u_2)$.*

Proof of Theorem 2.7. Let $X_1 = N_{k+1}$ and $X_2 = N_{k+1}^{\perp}$. We come to verify the conditions of Lemma 5.1. Clearly, $\dim N_{k+1}$ is of finiteness. By (2.5) it follows that

$$J(w) \geq \int_\Omega \frac{1}{2}[|\nabla w|^2\, dx - \lambda_{k+2} w^2] dx \geq 0, \quad \text{for all } w \in N_{k+1}^{\perp},$$

which implies (I_2) holds. Lemma 3.3(a) implies that (I_3) is also satisfied. In addition, by Lemma 3.1, the functional J satisfies the (PS) condition. Now it suffices to prove that (I_1) is true. In fact, by (2.4), it follows that, for any $\varepsilon > 0$, $q \in (2, 2n/(n-2))$ if $n > q$ or $q \in (2, \infty)$ if $1 \leq n \leq q$, there exists $C_\varepsilon > 0$ such that

$$F(x, s) \geq \frac{1}{2}\eta_2 s^2 - C_\varepsilon |s|^p, \quad \text{a.e. } x \in \Omega,\ \text{for all } s \in \mathbb{R}.$$

Then, by the Poincaré inequality as well as the Sobolev inequality $\|u\|_q^q \leq K\|u\|^q$, we have for $u \in N_{k+1}$,

$$\begin{aligned} J(u) &\leq \frac{1}{2}\|u\|^2 - \frac{1}{2}\eta_2 \int_\Omega |u|^2 dx + C_\varepsilon \int_\Omega |u|^q\, dx \\ &\leq -\frac{\eta_2 - \lambda_{k+1}}{2\lambda_{k+1}}\|u\|^2 + KC_\varepsilon\|u\|^q = \bigg[-\frac{\eta_2 - \lambda_{k+1}}{2\lambda_{k+1}} + KC_\varepsilon\|u\|^{q-2}\bigg]\|u\|^2. \end{aligned}$$

Hence, if taking $\overline{\rho} = ((\eta_2 - \lambda_{k+1})/(2\lambda_{k+1C_1C_\varepsilon}))^{1/(q-2)}$, we can obtain

$$J(u) < 0 \quad \text{for } u \in N_{k+1} \text{ with } \|u\| = \rho \in (0, \overline{\rho}),$$

which implies that (I_1) is satisfied. Then, by Lemma 5.1, it follows that J has at least one nontrivial critical point u with $J(u) < 0$. □

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Manuscript received January 6, 2009

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