Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 36, 2010, 197–262

RIGOROUS NUMERICS FOR DISSIPATIVE PDES III. AN EFFECTIVE ALGORITHM FOR RIGOROUS INTEGRATION OF DISSIPATIVE PDES

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Abstract. We describe a Lohner-type algorithm for rigorous integration of dissipative PDEs. Using it for the Kuramoto–Sivashinsky PDE on the line with odd and periodic boundary conditions we give a computer assisted proof the existence of multiple periodic orbits.

1. Introduction

In the study of nonlinear PDEs there is a huge gap between what we can observe in the numerical simulations and what we can prove rigorously. One possibility to overcome this problem are the computer assisted proofs. This paper is an attempt in this direction. We give a computer assisted proofs of the existence of multiple periodic orbits, both stable and unstable ones, for the Kuramoto–Sivashinsky PDE on the line with periodic and odd boundary conditions. The approach is a mixture of rigorous numerics and topological methods and does not make any use of any special features of Kuramoto–Sivashinsky PDE, or any global existence results nor spectral gap etc and therefore should be applicable to

2010 *Mathematics Subject Classification.* 35B40, 35B45, 65G30, 65N30.

Key words and phrases. Periodic orbits, dissipative PDEs, Galerkin projection, rigorous numerics, computer assisted proof.

Research supported in part by Polish State Committee for Scientific Research grant 2 P03A 019 22 and MNiI grant N201 024 31/2163.

other systems of dissipative PDEs, like for example Navier–Stokes or Ginzburg–Landau equations.

One of the main goals of this paper is to present the algorithm for a rigorous numerical integration of a certain class of dissipative PDEs. To be more specific we consider PDEs of the following type

$$u_t = Lu + N(u, Du, \dots, D^r u), \tag{1.1}$$

where $u \in \mathbb{R}^n$, $x \in \mathbb{T}^d$, ($\mathbb{T}^d = (\mathbb{R}/2\pi)^d$ is an d-dimensional torus), L is a linear operator, N – a polynomial and by $D^s u$ we denote s-th order derivative of u, i.e. the collection of all partial derivatives of u of order s. In fact N might contain in constant term a time independent forcing, smooth enough (time dependence also does not hurt), but we will not consider it here. We require that L is diagonal in the Fourier basis $\{e^{kx}\}_{k\in\mathbb{Z}^d}$, namely $Le^{ikx} = \lambda_k e^{ikx}$, and the eigenvalues λ_k satisfy:

$$\lambda_k = -v(|k|)|k|^p, \tag{1.2}$$

$$0 < v_0 \le v(|k|) \le v_1, \quad \text{for } |k| > K_-, \tag{1.3}$$

$$p > r. \tag{1.4}$$

The fact that we are considering functions on the torus means that we impose periodic boundary conditions. We may eventually seek odd or even solutions or impose some other conditions.

Our approach starts with replacing (1.1) by an infinite ladder of ordinary differential equations for Fourier coefficients of $u(t,x) = \sum_k u_k(t)e^{ikx}$. We obtain

$$\frac{du_k}{dt} = \lambda_k u_k + N_k(u), \quad \text{for all } k \in \mathbb{Z}^d. \tag{1.5}$$

The next step is to split the phase space for (1.5) into two parts: the finite dimensional part, X, containing the Fourier modes most relevant for the dynamics of (1.1) and the tail $T \subset X^\perp$. After this splitting the problem (1.5) is replaced by two problems (1.6) and (1.7). The first part consists of a finite dimensional differential inclusion for $p \in X$, given by

$$\frac{dp}{dt} \in P(Lp + N(p+T)), \quad p \in X \tag{1.6}$$

where P is a projection onto X. The second part is concerned with the evolution of T, which is governed by an infinite set of inequalities of the form

$$\lambda_k u_{k,j} + N^-_{k,j} < \frac{du_{k,j}}{dt} < \lambda_k u_{k,j} + N^+_{k,j}, \quad j = 1, \dots, n \text{ and for } k \text{ not in } X \tag{1.7}$$

where $N^{\pm}_{k,j}$ are suitably chosen constants. Obviously, to infer from (1.6) and (1.7) any information on the behavior of solutions of the full system (1.5) one needs some consistency conditions. A systematic treatment of this issue is at the heart of our *method of self-consistent bounds*, which was introduced in [31] and later developed in [25], [27]–[29].

The main example treated in this paper is the Kuramoto–Sivashinsky PDE [12], [21] (in the sequel we will refer to it as the KS equation)

$$u_t = -\nu u_{xxxx} - u_{xx} + (u^2)_x, \quad \nu > 0 \tag{1.8}$$

where $x \in \mathbb{R}$, $u(t,x) \in \mathbb{R}$ and we impose odd and periodic boundary conditions

$$u(t,x) = -u(t,-x), \qquad u(t,x) = u(t,x+2\pi). \tag{1.9}$$

The choice of the KS equation for this study is motivated by the following facts:

- The existence theory and asymptotic properties of solutions of (1.8) are well established, see for example [4]–[6] and the literature cited there. It should be stressed that we are not using these results in our work, but they assure us that all interesting dynamics is 'finite dimensional' and should be accessible using the method of self-consistent bounds combined with topological tools.
- There exists a lot of numerical studies of the dynamics of the KS equation (see for example [3], [7]–[9]), where it was shown that the dynamics of the KS equation is highly nontrivial and it is well represented by relatively small number of modes.
- We believe that the experience gained and new tools developed in the study of the KS equation may help in the rigorous study of the dynamics of the Navier–Stokes equations or the Ginzburg–Landau equation [20].

We implemented the proposed algorithm for the KS equation (1.8) with the odd and periodic boundary conditions (1.9). Using it we proved the existence of several periodic orbits, both attracting and unstable ones, for various parameter values of ν in the interval $[0.02991, 0.128]$. Proofs are topological and are based on the Brouwer Theorem in case of attracting orbits and on the Miranda Theorem [15] in case of unstable ones.

The main difference between this paper and [29] is the generality and the efficiency of the algorithm for a rigorous integration of (1.8). The algorithm described in [29] required some preparatory work to construct the a priori bounds, which have to be verified during the computation, moreover the tail was fixed in the computation. The present algorithm allows for the tail evolution and do not require any a priori bounds to start the computation, hence it could be used also to obtain rigorous bounds for the forward orbit of any initial condition with a finite description, this was not possible using the previous algorithm. Other improvements, while rather technical, are also of great importance for the performance of the algorithm. They include a new function for the generation of the rough enclosure for differential inclusions, which allows to use much larger time steps. All these improvements taken together result in more than 6 fold speed up of the proof of the existence of periodic orbit for $\nu = 0.127$. This orbit has the reflectional symmetry and this fact was essential in the proof because it allowed us to consider the half-Poincaré map instead of the full Poincaré map. Our attempts to compute the full Poincaré map along this periodic orbit using the previous algorithm failed due to blow-up of rigorous enclosures produced, which was due to the instability of the numerical estimates in the flow direction. Using the current algorithm we were able to overcome this problem and also treat smaller values of ν, which are more difficult computationally and are more interesting from the dynamics standpoint.

The choice of odd boundary conditions was motivated by earlier numerical studies [3], [9], but the basic mathematical reason is: equation (1.8) with periodic boundary conditions has the translational symmetry, which implies that for fixed value of ν periodic orbits (fixed points, etc.) are members of one-parameter families of periodic orbits (fixed points, etc.). The restriction to the subspace of odd functions breaks this symmetry and gives a hope that the dynamically interesting objects are topologically isolated, which is later confirmed by proofs.

The content of this paper can be described as follows: in Sections 2 and 3 we outline the method of self-consistent bounds and discuss how it can be used for the study of dynamics of dissipative PDEs. This material is based on [31], [27], [28], but some new theorems about the applicability to other dissipative PDEs are added in Section 3. In Section 4 we present a Lohner-type algorithm for the integration of ordinary differential inclusions, which in the context of the rigorous integration of PDEs is used to provide enclosures for (1.6). In Section 5 we present a new effective enclosure theorem for ordinary differential inclusions and an enclosure algorithm based on it. In Sections 6, 7 and 8 we discuss the algorithm with an evolving tail for the rigorous integration of dissipative PDEs with periodic boundary conditions. In Section 9 we treat the issue of Poincaré maps. In the remaining sections we report on the computer assisted proofs of the existence of periodic orbits for the KS equation, both apparently stable and unstable ones. We say "apparently" to indicate that we were unable to establish rigorously, whether these orbits are stable or unstable, but the nonrigorous simulation clearly indicates their dynamical character and the set-up of the proof takes this into account.

1.1. Notation. Let (T,ρ) be a metric space. For a set $X \subset T$ by $\operatorname{int} X$, $\overline{X}$ and ∂X we denote the interior, the closure and the boundary of X, respectively. If $X \subset Y \subset T$, then by $\operatorname{int}_Y X$ and by $\partial_Y X$ we will denote respectively the interior and the boundary of X with respect to the metric space (Y,ρ). By $B(c,r) = \{x \mid \rho(c,x) < r\}$ we will denote the ball of radius r. For a point $p \in T$

put $\rho(p,X) = \inf\{\rho(p,q) \mid q \in X\}$. We define $B(X,\varepsilon) = \{y \mid \rho(y,X) < \varepsilon\}$. The Hausdorff distance, $\operatorname{dist}(G,H)$, between two closed sets G and H is defined by the formula

$$\operatorname{dist}(G,H) = \max\Big\{\sup_{q\in G}\rho(q,H), \sup_{h\in H}\rho(h,G)\Big\}.$$

For $-\infty \le t_0 < t_1 \le \infty$ by $C([t_0,t_1],\mathbb{R}^s)$ we will denote the set of all continuous functions defined on $[t_0,t_1]$ with the values in $\mathbb{R}^s$ and by $C_b([t_0,t_1],\mathbb{R}^s)$ we will denote the set of all bounded and continuous functions defined on $[t_0,t_1]$ with the values in $\mathbb{R}^s$.

Let $\mathrm{cf}(\mathbb{R}^n)$ denotes the set of all nonempty, convex and compact subsets of $\mathbb{R}^n$. A multivalued map $f\colon \mathbb{R}^n \to \mathrm{cf}(\mathbb{R}^n)$ is said to be *continuous* if it is continuous with respect to the Hausdorff distance.

For an ordinary differential equation

$$x' = f(x), \quad x \in \mathbb{R}^n \tag{1.10}$$

where $f \in \mathcal{C}^1$, by φ we will denote the local flow induced by (1.10). We set $\varphi(t,x_0) = x(t)$ where $x(t)$ is the unique solution of (1.10) with the initial condition $x(0) = x_0$.

Let $f\colon \mathbb{R}^n \to \mathrm{cf}(\mathbb{R}^n)$ be continuous. Consider a differential inclusion

$$x' \in f(x), \tag{1.11}$$

By a solution of (1.11) through x_0 we will understand a C^1 function $x\colon (t_0,t_1) \to \mathbb{R}^n$, such that $0 \in (t_0,t_1)$, $x(0) = x_0$ and (1.11) holds for $t \in (t_0,t_1)$. Moreover, we will always assume that the solution is defined on the maximal existence interval.

We define the local flow, φ, induced by (1.11) as follows: $(t,x_0) \in \mathbb{R}\times\mathbb{R}^n$ is in the domain of φ if for all solutions x through x_0 the value of $x(t)$ is defined and then

$$\varphi(t,x_0) = \{x(t) \mid x\colon (t_0,t_1) \to \mathbb{R}^n \text{ is a solution through } x_0\}.$$

While we will use the same symbol $\varphi(t,x)$ to indicate the local flow induced both by an ODE or an inclusion it will be always clear from the context what type of the flow we are considering.

In the sequel we will use an expression of the form $\varphi([0,h],x_0) \subset Z$. Such expression means that $\varphi([0,h],x_0)$ is defined for $t \in [0,h]$ and the stated inclusion holds, i.e. $\varphi(t,x_0) \subset Z$ for $t \in [0,h]$.

2. The method of self-consistent bounds

We begin with an abstract nonlinear evolution equation in a real Hilbert space H (L^2 or some its subspaces in our treatment of dissipative PDEs) of the form

$$\frac{du}{dt} = F(u) \tag{2.1}$$

where the domain of F is dense in H. By a solution of (2.1) we understand a function $u\colon [0, t_{\max}) \to \mathrm{dom}\,(F)$, such that u is differentiable and (2.1) is satisfied for all $t \in [0, t_{\max})$. (For the discussion of classical solutions of dissipative PDEs see Section 3.3)

The scalar product in H will be denoted by $(u \mid v)$. Throughout the paper we assume that there is a set $I \subset \mathbb{Z}^d$ and a sequence of subspaces $H_k \subset H$ for $k \in I$, such that $\dim H_k = d_1 < \infty$ and H_k and $H_{k'}$ are mutually orthogonal for $k \neq k'$. Let $A_k\colon H \to H_k$ be the orthogonal projection onto H_k. We assume that for each $u \in H$ holds

$$u = \sum_{k \in I} u_k = \sum_{k \in I} A_k u. \tag{2.2}$$

The above equality for a given $u \in H$ and $k \in I$ defines u_k. Analogously if B is a function with the range in H, then $B_k(u) = A_k B(u)$. Equation (2.2) implies that $H = \overline{\bigoplus_{k \in I} H_k}$.

For $k \in \mathbb{Z}^d$ we define

$$|k| = \sqrt{\sum_{i=1}^{d} k_i^2}.$$

For $n > 0$ we set

$$X_n = \bigoplus_{|k| \leq n,\, k \in I} H_k, \qquad Y_n = X_n^{\perp},$$

by $P_n\colon H \to X_n$ and $Q_n\colon H \to Y_n$ we will denote the orthogonal projections onto X_n and onto Y_n, respectively.

Definition 2.1. We say that $F\colon H \supset \mathrm{dom}\,(F) \to H$ is admissible if, for any $i \in \mathbb{R}$ such that $\dim X_i > 0$, the following conditions are satisfied:

(a) $X_i \subset \mathrm{dom}\,(F)$,
(b) $P_i F\colon X_i \to X_i$ is a C^1 function.

Definition 2.2. Assume F is admissible. For a given number $n > 0$ the ordinary differential equation

$$x' = P_n F(x), \quad x \in X_n, \tag{2.3}$$

will be called *the n-th Galerkin projection* of (2.1). By $\varphi^n(t, x)$ we denote the local flow on X_n induced by (2.3).

Definition 2.3. Assume F is an admissible function. Let $m, M \in \mathbb{R}$ with $m \leq M$. Consider an object consisting of a compact set $W \subset X_m$ and a sequence of compact sets $B_k \subset H_k$ for $|k| > m$, $k \in I$. We define following conditions:

(C1) For $|k| > M$, $k \in I$ holds $0 \in B_k$.

(C2) Let $\widehat{a}_k := \max_{a \in B_k} \|a\|$ for $|k| > m$, $k \in I$ and then $\sum_{|k|>m,\, k\in I} \widehat{a}_k^2 < \infty$. In particular, $W \oplus \prod_{|k|>m} B_k \subset H$ and, for every $u \in W \oplus \prod_{k\in I,\, |k|>m} B_k$, holds $\|Q_n u\| \leq \sum_{|k|>n,\, k\in I} \widehat{a}_k^2$.

(C3) The function $u \mapsto F(u)$ is continuous on $W \oplus \prod_{k\in I,\, |k|>m} B_k \subset H$. Moreover, if we define for $k \in I$, $f_k = \max_{u \in W \oplus \prod_{k\in I,\, |k|>m} B_k} |F_k(u)|$, then $\sum f_k^2 < \infty$.

(C4) For $|k| > m$, $k \in I$, B_k is given by

$$B_k = \overline{B(c_k, r_k)}, \qquad r_k > 0 \tag{2.4}$$

$$\text{or} \quad B_k = \prod_{s=1}^{d} [a_s^-, a_s^+], \quad a_s^- < a_s^+,\ s = 1, \ldots, d_1 \tag{2.5}$$

Let $u \in W \oplus \prod_{|k|>m} B_k$. Then for $|k| > m$ holds:

- if B_k is given by (2.4) then

$$u_k \in \partial_{H_k} B_k \Rightarrow (u_k - c_k | F_k(u)) < 0. \tag{2.6}$$

- if B_k is given by (2.5) then

$$u_{k,s} = a_{k,s}^- \Rightarrow F_{k,s}(u) > 0, \tag{2.7}$$

$$u_{k,s} = a_{k,s}^+ \Rightarrow F_{k,s}(u) < 0. \tag{2.8}$$

In the sequel we will refer to equations (2.6) and (2.7)–(2.8) as *isolation equations*.

Definition 2.4 ([31, Definitions 2.1, 2.11]). Assume F is an admissible function. Let $m, M \in \mathbb{R}$ with $m \leq M$. Consider an object consisting of a compact set $W \subset X_m$ and a sequence of compacts $B_k \subset H_k$ for $|k| > m$, $k \in I$. We say that set $W \oplus \prod_{k\in I,\, |k|>m} B_k$ forms *self-consistent bounds for* F if conditions (C1)–(C3) are satisfied.

If additionally condition (C4) holds, then we say that $W \oplus \prod_{k\in I,\, |k|>m} B_k$ forms *topologically self-consistent bounds for* F.

If F is clear from the context, then we will often drop F, and we will speak simply about *self-consistent bounds* or *topologically self-consistent bounds*.

In our previous works on the KS equation [31], [25], [29], we had $I = \mathbb{Z}_+$, $H_k = \mathbb{R}$ and $B_k = [a_k^-, a_k^+]$. The conditions from Definition 2.3 are generalizations of the conditions given there to a more general setting.

Reader familiar with our earlier works should be also warned that in the terminology of [29, Definition 2] conditions (C1)–(C4) defined *self-consistent a-priori bounds*. In this paper we returned to the terminology used in [31] and we dropped the phrase *a-priori*.

Given self-consistent bounds W and $\{B_k\}_{k\in I,\,|k|>m}$, by T (the tail) we will denote

$$T := \prod_{|k|>m} B_k \subset Y_m.$$

Here are some useful lemmas illustrating the implications of (C1)–(C3). From condition (C2) it follows immediately that:

Lemma 2.5. *If $W\oplus T$ forms self-consistent bounds, then $W\oplus T$ is a compact subset of H.*

The following lemma is an immediate consequence of (C2) and (C3).

Lemma 2.6. *Given self-consistent bounds $W\oplus T$, then*

$$\lim_{n\to\infty} P_n(F(u)) = F(u), \quad \text{uniformly for } u\in W\oplus T.$$

The lemma below was proved in [29, Lemma 5], where the definition of self-consistent bounds required conditions (C1)–(C4) and $\dim H_k = 1$, but the condition (C4) and the dimension of H_k were not used in the proof. Hence we can write this lemma as follows:

Lemma 2.7. *Let $W\oplus T$ forms self-consistent bounds for* (2.1). *Let $\{d_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ be a sequence, such that $\lim_{n\to\infty} d_n = \infty$. Assume that $x_n\colon[t_1,t_2]\to W\oplus T$, for all n, is a solution of*

$$\frac{dp}{dt} = P_{d_n}(F(p)), \quad p(t)\in X_{d_n}.$$

Then there exists a convergent subsequence $\{d_{n_l}\}_{l\in\mathbb{N}}$ such that, $\lim_{l\to\infty} x_{n_l} = x^*$, where $x^*\colon[t_1,t_2]\to W\oplus T$ and the convergence is uniform on $[t_1,t_2]$. Moreover, x^* satisfies (2.1).

Later we will need a slightly stronger version of the above lemma, which we state without a proof, because the proof of Lemma 2.7 works also for this version.

Lemma 2.8. *Let $W_i\oplus T_i$, $i=1,\dots,k$ forms self-consistent bounds for* (2.1). *Let $\{d_n\}_{n\in\mathbb{N}}\subset\mathbb{R}$ be a sequence, such that $\lim_{n\to\infty} d_n=\infty$. Assume that, for all n, $x_n\colon[t_1,t_2]\to\bigcup_{i=1}^k W_i\oplus T_i$ is a solution of*

$$\frac{dp}{dt} = P_{d_n}(F(p)), \quad p(t)\in X_{d_n}.$$

Then there exists a convergent subsequence $\{d_{n_l}\}_{l\in\mathbb{N}}$ such that, $\lim_{l\to\infty} x_{n_l} = x^*$, where $x^*\colon[t_1,t_2]\to\bigcup_{i=1}^k W_i\oplus T_i$ and the convergence is uniform on $[t_1,t_2]$. Moreover, x^* satisfies (2.1).

3. The existence of uniform bounds for all Galerkin projections for short time steps

Consider equation (1.1). We assume that conditions (1.2)–(1.4) are satisfied. If $a(t,x)$ is a sufficiently regular solution of (1.1), then we can expand it in Fourier series $a(t,x) = \sum_{k\in\mathbb{Z}^d} a_k(t)e^{ik\cdot x}$ to obtain an infinite ladder of ordinary differential equations for the coefficients a_k

$$\frac{da_k}{dt} = \lambda_k a_k + N_k(a), \quad k \in \mathbb{Z}^d, \tag{3.1}$$

where $N_k(a)$ is k-th Fourier coefficient of function $N(a, Da, \ldots, D^r a)$.

Observe that $a_k \in \mathbb{C}^n$ and (3.1) are not independent, because the reality of a imposes the following condition

$$a_{-k} = \overline{a}_k. \tag{3.2}$$

To put (3.1) in the context of the previous sections we define

$$I = \mathbb{Z}^d, \quad H = \left\{ (a_k)_{k\in I} \;\middle|\; \sum_{k\in I} |a_k|^2 < \infty \right\}$$

and consider the subspace defined by condition (3.2). This subspace is invariant for all Galerkin projections of (1.1) onto X_n. Other constraints like oddness or evenness of $a(t,x)$ may cause the change of I, moreover also the basis in our Hilbert space may change accordingly, for example, for the KS equation (1.8) with odd and periodic boundary conditions (1.9), we have $I = \mathbb{Z}_+$ and $u(t,x) = \sum_{k\in I} -2a_k(t)\sin(kx)$, where $a_k \in \mathbb{R}$ and equation (3.1) becomes

$$\frac{da_k}{dt} = k^2(1-\nu k^2)a_k - k\sum_{n=1}^{k-1} a_n a_{k-n} + 2k\sum_{n=1}^{\infty} a_n a_{n+k}, \quad k = 1, 2, \ldots \tag{3.3}$$

(see [3], [31]).

Observe that conditions (1.2)–(1.4) are satisfied for the KS equation. Namely, we have $v(|k|) = \nu - 1/|k|^2$, $p = 4$, $r = 1$. For the Navier–Stokes equations with periodic boundary conditions $p = 2$, $r = 1$ and $v(k) = \nu$, where ν is the viscosity.

3.1. Estimates. In this subsection our goal is to prove the following

Lemma 3.1. *Let $s > s_0 = d + r$. If $|a_k| \le C/|k^s|$, $|a_0| \le C$, then there exists $D = D(C,s)$ such that*

$$|N_k| \le \frac{D}{|k|^{s-r}}, \quad |N_0| \le D.$$

Before we proceed with the proof we need several lemmas. To make expression of some formulas less cumbersome in this subsection for $0 = \{0\}^d \in \mathbb{Z}^d$ we redefine its norm by setting $|0| = 1$.

LEMMA 3.2. *Let $\gamma > 1$. For any $a, b \geq 0$ the following inequality is satisfied*

$$(a+b)^\gamma \leq 2^{\gamma-1}(a^\gamma + b^\gamma)$$

PROOF. This is an easy consequence of the convexity of function $x \mapsto x^\gamma$ for $\gamma > 1$. Namely

$$(a+b)^\gamma = 2^\gamma \left(\frac{a+b}{2}\right)^\gamma \leq 2^\gamma \left(\frac{a^\gamma + b^\gamma}{2}\right) = 2^{\gamma-1}(a^\gamma + b^\gamma). \qquad \square$$

The following lemma was proved in [19]:

LEMMA 3.3. *Assume that $\gamma > d$. Then there exists $C_Q(d,\gamma) \in \mathbb{R}$ such that, for any $k \in \mathbb{Z}^d \setminus \{0\}$, holds:*

$$\sum_{k_1 \in \mathbb{Z}^d \setminus \{0,k\}} \frac{1}{|k_1|^\gamma |k-k_1|^\gamma} \leq \frac{C_Q(d,\gamma)}{|k|^\gamma}.$$

PROOF. From the triangle inequality and Lemma 3.2 we have

$$\frac{|i|^\gamma}{|k-i|^\gamma |k|^\gamma} \leq \frac{(|k-i|+|k|)^\gamma}{|k-i|^\gamma |k|^\gamma} \leq \frac{2^{\gamma-1}(|k-i|^\gamma + |k|^\gamma)}{|k-i|^\gamma |k|^\gamma} = 2^{\gamma-1}\left(\frac{1}{|k|^\gamma} + \frac{1}{|k-i|^\gamma}\right).$$

Hence

$$\sum_{k \in \mathbb{Z}^d \setminus \{0,i\}} \frac{1}{|k|^\gamma |i-k|^\gamma} \leq \sum_{k \in \mathbb{Z}^d \setminus \{0,i\}} \frac{2^{\gamma-1}}{|i|^\gamma}\left(\frac{1}{|k|^\gamma} + \frac{1}{|i-k|^\gamma}\right) < \frac{2^\gamma}{|i|^\gamma} \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \frac{1}{|k|^\gamma}.$$

$\square$

Now we want to include also the vectors of zero length in the sum appearing in Lemma 3.3.

LEMMA 3.4. *Assume that $\gamma > d$. Then there exists $C_2(d,\gamma) \in \mathbb{R}$ such that for any $k \in \mathbb{Z}^d$ holds*

$$\sum_{k_1,k_2 \in \mathbb{Z}^d,\, k_1+k_2=k} \frac{1}{|k_1|^\gamma |k_2|^\gamma} \leq \frac{C_2(d,\gamma)}{|k|^\gamma}.$$

PROOF. Consider two cases $k = 0$ and $k \neq 0$.

If $k = 0$, then there exists $\widetilde{C}(d,\gamma) \in \mathbb{R}$ such that

$$\sum_{k_1,k_2 \in \mathbb{Z}^d,\, k_1+k_2=k} \frac{1}{|k_1|^\gamma |k_2|^\gamma} = 1 + \sum_{k_1 \in \mathbb{Z}^d \setminus \{0\}} \frac{1}{|k_1|^{2\gamma}} = \widetilde{C}(d,\gamma).$$

If $k \neq 0$, then from Lemma 3.3 it follows that

$$\sum_{k_1,k_2 \in \mathbb{Z}^d,\, k_1+k_2=k} \frac{1}{|k_1|^\gamma |k_2|^\gamma} = \frac{2}{|k|^\gamma} + \sum_{k_1,k_2 \in \mathbb{Z}^d \setminus \{0\},\, k_1+k_2=k} \frac{1}{|k_1|^\gamma |k_2|^\gamma} \leq \frac{C_Q(d,\gamma)+2}{|k|^\gamma}.$$

Hence the assertion holds for $C_2(d,\gamma) = \max(\widetilde{C}(d,\gamma), C_Q(d,\gamma)+2)$. $\square$

LEMMA 3.5. *Assume $\gamma > d$. For any $n \in \mathbb{Z}_+$, $n > 1$ there exists $C_n(d,\gamma) \in \mathbb{R}$ such that for any $k \in \mathbb{Z}^d$ holds*

$$\sum_{k_1,\dots,k_n \in \mathbb{Z}^d,\, \sum_{i=1}^n k_i = k} \frac{1}{|k_1|^\gamma \cdot \ldots \cdot |k_n|^\gamma} \leq \frac{C_n(d,\gamma)}{|k|^\gamma}.$$

PROOF. By induction. Case $n = 2$ is contained in Lemma 3.4. Assume now that the assertion holds for n. We have

$$\begin{aligned}
&\sum_{k_1,\dots,k_{n+1} \in \mathbb{Z}^d,\, \sum_{i=1}^{n+1} k_i = k} \frac{1}{|k_1|^\gamma \cdot \ldots \cdot |k_{n+1}|^\gamma} \\
&\quad = \sum_{k_{n+1} \in \mathbb{Z}^d} \left(\frac{1}{|k_{n+1}|^\gamma} \sum_{k_1,\dots,k_n \in \mathbb{Z}^d,\, \sum_{i=1}^n k_i = k - k_{n+1}} \frac{1}{|k_1|^\gamma \cdot \ldots \cdot |k_n|^\gamma} \right) \\
&\qquad \leq \sum_{k_{n+1} \in \mathbb{Z}^d} \frac{1}{|k_{n+1}|^\gamma} \cdot \frac{C_n(d,\gamma)}{|k - k_{n+1}|^\gamma} \leq \frac{C_2(d,\gamma) C_n(d,\gamma)}{|k|^\gamma}.
\end{aligned}$$

□

PROOF OF LEMMA 3.1. For the proof it is enough to assume that N is a monomial. After formally inserting the Fourier expansion for $u, Du, \dots, D^r u$ we obtain the expression of the following type

$$N_k(u) = \sum_{k_1 + \ldots + k_l = k} v_{k_1} \cdot ldots \cdot v_{k_l}, \tag{3.4}$$

where each of the variables v_{k_i}, $i = 1, \dots, l$ is some Fourier coefficient of one the components of u or its partial derivatives of the order less than or equal to r.

Observe that for the Fourier coefficients of partial derivatives up to order r we have the following estimates

$$\left| \frac{\partial^{\beta_1 + \ldots + \beta_l} u}{\partial x_1^{\beta_1} \ldots \partial x_d^{\beta_l}} \right| \leq \frac{C}{|k|^{s - (\beta_1 + \ldots + \beta_l)}} \leq \frac{C}{|k|^{s-r}}. \tag{3.5}$$

From conditions (3.4) and (3.5), and Lemma 3.5 we obtain

$$|N_k(u)| \leq \sum_{k_1 + \ldots + k_n = k} \frac{C^n}{|k_1|^{s-r} \cdot \ldots \cdot |k_n|^{s-r}} \leq \frac{C^n C_n(d, s-r)}{|k|^{s-r}}.$$ □

3.2. Existence theorems. The main result in this section is Theorem 3.7, which states that equation (3.1) satisfying conditions (1.2)–(1.4) has solutions within self-consistent bounds for a sufficiently short time.

THEOREM 3.6. *Consider* (3.1). *Assume that conditions* (1.2)–(1.4) *hold. Let be $s_0 = p + d + 1$ and $m \in \mathbb{R}$. Consider compact set $W \subset X_m$ and a sequence* *of compact sets* $B_k \subset H_k$ *for* $|k| > m$*, such that there exist* $s \geq s_0$ *and* $C \in \mathbb{R}$ *and the following condition is satisfied*

$$|B_k| \leq \frac{C}{|k|^s}, \quad |k| > m, \ k \in I.$$

Then $W \oplus \prod_{k \in I,\, |k|>m} B_k$ *satisfies conditions* (C2) *and* (C3)*.*

Proof. Condition (C2) is obvious. It remains to prove (C3). Let $T = \prod_{k \in I,\, |k|>m} B_k$.

The first question is whether $W \oplus T \subset \operatorname{dom} F$. Consider $u \in W \oplus T$. From Lemma 3.1 it follows that $F_k(u)$ is defined and, for $|k| > m$, holds:

$$|F_k(u)| \leq v_1 C |k|^{p-s} + D|k|^{r-s} \leq \frac{D_2}{|k|^{s-p}}$$

for some constants D and D_2. Hence

$$f_k = \max_{u \in W \oplus T} |F_k(u)| \leq \frac{D_2}{|k|^{s-p}}, \quad |k| > m, \ k \in I. \tag{3.6}$$

For $s \geq s_0$ we have $\sum_{|k|>m,\, k \in I} f_k^2 < \infty$. From this it follows that $W \oplus T \subset \operatorname{dom}(F)$.

It remains to prove the continuity of $F: W \oplus T \to H$. From condition (3.6) it follows that

$$\lim_{n \to \infty} \sum_{|k|>n} |A_k F(x)|^2 = 0$$

uniformly on $W \oplus T$. Hence it is enough to prove that $F_k: W \oplus T \to H_k$ is continuous.

Let us fix $k \in I$ and assume $u^n, u^* \in W \oplus T$, for $n \in \mathbb{N}$ and $u^n \to u^*$ for $n \to \infty$. We have (compare the proof of Lemma 3.1)

$$F_k(u) = \lambda_k u_k + N_k(u) = \lambda_k u_k + \sum_{i \in J} N_{k,i}(u),$$

where J is some set of multindices and for each $i \in J$, $N_{k,i}$ is monomial depending on the finite number of u_l, i.e.

$$N_{k,i} = a\ u_{k_1} \cdot \ldots \cdot u_{k_l}, \quad \text{for some } a \in \mathbb{C} \text{ and } k_1 + \ldots + k_l = k.$$

The term $\lambda_k u_k$ is continuous, hence it is enough to consider N_k, only. Let us fix $\varepsilon > 0$. From Lemma 3.1 it follows that there exists a finite set $S \subset J$, such that

$$\sum_{i \in J \setminus S} |N_{k,i}(u)| < \frac{\varepsilon}{3}, \quad \text{for all } u \in W \oplus T. \tag{3.7}$$

There exists L, such that for all $i \in S$ monomials $N_{k,i}(u)$ depend in fact on the variables u_l for $|l| \leq L$, hence $\sum_{i\in S} N_{k,i}(u)$ is continuous on $W \oplus T$. Therefore there exists n_0, such that

$$\left|\sum_{i\in S} N_{k,i}(u^n) - \sum_{i\in S} N_{k,i}(u^*)\right| < \frac{\varepsilon}{3}. \tag{3.8}$$

From (3.8) and (3.7) we obtain for $n > n_0$

$$\begin{aligned}|N_k(u^n) - N_k(u^*)| \leq{}& \left|\sum_{i\in S} N_{k,i}(u^n) - \sum_{i\in S} N_{k,i}(u^*)\right| \\ &+ \sum_{i\in J\setminus S} |N_{k,i}(u^n)| + \sum_{i\in J\setminus S} |N_{k,i}(u^*)| < \varepsilon.\end{aligned}$$

Hence $\lim_{n\to\infty} N_k(u^n) = N_k(u^*)$. □

Now we are ready to state and prove our main theorem in this section.

Theorem 3.7. *Consider* (3.1). *Assume that conditions* (1.2)–(1.4) *hold. Let* $s_0 = p + d + 1$. *Let* $Z \oplus T_0$ *form self-consistent bounds for* (3.1) *such that, for some* C_0 *and* $s \geq s_0$, *holds:*

$$|T_{0,k}| \leq \frac{C_0}{|k|^s}, \quad |k| > m,\ k \in I,\ s > s_0.$$

Then there exist $h > 0$, $W \oplus T_1$ *– self-consistent bounds for* (3.1) *and* $L > 0$, *such that, for all* $l > L$ *and* $u \in P_l(Z \oplus T_0)$,

$$\varphi^l([0,h],u) \subset W \oplus T_1 \quad \textit{and} \quad |T_{1,k}| \leq \frac{C_1}{|k|^s}, \quad |k| > m,\ k \in I.$$

Proof. Let $W \subset X_m$ be a compact set, such that $Z \subset \operatorname{int}_{X_m} W$. By eventually increasing C_0 we can assume that

$$|u_k| \leq \frac{C_0}{|k|^s}, \quad \text{for all } u \in W \oplus T_0 \text{ and } k \in I.$$

We set $C_1 = 2C_0$ and define the tail T_1 by

$$T_1 = \prod_{|k|>m,\, k\in I} \overline{B}\left(0, \frac{C_1}{|k|^s}\right).$$

From Lemma 3.1 it follows that there exists $D = D(C_1, s)$, such that

$$|N_k(u)| < \frac{D}{|k|^{s-r}}, \quad \text{for all } u, \text{ such that } |u_k| \leq \frac{C_1}{|k|^s}.$$

Let $u \in W \oplus T_1$ and $|u_{k_0}| = C_1/|k_0|^s$ for some $|k_0| > K_-$.

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(u_{k_0}|u_{k_0}) &< -v_0|k_0|^p|u_{k_0}|^2 + |u_{k_0}||N_{k_0}(u)| \\ &\leq (-v_0C_1|k_0|^{p-s} + D|k_0|^{r-s})|u_{k_0}|\frac{d|u_{k_0}|^2}{dt} < 0,\end{aligned} \tag{3.9}$$

$|k_0| > L$, for L sufficiently large.

Consider now the differential inclusion

$$x' \in P_L F(x) + \Delta, \quad x \in X_L,\ \Delta \subset X_L \tag{3.10}$$

where the set Δ represents the Galerkin projection errors on $W \oplus T_1$ and is given by

$$\Delta = \{P_L F(u) - P_L F(P_L u) \mid u \in W \oplus T_1\}.$$

As it was mentioned in the introduction, by a solution of differential inclusion (3.10) we will understand any C^1 function $x\colon [0, t_m] \to X_L$ satisfying condition (3.10).

It is easy to see that there exists $h > 0$, such that if $x\colon [0, t_m] \to X_L$, where $t_m \le h$, is a solution of (3.10) and $x(0) \in P_L(Z \oplus T(0)))$, then

$$x(t) \in \operatorname{int}_{X_L} P_L(W \oplus T_1), \quad t \in [0, h]. \tag{3.11}$$

Namely, it is enough to take $h > 0$ satisfying the following condition

$$h \cdot \left(\max_{u \in W \oplus T_1 \in} |P_L F(u)| + \max_{\delta \in \Delta} |\delta| \right) < \operatorname{dist}(P_L(Z \oplus T_0), \partial_{X_L} P_L(W \oplus T_1)).$$

Let $l > L$ and let $u\colon [0, t_1) \to X_l$ be a solution of

$$u' = P_l F(u), \quad u(0) = u_0 \in P_l(X \oplus T_0).$$

By changing the vector field in the complement of $P_l(W \oplus T_1)$ we can assume that $t_1 = \infty$. Let

$$t_m = \sup\{t > 0 \mid u([0, t]) \subset P_l(W \oplus T_1)\}.$$

Obviously $t_m > 0$. It is enough to prove that $t_m \ge h$. Observe that for $t \in [0, t_m]$ $P_L u(t)$ is a solution of (3.10), hence from (3.11) we obtain

$$P_L u([0, t_m]) \subset \operatorname{int}_{X_L} P_L(W \oplus T_1).$$

From (3.9) it follows immediately that $Q_L u([0, t_m]) \subset \operatorname{int}_{Y_l} P_l Q_L(W \oplus T_1)$. Hence $u(t_m) \in \operatorname{int}_{X_l} P_l(W \oplus T_1)$. From above condition and the continuity of u it follows that, for some $\delta > 0$, holds:

$$u(t_m + t') \in \operatorname{int}_{X_l} P_l(W \oplus T_1), \quad t' \in [0, \delta]$$

hence $t_m = h$. $\square$

3.3. Classical solutions from self-consistent bounds. The goal of this section is to address the question, whether the solutions of (1.1) obtained through the method of self-consistent bounds are classical solutions.

To formulate the answer in an abstract setting we need some assumptions about the behavior of the derivatives for functions from H_k.

For $s \geq 0$ let $C^s_{\text{per}}(n) = C^s(\mathbb{T}^d, \mathbb{R}^n)$ denote the space of functions on the d-torus of class C^s. For $u \in C^0_{\text{per}}(n)$ we set $|u|_0 = \sup_{x \in \mathbb{T}^d} |u(x)|$, where on $\mathbb{R}^n$ we use any fixed norm.

DEFINITION 3.8. Let $H = \overline{\bigoplus_{k \in I} H_k}$. We say that *the decomposition of H into H_k* is r-smooth, when the following conditions are satisfied:

(a) there exists a partial linear map $\iota : H \supset \operatorname{dom}(\iota) \to C^r_{\text{per}}(n)$, such that $\bigoplus_{k \in I} H_k \subset \operatorname{dom}(\iota)$ and $\ker(\iota) = \{0\}$,
(b) there exists constant R such that, for each $k \in I$, $u \in H_k$ and for $l = 1, \ldots, r$, holds:
$$\left| \frac{\partial^l \iota(u)}{\partial x_{i_1} \ldots \partial x_{i_l}} \right|_0 \leq R|k|^l |u|,$$
for any $(i_1, \ldots, i_l) \in \{1, \ldots, d\}^l$.

Observe that the Fourier expansion, which means that H_k is the space spanned by $e_j \cdot \exp ikx$, where $\{e_j\}_{j=1,\ldots,n}$ is a canonical basis in $\mathbb{R}^n$, is obviously an r-smooth decomposition of $L_2([0, 2\pi], \mathbb{R}^n)$ for any r.

THEOREM 3.9. *Consider* (3.1). *Assume that conditions* (1.2)–(1.4) *hold. Let $s_0 = d + p + 1$. Assume that $H = \overline{\bigoplus_{k \in I} H_k}$ is an s-smooth decomposition of H, for $s \geq s_0$. Let $u\colon [t_1, t_2] \to W \oplus T \subset H$, where $W \oplus T$ are self-consistent bounds for* (3.1) *such that, for some constants $m, C \in \mathbb{R}$, $s \geq s_0$,*
$$|T_k| \leq \frac{C}{|k|^s}, \quad |k| > m, \ k \in I,$$
holds, then u is a classical solution of (1.1).

PROOF. We define $a(t, x)$ by
$$a(t, x) = \sum_{k \in I} \iota(u_k(t))(x).$$
From our assumptions it follows that the above series is converging uniformly on $[t_1, t_2] \times \mathbb{T}^d$. Also for any partial derivative of order less than or equal to p holds
$$\frac{\partial^l a}{\partial x_{i_1} \ldots \partial x_{i_l}} = \sum_{k \in I} \frac{\partial^l \iota(u_k)}{\partial x_{i_1} \ldots \partial x_{i_l}}.$$
Moreover, the convergence is uniform on $[t_1, t_2] \times \mathbb{T}^d$. Since $s \geq s_0$, hence also the Fourier expansions for La and $N(a)$ (see Lemma 3.1) are converging uniformly on $[t_1, t_2] \times \mathbb{T}^d$. This finishes the proof. □

In fact from the proof of the above theorem one can obtain the information about the regularity of solutions. For the results of this type using this approach for Navier–Stokes equation we refer the reader [27].

3.4. Analyticity of solutions. The goal of this section is to prove using the self-consistent bounds approach the results from [6], [19] about the analyticity of solutions.

To discuss the analyticity we need first several lemmas.

Lemma 3.10. *Assume that for some $\gamma > 0$, $a > 0$ and $D > 0$ there is $|u_k| \le De^{-a|k|}/|k|^\gamma$ for $k \in \mathbb{Z}^d \setminus \{0\}$. Then the function $u(x) = \sum_{k\in\mathbb{Z}^d} u_k e^{ikx}$ is analytic.*

Lemma 3.11. *Let $s > s_0 = d+r$ and $q > 0$. If $|a_k| \le Ce^{-q|k|}/|k^s|$, $|a_0| \le C$, then there exists $D = D(C,s)$ such that*

$$|N_k| \le \frac{De^{-q|k|}}{|k|^{s-r}}, \quad |N_0| \le D.$$

Proof. From the triangle inequality and Lemma 3.5 we have

$$\sum_{k_1,\dots,k_n\in\mathbb{Z}^d,\ \sum_{i=1}^n k_i = k} \frac{e^{-q|k_1|}\cdot\ldots\cdot e^{-q|k_n|}}{|k_1|^\gamma\cdot\ldots\cdot|k_n|^\gamma}$$
$$\le e^{-q|k|} \sum_{k_1,\dots,k_n\in\mathbb{Z}^d,\ \sum_{i=1}^n k_i = k} \frac{1}{|k_1|^\gamma\cdot\ldots\cdot|k_n|^\gamma} \le \frac{C_n(d,\gamma)e^{-q|k|}}{|k|^\gamma}.$$

The remainder of the proof is essentially the same as the proof of Lemma 3.1.□

Lemma 3.12. *Consider* (3.1). *Assume that conditions* (1.2)–(1.4) *hold. Let $s_0 = p+d+1$. Let $Z \oplus T_0$ form self-consistent bounds for* (3.1), *such that for some C_0 and $s \ge s_0$ holds*

$$|T_{0,k}| \le \frac{C_0}{|k|^s}, \quad |k| > m,\ k\in I,\ s > s_0.$$

Then there exists $h > 0$, $q > 0$, C_2 and $L > 0$ such that, for $l > L$ and $u \in P_l(Z\oplus T_0)$ holds: $\varphi^l(t,u) = \sum_{k\in I} u_k(t)$ is defined for $t\in[0,h]$ and

$$|u_k(t)| \le \frac{C_2 e^{-q|k|t}}{|k|^s}, \quad k\in I,\ t\in[0,h]. \tag{3.12}$$

Proof. Let $h > 0$, $W\oplus T_1$, $L > 0$ be as obtained in Theorem 3.7. It remains to prove (3.12). Let us choose C_2 so that

$$\frac{C_2}{|k|^s} > \|(W\oplus T_1)_k\|, \quad \text{for all } k\in I. \tag{3.13}$$

Let $D = D(C_2, s)$ be as obtained in Lemma 3.11. Let us fix

$$K_e > \left(\frac{D}{C_2 v_0}\right)^{1/(p-r)}.$$

From (3.13) it follows that there exists $0 < q_0 \leq (v_0 - D/(C_2 K_e^{p-r}))$, such that for $0 < q < q_0$ holds

$$\frac{C_2 e^{-q|k|h}}{|k|^s} > \|(W \oplus T_1)_k\|, \quad \text{for all } k \in I,\ |k| < K_e. \tag{3.14}$$

We will show now that for any Galerkin projection P_l with $l > L$ and $u \in P_l(Z \oplus T_0)$, if $|u_k(t_0)| \leq C_2 e^{-q|k|t_0}/|k|^s$ for $|k| > K_e$ and $|u_{k_0}(t_0)| = C_2 e^{-q|k_0|t_0}/|k_0|^s$ for some $|k_0| > K_e$ and $t_0 \in [0, h]$, then

$$\frac{d|u_{k_0}|}{dt}(t_0) < -q|k_0||u_{k_0}(t_0)|. \tag{3.15}$$

From Lemma 3.11 we have

$$\frac{d|u_{k_0}|}{dt}(t_0) \leq -v_0|k_0|^p|u_{k_0}(t_0)| + \frac{De^{-q|k_0|t_0}}{|k_0|^{s-r}}.$$

Hence to obtain (3.15) it is enough to show that

$$\frac{De^{-q|k_0|t_0}}{|k_0|^{s-r}} < (v_0|k_0|^p - q|k_0|)\frac{C_2 e^{-q|k_0|t_0}}{|k_0|^s}.$$

Since $(v_0|k|^p - q|k|) \leq (v_0 - q)|k|^p$, then it is enough to prove that, for $|k| > K_e$ and q small enough, $D < (v_0 - q)|k|^{p-r} C_2$. This, for $q < v_0$, is equivalent to

$$q < v_0 - \frac{D}{C_2 K_e^{p-r}} = \overline{q}_0.$$

Observe that, with our choice of K_e, we have $\overline{q}_0 > 0$. It is now easy to observe that (3.15) and (3.14) implies (3.12). $\square$

Lemma 3.13. *Consider* (3.1)*. Assume that conditions* (1.2)–(1.4) *hold. Let $s_0 = p + d + 1$. Let $Z \oplus T_0$ form self-consistent bounds for* (3.1)*, such that for some C_0, $q > 0$ and $s \geq s_0$ holds*

$$|T_{0,k}| \leq \frac{C_0 e^{-q|k|}}{|k|^s}, \quad |k| > m,\ k \in I,\ s > s_0.$$

Then there exists $h > 0$, $q_1 > 0$ and $L > 0$ such that, for $l > L$ and $u \in P_l(Z \oplus T_0)$, holds:

$$\varphi^l(t, u) = \sum_{k \in I} u_k(t)$$

is defined for $t \in [0, h]$ and

$$|u_k(t)| \leq \frac{C_2 e^{-q_1|k|}}{|k|^s}, \quad k \in I,\ t \in [0, h] \tag{3.16}$$

Proof. Let $h > 0$, $W \oplus T_1$, $L > 0$ be as obtained in Theorem 3.7 for $Z \oplus \widetilde{T}_0$, where

$$|\widetilde{T}_{0,k}| \leq \frac{C_0}{|k|^s}, \quad |k| > m,\ k \in I,\ s > s_0.$$

Since $T_0 \subset \widetilde{T}_0$, then to complete the proof it remains to prove (3.16).

Let us choose C_2 so that

$$\frac{C_2}{|k|^s} > \|(W \oplus T_1)_k\|, \quad \text{for all } k \in I.$$

Let $D = D(C_2, s)$ be as obtained in Lemma 3.11. Let us fix

$$K_e > \left(\frac{D}{C_2 v_0}\right)^{1/(p-r)}. \tag{3.17}$$

There exists $0 < q_0$ such that, for $0 < q_1 < q_0$, holds

$$\frac{C_2 e^{-q_1|k|}}{|k|^s} > \|(W \oplus T_1)_k\|, \quad \text{for all } k \in I,\ |k| < K_e. \tag{3.18}$$

We will show now that for any Galerkin projection P_l with $l > L$ and $u \in P_l(Z \oplus T_0)$, if $|u_k(t_0)| \leq C_2 e^{-q_1|k|}/|k|^s$ for $|k| > K_e$ and $|u_{k_0}(t_0)| = C_2 e^{-q_1|k_0|}/|k_0|^s$ for some $|k_0| > K_e$ and $t_0 \in [0, h]$, then

$$\frac{d|u_{k_0}|}{dt}(t_0) < 0. \tag{3.19}$$

From Lemma 3.11 we have

$$\frac{d|u_{k_0}|}{dt}(t_0) \leq -v_0|k_0|^p |u_{k_0}(t_0)| + \frac{D e^{-q_1|k_0|}}{|k_0|^{s-r}}.$$

Hence to obtain (3.19) it is enough to show, for $|k| > K_e$ and $q_1 > 0$ small enough, that

$$\frac{D e^{-q_1|k_0|}}{|k_0|^{s-r}} < v_0 |k_0|^p \frac{C_2 e^{-q_1|k_0|}}{|k_0|^s}.$$

This is equivalent to $D < v_0 K_e^{p-r} C_2$, which is satisfied due to (3.17). We choose $q_1 < \min(q, q_0)$. It is now easy to observe that (3.19) and (3.18) implies (3.16).$\square$

From Lemmas 3.12 and 3.16 we obtain easily the following

Theorem 3.14. *Consider* (3.1). *Assume that conditions* (1.2)–(1.4) *hold. Let* $s_0 = d + p + 1$. *Assume that* $H = \overline{\bigoplus_{k \in I} H_k}$ *is an* s*-smooth decomposition of* H, *for* $s \geq s_0$. *Let* $u\colon [t_1, t_2] \to W \oplus T \subset H$, *where* $W \oplus T$ *are self-consistent bounds for* (3.1), *be a solution of* (3.1) *such that for some constants* $m, C \in \mathbb{R}$, $s \geq s_0$

$$|T_k| \leq \frac{C_0}{|k|^s}, \quad |k| > m,\ k \in I,$$

holds, then u *is a classical solution of* (1.1), *which for any* $t \in (t_1, t_2]$ *is an analytic function of* x. *Moreover, for some* $q > 0$, *the following holds:*

$$|u_k(t)| \leq \frac{C_1 e^{-q|k|(t-t_1)}}{|k|^s}.$$

4. The algorithm for rigorous enclosure of solutions of perturbations of ODEs and differential inclusions

4.1. The interval arithmetic and notation used in the description of algorithms. The interval arithmetic [16], [17] is a suitable tool to deal with the non-rigorous computer arithmetic, because it replaces a mathematical object, r, a real number or a collection of reals composing a vector, a matrix etc., by an interval or a collection of intervals $\mathbf{r}$, such that $r \in \mathbf{r}$. Moreover, the arithmetic operations on the interval objects can be defined so that the result of the interval computation always contains the result of the corresponding real operation.

In the description of algorithms we will use the same conventions as in [26] regarding the notation of single valued and multivalued (interval) objects. In the sequel, by arabic letters we denote single valued objects like vectors, real numbers, matrices. Quite often we will use square brackets, for example $[r]$, to denote sets. Usually this will be some set constructed by some algorithm. Sets will also be denoted by single letters, for example S, when it is clear from the context that it represents a set. In situations when we want to stress (for example in the detailed description of an algorithm) that we have a set in a formula involving both single-valued objects and sets we will rather use the square bracket, hence we prefer to write $[S]$ instead of S to represent the set. From this point of view $[S]$ and S are different symbols in the alphabet used to name variables and formally speaking there is no relation between the set represented by $[S]$ and the object represented by S. If in the description of an algorithm we will have a situation that both variables, $[S]$ and S, are used simultaneously, then usually $S \in [S]$, but this is always stated explicitly.

For a set $[S]$ by $[S]_I$ we denote the interval hull of $[S]$, i.e. the smallest product of intervals containing $[S]$. The symbol $\operatorname{hull}(x_1, \dots, x_k)$ will denote the interval hull of intervals $x_1, \dots, x_k$. The set $Y \subset \mathbb{R}^m$ will be called an interval set if $Y = \prod_{i=1}^m Y_i$, where Y_i are closed intervals (we will allow also for degenerate intervals $I = [a, a]$).

For any interval $I = [a, b]$ we define a diameter of I $\operatorname{diam}(I)$ and the functions $\operatorname{left}(I)$, $\operatorname{right}(I)$, I^+ and I^- by

$$\operatorname{diam}(I) = b - a, \quad I^- = \operatorname{left}(I) = a, \quad I^+ = \operatorname{right}(I) = b.$$

For $c > 0$ and $X = [a - \delta/2, a + \delta/2]$, where $\delta \geq 0$ we define

$$\operatorname{inflate}(X, c) = [a - c\delta/2, a + c\delta/2].$$

For any interval set (vector, matrix) $[S]$ by $\operatorname{m}([S])$ we will denote a center point of $[S]$ and by $\operatorname{diam}([S])$ we will denote the maximum of diameters of its components.

In the description of algorithm we will use the expression $a \in \mathrm{bool}$ to indicate that a is a boolean variable with the possible values *false* and *true*. Sometimes integer constants 0 and 1 might used for *false* and *true*, respectively.

4.2. An outline of the algorithm. For the purpose of the rigorous integration of dissipative PDEs we will study the following nonautonomous ODE,

$$x'(t) = f(x(t), y(t)) \tag{4.1}$$

where $x \in \mathbb{R}^m$ and $y: \mathbb{R} \supset D \to \mathbb{R}^n$ is bounded and continuous, and f is C^1. Assume that we have some knowledge about $y(t)$, for example $|y(t)| < \varepsilon$ for $0 \le t \le t_1$. We would like to find a rigorous enclosure for $x(t)$.

What we describe below is basically the algorithm for the rigorous enclosure of the solutions of the differential inclusion

$$\frac{dx}{dt}(t) \in f(x) + [\delta], \tag{4.2}$$

where $[\delta] \subset \mathbb{R}^m$. In the context of the rigorous integration of dissipative PDEs the function $y(t)$ in (4.1) represents the tail and $[\delta]$ in (4.2) is the Galerkin projection error.

For a bounded and continuous function $y: [0,\infty) \to \mathbb{R}^n$ let $\varphi(t, x_0, y)$ denotes a solution of equation (4.1) with the initial condition $x(0) = x_0$. For a given $y_0 \in \mathbb{R}^n$ let $\overline{\varphi}(t, x_0, y_0)$ be a solution of the following Cauchy problem

$$x' = f(x, y_0), \quad x(0) = x_0 \tag{4.3}$$

with the same initial condition $x(0) = x_0$. Observe that system (4.3) is a particular case of (4.1) with $y(t) = y_0$.

We are interested in finding rigorous bounds for $\varphi(t, [x_0], [y_0])$, where $[x_0] \subset \mathbb{R}^m$ and $[y_0] \subset C_b([0,\infty), \mathbb{R}^n)$. The set $[y_0]$ might be defined be some dynamical process, in this case we may need to compute the range of $[y_0]$ during each time step or be given explicitly, for example: $y \in [y_0]$ if and only if y is bounded and continuous and $y(t) \in [-\varepsilon, \varepsilon]^n$.

To achieve the above mentioned goal we propose a modification of the original Lohner algorithm [13], [14]. Our presentation and notation follows a description of the C^0-Lohner algorithm presented in [26] and almost coincide with the content of Section 6 from [29]. The main difference compared to [29] is in how the first and the fifth parts are realized in the context of dissipative PDEs. This is described in the subsequent sections.

4.3. A fundamental estimate. The following lemma is a particular case of Theorem 1 in Section 13 in [22] (see subsection IV "The Lipschitz condition"), a self-contained proof (with precisely specified assumptions) can also be found in [10].

Lemma 4.1. *Assume $t_0, h \in \mathbb{R}$ and $h > 0$. Let $f\colon \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_1}$ be a $\mathcal{C}^1$-function. For a fixed $y_c \in \mathbb{R}^{n_2}$ and a bounded and continuous function $y\colon [t_0, t_0+h] \to \mathbb{R}^{n_2}$ consider*

$$x' = f(x, y_c), \qquad x(t_0) = x_0 \tag{4.4}$$

$$x' = f(x, y_c) + (f(x, y(t)) - f(x, y_c)), \quad x(t_0) = x_0. \tag{4.5}$$

Let $x_1, x_2 : [t_0, t_0+h] \to \mathbb{R}^{n_1}$ be solutions of (4.4) *and* (4.5)*, respectively. We assume that*

(a) *$W_y \subset \mathbb{R}^{n_2}$ is a convex set and $y([t_0, t_0+h]) \subset W_y$.*
(b) *Let $W_1 \subset W_2 \subset \mathbb{R}^{n_1}$ be convex and compact, such that, for $s \in [t_0, t_0+h]$, holds:*

$$x_1(s) \in W_1 \quad \text{and} \quad x_2(s) \in W_2.$$

Then, for $t \in [t_0, t_0+h]$ and for $i = 1, \ldots, n_1$, holds:

$$|x_{1,i}(t) - x_{2,i}(t)| \leq \left(\int_{t_0}^{t} e^{J(t-s)} C \, ds \right)_i,$$

provided $C \in \mathbb{R}^{n_1}$ and $J \in \mathbb{R}^{n_1 \times n_1}$ satisfy the following conditions:

$$C_i \geq \sup\{|f_i(x, y_c) - f_i(x, y)|, \ x \in W_1, \ y \in W_y\}, \quad i = 1, \ldots, n_1,$$

$$J_{ij} \geq \begin{cases} \sup \dfrac{\partial f_i}{\partial x_j}(W_2, W_y) & \text{if } i = j, \\[2ex] \sup \left| \dfrac{\partial f_i}{\partial x_j}(W_2, W_y) \right| & \text{if } i \neq j. \end{cases}$$

Comment. It is very important for the application to dissipative PDEs, that in the above lemma the terms on the diagonal in matrix J can be negative. As a result of this fact the increasing of the dimension of the Galerkin projections does not result in a significant increase of $\|e^{Jt}\|$ for $t > 0$. This fact allows also to obtain the equicontinuity of all Galerkin projections, which can be later used to obtain an ODE-type uniqueness proof for dissipative PDEs (see [28]).

4.4. One step of the algorithm. The basic outline of the algorithm is nearly the same as in [29]. The only, but essential, difference is that in the case of dissipative PDEs we have an efficient procedure for the computation of the evolution of the tail.

In the description below the objects with an index k refer to the current values and those with an index $k+1$ are the values after the next time step.

We define

$$[y_k] = \{y \in C_b([0, \infty], \mathbb{R}^n) \mid y(t) = z(t_k + t) \text{ for some } z \in [y_0]\}.$$

We will also use the following notation for $[y] \subset C_b([0, \infty), \mathbb{R}^s)$

$$[y]([t_1, t_2]) = \{z(t) \mid z \in [y], \ t \in [t_1, t_2]\}.$$

One step of the Lohner algorithm is a shift along the trajectory of system (4.1) with the following input and output data:

Input data:

- t_k is a current time,
- h_k is a time step,
- $[x_k] \subset \mathbb{R}^m$, such that $\varphi(t_k, [x_0], [y_0]) \subset [x_k]$,
- bounds for $[y_k]$,

Output data:

- $t_{k+1} = t_k + h_k$ is a new current time,
- $[x_{k+1}] \subset \mathbb{R}^m$, such that $\varphi(t_{k+1}, [x_0], [y_0]) \subset [x_{k+1}]$,
- bounds for $[y_{k+1}]$.

We do not specify here the representation of sets $[x_k]$. This issue is very important in the handling of the wrapping effect and is discussed in detail in [13], [14], [16], [17] (see also Section 3 in [26]).

One step of the algorithm consists from the following parts:

1. **Generation of a-priori bounds for φ and $[y_0]([t_k, t_{k+1}])$.** We find convex and compact set $[W_2] \subset \mathbb{R}^m$ and convex set $[W_y] \subset \mathbb{R}^n$, such that
$$\varphi([0, h_k], [x_k], [y_k]) \subset [W_2], \qquad [y_k]([0, h_k]) \subset [W_y].$$
2. We fix $y_c \in [W_y]$.
3. **Computation of an unperturbed x-projection.** We apply one step of the C^0-Lohner algorithm to (4.3) with the time step h_k and the initial condition given by $[x_k]$ and $y_0 = y_c$. As a result we obtain $[\overline{x}_{k+1}] \subset \mathbb{R}^m$ and convex and compact set $[W_1] \subset \mathbb{R}^m$, such that
$$\overline{\varphi}(h_k, [x_k], y_c) \subset [\overline{x}_{k+1}], \qquad \overline{\varphi}([0, h_k], [x_k], y_c) \subset [W_1].$$
4. **Computation of the influence of the perturbation.** Using formulas from Lemma 4.1 we find set $[\Delta] \subset \mathbb{R}^m$, such that
$$\varphi(t_{k+1}, [x_0], [y_0]) \subset \overline{\varphi}(h_k, [x_k], y_c) + [\Delta].$$
Hence
$$\varphi(t_{k+1}, [x_0], [y_0]) \subset [x_{k+1}] = [\overline{x}_{k+1}] + [\Delta] \tag{4.6}$$
5. Computation of $[y_{k+1}]$.

4.5. Part 1 – comments. In the context of an ordinary differential inclusion (4.2) we can set $[W_y] = [\delta]$. The question of finding $[W_2]$ is treated in Section 5.

In the context of a dissipative PDE we cannot find $[W_y]$ and $[W_2]$ using independent routines, some consistency conditions are necessary. This question is treated in detail in Sections 6–8.

4.6. Part 4 – details. We use Lemma 4.1.

1. We set

$$\begin{aligned}
[\delta] &= [\{f(x,y_c) - f(x,y) \mid x \in [W_1],\ y \in [W_y]\}]_I \\
C_i &= \mathrm{right}(|[\delta_i]|), \quad i = 1, \dots, n \\
J_{ij} &= \begin{cases} \mathrm{right}\left(\dfrac{\partial f_i}{\partial x_i}([W_2],[W_y])\right) & \text{if } i = j, \\ \mathrm{right}\left(\left|\dfrac{\partial f_i}{\partial x_j}([W_2],[W_y])\right|\right) & \text{if } i \neq j. \end{cases}
\end{aligned}$$

2. $D = \int_0^h e^{J(h-s)} C\, ds$.
3. $[\Delta_i] = [-D_i, D_i]$, for $i = 1, \dots, n$.

For the computation of $\int_0^t e^{A(t-s)} C\, ds$, see Section 6.5 in [29].

After we compute Δ to avoid the wrapping effect we perform a rearrangement, see Section 6.6 in [29].

4.7. Part 5 – comments. For ordinary differential inclusions (4.2) we don't have to do anything. In the context of PDEs this is a very important issue and it is treated in Section 7.4.

5. Generation of a-priori bounds for ordinary differential inclusions

The goal of this section is to present an algorithm for the generation of a-priori bounds for ordinary differential inclusions. We will frequently refer to such a-priori bounds as *the rough enclosure*. The main result is Theorem 5.5 and the algorithm based on it is presented in Section 5.3. These developments realize for differential inclusions Part 1 of the algorithm outlined in Section 4.

5.1. A naive rough enclosure function. We start with the following easy theorem.

Theorem 5.1. *Consider a differential equation*

$$x' = f(x), \quad x \in \mathbb{R}^n \tag{5.1}$$

where $f \in C^1$. Let φ be a local flow induced by (5.1)*, $h \in \mathbb{R}$ and X, Z be interval sets, $X \subset \operatorname{int} Z$. Suppose that*

$$Y := \textit{interval hull}(X + [0,h] f(Z)) \subset \operatorname{int} Z \tag{5.2}$$

then $\varphi([0,h], X) \subset Y$.

An easy proof is left to the reader. Above theorem can be also derived from Theorem 5.5 with $D = \emptyset$.

Let Y be as in the above theorem, we will refer to it as the *first order enclosure*, because it is based on the first order Taylor formula. Analogous theorems using higher order Taylor formulas are possible, but our experience show that they are not much better.

Remark 5.2. From Theorem 5.1 it follows immediately that, if we take h sufficiently small, then there exists the first order enclosure. In fact any interval set Z, such that $X \subset \operatorname{int} Z$ is good for sufficiently small h.

Observe that condition (5.2) imposes severe restrictions on the size of h even in the situation, when it is obvious that the enclosure should exists for any $h > 0$. As an example we consider a single linear equation

$$x' = f(x) = -Lx, \quad L > 0, \quad x \in \mathbb{R}.$$

Assume that (5.2) holds for some intervals X, Y, Z and $f(x) = -Lx$. By taking diameters of both sides of (5.2) we obtain

$$hL \cdot \operatorname{diam}(Z) < \operatorname{diam}(Z), \tag{5.3}$$

$$hL < 1. \tag{5.4}$$

On the other side is easy to see that the interval $Y = [-\max|X|, \max|X|]$ is the enclosure for any $h > 0$.

An natural generalization of (5.4) to multidimensional nonlinear system is

$$h|df| < 1, \tag{5.5}$$

where $|df|$ is the maximum of the norm of $df(x)$ for x over the region of interest.

In the context of the Galerkin projection of dissipative PDE from condition (5.5) it follows that in order to obtain the first rough enclosure for n-th Galerkin projection the time step must satisfy

$$h|\lambda_k| < 1, \quad \text{for } |k| \leq n.$$

This usually leads to unreasonably small time steps, which is dictated not by the dynamics of the system under the consideration, but by the inclusion in the integration of highly damped variables of little relevance for the dynamics. In the next section we will present an enclosure theorem and an algorithm based on it, which allows to use considerably larger time steps.

5.2. The rough enclosure algorithm based on isolation. Our goal is to devise a rough enclosure routine, which will take into account the strong damping for some variables and will overcome the restriction on h given by (5.5).

Before we proceed further we need a few easy lemmas.

Lemma 5.3. *Let N be a constant. Let $x(t)$ be a C^1 function. If $dx/dt < \lambda x + N$, then, for $t > 0$, holds:*

$$x(t) < \left(x(0) - \frac{N}{-\lambda}\right)e^{\lambda t} + \frac{N}{-\lambda}.$$

Similarly, if $dx/dt > \lambda x + N$, then, for $t > 0$, holds:

$$x(t) > \left(x(0) - \frac{N}{-\lambda}\right)e^{\lambda t} + \frac{N}{-\lambda}.$$

Lemma 5.4. *Let N be a constant. Let $x(t)$ be a C^1-function. If $dx/dt < \lambda x + N$, then, for $t > 0$, holds:*

$$x(t) < \frac{N}{-\lambda}, \quad \textit{if } x(0) < \frac{N}{-\lambda}, \qquad x(t) < x(0), \quad \textit{if } x(0) \geq \frac{N}{-\lambda}.$$

Similarly, if $dx/dt > \lambda x + N$, then, for $t > 0$, holds

$$x(t) > \frac{N}{-\lambda}, \quad \textit{if } x(0) > \frac{N}{-\lambda}, \qquad x(t) > x(0), \quad \textit{if } x(0) \leq \frac{N}{-\lambda}.$$

We assume that our problem can be written as

$$\frac{dx_i}{dt} \in f_i(x) = \lambda_i x_i + N_i(x), \quad i = 1, \dots, n \tag{5.6}$$

where $N_i: \mathbb{R}^n \to \mathrm{cf}(R)$ is a multivalued continuous function and by φ we will denote *the* (*local*) *flow* induced by (5.6) (see Subsection 1.1 for the definition).

Now we state a theorem which is a basis of our improved enclosure function.

Theorem 5.5. *Consider* (5.6). *Let $h > 0$ and $X \subset Z \subset \mathbb{R}^n$ be interval sets. Let $D \subset \{1, \dots, n\}$* (*the set of dissipative* (*damped*) *directions*)*, such that if $k \in D$, then*

$$\lambda_k < 0, \quad \lambda_k a_k + N_k^- < \frac{da_k}{dt} < \lambda_k a_k + N_k^+,$$

where $N_k(Z) \subset (N_k^-, N_k^+)$. For $k \in D$ we set

$$b_k^\pm = \frac{N_k^\pm}{-\lambda_k}, \quad g_k^\pm = (X_k^\pm - b_k^\pm)e^{\lambda_k h} + b_k^\pm.$$

Let $Y = \prod_{i=1}^n Y_i$ be such that

$$\begin{aligned} Y_i &= X_i + [0,h] f_i(Z) && \textit{if } i \notin D, \\ Y_i &= Z_i && \textit{if } i \in D. \end{aligned}$$

Then $\varphi([0,h], X) \subset Y$, provided the following conditions are satisfied for $i = 1, \dots, n$:

(a) *if $i \notin D$, then $Y_i \subset \operatorname{int} Z_i$,*
(b) *upper bounds: for $i \in D$, if $Z_i^+ < b_i^+$, then $Z_i^+ \geq g_i^+$,*
(c) *lower bounds: for $i \in D$, if $Z_i^- > b_i^-$, then $Z_i^- \leq g_i^-$.*

Proof. After a modification of the right-hand side of (5.6) outside a sufficiently large ball we can assume that all solutions of (5.6) are defined for $t \in \mathbb{R}_+$.

By a small stretching of Z in dissipative directions we can construct a new interval set $\widetilde{Z}$, such that

$$\begin{aligned} \widetilde{Z}_i &= Z_i, & i &\notin D, \\ Z_i &\subset \operatorname{int} \widetilde{Z}_i, & i &\in D, \\ N_k(\widetilde{Z}) &\subset (N_k^-, N_k^+), & k &\in D, \\ X_i + [0,h] f_i(\widetilde{Z}) &\subset \operatorname{int} Z_i = \operatorname{int} \widetilde{Z}_i, & i &\notin D. \end{aligned}$$

Obviously $Y \subset \operatorname{int} \widetilde{Z}$.

Let us fix $x_0 \in X$ and $x(t)$ be a solution of (5.6) through x_0 and let

$$T = \sup\{t \in [0,h] \mid \text{ for all } s \in [0,t],\ x(s) \in Y\}.$$

To finish the proof it is enough to show that $T = h$.

If $T < h$, then there exists $\delta > 0$, such that $T + \delta < h$ and $x(T+t) \in \widetilde{Z}$. Hence, from Lemma 5.3, it follows that

$$x_i(T+t) \in \operatorname{int} Y_i \subset Z_i, \quad \text{for } i \in D \text{ and } t \in (0,\delta]. \tag{5.7}$$

Hence $x(s) \in Z$, for $s \in [0, T+\delta]$. Applying the Mean Value Theorem to x_i for $i \notin D$ for $t \in [0,\delta]$ we obtain

$$x_i(T+t) \in x_{0,i} + (T+t) \cdot f_i(Z) \subset x_{0,i} + [0,h] f_i(Z) \subset Y_i. \tag{5.8}$$

From (5.8) and (5.7) it follows that $x([0,T+\delta]) \subset Y$. This is in a contradiction with the definition of T, hence $T = h$. □

5.3. The algorithm for rough enclosure for differential inclusions. The initial guess. We define

$$\begin{aligned} Z_i &= X_i + [0,h] f(X), && \text{if } \lambda_i \geq 0, \\ Z_i &= X_i, && \text{if } \lambda_i < 0, \\ N_i &= N_i(Z), \\ b_i &= \frac{N_i}{-\lambda_i}. \end{aligned}$$

We define set D, the set of dissipative coordinates, as follows: $i \in D$ if and only if $\lambda_i < -0.01$.

It appears to me that the natural size of the enclosure in i-th direction will be given by $\operatorname{diam}(X_i \cup g_i)$. We will use it to modify the set Z in the dissipative directions.

We choose two real constants $c > 1$ and $0 < c_d < 1$ (we use $c = 1.1$, $c_d = 0.1$) and we redefine Z_i by setting:

$$Z_i = \mathrm{inflate}(Z_i, c), \quad i \notin D.$$

For $i \in D$ we recompute N_i and b_i and we set

$$\begin{aligned}
g_i^{\pm} &= (X_i^{\pm} - b_i^{\pm})e^{\lambda_i h} + b_i^{\pm},\\
w_i &= \mathrm{diam}\,([g_i^-, g_i^+] \cup X_i),\\
Z_i^+ &= \begin{cases} X_i^+ & \text{if } X_i^+ \geq b_i^+,\\ \min(b_i^+, g_i^+ + c_d w_i) & \text{if } X_i^+ < b_i^+,\end{cases}\\
Z_i^- &= \begin{cases} X_i^- & \text{if } X_i^- \leq b_i^-,\\ \max(g_i^- - c_d w_i, b_i^-) & \text{if } X_i^- > b_i^-.\end{cases}
\end{aligned}$$

Validation and a new guess. For each i we initialize the array validated, by $validated[i] = true$.

For each $i \notin D$, we set $Y_i = X_i + [0, h]f_i(Z)$.

If not $Y_i \subset \mathrm{int}\, Z_i$, then we set $validated[i] = false$ and define a new guess by

$$Z_i = \mathrm{inflate}(Y_i \cup Z_i, c).$$

For each $i \in D$ we do the following:

- We compute N_i and b_i. If not $b_i \subset X_i$, then we compute $g_i^{\pm}$ and w_i.
- If not $Z_i^+ > b_i^+$ (this implies that $X_i^+ \leq b_i^+$) and if $Z_i^+ < g_i^+$, then we set $validated[i] = false$ and we define a new guess by setting

$$Z_i^+ = \min(b_i^+, g_i^+ + c_d w_i).$$

- With Z_i^- we proceed symmetrically, i.e. if not $Z_i^- < b_i^-$ (this implies that $X_i^- \geq b_i^-$) and if $Z_i^- > g_i^-$, then we set $validated[i] = false$ and we produce a new guess by setting

$$Z_i^- = \max(g_i^- - c_d w_i, b_i^-).$$

Finally the enclosure is *validated* if $validated[i] = true$ for all i.

We iterate the above step until we achieve the validation or the number of steps is larger than some limit (equal to $\max(5, n/2)$ in my program, where n is the dimension of the phase space).

If we achieved the validation then, we refine the obtained enclosure as follows. We compute for all i $N_i = N_i(Y)$, b_i and b_i and we set

$$\begin{aligned}
Y_i &= X_i + [0, h]f_i(Y), && \text{for } i \notin D,\\
Y_i^+ &= \begin{cases} X_i^+ & \text{if } X_i^+ \geq b_i^+,\\ g_i^+ & \text{if } X_i^+ < b_i^+,\end{cases} && \text{for } i \in D,
\end{aligned}$$

$$Y_i^- = \begin{cases} X_i^- & \text{if } X_i^- \le b_i^-, \\ g_i^- & \text{if } X_i^- > b_i^-. \end{cases} \quad \text{for } i \in D.$$

We can iterate the refinement a few times.

6. Treatment of the tail for dissipative PDEs

In this section we discuss how to realize Parts 1 and 5 of the algorithm for the rigorous integration of dissipative PDEs. The algorithm itself is presented in the next section.

We consider the problem (3.1) derived from (1.1) and we adopt the notation used in Sections 2 and 3. Let us stress that we do not assume the local existence of solutions of (1.1), it is a byproduct of the algorithm.

During the computation we want the bounds for solutions to be given by self-consistent bounds. These bounds will be valid for sufficiently high dimensional Galerkin projections of (3.1), so we can use Lemma 2.7 to obtain the existence of solutions of (3.1).

Notations. For self-consistent bounds $W \oplus T$ we will denote by $m(T)$ and $M(T)$ the numbers m and M from Definition 2.3, respectively. In the sequel we will often use variables T, $T(h)$, $T([0,h])$ to indicate the tail. By $T(0)$ we will usually denote the initial tail (or a candidate for such set), by $T(h)$ the tail at time $t = h$ (or a candidate) and by $T([0,h])$ the tail for $t \in [0,h]$ (or a candidate). For tail T, by $\varphi(t, x_0, T)$, where $t \in \mathbb{R}$ and $x_0 \in X_{m(T)}$, we will denote the set all possible values of $x(t)$, where x is a solution of differential inclusion (6.1) (a C^1-function) defined on the maximum interval of the existence

$$x' \in P_{m(T)}F(x+T), \quad x(0) = x_0. \tag{6.1}$$

If some particular $x(t)$ does not exists for some t, then also $\varphi(t, x_0, T)$ is undefined. In the sequel we will use expression of the form

$$\varphi([0,h], x_0, T) \subset Z.$$

It means that $\varphi([0,h], x_0, T)$ defined, hence any solution of (6.1) is defined for $t \in [0,h]$, and the stated inclusion holds.

Standing assumptions. In this section we assume that $I = \mathbb{Z}_+$ and $H_k = \mathbb{R}$, hence the sets B_k in self-consistent bounds can be represented as $[a_k^-, a_k^+]$, where $a_k^- \le a_k^+$, $a_k^\pm \in \mathbb{R}$. In this situation we can also assume that $m, M \in \mathbb{N}$. The generalization to a more general situation is straightforward, it is enough to take $B_k = \prod_{i=1}^{d_1} [a_{k,i}^-, a_{k,i}^+]$ for $m < |k| \le M$ and $B_k = \overline{B_{H^k}}(0, r_k)$ for $|k| > M$.

Moreover, we assume that conditions (1.2)–(1.4) are satisfied.

LEMMA 6.1. *Assume that $W_2 \subset \mathbb{R}^m$ and T are self-consistent bounds for* (3.1). *Let $X_0 \subset W_2$ and $T(0) \subset T$ be self-consistent bounds for* (3.1), *such that $m(T) = m(T(0))$ and $M(T(0)) = M(T)$. Assume that*

$$\varphi([0,h], X_0, T) \subset W_2. \tag{6.2}$$

Let $N_k^{\pm}$ be such that

$$N_k^- < N_k(x+q) < N_k^+, \quad \text{for all } k > m,\ x \in W_2 \text{ and } q \in T\ . \tag{6.3}$$

Assume that, for $k > m$, $\lambda_k < 0$. For $k > m$ we define $b_k^{\pm}$, $g_k^{\pm}$, $T(h)_k^{\pm}$ and $T([0,h])_k^{\pm}$ as follows

$$\begin{aligned}
b_k^{\pm} &= \frac{N_k^{\pm}}{-\lambda_k}, \\
g_k^{\pm} &= (T(0)_k^{\pm} - b_k^{\pm})e^{\lambda_k h} + b_k^{\pm}, \\
T(h)_k^{\pm} &= g_k^{\pm},
\end{aligned}$$

$$T([0,h])_k^+ = \begin{cases} T(0)_k^+ & \text{if } T(0)_k^+ \geq b_k^+, \\ g_k^+ & \text{if } T(0)_k^+ < b_k^+, \end{cases} \tag{6.4}$$

$$T([0,h])_k^- = \begin{cases} T(0)_k^- & \text{if } T(0)_k^- \leq b_k^-, \\ g_k^- & \text{if } T(0)_k^- > b_k^-. \end{cases} \tag{6.5}$$

If

$$T([0,h]) \subset T, \tag{6.6}$$

then for any $n > M$ holds

$$\varphi^n([0,h], X_0 \oplus P_n T(0)) \subset W_2 \oplus P_n T([0,h]), \tag{6.7}$$

$$\varphi^n(h, X_0 \oplus P_n T(0)) \subset W_2 \oplus P_n T(h). \tag{6.8}$$

Moreover, for any $u_0 \in X_0 \oplus T(0)$ there exists $u\colon [0,h] \to W_2 \oplus T([0,h])$, a solution of (3.1), *such that $u(0) = u_0$ and $u(h) \in W_2 \oplus T(h)$.*

PROOF. It is enough to prove (6.7), because (6.8) follows then immediately from Lemma 5.3 and the last assertion is a consequence of Lemma 2.7 applied to self-consistent bounds $W_2 \oplus T$ and conditions (6.7) and (6.8).

To prove (6.7) let us fix $n > M$, $p \in X_0$ and $y \in P_n T(0)$. For sufficiently small $\varepsilon > 0$ let $W(\varepsilon) \subset \mathbb{R}^m$ and $V(\varepsilon) \subset \mathbb{R}^{n-m}$ be such that

$$\begin{aligned}
W_2 &\subset \operatorname{int} W(\varepsilon), & W(\varepsilon) &\subset B(W_2, \varepsilon), \\
P_n T([0,h]) &\subset \operatorname{int} V(\varepsilon), & V(\varepsilon) &\subset B(P_n(T([0,h])), \varepsilon)
\end{aligned}$$

and $N_k^- < N_k(x+q) < N_k^+$, for $x + q \in W(\varepsilon) \oplus V(\varepsilon)$ and $k = m, \ldots, n$, where $N_k^{\pm}$ are the constants from condition (6.3). We define

$$t_1 = \sup\{t \in [0,h] \mid \varphi^n([0,t], p+y) \subset W_2 \oplus P_n T([0,h])\}.$$

To finish the proof it is enough to show that $t_1 = h$. If this is not the case, then there exists $\delta > 0$, $t_1+\delta \le h$, such that we have $\varphi^n([0,t+\delta],p+y) \subset W(\varepsilon)\oplus V(\varepsilon)$. Hence we can use the constants $N_k^{\pm}$ in Lemma 5.3 for $t \in [0,t_1+\delta]$ to obtain

$$Q_m\varphi^n([0,t_1+\delta],p+y) \subset \operatorname{int} P_nT([0,h])).$$

From conditions (6.2) and (6.6) it follows that

$$P_m\varphi^n([0,t_1+\delta],p+y) \subset W_2.$$

Hence

$$\varphi^n([0,t_1+\delta],p+y) \subset W_2 \oplus P_nT([0,h].$$

But this is in the contradiction with the definition of t_1 and our assumption that $t_1+\delta < h$. Hence $t_1 = h$. □

Lemma 6.2. *Same assumptions and definitions as in Lemma* 6.1*. If additionally $N_k^- = -N_k^+$, $T(0)_k^- = -T(0)_k^+$ for $k > M(T)$, then $W_2 \oplus T(h)$ are self-consistent bounds for F.*

Proof. Observe that $T(h)_k \subset T([0,h])_k \subset T_k$ for all $k > m$, hence $W_2 \oplus T(h) \subset W_2 \oplus T$. From this it follows that conditions (C2) and (C3) are satisfied on $W_2 \oplus T(h)$. To finish the proof is enough to notice that $T(h)_k^- = -T(h)_k^+$ for $k > M$. □

6.1. Uniform treatment of the tail, polynomial bounds. In a computer program we cannot work directly with an infinite sequence of intervals $[a_k^-, a_k^+]$. We need to have a finite number of formulas describing $[a_k^-, a_k^+]$.

Definition 6.3. Let $m \le M$ be positive integers. The structure, T, consisting of the sequence of pairs $\{a_k^-, a_k^+\}_{k\in I,\ k>m}$, such that

(a) $a_k^- \le a_k^+$ for all $k \in I$,
(b) there exists $C \ge 0$ and $s \ge 0$, such that

$$a_k^+ = -a_k^- = \frac{C}{|k|^s}, \quad \text{for } k > M,$$

will be called *the polynomial bound.*

For polynomial bound T, by $m(T)$, $M(T)$, $s(T)$ and $C(T)$ we will denote the numbers m, M, s and C, respectively.

We define $T_k^{\pm}$ by $T_k^{\pm} = a_k^{\pm}$. When discussing algorithms we will also use the expression $T \in \mathrm{PolyBd}$ to say that T *is a polynomial bound.*

We define *the near tail of* T by $\mathrm{nearTail}(T) = \prod_{m<k\le M}[a_k^-, a_k^+]$ and *the far tail of* T by $\mathrm{farTail}(T) = \prod_{k>M}[a_k^-, a_k^+]$.

In my implementation for the KS-equation we consider polynomial bounds with fixed values of m and M. For such class of tails it is easy to define and implement the arithmetic and set theoretic operations.

For example the question of the verification of inclusion $T_k \subset b_k$ for $|k| > M$, where $T, b \in \mathrm{PolyBd}$ and $M(b) = M(T) = M$ can be handled as follows:

- If $C(b) = 0$, then $T_k \subset b_k$ for $|k| > M$, if and only if $C(T) = 0$.
- If $C(b) \neq 0$ and $C(T) = 0$, then $T_k \subset b_k$ for $|k| > M$.
- If $C(b) \neq 0$. Let $K = \min\{|k| \mid k \in I,\ |k| > M\}$. Then $T_k \subset b_k$ for $|k| > M$, ifand only if the following two conditions are satisfied
$$s(T) \geq s(b), \qquad \frac{C(T)}{K^{s(T)}} \leq \frac{C(b)}{K^{s(b)}}.$$

6.2. Uniform computation of b_k. In the context of a computer assisted proof using the enclosure function based on Lemma 6.1 we have to explain how the expressions for $b_k^\pm$ and $g_k^\pm$ can be handled using polynomial bounds. In the remainder of this section we will use the notations from Lemma 6.1. Moreover, we assume that T and N are polynomial bounds such that $m(T) = m(N) = m$ and $M(T) = M(N) = M$ and for all other polynomial bounds introduced below we have these values of m and M.

For the further discussion we assume that λ_k satisfies conditions (1.2) and (1.3). We define an auxiliary function $V\colon \mathbb{R}_+ \to \mathbb{R}_+$ by
$$V(x) = \inf\{v(|k|) \mid k \in I,\ |k| > x\}.$$
Observe that the assumption $\lambda_k < 0$ for $|k| > m$ implies that $0 < v(m) \leq v_1$.

Now we are ready to explain how the formula
$$b_k^\pm = \frac{N_k^\pm}{-\lambda_k} \tag{6.9}$$
can be treated in a finite programmable way in terms of polynomial bounds.

We define $b \in \mathrm{PolyBd}$ as follows:

- (near tail) to calculate the near tail of b we evaluate (6.9) for $k \in I$, $m < k \leq M$,
- (far tail) for $k > M$ we set
$$b_k^+ = -b_k^- = \frac{C(b)}{|k|^{s(b)}} = \frac{C(N)}{V(M)|k|^{s(N)+p}}.$$

Observe that with such definition we have $b_k^+ \geq N_k^+/-\lambda_k$ for all $|k| > M$ (with a reversed inequality for b_k^-) and this change corresponds to taking bigger value for N_k^+ in an application of Lemma 5.3. Hence the formulas for $T(h)$ and $T([0,h])$ give valid enclosures, when we use the polynomial bound $b_k^\pm$ defined above.

6.3. Uniform computation of $T(h)$. In Lemma 6.1 the following expression was obtained
$$T(h)_k^\pm = (T(0)_k^\pm - b_k^\pm)e^{\lambda_k h} + b_k^\pm. \tag{6.10}$$

We want to represent $T(h)$ as the polynomial bound. This is achieved by finding a larger set which is a polynomial bound and contains the product of intervals defined by equation (6.10).

The near tail of $T(h)$ is defined by a direct evaluation of (6.10). The far tail requires some analytical work. We have

Lemma 6.4. *Let I, m, M, λ_k be as above. For any $r \in \mathbb{R}$ and $h > 0$, there exists $E = E(r, h, M) > 0$, such that*

$$e^{\lambda_k h} \leq \frac{E}{|k|^r}, \quad \text{for } |k| > M.$$

Proof. It is enough to observe that the function $|k|^r e^{-a|k|^p h}$, where $a = \inf\{v(|k|), |k| > M\} > 0$, is bounded. □

Now we are ready to give a formula for $T(h)^{\pm}_k$, for $k > M$,

$$\begin{aligned} T(h)^+_k &\leq T(0)^+_k e^{\lambda_k} + b^+_k \\ &\leq \frac{C(T(0))}{|k|^{s(T(0))}} \cdot \frac{E(s(b) - s(T(0)), h, M)}{|k|^{s(b)-s(T(0))}} + \frac{C(b)}{|k|^{s(b)}} \\ &= \frac{C(T(0)) \cdot E(s(b) - s(T(0)), h, M) + C(b)}{|k|^{s(b)}}. \end{aligned}$$

Hence we set

$$\begin{aligned} C(T(h)) &= C(T(0)) \cdot E(s(b) - s(T(0)), h, M) + C(b), \\ s(T(h)) &= s(b). \end{aligned}$$

7. The enclosure procedure for the tail

The goal of this section is to describe the function, which constructs the rough enclosure (Part 1 of the algorithm) and computes the tail after the time step (Part 5) for dissipative PDEs. The proposed function is based on Lemma 6.1 and uses the notion of the polynomial bound introduced in Section 6.1.

As in Section 6, throughout this section we assume that the range of k is $I = \mathbb{Z}_+$ and $\dim H_k = 1$. The modification required for other dissipative equations with periodic boundary conditions is obvious and will not be discussed. We have $m, M \in \mathbb{Z}_+$ fixed in advance and all polynomial bounds will use these values.

We assume that we have the enclosure function for the differential inclusion (see Section 5.3) $x' \in P_m F(x + T)$, where $x \in \mathbb{R}^m$ and T is the tail.

We assume that this function has the following declaration:
function incl_enclosure$(h \in \mathbb{R}, [x] \subset \mathbb{R}^m, T \in \text{PolyBd}, [W_2] \subset \mathbb{R}^m) \in$ *bool*.

This function constructs the set $[W_2] \subset \mathbb{R}^m$, such that $\varphi([0, h], [x], T) \subset [W_2]$. If it succeeds then *true* is returned and $[W_2]$ is updated, otherwise it returns *false*. In both cases the parameter T is unchanged.

7.1. Case of an a-priori given tail. We have the set $A \subset \mathbb{R}^m$ representing the a-priori bounds used to compute the global tail, $T_G = \prod_{k>m}[T^-_{G,k}, T^+_{G,k}]$.

We generate $[W_2]$ by calling **function incl_enclosure**$(h, [x], T_G, [W_2])$: *bool* and we check whether $[W_2] \subset A$. If this is the case, then the pair $([W_2], T_G)$ is validated.

This is the approach used in [29]. It turned out to be ineffective when compared to the one with the evolving tail described below.

7.2. Basic functions. We assume that we have a function computing the nonlinear term in (3.1) with the following declaration:
function $N([z] \subset \mathbb{R}^m, T \in \text{PolyBd}) \in \text{PolyBd}$, where $[z]$ and T are such that, for all $k > m$, holds

$$\inf_{(x,y)\in[z]\oplus T} N_k(x,y) > N^-([z],T), \qquad \sup_{(x,y)\in[z]\oplus T} N_k(x,y) < N^+([z],T),$$

and, for $k > M$, we have

$$N^+_k([z],T) = -N^-_k([z],T) = \frac{C(N)}{k^{s(N)}}.$$

For the KS equation in our implementation we have $s(N) = s(T) - 2$, but it is possible to obtain $s(N) = s(T) - 1$ (see Lemma 3.1).

There is an unpleasant feature of our implementation of $N([z],T)$ (but it appears to be a rather inherent for such approach): it happens that (see formula (8.2) in Section 8): we have two tails $T_2 \subset T_1$, such that $T_{1,k} = T_{2,k}$ for $m < k \leq M+1$ (the near-tails of T_1 and T_2 are the same), but $s(T_1) < s(T_2)$ (the far tail in T_2 is decaying faster than that in T_1), but nevertheless

$$N^+_{M+1}([z],T_1) < N^+_{M+1}([z],T_2),$$

which later produces worse isolation intervals (for $k \approx M+1$) for T_2 than for T_1. This phenomenon results from the following fact, when we try to bound N_k by C/k^s, then taking larger s forces larger C, which may result in larger value of N_k for $k \approx M+1$.

To handle the above issue we introduce the function (the method) *decpower*, which for the polynomial bound T will produce a new polynomial bound T' with a slower decay rate for the far tail. Namely if $T' = T.\mathit{decpower}(d)$, then

$$\begin{aligned} T &\subset T', \\ T'_k &= T_k, \qquad\qquad \text{for } m < k \leq M+1, \\ s(T') &= s(T) - d. \end{aligned}$$

The following obvious lemma tells how to check condition $T([0,h]) \subset T$ from Lemma 6.1.

LEMMA 7.1. *The same assumptions and definitions as in Lemma* 6.1. *Assume that:*

(a) $T(0) \subset T$,
(b) *if* $T(0)^+_k < b^+_k$, *then* $T^+_k \geq g^+_k$ *for* $k > m$,
(c) *if* $T(0)^-_k > b^-_k$, *then* $T^-_k \leq g^-_k$ *for* $k > m$.

Then $T([0,h]) \subset T$.

Now we are ready to describe the algorithm for the tail validation.

function validate_tail($h \in \mathbb{R}, [z] \subset \mathbb{R}^m, T(0) \in \text{PolyBd}, T \in \text{PolyBd}$, *gen_new* $\in$ *bool*) $\in$ *bool*

Input parameters:

- $h > 0$ is the time step,
- $[z] \subset \mathbb{R}^m$ represents the a-priori bounds for $x([0,h])$,
- $T(0)$ is the initial condition for the tail,
- T is the candidate for $T([0,h])$,
- *gen_new* tells whether to generate (update) T.

Output: *true* is returned if T is validated, otherwise *false* is returned. Additionally if *gen_new* is equal to *true*, then T is updated as follows: in case it is validated, then we find better (smaller) T, otherwise we produce the new guess for T. If *gen_new* is equal to *false*, then T is left unchanged. For the precise meaning of validation see Theorem 7.3.

The body of the function:

- we set $validated = false$, $farTailValidated = false$, $kvalidated[k] = false$ for $m < k \leq M$,
- computation of $N \in \text{PolyBd}$, $b \in \text{PolyBd}$ and g_k for $m < k < M$

$$\begin{aligned} N^\pm_k &= N^\pm_k([z], T), \\ b^\pm_k &= \frac{N^\pm_k}{-\lambda_k}, && \text{for } m < k \leq M, \\ g^\pm_k &= (T(0)^\pm_k - b^\pm_k)e^{\lambda_k h} + b^\pm_k, && \text{for } m < k \leq M. \end{aligned}$$

To define b_k, for $k > M$, we proceed along the lines described in Section 6.2. We set

$$b^+_k = -b^-_k = \frac{C(N)}{V(M)k^{s(N)+p}} = \frac{C(b)}{k^{s(b)}},$$

where for the KS equation from (3.3) we have $V(M) = \nu - 1/(M+1)^2$.

Observe that with such b_k we have, for $k > M$, $N_k^+/-\lambda_k \leq b_k^+$ and the equality holds for $k = M+1$ only.

- (validation) we set $validated = true$ if the assumptions in Lemma 7.1 are satisfied, because if it is the case, then from Lemma 6.1 we obtain the desired enclosure.

Below we discuss this verification in some detail.

The first check $T(0) \subset T$ is discussed in Section 6.1. In it does not hold then we exit the function returning *false*.

Next we have to check, for all $k > m$, the following conditions:

$$\text{if } T(0)_k^+ < b_k^+, \text{ then } T_k^+ \geq g_k^+, \tag{7.1}$$

$$\text{if } T(0)_k^- > b_k^-, \text{ then } T_k^- \leq g_k^-. \tag{7.2}$$

For the near tail ($m < k \leq M$) we verify the above conditions one by one, setting $kvalidated[k] = true$ when (7.1) and (7.2) are satisfied for k and $kvalidated[k] = false$, otherwise.

For the far tail ($k > M$) we proceed as follows. First of all observe that due to symmetry of all polynomial bounds involved it is enough to verify condition (7.1), only.

We have three cases:

(I) $s(b) > s(T(0))$ and $C(T(0)) \neq 0$.

We check that $T_k^+ \geq g_k^+$, for $M+1 \leq k \leq L$, where

$$L = \left(\frac{C(b)}{C(T(0))}\right)^{1/(s(b)-s(T(0)))}.$$

If (7.3) is satisfied we set $farTailValidated = true$.

To justify the above condition let us notice that if $k \geq M+1$ and $k \geq L$, then $T(0)_k^+ \geq b_k^+$ for $k \geq M+1$. Observe also that if $L < M+1$, then condition (7.3) is satisfied, because there are no k's in this range.

(II) $s(b) = s(T(0))$ or $C(T(0)) = 0$.

If

$$C(T(0)) \geq C(b), \tag{7.4}$$

then we set $farTailValidated = true$, because in this situation we have $T(0)_k^+ \geq b_k^+$ for $k \geq M+1$, hence there is nothing more to check.

If (7.4) does not hold, then we should check whether $T_k^+ \geq g_k^+$. Since in this case we have $T(0)_k^+ < g_k^+ < b_k^+$, we will instead check the stronger condition

$$T_k^+ \geq b_k^+, \quad \text{for } k > M,$$

which is equivalent to the following two conditions

$$s(T) \leq s(b), \quad T^+_{M+1} \geq b^+_{M+1}.$$

If the above conditions are satisfied then we set $farTailValidated = true$.

(III) $s(b) < s(T(0))$ and $C(T(0)) \neq 0$.

Let us define

$$L = \left(\frac{C(T(0))}{C(b)}\right)^{1/(s(T(0))-s(b))}.$$

It is easy to see that

$$\begin{aligned} T(0)^+_k &\geq b^+_k \quad \text{for } M < k \leq L, \\ T(0)^+_k &< b^+_k \quad \text{for } k > L \text{ and } k > M. \end{aligned}$$

Hence, for $k > M$ and $k > L$, we have to check that $T^+_k \geq g^+_k$. Like in the previous case we replace g^+_k by b^+_k and we obtain the following two conditions

$$s(T) \leq s(b), \quad T^+_{p+1} \geq b^+_{p+1},$$

where $p = max(M, \mathrm{int}(L))$ and $\mathrm{int}(L)$ is the largest integer less than or equal to L.

If the above conditions are satisfied then we set $farTailValidated = true$.

- *update of* T. There are two update modes depending on the current value of the boolean variable *validated*.

If $validated = true$, then we proceed as follows (compare formulas (6.4) and (6.5) in Lemma 6.1):
For $i = m + 1$ to M we update $T^\pm_k$ as follows:

$$\begin{aligned} &\text{if } b^+_k \leq T(0)^+_k \text{ then } T^+_k = T(0)^+_k, \\ &\text{if } b^+_k > T(0)^+_k \text{ then } T^+_k = g^+_k, \\ &\text{if } b^-_k \geq T(0)^-_k \text{ then } T^-_k = T(0)^-_k, \\ &\text{if } b^-_k < T(0)^-_k \text{ then } T^-_k = g^-_k. \end{aligned}$$

For the far tail we perform the modification only if $b_k \subset T(0)_k$, for $k > M$ and $C(T(0)) \neq 0$. If this is the case, then we leave $s(T)$ unchanged and we set

$$C(T) = C(T(0))(M+1)^{s(T)-s(T(0))}.$$

With this modification we obtain $farTail(T(new)) \subset fartail(T(old))$ and $T_{M+1} = T(0)_{M+1}$.

Now, if $validated = false$, then we modify only these coordinates in the near tail, for which the validation failed ($kvalidated[k] = false$). Below we present the details.

We have two parameters $0 < d_g < 1$ and $d_2 > 1$ (in my program $d_g = 0.1$, $d_2 = 1.01$). For $k = m+1$ to M, such that $kvalidated[k] = false$ we do the following

$$\begin{aligned} &\text{if } b_k^+ > T_k^+ \text{ then } T_k^+ = (1-d_g)g_k^+ + d_g b_k^+,\\ &\text{if } b_k^- < T_k^- \text{ then } T_k^- = (1-d_g)g_k^- + d_g b_k^-,\\ &\quad T_k = inflate(T_k, d_2). \end{aligned}$$

If $farTailValidated = true$, $b_k \subset T(0)_k$ for $k > M$ holds and $C(T(0)) \neq 0$, then we modify T as follows: we leave $s(T)$ unchanged and we set

$$C(T) = C(T(0))(M+1)^{s(T)-s(T(0))}.$$

If $farTailValidated = false$, then we define the new far tail so that $b_k \cup T(0)_k \subset T_k$. For this end we leave $s(T)$ unchanged and we set

$$C(T) = \max(d_2 C(b)(M+1)^{s(T)-s(b)}, C(T(0))(M+1)^{s(T)-s(T(0))}).$$

The above situation happens for an empty (zero) tail.

Remark 7.2. Let us remark that it is essential for our function to work that we keep $s(T)$ unchanged instead of setting $s(T) = s(T(0))$, because increasing s may result in worse estimates for N_k for $k \approx M+1$, see comments at the begin of this subsection.

- return *validated*.

End of the function validate_tail. It turns out that it makes sense to define a separate function for the validation of the far tail.
function validate_far_tail$(h \in \mathbb{R}, [z] \subset \mathbb{R}^m, T(0) \in \text{PolyBd}, T \in \text{PolyBd}) \in$ *bool*
Input parameters:

- $h > 0$ is the time step,
- $[z] \subset \mathbb{R}^m$ represents the a-priori bounds for $x(0,h])$,
- $T(0)$ is the initial condition for the tail,
- T is the candidate for $T([0,h])$.

Output: *true* is returned if and only if $T(0) \subset T$ and conditions (7.1)–(7.2) for $k \geq M+1$, otherwise *false* is returned.

We omit the discussion of this function because it is really contained in the description of function **validate_tail** (see variable *farTailValidated*).

Theorem 7.3. *Assume that* **validate_tail**$(h, [z], T(0), T, gen\_new)$ *returns true. Let $n > M$, let $(x(t), y(t)) \in \mathbb{R}^m \times \mathbb{R}^{n-m}$ for $t \in [0,h]$ be a solution of*

$$x' = P_m F(x,y), \quad y' = P_n Q_m F(x,y),$$

such that $x(t) \in [z]$ *for* $t \in [0,h]$ *and* $y(0) \in T(0)$*, then* $y(t) \in T$ *for* $t \in [0,h]$.

Proof. For the proof it is enough to compare the checks performed in the validation part with Lemmas 7.1 and 6.1. In particular it is easy to see, that if $validated = true$, then $T([0,h]) \subset T$, where $T([0,h])$ is defined as in Lemma 6.1. In the update part the substitution $neartail(T) = neartail(T([0,h]))$ is performed for the near tail and for the far tail we substitute it with some enclosure of $T([0,k])_k$ for $k > M$. □

7.3. The enclosure algorithm. We assume that we have the function $guessfarTail$, which produces a reasonable initial guess for the far tail. For the KS equation on the line with odd and periodic boundary conditions such a function is given in Section 8.2.

function enclosure_with_tail($h : real, [x] \subset \mathbb{R}^m, T(0)$: PolyBd, $[W_2] \subset \mathbb{R}^m$,
T : PolyBd, $T\_is\_good\_init\_guess$: *bool*) : *bool*;
begin
  $max\_iter = T(0).M/2$;
  $maxdcount = 3$;
  if not $T\_is\_good\_init\_guess$
    $Tinitial = guessfarTail([x], T(0))$;
  else Tinitial=T;
  $validated = false$;
  $dcount = 0$;
  while(!validated **and** $(dcount \leq maxdcount)$) **do**
    $[W_2] = [x] + [0,h] \cdot P_m F([x])$;
    $T = Tinitial$;
    $T.decpower(dcount)$; // now $Tinitial \subset T$ with slower decay
    if validate_far_tail($h, [W_2], T(0), T$) **then**
      validate_tail($h, [W_2], T(0), T, true$);
    // we have now the initial guess for the tail in variable T
      $i = 1$;
      while ((!validated) **and** $(i \leq max\_iter)$) **do**
        if incl_enclosure($h, [x], T, [W_2]$) **then**
          validated=**validate_tail**($[W_2], T(0), T, true$);
        $i = i + 1$;
      end while;
    end if;
    $dcount = dcount + 1$
  end while
  if not *validated* **return** false

  $i = 1$; /* the refinement loop */

$max\_iter = 1$;
while $(i \le max\_iter)$ **do**
incl_enclosure$(h, [x], T, [W_2])$;
validate_tail$([W_2], T(0), T, true)$;
$i = i + 1$;
end while;
return *validated*;
end

From Theorem 7.3 and the above algorithm we obtain immediately the following

Theorem 7.4. *Let $h > 0$, assume that $[x] \oplus T(0)$ are self-consistent bounds, $m = m(T(0))$ and $M = M(T(0))$. Assume that:*

$$\textbf{enclosure\_with\_tail}(h, [x], T(0), [W_2], T, Tisgoodinitialguess)$$

returns true. Then, for any $n > M$, $x(0) \in [x]$ and $y(0) \in P_nT(0)$, holds:

$$\varphi^n([0,h], x(0) \oplus P_n y(0)) \subset [W_2] \oplus P_n T.$$

Moreover, $[W_2] \oplus T$ are self-consistent bounds.

7.4. Computation of $T(h)$. Assume that $[x_0] \oplus T(0)$ and $[W_2] \oplus T([0,h])$ are self-consistent bounds, such that for $n > M$ holds:

$$\varphi^n([0,h], x(0) \oplus P_n T(0)) \subset [W_2] \oplus P_n T([0,h]).$$

From Lemma 6.1 it follows that

$$T(h)^{\pm}_k = (T(0)^{\pm}_k - b^{\pm}_k)e^{\lambda_k h} + b^{\pm}_k, \quad \text{for } k > m,$$

where $b \in \text{PolyBd}$ satisfies:

$$N = N([W_2], T([0,h])) \in \text{PolyBd}, \qquad \frac{N_k}{-\lambda_k} \subset b_k, \quad \text{for } m < k.$$

To enclosure $T(h)$ we proceed along the lines outlined in Section 6.3. We need to find $E = E(r, h, M)$ defined in Lemma 6.4 for the KS equation given by (3.3).

As the first step in this direction we prove the following lemma:

Lemma 7.5. *Assume $\lambda_k = -\nu k^4 + k^2$, where $\nu > 0$. Let $r, E, h \in \mathbb{R}$, $E > 0$, $h > 0$. Assume that for some $K > 0$ holds:*

$$e^{h\lambda_K} \le \frac{E}{K^r}, \quad -4h\nu K^4 + 2hK^2 + r \le 0, \quad 4\nu K^2 \ge 1. \tag{7.5}$$

Then, for any $k > K$, $e^{h\lambda_k} \leq E/k^r$.

Proof. It is enough to show that the function $f(k) = e^{h\lambda_k} k^r$ is nonincreasing for $k \geq K$. We have

$$f'(k) = \lambda'_k h e^{h\lambda_k} k^r + r e^{h\lambda_k} k^{r-1} = (\lambda'_k k h + r) k^{r-1} e^{h\lambda_k}.$$

Hence $f'(k) \leq 0$ if

$$g(k) = -4h\nu k^4 + 2hk^2 + r \leq 0.$$

We want the above condition to hold for $k \geq K$. We will show it by proving that $g'(k) \leq 0$ for $k > K$, because in view of (7.5) we know that $g(K) \leq 0$.

Observe that $g'(k) < 0$ if and only if the following condition holds:

$$4hk(-4\nu k^2 + 1) \leq 0,$$

since we are interested in $k > 0$, hence we obtain $4\nu k^2 \geq 1$, for $k \geq K$. □

We look for $C(T(h))$, such that for $k > M$ we have

$$T(h)^+_k < T(0)^+_k e^{\lambda_k h} + b^+_k \leq \frac{C(T(h))}{k^{s(b)}}.$$

To compute $C(T(h))$ we use Lemma 7.5 with $r = s(b) - s(T(0))$. We check whether $K \leq M+1$ (if this is not the case we return the *failure* message). Hence we have to verify that

$$-4h\nu(M+1)^4 + 2h(M+1)^2 + s(b) - s(T(0)) \leq 0, \tag{7.6}$$

$$4\nu(M+1)^2 \geq 1. \tag{7.7}$$

If above conditions are satisfied then we set $E = e^{h\lambda_{M+1}}(M+1)^{s(b)-s(T(0))}$. Now from (6.10) it follows that we can set

$$T^\pm_k(h) = \pm\frac{C(T(0))E + C(b)}{k^{s(b)}}.$$

Observe that with the above definition of $T(h)$ there is no guarantee that $T(h)_{k>M} \subset T_{k>M}$, hence in the final step we set $T(h) := T[0,h] \cap T(h)$.

Remark 7.6. To obtain some intuitions about conditions (7.6) and (7.7) let us consider the typical numbers for the KS equation (3.3). For example for the possible chaotic case for the KS we have $r = 2$, $v \approx \nu \approx 0.03$, $M = 3m = 36$, $h \approx 1/(2\lambda_m)$. We obtain

$$\begin{aligned} 4\nu M^2 &\approx 155.5, \\ -4h\nu M^4 + 2hM^2 + r &= -4\frac{M^4\nu}{2\nu(M/3)^4} + \frac{2M^2}{2\nu(M/3)^4} + 2 \\ &= -162 + \frac{81}{\nu M^2} + 2 \approx -158. \end{aligned}$$

So conditions (7.6) and (7.7) are satisfied with large margin.

7.5. Estimates during the time step. From the results of Section 6 (the monotonicity of the bounds) it follows that we have the following refinement of the enclosure $T = T([0,h])$, $T([0,h]) \subset T(0) \cup T(h)$.

We will use it in the section region in the computation of the bounds for the Poincaré map (see [26, Section 5]).

8. Finding a good guess for the far tail for the KS equation

We will discuss here the question: *How to obtain a good initial guess for the far tail*?

By a good guess we understand $T \in \mathrm{PolyBd}$, such that condition (6.6) in Lemma 6.1 ($T([0,h]) \subset T$) is likely to be satisfied. In this section we consider the KS equation (3.3) and we derive heuristic conditions, which will guarantee that

$$\frac{N_k([z],T)}{-\lambda_k} \subset T_k, \quad \text{for } k > M. \tag{8.1}$$

Observe that (8.1) together with condition $T(0)_k \subset T_k$ for $k > M$ (this is a minimal requirement for T being the tail enclosing evolution of $T(0)$) implies that $T([0,h])_k \subset T_k$, for $k > M$.

In this section as the result of the analysis of (8.1) we will obtain:

- the relation between possible values of M and s for T, (see condition (8.4))
- the function realizing the guess of the far tail (see Section 8.2).

The KS equation with odd and periodic boundary conditions in the Fourier domain can be written as (compare (3.3))

$$a_k' = k^2(1-\nu k^2)a_k - k(FS(k) - 2 \cdot IS(k)), \quad k = 1,2,\dots$$

where

$$\begin{aligned} FS(k) &= \sum_{n=1}^{k-1} a_n a_{k-n}, & IS(k) &= \sum_{n=1}^{\infty} a_n a_{n+k}, \\ B_k &= -FS(k) + 2IS(k), & N_k &= kB_k. \end{aligned}$$

We fix $T \in \mathrm{PolyBd}$. Let $N = N(W,T) \in \mathrm{PolyBd}$, where $W \subset \mathbb{R}^m$ is a compact set, which will not be important in the following discussion. In the sequel we assume that $C = C(T)$ and $s = s(T)$. First we need to find $D = C(N)$ (see [31, Corollary 3.7]), such that $|B_k| \leq D/k^{s-1}$.

Here we will organize the computation of D slightly differently than in [31] to get a better feeling about the dependence of D on C and s. There are also some mistakes in the printed version of [31] on page 279 in formulas for D_1 and D_2, which where derived from (correct) Lemma 3.6 in [31] (but fortunately these errors were not present in the actual programm, which was based on the correct Lemma 3.6).

First, we seek the bounds for $FS(k)$ and $IS(k)$:

$$|FS(k)| \leq \frac{D_1}{k^{s-1}}, \quad |IS(k)| \leq \frac{D_2}{k^{s-1}}$$

and then we obtain

$$|B_k| \leq \frac{D_1 + 2D_2}{k^{s-1}}.$$

The following lemmas has been proven in [31]:

LEMMA 8.1 ([31, Lemma 3.4]). *Let $M < k \leq 2M$. Then*

$$FS(k) \subset 2 \sum_{k-M \leq n < k/2} a_n a_{k-n} + e(k) a_{k/2}^2 + 2C \sum_{n=1}^{k-M-1} \frac{|a_n|}{(k-n)^s}[-1,1],$$

where $e(k) = 1$ if k is even and $e(k) = 0$ when k is odd.

LEMMA 8.2 ([31, Lemma 3.5]). *Let $k > 2M$. Then*

$$FS(k) \subset \frac{C}{k^{s-1}} \left(\frac{2^{s+1}}{2M+1} \sum_{n=1}^{M} |a_n| + \frac{C4^s}{(2M+1)^{s+1}} + \frac{C2^s}{(s-1)M^s} \right)[-1,1].$$

LEMMA 8.3 ([31, Lemma 3.6]). *Let $k > M$. Then*

$$IS(k) \subset \frac{C}{k^{s-1}(M+1)} \left(\frac{C}{(M+1)^{s-1}(s-1)} + \sum_{n=1}^{M} |a_n| \right)[-1,1].$$

Let us set

$$\begin{aligned} D_1(k \leq 2M) &= \max\{k^{s-1}|FS(k)|,\ M < k \leq 2M\}, \\ D_1(k > 2M) &= C\left(\frac{2^{s+1}}{2M+1} \sum_{n=1}^{M} |a_n| + \frac{C4^s}{(2M+1)^{s+1}} + \frac{C2^s}{(s-1)M^s} \right). \end{aligned}$$

From the above lemmas it follows immediately that

$$\begin{aligned} D_1 &= \max(D_1(k > 2M), D_1(k \leq 2M)), \\ D_2 &= \frac{C}{(M+1)} \left(\frac{C}{(M+1)^{s-1}(s-1)} + \sum_{n=1}^{M} |a_n| \right). \end{aligned}$$

8.1. Dependence of D_i on C and s. We will make several assumptions regarding the candidate tail (these numbers are typical for attracting periodic orbits for $\nu = 0.1 - 0.127$)

$$\sum_{n=1}^{M} \sup_{W \oplus T} |a_n| \approx A \approx 2,$$

$$C < 10^{15}, \quad s \approx 12, \quad M \approx 40, \quad \frac{C}{M^s} < 10^{-8}.$$

Consider first D_2. It is easy to see that the term linear in C is dominant, namely

$$\left(\sum_{n=1}^{M}|a_n|\right)\Big/\left(\frac{C}{(M+1)^{s-1}(s-1)}\right)\approx\frac{A(M+1)^{s-1}(s-1)}{C}$$
$$<\frac{2\cdot 40^{10}\cdot 10}{10^{15}}=2^{21}10^{-4}\approx 2\cdot 10^{6}\cdot 10^{-4}=200.$$

Hence we have

$$D_2\approx\frac{A}{M+1}\cdot C<\frac{C}{10}.$$

Remark 8.4. D_2 appears to depend linearly on C. The dependence on s appears to be insignificant.

Now we take a closer look at $D_1(k>2M)$ observe first that the third term is considerably larger than the second one. Namely we have

$$\frac{C4^s}{(2M+1)^{s+1}}<\frac{C2^s}{M^s(2M+1)}=\frac{s-1}{2M+1}\cdot\frac{C2^s}{(s-1)M^s}.$$

The first term is dominating the other two. Namely we have

$$\frac{2^{s+1}/(2M+1)\sum_{n=1}^{M}|a_n|}{C2^s/((s-1)M^s)}\approx\frac{2A(s-1)}{(2M+1)C/M^s}\approx 10^8.$$

Hence we obtain the following:

Remark 8.5. It appears that

$$D_1(k>2M)\approx\frac{2^{s+1}A}{2M+1}C\approx 2^{12}C.$$

Moreover, it is also clear that $D_1(k>2M)$ is several orders of magnitude larger than D_2. There is also the significant dependence on s.

The expression for $D_1(k\leq 2M)$ appears to be more difficult to analyze. For further discussion let us define

$$\begin{aligned}FS_1(k)&=\left|2\sum_{k-M\leq n<k/2}a_na_{k-n}+e(k)a_{k/2}^2\right|,\\ FS_C(k)&=2C\sum_{n=1}^{k-M-1}\frac{|a_n|}{(k-n)^s}.\end{aligned}$$

We have

$$|FS(k)|\leq FS_1(k)+FS_C(k).$$

Let us make a few observations:

- $FS_1(k)$ contains the largest number of terms for $k=M+1$, hence we expect that it achieves is the maximum value for $k=M+1$. It is much less obvious where the maximum for $k^{s-1}FS_1(k)$ will be, but in my experiments it turns out that it is achieved also for $k=M+1$,

- in the term $FS_C(k)$ the number of terms increases with k, but since there are only a few dominating modes (say d) we can approximate $FS_C(k)$ and $k^{s-1}FS_C(k)$ for $k > M-1+d$ by

$$\begin{aligned} FS_C(k) &\approx \frac{2C}{k^s}\left(\sum |a_n|\right), \\ k^{s-1}FS_C(k) &\approx \frac{2C}{k}\left(\sum |a_n|\right) \le \frac{2A}{M+1}C. \end{aligned}$$

From above considerations it follows that

$$D_1(k \le 2M) \approx (M+1)^{s-1}FS_1(M+1) + \frac{2A}{M+1}C.$$

Summarizing, it appears that

$$D(C,s) \approx \max\left(D_1(k \le 2M), \frac{2^{s+1}A}{2M+1}C\right), \tag{8.2}$$

hence, for large C,

$$D(C,s) \approx \frac{2^{s+1}A}{2M+1}C. \tag{8.3}$$

Consider now isolation equation (8.1) for $k > M$

$$\frac{C}{k^s} \ge \frac{D}{k^{s-2}k^4(\nu - 1/k^2)}.$$

An easy computation shows that it is equivalent to

$$k^2\left(\nu - \frac{1}{k^2}\right)C \ge D, \quad k > M.$$

It turns out that it is enough to check the above inequality for $k = M+1$, only. Hence we obtain

$$(M+1)^2\left(\nu - \frac{1}{(M+1)^2}\right)C \ge D.$$

After using (8.3) we obtain

$$\frac{(M+1)^2(2M+1)(\nu - 1/(M+1)^2)}{A} \ge 2^{s+1}. \tag{8.4}$$

8.2. The function for generation of the initial guess. It appears that equation (8.4) should serve as the basic test, whether the enclosure is possible with the given values of s and M, because it guarantees that a "large" enclosure for the far tail always exists. In this situation it is enough to increase C in some geometric fashion until we enter into the linear regime for $D(C)$.

It is also easy to compute C_L, where the linear regime approximately begin. From (8.2) we obtain

$$C_L = \frac{D_1(k \le 2M)\cdot(2M+1)}{2^{s+1}A}. \tag{8.5}$$

The above considerations lead to the following procedure for guessing the enclosure for the far tail.

Guess of the enclosure for the tail

Input: $[x] \subset \mathbb{R}^m$, $T(0) \in \mathrm{PolyBd}$

Output: $T \in \mathrm{PolyBd}$, this is a candidate for $T([0,h])$.

0. $m(T) = m(T(0))$, $M(T) = M(T(0))$,
1. Computation of A, $D_1(k \leq 2M)$,
2. Computation of $s(T)$. We seek the largest integer, $s_{\max}$, such that condition (8.4) holds and $s_{\max} \leq s(T(0))$. We set $s(T) = s_{\max}$.
3. Computation of $C(T)$. We take the maximum of $C(T(0))$ and $3C_L$, where C_L is given by (8.5).

Warning. If we start with an empty near tail and we just evaluate A and D_1 on the initial condition, then we end up with $D_1(k \leq 2M) = 0$, which leads $C_L = c(T) = 0$. But even in this case a correct value of $s(T)$ is generated, and while this is really not a good guess since the tail is empty, the enclosure function works, because the update step in function **validate_tail** produces a new candidate, T, such that $C(T) \neq 0$.

8.4. Other equations. The analysis presented in this section was restricted to the KS equation (1.8) and used heavily the fact that N was quadratic. But it is quite obvious that this approach could be generalized to a general polynomial function N. Observe that in this case we should obtain the following expression

$$|N_k| \leq \frac{CA_1 + C^2A_2 + \ldots + C^pA_p}{k^{s-r}}$$

where A_i are functions of M, s. Just as in the case of the KS equation the functions A_i for $i > 1$ will contain positive powers M in the denominator, hence for bounded set of possible C the terms A_iC^i for $i > 1$ could be made as small as need by taking sufficiently large M and then we are in the situation already considered for the KS equation earlier in this section.

9. Computation of the Poincaré map

The goal of this section is to discuss the question of the computation of the Poincaré map for (3.1).

We fix parameters m and M for all self-consistent bounds appearing in the sequel.

To compute the Poincaré map we need the estimates for the trajectory during the time step. As such estimates one can use the rough enclosure obtained in Part 1 of the algorithm, but these estimates are usually too crude and can be easily improved. For the tail this was discussed in Section 7.5 and for the main variables ($x \in X_m$) the procedure is described in Section 6.7 in [29].

Consider a sequence $0 = t_0 < t_1 < \ldots < t_N$ and let $h_i = t_i - t_{i-1}$ be the corresponding time steps. Assume that we apply our algorithm for the rigorous integration of (3.1) to some initial condition $X_0 \oplus T_0$ using the time steps h_i. To facilitate the further discussion we introduce the following notation:

- by $\widehat{\varphi}(t_i, X_0 \oplus T_0)$ we will denote the result of i-th iteration of our algorithm for the sequence of time steps $h_1, \ldots, h_i$,
- for any $h > 0$ and the self-consistent polynomial bounds V by $\widehat{\varphi}([0,h], V)$ we will denote the enclosure for $\varphi([0,h], V)$ obtained by our algorithm.

Using the above conventions we have, for any $n > M$ and $x_0 \in X_0 \oplus T_0$,

$$\begin{aligned} \varphi^n(t_i, P_n(x_0)) &\in \widehat{\varphi}(t_i, X_0 \oplus T_0), && i = 1, \ldots, N, \\ \varphi^n([t_{i-1}, t_i], P_n(x_0)) &\subset \widehat{\varphi}([0, h_i], \widehat{\varphi}(t_{i-1}, X_0 \oplus T_0)), && i = 1, \ldots, N. \end{aligned}$$

Extending in a natural way the above notation we set

$$\widehat{\varphi}([t_i, t_{i+k}], X_0 \oplus T_0) = \bigcup_{l=1}^{k} \widehat{\varphi}([0, h_{i+l}], \widehat{\varphi}(t_{i+l-1}, X_0 \oplus T_0)).$$

We have, for any $n > M$,

$$\varphi^n([t_i, t_{i+k}], P_n(X_0 \oplus T_0)) \subset \widehat{\varphi}([t_i, t_{i+k}], X_0 \oplus T_0).$$

In the computer assisted proofs of the existence of periodic orbits we consider the Poincaré maps for all Galerkin projections P_n of (3.1) for $n > M$. Moreover, we want to obtain such bounds in a single application of the algorithm as in Theorem 7.4. For this purpose we will always define the section of (3.1) in terms of X_m.

Let $\alpha: X_m \to \mathbb{R}$ be a C^1-function. We define the section $\theta \subset H$ as follows:

- $\theta = \{x \in H \mid \alpha(P_m x) = 0\}$,
- $P_m(\theta)$ is a submanifold in X_m of the codimension one.

In our computation for the KS equation we always use α linear (affine). For the purpose of the computation of the Poincaré map we need to add some transversality condition with respect to (3.1). But since the vector field defined by (3.1) might be not defined on θ, we rather formulate an easy theorem containing the transversality condition as an assumption, which has to be verified during the execution of the algorithm.

In paper [29] we considered the Poincaré map on section θ, denoted there by $\mathcal{G}_\theta$, as a multivalued map defined on $P_m(\theta)$ with values in $P_m(\theta)$. Here we will rather treat $\mathcal{G}_\theta$ be as a multivalued map with both the domain and the range being infinite dimensional.

Definition 9.1. Consider (3.1) and the section θ. For $n > M$ let us denote the Poincaré map for φ^n by $G_{n,\theta}$. Then we define the Poincaré map $\mathcal{G}_\theta$ as follows:

$$\begin{gathered} x \in \operatorname{dom} \mathcal{G}_\theta \text{if and only if } P_n x \in \operatorname{dom} G_{n,\theta} \quad \text{for all } n > M, \\ \mathcal{G}_\theta(x) = \text{convex hull}(\{G_{n,\theta}(P_n(x)) \mid n > M\}), \quad x \in \operatorname{dom} G_\theta. \end{gathered}$$

For two sections θ_1 and θ_2 analogously define $G_{n,\theta_1\to\theta_2}$ and $\mathcal{G}_{\theta_1\to\theta_2}$. In this notation we have $\mathcal{G}_\theta = \mathcal{G}_{\theta\to\theta}$.

Theorem 9.2. *Consider* (3.1). *Let $X_0 \oplus T_0$ be self-consistent bounds, such that there exists N, a sequence of real numbers t_i for $i = 1,\dots,N$ and two sequences of self-consistent bounds $X_i \oplus T_i$ for $i = 1,\dots,N$ and $W_i \oplus V_i$ for $i = 1,\dots,N-1$ such that, for $0 < t_1 < \dots < t_N$,*

$$\widehat{\varphi}(t_1, X_0 \oplus T_0) \subset X_1 \oplus T_1, \tag{9.1}$$

$$\widehat{\varphi}([0, t_{i+1} - t_i], X_i \oplus T_i) \subset W_i \oplus V_i, \qquad i = 1,\dots,N-1, \tag{9.2}$$

$$\widehat{\varphi}(t_{i+1} - t_i, X_i \oplus T_i) \subset X_{i+1} \oplus T_{i+1}, \quad i = 1,\dots,N-1. \tag{9.3}$$

Assume that $\alpha(X_1) < 0$, $\alpha(X_N) > 0$ and

$$\nabla\alpha(P_m(x)) \cdot P_m F(x) > 0, \quad \textit{for all } x \in \bigcup_{i=1}^{N-1} W_i \oplus V_i,$$

then, for any $n > M$ and any $x \in P_n(X_0 \oplus V_0)$, there exists a uniquely defined $t^n_\theta(x)$, such that $t_1 < t^n_\theta(x) < t_N$ and $\varphi^n(t^n_\theta(x), x) \in \theta$. Moreover, the map $t_{n,\theta}: P_n(X_0 \oplus T_0) \to \mathbb{R}$ is continuous. Consequently, the map $G_{n,\theta}: P_n(X_0 \oplus T_0) \to P_n\theta$, given by $G_{n,\theta}(x) = \varphi^n(t_{n,\theta}(x), x)$ is well defined and continuous and

$$G_{n,\theta}(P_n(X_0 \oplus T_0)) \subset P_n\biggl(\theta \cap \bigcup_{i=1}^{N-1} W_i \oplus V_i\biggr).$$

Proof. From conditions (9.1)–(9.3) it follows that

$$\begin{aligned} \varphi^n(t_1, P_n(X_0 \oplus T_0)) &\subset X_1 \oplus T_1, \\ \varphi^n([0, t_{i+1} - t_i], P_n(X_i \oplus T_i)) &\subset W_i \oplus V_i, \qquad i = 1,\dots,N-1, \\ \varphi^n(t_{i+1} - t_i, P_n(X_i \oplus T_i)) &\subset X_{i+1} \oplus T_{i+1}, \quad i = 1,\dots,N-1. \end{aligned}$$

To finish the proof observe that

$$\frac{d\alpha \circ \varphi^n(t,x)}{dt} = \nabla\alpha(P_m(\varphi^n(t,x))) \cdot P_m F(\varphi^n(t,x)) > 0. \qquad \square$$

In the context of Theorem 9.2 the algorithm computing $\mathcal{G}_\theta$ will give

$$\mathcal{G}_\theta(X_0 \oplus T_0)) \subset \widehat{\mathcal{G}_\theta}(X_0 \oplus T_0) = \theta \cap \bigcup_{i=1}^{N-1} W_i \oplus V_i, \tag{9.4}$$

where by $\widehat{\mathcal{G}_\theta}$ we denote the bounds for $\mathcal{G}_\theta$ computed by our algorithm.

We also introduce the following:

Definition 9.3. Same assumptions as in Theorem 9.2. We define the *transition time* $t_{\mathcal{G}_\theta}$ by $t_{\mathcal{G}_\theta} = (t_1, t_N)$.

The extension of Theorem 9.2 and Definition 9.3 to maps obtained as the transition between sections θ_1 and θ_2 is straightforward and is left to the reader.

An important issue in this context is the realization of the intersection appearing in formula (9.4). In our implementation θ is always linear. We have found the following approach to be the most efficient: we introduce a new coordinate system (an affine transformation) in X_m, such that if $(z_1, \dots, z_t)$ denote the new coordinates, then $\theta = \{z_1 = 0\}$. We will refer to these coordinates as *the section coordinates*. Moreover, if we are close to the section (in the section region), then we express all enclosures $W_i \oplus V_i$ in these coordinates. This means that formulas for rigorous estimates during the time steps from [29, Section 6.7] have to be evaluated directly in the section coordinates. In this situation the intersection (9.4) is just a projection onto $(z_2, \dots, z_t)$ of all sets $V_i \oplus V_i$.

This approach has also an another advantage. When one uses the Brouwer Theorem to prove the existence of a fixed point for the smooth map P (Poincaré map) one needs, B, a set homeomorphic to a ball such that $P(B) \subset B$. Usually the shape of B has to be carefully chosen. Assume that x_0 is a good approximation of this fixed point and $v_1, \dots, v_n$ are approximate eigenvectors of $dP(x_0)$. Then good candidate set for B is given by

$$B = \left\{ x_0 + \sum_{i=1}^{n} a_i v_i \;\middle|\; a_i \in [-\delta_i, \delta_i] \right\},$$

for some $\delta_i > 0$ for $i = 1, \dots, n$.

It is then desirable to express the computed value of $P(B)$ directly using the linear coordinate frame induced by $v_1, \dots, v_n$.

10. Periodic orbits for the KS equation – topological theorems

In this section and the following ones we report on the computer assisted proofs of the existence of multiple periodic orbits for the KS equation (1.8) with periodic and odd boundary conditions (1.9). These orbits are obtained using the algorithm for the rigorous integration of dissipative PDEs described in earlier sections.

In the Fourier domain the system (1.8)–(1.9) is given by (3.3) and has the reflectional symmetry R, which acts as follows:

$$a_{2k} \to a_{2k}, \quad a_{2k+1} \to -a_{2k+1}, \quad k \in \mathbb{Z}_+.$$

We consider $\nu \in [0.02991, 0.128]$. For the description of various periodic orbits and Silnikov connections, indicating the existence of the chaotic dynamics for $\nu \in (0.111, 0.133)$, the reader is referred to [8] and the literature cited there. One should be aware that in [8] the KS equation is written in a different form and the parameter $\alpha = 4/\nu$ is used. Let us focus on the periodic orbits branch denoted in [8] by γ_{Hopf}. This branch, consisting of R-symmetric attracting periodic orbits, bifurcates off the positive bimodal fixed point branch for $\nu \approx 0.13254$. As ν decreases the branch γ_{Hopf} undergoes the period doubling bifurcation at $\nu \approx 0.1223$ losing its stability, which is inherited by an asymmetric periodic orbit. Along the branch γ_{Hopf} we proved the existence of periodic orbits, both stable and unstable ones, respectively before and past the period doubling bifurcation. We proved also the existence of an orbit on the non-symmetric branch bifurcating from γ_{Hopf}.

Other periodic orbits, whose existence is proven in this paper are unrelated to γ_{Hopf} and were chosen, with the objective to be in the chaotic region [3] or close to it.

To prove the existence of periodic orbits, which in numerical simulations appears to be attracting, a Brouwer-type theorem was used – see Section 10.1. For the apparently unstable orbits we use the covering relations [30] and the Miranda Theorem [15] – see Section 10.2.

10.1. Brouwer-type existence theorem. We fix parameters m and M, and we assume that these parameters are used for all self-consistent bounds appearing in the computations.

Theorem 10.1. *Consider* (3.1), *assume that conditions* (1.2)–(1.4) *hold. Let* $s_0 = d + p + 1$. *Let* θ *be a section. Assume that there exists set* $B \oplus T_0 \subset P_m(\theta) \oplus Y_m$, *such that*

(a) $B \oplus T_0$ *are self-consistent bounds,*
(b) B *is homeomorphic to* $(m-1)$*-dimensional closed ball,*
(c) $\mathcal{G}_{\theta\to\theta}(B \oplus T_0) \subset B \oplus T_0$,
(d) *there exists* $a, b > 0$, *such that* $a < t < b$ *for all* $t \in t_{\mathcal{G}_{\theta\to\theta}}(B \oplus T_0)$,
(e) *for all* $0 < t < t_{\mathcal{G}_{\theta\to\theta}}(B \oplus T_0)$ *holds* $\varphi(t, B \oplus T_0) \cap \theta = \emptyset$.

Then there exists $t^* \in t_{\mathcal{G}_{\theta\to\theta}(B\oplus T_0)}$, $u\colon \mathbb{R} \to H$ *a solution of* (3.1), *such that* $u(0) \in B \oplus T_0$ *and* $u(t^*) = u(0)$ (*hence* $u(t)$ *is periodic*). *Moreover, if all self-consistent bounds used in the computation of* $\mathcal{G}_{\theta\to\theta}(B \oplus T_0)$ *were polynomial bounds with* $s \geq s_0$, *then* u *defines a classical solution of* (1.1).

Proof. Let $t_{n,\theta\to\theta}(x)$ be the Poincaré return time to section $P_n(\theta)$ for $x \in X_n$ for φ^n. From Theorem 9.2 we obtain for $n > M$

$$t_{n,\theta\to\theta}(P_n(B \oplus T_0)) \subset t_{\mathcal{G}_{\theta\to\theta}}, \quad G_{n,\theta\to\theta}(P_n(B \oplus T_0)) \subset P_n(B \oplus T_0).$$

From the Brouwer Theorem applied to $G_{n,\theta\to\theta}$ on $P_n(B\oplus T_0)$ for each $n > M$ we obtain a periodic orbit for n-th Galerkin projection. Let us denote this orbit by u^n. We have $u^n:\mathbb{R}\to X_n$ and t^n, such that $u(0)=u(t^n)\in P_n(B\oplus T_0)$, $t^n\in t_{\mathcal{G}_{\theta\to\theta}}$. By picking up a subsequence we can assume that $t^n\to t^*$.

Let $t_{\max}=right(t_{\mathcal{G}_{\theta\to\theta}})$. Observe that the set $\varphi([0,t_{\max}],B\oplus T_0)$ is a finite sum of self-consistent bounds (one for each time step), hence from Lemma 2.8, it follows that we can pick up in (u^n) a convergent subsequence, which is converging to u^* a solution of (3.1). It is easy to see (see the proof of Theorem 8 in [29]) that u^* is periodic of period t^*.

The assertion regarding the classical solution is an immediate consequence of Theorem 3.9. □

To obtain orbits with the reflectional symmetry, R, we will use the following obvious modification of Theorem 10.1.

THEOREM 10.2. *Consider* (3.1), *assume that conditions* (1.2)–(1.4) *hold. Let* $s_0=d+p+1$. *Assume that there exists a symmetry* $R:H\to H$, *such that* $R(\mathrm{dom}\,(F))=\mathrm{dom}\,(F)$ *and* $F\circ R=R\circ F$ *on* $\mathrm{dom}\,(F)$. *Let* θ *be a section. Assume that there exists set* $B\oplus T_0\subset P_m(\theta)\oplus Y_m$, *such that*

(a) $B\oplus T_0$ *are self-consistent bounds,*
(b) B *is homeomorphic to* $(m-1)$*-dimensional closed ball,*
(c) $R\circ\mathcal{G}_{\theta\to R\theta}(B\oplus T_0)\subset B\oplus T_0$,
(d) *there exists* $a,b>0$ *such that* $a<t<b$ *for all* $t\in t_{\mathcal{G}_{\theta\to R\theta}}(B\oplus T_0)$,
(e) *for all* $0<t<t_{\mathcal{G}_{\theta\to R\theta}}(B\oplus T_0)$ *holds* $\varphi(t,B\oplus T_0)\cap R\theta=\emptyset$.

Then there exists $t^*\in t_{\mathcal{G}_{\theta\to R\theta}(B\oplus T_0)}$, $u:\mathbb{R}\to H$ *a solution of* (3.1), *such that* $u(0)\in B\oplus T_0$ *and* $u(t^*)=Ru(0)$, *hence* u *is* R*-symmetric periodic orbit. Moreover, if all self-consistent bounds used in the computation of* $\mathcal{G}_{\theta\to\theta}(B\oplus T_0)$ *were polynomial bounds with* $s\geq s_0$, *then* u *defines a classical solution of* (1.1).

10.2. Covering relations and the Miranda Theorem. The notion of the covering relation was introduced in papers [23], [24]. Here we follow the most recent and the most general version introduced in [30] and the reader is referred there for proofs.

DEFINITION 10.3. A h-set, N, is the object consisting of the following data

(a) $|N|$ – a compact subset of $\mathbb{R}^n$, a support of N,
(b) $u(N),s(N)\in\{0,1,\dots\}$, such that $u(N)+s(N)=n$,
(c) a homeomorphism $c_N:\mathbb{R}^n\to\mathbb{R}^n=\mathbb{R}^{u(N)}\times\mathbb{R}^{s(N)}$, such that

$$c_N(|N|)=\overline{B_{u(N)}}(0,1)\times\overline{B_{s(N)}}(0,1).$$

We set

$$\begin{aligned} N_c &= \overline{B_{u(N)}}(0,1) \times \overline{B_{s(N)}}(0,1), & N^- &= c_N^{-1}(N_c^-), \\ N_c^- &= \partial\overline{B_{u(N)}}(0,1) \times \overline{B_{s(N)}}(0,1), & N^+ &= c_N^{-1}(N_c^+), \\ N_c^+ &= \overline{B_{u(N)}}(0,1) \times \partial\overline{B_{s(N)}}(0,1). \end{aligned}$$

Hence a h-set, N, is a product of two closed balls in some coordinate system. The numbers, $u(N)$ and $s(N)$, stand for the dimensions of nominally unstable and stable directions, respectively. The subscript c refers to the new coordinates given by homeomorphism c_N. Usually we will identify the h-set with its support. According to this convention $|N| = N$.

For the unstable periodic orbits for the KS-equation considered in this paper it is enough to consider h-sets with $u = 1$, so we have only one nominally expanding direction. This restriction enables us to give sufficient conditions for the existence of covering relations, which are easy to verify.

Definition 10.4. Let N be a h-set, such that $u(N) = 1$. We set

$$\begin{aligned} N_c^{le} &= \{-1\} \times \overline{B_{s(N)}(0,1)}, & S(N)_c^l &= (-\infty,-1) \times \mathbb{R}^{s(N)}, \\ N_c^{re} &= \{1\} \times \overline{B_{s(N)}(0,1)}, & S(N)_c^r &= (1,\infty) \times \mathbb{R}^{s(N)}. \end{aligned}$$

We define

$$\begin{aligned} N^{le} &= c_N^{-1}(N_c^{le}), & S(N)^l &= c_N^{-1}(S(N)^l), \\ N^{re} &= c_N^{-1}(N_c^{re}), & S(N)^r &= c_N^{-1}(S(N)^r). \end{aligned}$$

We will call N^{le}, N^{re}, $S(N)^l$ and $S(N)^r$ the left edge, the right edge, the left side and right side of N, respectively.

It is easy to see that $N^- = N^{le} \cup N^{re}$.

We will not recall here the definition of covering relation in full generality, as we restrict ourselves to the case of $u = 1$, and will reformulate Theorem 16 from [30] as the definition.

Definition 10.5. Let N, M be two h-sets in $\mathbb{R}^n$, such that $u(N) = u(M) = 1$ and $s(N) = s(M) = s = n - 1$. Let $f\colon |N| \to \mathbb{R}^n$ be continuous. We say that N f-covers M with degree $w = \pm 1$, denoted by

$$N \overset{f,w}{\Longrightarrow} M,$$

if there exists $q_0 \in \overline{B}_s(0,1)$, such that the following conditions are satisfied

$$\begin{aligned} f(c_N([-1,1] \times \{q_0\})) &\subset \operatorname{int}(S(M)^l \cup |M| \cup S(M)^r), \\ f(|N|) \cap M^+ &= \emptyset, \end{aligned}$$

and one of the following two conditions holds:

$$f(N^{le}) \subset S(M)^l \quad \text{and} \quad f(N^{re}) \subset S(M)^r, \tag{10.1}$$

$$f(N^{le}) \subset S(M)^r \quad \text{and} \quad f(N^{re}) \subset S(M)^l. \tag{10.2}$$

$w = 1$ if condition (10.1) is satisfied and $w = -1$ if condition (10.2) holds.

Quite often we will drop w in the symbol of the covering relation.

REMARK 10.6. A usual picture of a h-set on the plane with $u(N) = s(N) = 1$ is given in Figure 1. A typical picture illustrating covering relation on the plane with one "unstable" direction is given on Figure 2.

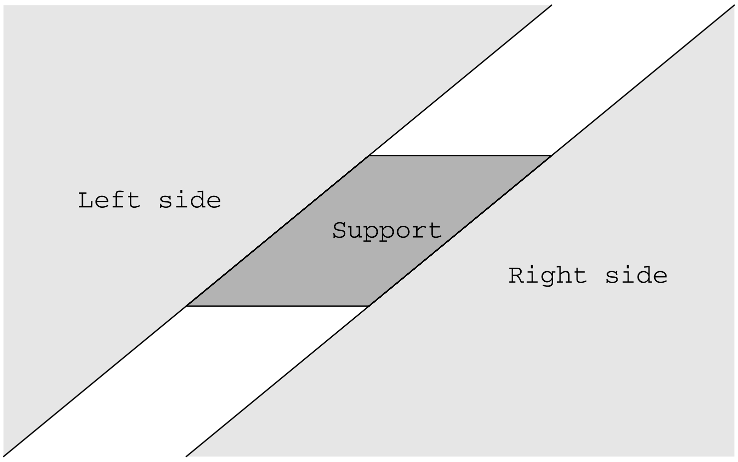

FIGURE 1. An example of an h-set on the plane

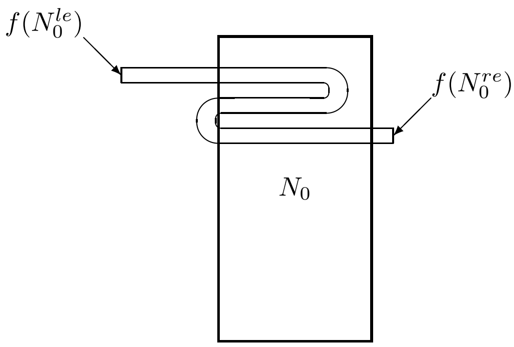

FIGURE 2. An example of an f−covering relation: $N_0 \overset{f,1}{\Longrightarrow} N_0$

From Theorem 9 in [30] we immediately obtain the following Miranda Theorem [15].

Theorem 10.7. *If* $N \overset{f,w}{\Longrightarrow} N$, *then there exists* $x \in \operatorname{int} N$ *such that* $f(x) = x$.

In the context of computation of the Poincaré map for (2.1) we have the following easy

Lemma 10.8. *Assume that* $B_1 \oplus T_1$, $B_2 \oplus T_2$ *are self-consistent bounds and* B_1, B_2 *are* $(m-1)$*-dimensional h-sets.* $M = M(T_1) = M(T_2)$ *and* $u(B_1) = u(B_2) = 1$. *Assume that*

$$\mathcal{G}_{\theta_1 \to \theta_2}(B_1 \oplus T_1) \subset (\operatorname{int} S(B_2)^l \cup B_2 \cup S(B_2)^r) \oplus \prod_{k>m} (T^-_{2,k}, T^+_{2,k})$$

and one of the following two conditions holds:

$$\begin{aligned} &P_m \mathcal{G}_{\theta_1 \to \theta_2}(B_1^{le} \oplus T_1) \subset S(B_2)^l \quad \text{and} \quad P_m \mathcal{G}_{\theta_1 \to \theta_2}(B_1^{re} \oplus T_1) \subset S(B_2)^r, \\ &P_m \mathcal{G}_{\theta_1 \to \theta_2}(B_1^{le} \oplus T_1) \subset S(B_2)^r \quad \text{and} \quad P_m \mathcal{G}_{\theta_1 \to \theta_2}(B_1^{re} \oplus T_1) \subset S(B_2)^l. \end{aligned}$$

Then, for every $n > M$,

$$B_1 \oplus P_n(T_1) \overset{G_{n,\theta_1 \to \theta_2}}{\Longrightarrow} B_2 \oplus P_n(T_2).$$

Definition 10.9. We say that

$$B_1 \oplus T_1 \overset{\mathcal{G}_{\theta_1 \to \theta_2}}{\Longrightarrow} B_2 \oplus T_2$$

if $B_1 \oplus T_1$, $B_2 \oplus T_2$ satisfy assumptions of Lemma 10.8 for the map $\mathcal{G}_{\theta_1 \to \theta_2}$.

Now we are ready to state

Theorem 10.10. *Consider* (3.1), *assume that conditions* (1.2)–(1.4) *hold. Let* $s_0 = d + p + 1$. *Let* θ *be a section. Assume that there exists set* $B \oplus T \subset P_m(\theta) \oplus Y_m$, *such that*

(a) $B \oplus T$ *are self-consistent bounds,*
(b) B *is an* $(m-1)$*-dimensional h-set,*
(c) $B \oplus T_0 \overset{\mathcal{G}_{\theta \to \theta}}{\Longrightarrow} B \oplus T_0$,
(d) *there exists* $a, b > 0$, *such that* $a < t < b$ *for all* $t \in t_{\mathcal{G}_{\theta \to \theta}}(B \oplus T_0)$,
(e) *for all* $0 < t < t_{\mathcal{G}_{\theta \to \theta}}(B \oplus T_0)$ *holds* $\varphi(t, B \oplus T_0) \cap \theta = \emptyset$.

Then there exists $t^* \in t_{\mathcal{G}_{\theta \to \theta}(B \oplus T_0)}$, $u \colon \mathbb{R} \to H$ *a solution of* (3.1), *such that* $u(0) \in B \oplus T_0$ *and* $u(t^*) = u(0)$. *Moreover, if all self-consistent bounds used in the computation of* $\mathcal{G}_{\theta \to \theta}(B \oplus T_0)$ *were polynomial bounds with* $s \geq s_0$, *then* u *defines a classical periodic solution of* (1.1).

Proof. For each n the existence of periodic orbit for n-th Galerkin projection follows directly from Lemma 10.8 and Theorem 10.7. Then we continue as in the proof of Theorem 10.10. $\square$

To obtain orbits with the reflectional symmetry, R, we will use the following modification of Theorem 10.1.

THEOREM 10.11. *Consider* (3.1), *assume that conditions* (1.2)–(1.4) *hold. Let* $s_0 = d + p + 1$. *Assume that there exists a symmetry* $R: H \to H$, *such that* $R(\mathrm{dom}\,(F)) = \mathrm{dom}\,(F)$ *and* $F \circ R = R \circ F$ *on* $\mathrm{dom}\,(F)$. *Let* θ *be a section. Assume that there exists set* $B \oplus T \subset P_m(\theta) \oplus Y_m$, *such that*

(a) $B \oplus T$ *are self-consistent bounds,*
(b) B *is an* $(m-1)$*-dimensional* h*-set,*
(c) $B \oplus T_0 \overset{R\circ\mathcal{G}_{\theta\to R\theta}}{\Longrightarrow} B \oplus T_0$,
(d) *there exists* $a, b > 0$ *such that* $a < t < b$ *for all* $t \in t_{\mathcal{G}_{\theta\to R\theta}}(B \oplus T_0)$,
(e) *for all* $0 < t < t_{\mathcal{G}_{\theta\to R\theta}}(B \oplus T_0)$ *holds* $\varphi(t, B \oplus T_0) \cap R\theta = \emptyset$.

Then there exists $t^* \in t_{\mathcal{G}_{\theta\to R\theta}(B\oplus T_0)}$, $u: \mathbb{R} \to H$ *a solution of* (3.1), *such that* $u(0) \in B \oplus T_0$ *and* $u(t^*) = Ru(0)$, *hence* u *is* R*-symmetric periodic orbit. Moreover, if all self-consistent bounds used in the computation of* $\mathcal{G}_{\theta\to\theta}(B \oplus T_0)$ *were polynomial bounds with* $s \geq s_0$, *then* u *defines a classical solution of* (1.1).

11. The outline the computer assisted proofs of the existence of periodic orbits

In our computations we used the formulas for the Galerkin errors developed in [31], [25]. The programm was written in c++, the *gnu* compiler was used. We tested our program under Linux and Windows operating systems. The source code is available at [2]. The computations have been performed using the interval arithmetics from the CAPD package developed at the Jagiellonian University, Kraków, Poland [1]. This interval package was based on the double precision arithmetic.

The general scheme of the proof is the same as in [29]. Since we want to discuss both symmetric and non-symmetric orbits at the same time we set $R = \mathrm{id}$ for non-symmetric orbits. The proof consists of the following steps:

1. (the initialization) setting up the parameters: dimensions m and M, finding an approximate periodic orbit, choosing the section θ_1, finding suitable coordinates on θ_1,
2. the construction of initial tail T,
3. the construction of a set $N_0 \oplus T_0$, such that for attracting orbits holds:

$$R \circ \mathcal{G}_{\theta_1\to R\theta_1}(N_0 \oplus T_0) \subset N_0 \oplus T_0. \tag{11.1}$$

For unstable orbits we require that

$$R \circ \mathcal{G}_{\theta_1\to R\theta_1}(N_0 \oplus T_0) \subset \mathrm{int}\,(S(N_0)^l \cup N_0 \cup S(N_0)^r) \oplus \prod_{k>m} (T^-_{0,k}, T^+_{0,k}). \tag{11.2}$$

This step includes the rigorous integration of (3.3).

4. for unstable orbits only, the verification that one of the following conditions are satisfied

$$P_m R\circ\mathcal{G}_{\theta_1\to R\theta_1}(N_0^{le}\oplus T_0)\subset S(N_0)^l \quad\text{and}\quad P_m R\circ\mathcal{G}_{\theta_1\to R\theta_1}(N_0^{re}\oplus T_0))\subset S(N_0)^r$$

$$P_m R\circ\mathcal{G}_{\theta_1\to R\theta_1}(N_0^{le}\oplus T_0)\subset S(N_0)^r \quad\text{and}\quad P_m R\circ\mathcal{G}_{\theta_1\to R\theta_1}(N_0^{re}\oplus T_0)\subset S(N_0)^l.$$

   This step includes rigorous integration of (3.3).
5. the conclusion of the proof, an application of Theorems 10.1, 10.2, 10.10 or 10.11.

11.1. Part 1 – the initialization. We set the values of m, the time step h, the order of numerical method r and d – the number of coordinates in the diagonalization of DG as in Tables 3 and 9. We set $M = 3m$.

Starting with x_0, a good candidate for periodic orbit for m-th Galerkin projection of the KS equation, we construct the section θ_1 and *the section coordinates* as in [29, Section 5.1]. θ_1 is a linear section through x_0 orthogonal to $P_mF(x_0)$. In our proofs we have found it most efficient to choose the point x_0 on the section $\sigma = \{a_1 - a_3 = 0, \quad (a_1 - a_3)' > 0\}$. We define section θ_1 as a section perpendicular to $P_mF(x_0)$ at x_0, given as follows

$$\alpha(x) = (P_mF(x_0)|x) - (P_mF(x_0)|x_0), \quad \alpha' > 0.$$

The main difference in this part of the proof, when compared to [29], is in the choice of *the section coordinates*, previously we had used *the orthogonalized eigenvectors*, now we use *the normalized eigenvectors*.

The *normalized eigenvectors coordinates* are constructed as follows: x_0 is an approximate fixed point for the map $g = RG_{m,\theta_1\to R\theta_1}\colon \theta_1 \supset U \to \theta_1$. We introduce a new orthogonal coordinate frame such that x_0 is at the origin. The first coordinate direction is $P_mF(x_0)$. To obtain the other directions we remove from the canonical basis $\{e_i\}_{i=1,\dots,m}$ the vector e_{i_0}, such that

$$|(P_m(F(x_0) \mid e_{i_0})| = \max_{i=1,\dots,m} |(P_m(F(x_0) \mid e_i)|.$$

Next we apply the Gram–Schmidt orthogonalization procedure to the system $P_mF(x_0), e_1, \dots, e_{i_0-1}, e_{i_0+1}, \dots, e_m$. The resulting vectors define the new coordinate directions. Observe that in these coordinates the section is given by condition $x_1 = 0$. On section θ we use $(y_1, \dots, y_{m-1}) = (x_2, \dots, x_m)$ as *the temporary coordinates.*

Next, we compute nonrigorously an approximate Jacobian matrix $Dg(x_0)$ using r-th order Taylor method and the time step h. The matrix $Dg(x_0) \in \mathbb{R}^{(m-1)\times(m-1)}$ is expressed using the temporary coordinates. From the matrix $Dg(x_0)$ we extract $\widetilde{D} \in \mathbb{R}^{d\times d}$ in an upper left corner, hence

$$\widetilde{D}_{ij} = Dg(x_0)_{ij}, \quad \text{for } i, j = 1, \dots, d.$$

Next, we apply to $\widetilde{D}$ a diagonalization procedure based on the QR-decomposition algorithm [18] to obtain the approximate eigenvectors $v_1, v_2, \dots, v_d$ corresponding to approximate eigenvalues $\lambda_1, \dots, \lambda_d$, which are ordered as follows

$$|\lambda_1| \geq \dots \geq |\lambda_d|.$$

The vectors v_i are normalized as follows. Let $|v|$ be the euclidian norm. If $\lambda_i \in \mathbb{R}$, then we require $|v_i| = 1$. If we have a pair of complex eigenvalues $\lambda_{j+1} = \overline{\lambda_j}$, then eigenvectors v_j and v_{j+1} are such that

$$Dg(x_0) \cdot (v_j + i(v_{j+1})) = \lambda_j \cdot (v_j + iv_{j+1}), \qquad \max(|v_j|, |v_{j+1}|) = 1.$$

Some of the diagonalization data for a symmetric periodic orbit for $\nu = 0.127$ can be found in Tables 3 and 4 in [29].

Vectors $\{v_1, \dots, v_d\}$ define a new coordinate system on $\mathbb{R}^d$ and together with coordinates $y_{d+1}, \dots, y_{m-1}$ define the new coordinates on θ_1, such that our candidate for the fixed point is at the origin, i.e. $x_0 = 0$. We will denote these coordinates by c_i and we will call them *the section coordinates.*

11.2. Part 2 – the construction of initial tail T. The initial tail was constructed using the routine described in [29, Section 5.2]. In this routine for all periodic orbits we used the following settings: the partition parameter $p = 50$, the stretching parameter $e = 1.25$ and $n_{\mathrm{iso}} = 0$. It produced $W \oplus \prod_{k>m} [a_k^-, a_k^+]$. In the present proof we used only $T_0 = \prod_{k>m} [a_k^-, a_k^+]$.

In the proofs it turns out that this initial tail shrinks usually by a huge factor, see Tables 5, 6, 11 and 12. Essentially the role of this step was to verify that for a given value of M finding the topologically self-consistent bounds is possible, i.e. there exists $s > s_0$ such that (8.4) holds.

11.3. Part 3 – the construction of $N_0 \oplus T_0$. Our goal is to construct a set $N \subset \theta_1$, such that $N_0 \oplus T_0 \subset \operatorname{dom}(\mathcal{G}_{\theta_1 \to R\theta_1})$ and either (11.1) or (11.2) holds, respectively for attracting and unstable orbits.

We constructed N_0 as a result of the following simple algorithm (the section coordinates are used to represent sets N_0 and N_1).

Algorithm.

1. *Initialization.* We assign the values for δ, h the time step, the order of numerical method for the computation of $\mathcal{G}_{\theta_1 \to R\theta_1}$ and *iter*, the number of times the loop consisting of steps 2 and 3 described below should be executed, as in Tables 3 and 9.
   We initialize $N_0 \oplus T_0$ as follows

$$N_0 \oplus T_0 = [-\delta, \delta]^{m-1} \oplus T,$$

   where T is the initial tail obtained in Section 11.2.

2. *Computation of the Poincaré map.* We compute

$$N_1 \oplus T_1 = R \circ \mathcal{G}_{\theta_1 \to R\theta_1}(N_0 \oplus T_0).$$

If the computation is terminated successfully then we go to step 3, otherwise the execution of the algorithm is interrupted and *false* is returned.

3. If

$$N_0 \oplus T_0 \subset N_1 \oplus T_1, \tag{11.3}$$

then the execution of the algorithm is interrupted and *false* is returned. If (11.3) holds, then we continue as follows:
• For an attracting orbit we check whether

$$N_1 \oplus T_1 \subset N_0 \oplus T_0, \tag{11.4}$$

then we set $N_0 \oplus T_0 = N_1 \oplus T_1$ we go to step 4.
• In the case of an unstable orbit we check whether (11.2) holds, which in terms of $N_1 \oplus T_1$ can be expressed as follows:

$$N_1 \oplus T_1 \subset \operatorname{int}\left(S(N_0)^l \cup N_0 \cup S(N_0)^r\right) \oplus \prod_{k>m} (T^-_{0,k}, T^+_{0,k}). \tag{11.5}$$

If it holds, then we set $N_0 \oplus T_0 = N_1 \oplus T_1$ we go to step 4.
If condition (11.4) or (11.5) is not satisfied then we set $N_0 = N_1 \cap N_0$, and if T_1 is not a subset of T_0, then we define a new value for T_0 as follows:

$$T_0 = \mathrm{PolyBd}(T_0 \cup T_1), \qquad T_0 = \mathrm{inflate}(T_0, 2), \tag{11.6}$$

where by $\mathrm{PolyBd}(T_0 \cup T_1)$ we denote the smallest polynomial bounds containing $T_0 \cup T_1$ and an inflation of polynomial bounds is understood componentwise.
Next, we jump back to step 2.

4. *Further refinement.* We iterate several times the computation of the Poincare map and set $N_0 \oplus T_0 = R \circ \mathcal{G}_{\theta_1 \to R\theta_1}(N_0 \oplus T_0)$ and return *true.*

End of algorithm. Let us comment about (11.6). In principle since in each iteration set N_0 is smaller we should always obtain that $T_1 \subset T_0$, which is a natural consequence following fact:

$$\text{if } A \subset B \text{ then } f(A) \subset f(B).$$

But our algorithm does not fulfill the above condition. Hence we increase the tail for the next iteration to make sure that $T_1 \subset T_0$, which was always the case.

11.4. Part 4 – the verification of conditions for image of the boundary for unstable orbits. We compute

$$R \circ \mathcal{G}_{\theta_1 \to R\theta_1}(N_0^{le} \oplus T_0) \quad \text{and} \quad R \circ \mathcal{G}_{\theta_1 \to R\theta_1}(N_0^{re} \oplus T_0).$$

In both cases we input the whole left and right edges as the initial condition.

12. Example theorems and data from the proofs

12.1. Exemplary theorems about the attracting orbits. In this section we present two exemplary theorems about the existence of the apparently attracting periodic orbits for the KS equation. We use the phrase *apparently attracting* to highlight the fact that in numerical simulations it is clearly visible that the orbit is attracting, but we are not able to prove that.

First theorem is about the orbit with the reflectional symmetry.

THEOREM 12.1. *Let* $u_0(x) = \sum_{k=1}^{10} -2a_k \sin(kx)$*, where* a_k *are given in Table* 1*. Then, for any* $\nu \in 0.127 + [-10^{-7}, 10^{-7}]$*, there exists function* $u^*(t,x)$*, a classical solution of* (1.8)–(1.9)*, such that*

$$\|u_0 - u^*(0,\,\cdot\,)\|_{L_2} < 3.27 \cdot 10^{-3}, \qquad \|u_0 - u^*(0,\,\cdot\,)\|_{C^0} < 2.35 \cdot 10^{-3}$$

and u^* *is periodic with respect to* t *with period* $T \in 2 \cdot [1.1216, 1.1227]$ *and has the reflectional symmetry,* R*.*

$a_1 = 2.012101\mathrm{e}-001$	$a_2 = 1.289980$
$a_3 = 2.012104\mathrm{e}-001$	$a_4 = -3.778664\mathrm{e}-001$
$a_5 = -4.230936\mathrm{e}-002$	$a_6 = 4.316156\mathrm{e}-002$
$a_7 = 6.940179\mathrm{e}-003$	$a_8 = -4.156467\mathrm{e}-003$
$a_9 = -7.944708\mathrm{e}-004$	$a_{10} = 3.316085\mathrm{e}-004$

TABLE 1. Coordinates of u_0 – the approximation of the initial condition for the periodic orbit in Theorem 12.1.

The theorem below present an example of the orbit without the reflectional symmetry.

THEOREM 12.2. *Let* $u_0(x) = \sum_{k=1}^{13} -2a_k \sin(kx)$*, where* a_k *are given in Table* 2*. Then, for* $\nu = 0.1215$*, there exists a function* $u^*(t,x)$*, a classical solution of* (1.8)–(1.9)*, such that*

$$\|u_0 - u^*(0,\,\cdot\,)\|_{L_2} < 1.3 \cdot 10^{-4}, \qquad \|u_0 - u^*(0,\cdot)\|_{C^0} < 8 \cdot 10^{-5}$$

$a_1 = 2.559307\mathrm{e}-001$	$a_2 = 1.096696$
$a_3 = 2.559308\mathrm{e}-001$	$a_4 = -3.079613\mathrm{e}-001$
$a_5 = -4.780290\mathrm{e}-002$	$a_6 = 3.002048\mathrm{e}-002$
$a_7 = 7.352651\mathrm{e}-003$	$a_8 = -2.530191\mathrm{e}-003$
$a_9 = -7.561954\mathrm{e}-004$	$a_{10} = 1.624854\mathrm{e}-004$
$a_{11} = 6.833019\mathrm{e}-005$	$a_{12} = -8.789133\mathrm{e}-006$
$a_{13} = -5.429533\mathrm{e}-006$	

Table 2. Coordinates of u_0 – the approximation of the initial condition for the periodic orbit in Theorem 12.2.

and u^* is periodic with respect to t with period $T \in [3.0744, 3.0745]$. Moreover, this orbit does not have the reflectional symmetry R.

Proof. The existence was obtained using Theorem 10.1. To prove that the orbit does not have the reflectional symmetry R, we checked that $R \circ G_{\theta_1 \to R\theta_1}(N_0 \oplus T_0) \cap (N_0 \oplus T_0) = \emptyset$. This check was performed by computer. $\square$

ν	Sym	m	h	order	δ	iter	d	comp. time
0.127 ± 10^{-7}	Yes	10	$8\mathrm{e}-4$	5	$4\mathrm{e}-4$	2	9	21 sec
0.127	Yes	10	$1\mathrm{e}-3$	4	$2\mathrm{e}-4$	1	9	14 sec
0.125	Yes	11	$1\mathrm{e}-3$	5	$2\mathrm{e}-4$	2	8	25 sec
0.1215	No	13	$4\mathrm{e}-4$	6	$5\mathrm{e}-5$	2	10	300 sec
0.032	Yes	23	$1.5\mathrm{e}-4$	5	$8\mathrm{e}-5$	2	12	457 sec

Table 3. The parameters used in the proofs of the existence of apparently attracting periodic orbits for the KS equation. The *Sym* column tells whether the periodic orbit has the reflectional symmetry R, iter contains the number of iterates in the algorithm required to fulfill assumptions of Theorem 10.1 or 10.2, comp. time – the computation time for one iterate on 1.73 GHz Windows machine. d is the number coordinates in the diagonalization of the Poincaré map, iter – the number of iterates in the construction of N. In the first row the expression 0.127 ± 10^{-7} means that the whole interval $[0.127 - 10^{-7}, 0.127 + 10^{-7}]$ was inserted for ν.

In fact more is proven than it is stated in the above theorems, because any detailed information about the tail is missing in the statement. Some partial information on it is contained in Tables 5 and 6 where the far tail described by C_e/k^{s_e} and from the near tail we have data for $a^{\pm}_{m+1}$. More comprehensive data are contained in the companion files, where also the complete results of each iteration of $R \circ \mathcal{G}_{\theta_1 \to R\theta_1}$ are given.

ν	period	L_2	H_1	C^0	C^1
0.127 ± 10^{-7}	$2 \cdot [1.1216, 1.1227]$	$3.22\mathrm{e}-03$	$9.00\mathrm{e}-03$	$2.10\mathrm{e}-03$	$6.30\mathrm{e}-03$
0.127	$2 \cdot [1.1218, 1.1225]$	$1.39\mathrm{e}-03$	$3.97\mathrm{e}-03$	$9.48\mathrm{e}-04$	$3.14\mathrm{e}-03$
0.125	$2 \cdot [1.2382, 1.2386]$	$6.59\mathrm{e}-04$	$1.92\mathrm{e}-03$	$4.40\mathrm{e}-04$	$1.44\mathrm{e}-03$
0.1215	$[3.0744, 3.0745]$	$1.24\mathrm{e}-04$	$3.63\mathrm{e}-04$	$7.53\mathrm{e}-05$	$2.42\mathrm{e}-04$
0.032	$2 \cdot [0.4092, 0.4094]$	$9.59\mathrm{e}-04$	$5.89\mathrm{e}-03$	$9.46\mathrm{e}-04$	$5.86\mathrm{e}-03$

FIGURE 4. Some data from the proof of the existence of apparently attracting periodic orbits for the KS equation. The columns L_2, H_1, C^0, C^1 contain the estimate on the distance in the corresponding norm, between the center of $P_m(N \oplus T_0)$ and the periodic orbit. In the first row the expression 0.127 ± 10^{-7} means that the whole interval $[0.127 - 10^{-7}, 0.127 + 10^{-7}]$ was inserted for ν.

ν	C_i	s_i	$a^+_{i,M+1}$	C_e	s_e	$a^+_{e,M+1}$
0.127 ± 10^{-7}	$1.35\mathrm{e}+11$	12	$1.71\mathrm{e}-7$	781	12	$9.91\mathrm{e}-16$
0.127	$1.35\mathrm{e}+11$	12	$1.71\mathrm{e}-7$	761	12	$9.66\mathrm{e}-16$
0.125	$1.02\mathrm{e}+11$	12	$4.25\mathrm{e}-8$	714	13	$8.80\mathrm{e}-18$
0.1215	$6.55\mathrm{e}+10$	12	$3.90\mathrm{e}-9$	0.632	13	$9.42\mathrm{e}-22$
0.032	$5.04\mathrm{e}+15$	12	$3.64\mathrm{e}-7$	$1.19\mathrm{e}+5$	12	$8.60\mathrm{e}-18$

FIGURE 5. Some data from the proof of the existence of apparently attracting periodic orbits for the KS equation. We compare the parameters describing the far tail, at the start of the proof (subscript i) and after the proof (subscript e). In each case $M = 3m$. In the first row the expression 0.127 ± 10^{-7} means that the whole interval $[0.127 - 10^{-7}, 0.127 + 10^{-7}]$ was inserted for ν. The numbers are rounded to three significant decimal digits.

ν	$[a^-_{i,m+1}, a^+_{i,m+1}]$	$[a^-_{e,m+1}, a^+_{e,m+1}]$
0.127 ± 10^{-7}	$-3.947040\mathrm{e}-7 + 2.308717\mathrm{e}-4 * [-1, 1]$	$7.954821\mathrm{e}-5 + 1.464614\mathrm{e}-6 * [-1, 1]$
0.127	$-3.947036\mathrm{e}-7 + 2.308715\mathrm{e}-4 * [-1, 1]$	$7.960494\mathrm{e}-5 + 1.089191\mathrm{e}-6 * [-1, 1]$
0.125	$2.658711\mathrm{e}-5 + 5.525438\mathrm{e}-5 * [-1, 1]$	$-1.563315\mathrm{e}-5 + 1.874870\mathrm{e}-7 * [-1, 1]$
0.1215	$1.536957\mathrm{e}-6 + 7.212824\mathrm{e}-6 * [-1, 1]$	$2.417373\mathrm{e}-7 + 2.491650\mathrm{e}-9 * [-1, 1]$
0.032	$-8.072115\mathrm{e}-5 + 1.308558\mathrm{e}-4 * [-1, 1]$	$-1.141919\mathrm{e}-5 + 1.749282\mathrm{e}-7 * [-1, 1]$

FIGURE 6. Some data from the proof of the existence of apparently attracting periodic orbits for the KS equation. We compare the $a^{\pm}_{m+1}$ in the near tail at the start of the proof (subscript i) and after the proof (subscript e). In the first row the expression 0.127 ± 10^{-7} means that the whole interval $[0.127 - 10^{-7}, 0.127 + 10^{-7}]$ was inserted for ν. The numbers are rounded to seven significant decimal digits.

We also have proved the existence of periodic orbits for $\nu = 0.127$, $\nu = 0.125$ and $\nu = 0.032$, see companion files for more details. There were several objectives behind these choices of the parameter value. For a given value of ν the main objective always was the minimalization of the computation time, having in mind that the eventual computer assisted proof of the existence of the symbolic dynamics, will require the partition of the domain into pieces for the computation of the Poincaré map.

- $\nu = 0.127$. This is a stable symmetric orbit on γ_{Hopf}. This case was done mainly for the comparison with [29]. The speed up factor (taking into account different speed of the machines used) is around 6.
- $\nu = 0.127 + [-10^{-7}, 10^{-7}]$. The same orbit as for $\nu = 0.127$. We tried to see how much we can extend the ν-interval of the existence of periodic orbit in single computation. We expected much larger interval of the diameter around 10^{-4}, but we could not do better than 10^{-7}.
- $\nu = 0.125$. This is an stable symmetric orbit on γ_{Hopf}. This case differs from $\nu = 0.127$ as follows: here the pair of leading eigenvalues is complex $\lambda_{1,2} \approx -0.051 \pm i * 0.0725$, there they were real. This case was the test for the choice of the good coordinates in case of complex eigenvalues.
- $\nu = 0.1215$. This is a non-symmetric stable periodic orbit on branch bifurcating from γ_{Hopf}. Contrary to all other cases this one required the rigorous integration of the full Poincaré map. Using the previous approach from [29] this was impossible. It turns out also that the leading eigenvalue is complex and is approximately equal to $-0.041 \pm i * 0.312$.
- $\nu = 0.032$. This is an stable symmetric orbit. This is the parameter value very close to the range $\nu \approx 0.0291$, where the chaotic dynamics was numerically observed in [3]. This computation required $m = 23$ and resulted in the longest computation time per iterate (see Table 3).

12.2. Two exemplary theorems about unstable orbits. Below we present two exemplary theorems on the existence of apparently unstable periodic orbits with and without the reflectional symmetry. We use the phrase *apparently unstable* to highlight the fact that in numerical simulations it is clearly visible that the orbit is unstable, but we are not able to prove that.

THEOREM 12.3. *Let* $u_0(x) = \sum_{k=1}^{11} -2a_k \sin(kx)$, *where* a_k *are given in* Table 7. *Then, for* $\nu = 0.1215$, *there exists a function* $u^*(t,x)$, *a classical solution of* (1.8)–(1.9), *such that*

$$\|u_0 - u^*(0,\cdot)\|_{L_2} < 1.4 \cdot 10^{-3}, \quad \|u_0 - u^*(0,\cdot)\|_{C^0} < 8.9 \cdot 10^{-4}$$

and u^* *is periodic with respect to* t *with period* $T \in 2 \cdot [1.5458, 1.5468]$ *and has the reflectional symmetry* R.

PROOF. We check the assumption of Theorem 10.11. □

$a_1 = 2.450030\mathrm{e}-01$	$a_2 = 1.041504$
$a_3 = 2.450008\mathrm{e}-01$	$a_4 = -2.760777\mathrm{e}-01$
$a_5 = -4.371381\mathrm{e}-02$	$a_6 = 2.531410\mathrm{e}-02$
$a_7 = 6.346057\mathrm{e}-03$	$a_8 = -1.996817\mathrm{e}-03$
$a_9 = -6.177255\mathrm{e}-04$	$a_{10} = 1.185220\mathrm{e}-04$
$a_{11} = 5.275889\mathrm{e}-05$	

TABLE 7. Coordinates of u_0 – the approximation of the initial condition for the periodic orbitin Theorem 12.3.

THEOREM 12.4. *Let* $u_0(x) = \sum_{k=1}^{13} -2a_k \sin(kx)$, *where* a_k *are given in Table* 8. *Then, for* $\nu = 0.1212$, *there exists a function* $u^*(t,x)$, *a classical solution of* (1.8)–(1.9), *such that*

$$\|u_0 - u^*(0,\,\cdot\,)\|_{L_2} < 2.6 \cdot 10^{-4}, \qquad \|u_0 - u^*(0,\,\cdot\,)\|_{C^0} < 1.6 \cdot 10^{-4}$$

and u^* *is periodic with respect to* t *with period* $T \in [3.1221, 3.1222]$. *Moreover, this orbit does not have the reflectional symmetry* R.

PROOF. We use Theorem 10.10 to obtain the existence of periodic orbit.

To prove that the orbit does not posses the reflectional symmetry R, we checked that $R \circ \widehat{G}_{\theta_1 \to R\theta_1}(N_0 \oplus T_0) \cap (N_0 \oplus T_0) = \emptyset$. This check was performed with computer assistance. □

$a_1 = 2.608268\mathrm{e}-01$	$a_2 = 1.115112$
$a_3 = 2.608267\mathrm{e}-01$	$a_4 = -3.208590\mathrm{e}-01$
$a_5 = -4.953884\mathrm{e}-02$	$a_6 = 3.199156\mathrm{e}-02$
$a_7 = 7.802341\mathrm{e}-03$	$a_8 = -2.766005\mathrm{e}-03$
$a_9 = -8.196012\mathrm{e}-04$	$a_{10} = 1.826998\mathrm{e}-04$
$a_{11} = 7.575075\mathrm{e}-05$	$a_{12} = -1.023717\mathrm{e}-05$
$a_{13} = -6.157452\mathrm{e}-06$	

TABLE 8. Coordinates of u_0 – the approximation of the initial condition for the periodic orbit in Theorem 12.4.

We have proved the existence of apparently unstable periodic orbits for several parameter values.

- $\nu = 0.1215$. This symmetric orbit belongs to γ_{Hopf}.

ν	Sym	m	h	order	δ	iter	d	comp. time
0.02991	Yes	25	1e − 4	5	5e − 5	3	12	600 sec
0.1212 a	No	13	4e − 4	5	2e − 5	1	12	185 sec
0.1212 s	Yes	14	5e − 4	7	5e − 5	1	9	190 sec
0.1215	Yes	11	2e − 3	8	1e − 4	2	10	18 sec

TABLE 9. The parameters used in the proofs of the existence of apparently unstable periodic orbits for the KS equation. The Sym column tells whether the periodic orbit has the reflectional symmetry R, iter contains the number of iterates in the algorithm required to fulfill assumptions of Theorem 10.10 or 10.11, (comp. time) – the computation time for one iterate on 3 GHz Linux machine. d is the number coordinates in the diagonalization of the Poincaré map. For the meaning of other columns see the text.

ν	period	L_2	H_1	C^0	C^1
0.02991	$2 \cdot [0.449023, 0.449067]$	6.9e − 04	4.4e − 03	7.0e − 04	4.4e − 03
0.1212 a	$[3.12211, 3.12219]$	2.6e − 04	7.5e − 04	1.6e − 04	5.1e − 04
0.1212 s	$2 \cdot [1.58136, 1.58192]$	5.8e − 04	1.9e − 03	4.0e − 04	1.38e − 03
0.1215	$2 \cdot [1.54587, 1.54679]$	1.4e − 03	4.1e − 03	8.7e − 04	2.8e − 03

TABLE 10. Some data from the proof of the existence of apparently unstable periodic orbits for the KS equation. The columns L_2, H_1, C^0, C^1 contain the estimate on the distance in the corresponding norm, between the center of $P_m(N \oplus T_0)$ and the periodic orbit.

ν	C_i	s_i	$a^+_{i,M+1}$	C_e	s_e	$a^+_{e,M+1}$
0.02991	9.23e + 15	12	2.48e − 07	$7.66e + 05$	13	2.71e − 19
0.1212 a	7.26e + 10	12	4.32e − 09	0.866	13	1.29e − 21
0.1212 s	5.35e + 10	12	1.34e − 09	0.224	14	3.02e − 24
0.1215	1.31e + 11	12	5.51e − 08	146	13	1.83e − 18

TABLE 11. Some data from the proof of the existence of apparently unstable periodic orbits for the KS equation. We compare the parameters describing the far tail at the start of the proof (subscript i) and after the proof (subscript e). In all cases $M = 3m$. The numbers are rounded to three significant decimal digits.

- $\nu = 0.1212$. For this parameter value we have proven the existence of three different periodic orbits. A pair on non-symmetric orbits (one is obtained from another by the application by R) and the symmetric one.

ν	$[a^-_{i,m+1}, a^+_{i,m+1}]$	$[a^-_{e,m+1}, a^+_{e,m+1}]$
0.02991	$-3.576929\mathrm{e}-05 + 7.912748\mathrm{e}-05 * [-1,1]$	$-1.403123\mathrm{e}-06 + 3.789903\mathrm{e}-08 * [-1,1]$
0.1212 a	$1.611572\mathrm{e}-06 + 7.856836\mathrm{e}-06 * [-1,1]$	$4.285918\mathrm{e}-07 + 6.233822\mathrm{e}-09 * [-1,1]$
0.1212 s	$3.098196\mathrm{e}-07 + 2.554750\mathrm{e}-06 * [-1,1]$	$2.533126\mathrm{e}-07 + 1.948557\mathrm{e}-09 * [-1,1]$
0.1215	$4.528402\mathrm{e}-06 + 7.478529\mathrm{e}-05 * [-1,1]$	$-5.638356\mathrm{e}-06 + 3.676028\mathrm{e}-07 * [-1,1]$

Table 12. Some data from the proof of the existence of apparently unstable periodic orbits for the KS equation. We compare the $a^{\pm}_{m+1}$ in the near tail at the start of the proof (subscript i) and after the proof (subscript e). The numbers are rounded to seven significant decimal digits.

The non-symmetric orbits apparently belong to the chaotic attractor, while the symmetric does not.

- $\nu = 0.02991$, this the parameter value considered in [3]. The orbit is on the chaotic attractor.

12.3. Final comments. Tables 5, 6, 11 and 12 show how much the tail has improved during the computation and this is the basic reason why the method proposed here is so much better than the one from [29]. When we compare the initial tail with the tail at the end of the proof we see the improvement of several orders of magnitude (2–3 orders for diameter of a_{m+1} and much more for the far tail). This results and is also a consequence of the significant decrease of the Galerkin projection errors, which is due to the fact that we allow the tail to evolve and the Galerkin errors are computed locally, while in [29] the tail was fixed and the Galerkin errors were computed globally.

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Manuscript received October 10, 2010

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