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PERIODIC SOLUTIONS OF THE PERTURBED SYMMETRIC EULER TOP

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Abstract. We study the problem of persistence of T-periodic solutions of the celebrated symmetric Euler top when subjected to a small T-periodic stimulus. All solutions of the unperturbed system are periodic (of different periods, including continua of equilibria). In the case that the perturbation depends also on the three components of the angular momentum (the unknowns of the system) we provide bifurcation functions whose simple zeros correspond to T-periodic solutions of the perturbed system.

1. Introduction

The aim of this paper is to determine if the oscillations of the celebrated symmetric Euler top

$$\dot{x} = -yz, \quad \dot{y} = xz, \quad \dot{z} = 0 \tag{1.1}$$

persists when subjected to a *small* external periodic stimulus. More exactly, we provide results on the existence of T-periodic solutions ($T > 0$) for differential systems of the form

$$\begin{aligned} \dot{x} &= -yz + \varepsilon p(t, x, y, z), \\ \dot{y} &= xz + \varepsilon q(t, x, y, z), \\ \dot{z} &= \varepsilon s(t, x, y, z), \end{aligned} \tag{1.2}$$

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where $\varepsilon > 0$ is small and the functions $p, q, s: \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ are continuous and T-periodic in their first variable, and sufficiently smooth in the last three variables.

The symmetric Euler top is a particular case of the Euler top

$$\dot{x} = \alpha yz, \quad \dot{y} = \beta xz, \quad \dot{z} = \gamma xy,$$

with real parameters

$$\alpha = \frac{\mu_2 - \mu_3}{\mu_2\mu_3}, \quad \beta = \frac{\mu_3 - \mu_1}{\mu_1\mu_3}, \quad \gamma = \frac{\mu_1 - \mu_2}{\mu_1\mu_2}.$$

Note that $\alpha + \beta + \gamma = 0$. System (1.3) describes the rotation of a rigid body with a fixed point and no torques acting on it. This classical system still plays an important role in Mechanics, see for instance [1]. Here, $(x, y, z) \in \mathbb{R}^3$ denotes the three components of the angular momentum, and the constants μ_i are the moments of inertia about the coordinate axes of the rigid body. The Euler top is a system with a large symmetry group, $SO(3) \times S^1$, and quasi-periodic dynamics on invariant tori of (generically) dimension two, see for instance [2]. Thus, the energy (kinetic energy in this case) $\mathcal{H}(x, y, z)$ is conserved for this system. Another conserved quantity during the system rotation is the square of the Euclidean norm of the angular momentum $\mathcal{D}(x, y, z)$, which is in fact one independent Casimir invariant of the Poisson formulation of system (1.3). In summary, the Euler top (1.3) is an integrable system having the first integrals

$$\mathcal{H}(x, y, z) = \frac{1}{2}\left(\frac{x^2}{\mu_1} + \frac{y^2}{\mu_2} + \frac{z^2}{\mu_3}\right), \quad \mathcal{D}(x, y, z) = x^2 + y^2 + z^2.$$

Analytical perturbations of (non necessarily symmetric) the Euler top (1.3) are considered in [8] under the restriction that Casimir invariants of the system remain invariant for the perturbed flow. By means of the Poincaré–Pontryagin theory, the existence of limit cycles on the Casimir invariants surfaces are investigated in [8].

We consider the *symmetric* Euler top, which is the case when two of the three moments of inertia are equal. Therefore, we will assume $\mu_1 = \mu_2$, or equivalently the condition $\alpha + \beta = \gamma = 0$. When $\mu_1 = \mu_2 < \mu_3$ the body is oblate whereas when $\mu_1 = \mu_2 > \mu_3$ the body is prolate. Hence, after rescaling of time, the symmetric Euler top has the form (1.1). In this symmetric case, two first integrals are

$$\mathcal{H}_1(x, y, z) = (x^2 + y^2)/2, \quad \mathcal{H}_2(x, y, z) = z,$$

and, moreover, the flow can be expressed in terms of elementary functions.

The phase portrait in $\mathbb{R}^3$ of (1.1) has the following features: the plane $z = 0$ and the z-axis are fulfilled of equilibria; each plane $z = \eta$, with $\eta \neq 0$, is invariant and foliated of periodic solutions of minimal period $2\pi/|\eta|$. Using the terminology of [11], we say that, for any $k \in \mathbb{Z}$,

$$\begin{aligned}\mathcal{Z}_k^v &= \{(0,0,\eta) : \eta \in (2k\pi/T, 2(k+1)\pi/T)\},\\ \mathcal{Z}_k^h &= \{(x_0,y_0,2k\pi/T) : (x_0,y_0) \in \mathbb{R}^2 \setminus \{(0,0)\}\},\end{aligned}$$

are *T-period manifolds* of (1.1), that is, the vertical $\mathcal{Z}_k^v$ and horizontal $\mathcal{Z}_k^h$ manifolds are foliated of T-periodic orbits of (1.1). Note that in our case T is not always the minimal period, and that, in fact, $\mathcal{Z}_k^v$ and $\mathcal{Z}_0^h$ contain only equilibria.

In Section 4 we treat system (1.2). The results obtained relays strongly on the fact that the perturbation depends on (x,y,z). The problem of existence of T-periodic solutions of (1.2) for $\varepsilon > 0$ small is reduced to finding simple zeros of two so-called *bifurcation functions* (one for each T-period manifold). This is a classical idea and many of the general methods encountered in the literature relay on it [5], [11], [3], [10]. We found that:

(1) the bifurcation function corresponding to the 1-dimensional T-period manifold $\mathcal{Z}_k^v$ (for any $k \in \mathbb{Z}$) is

$$\eta \mapsto \int_0^T s(t,0,0,\eta)\,dt;$$

(2) the bifurcation function corresponding to the 2-dimensional T-period manifold $\mathcal{Z}_k^h$ (for any $k \in \mathbb{Z}$) is

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \mapsto \begin{pmatrix} \displaystyle\int_0^T \varphi_k p(t,\varphi_k,\psi_k,\eta_k) + \psi_k q(t,\varphi_k,\psi_k,\eta_k)\,dt \\ \displaystyle\int_0^T s(t,\varphi_k,\psi_k,\eta_k)\,dt \end{pmatrix}$$

where $\eta_k = 2k\pi/T$ and we must replace

$$\begin{pmatrix} \varphi_k \\ \psi_k \end{pmatrix} = \begin{pmatrix} \cos(\eta_k t) & -\sin(\eta_k t) \\ \sin(\eta_k t) & \cos(\eta_k t) \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

The aim of Section 2 is to give the background of the general methods existed in the literature, mainly the ones presented in [11] and [10] (see also [3]) to study the persistence problem of periodic solutions of (1.2). Section 3 is devoted to the analysis of the symmetric Euler top and its weak perturbation (1.2), that is, the case when the perturbation field $(p(t),q(t),s(t))$ only depends on t. In Section 4 we present the main results of the paper which are the bifurcation functions corresponding to the problem of persistence of T-periodic solutions of (1.2). Finally, we conclude with several remarks and possible further work.

2. Some general considerations

In order to fix the notations, we consider in this section the differential equation

$$\dot{u} = f(u) + \varepsilon g(t, u, \varepsilon) \tag{2.1}$$

where $\varepsilon \geq 0$ is small, $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g: \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ are sufficiently smooth and g is T-periodic in its first variable. We assume that the unperturbed system

$$\dot{u} = f(u) \tag{2.2}$$

has a *T-period manifold* $\mathcal{Z} \subset \mathbb{R}^n$, that is, $\mathcal{Z}$ is a manifold consisting entirely of T-periodic orbits.

The problem of the persistence of the periodic orbits of $\mathcal{Z}$ have been treated by several authors, for example [9]–[12], [5], [3], [7]. As mentioned in [11] their results are obtained under the assumption that $\mathcal{Z}$ is *normally nondegenerate* (in fact in [11] the novelty consists in allowing the manifold to be *normally degenerate*).

A k-dimensional period manifold $\mathcal{Z}$ of some system (2.2) is called *normally nondegenerate* if for each $u_0 \in \mathcal{Z}$ the Floquet multiplier $+1$ of the first variational system

$$\dot{w} = Df(u(t, u_0))w$$

has geometric multiplicity equal to k (here $u(t, u_0)$ denotes the solution of (2.2) with $u(0, u_0) = u_0$).

We present now the recipe (as in [10], [3]) to obtain the so-called *Malkin bifurcation function* for system (2.1) with the normally nondegenerate k-dimensional T-period manifold $\mathcal{Z}$. These hypotheses assure the existence, for each $u_0 \in \mathcal{Z}$, of k linearly independent T-periodic solutions of the adjoint system

$$\dot{w} = -\left[Df(u(t, u_0))\right]^* w,$$

denoted $w_i(t, u_0)$, $i = \overline{1, k}$. Then the Malkin bifurcation function $M: \mathcal{Z} \to \mathbb{R}^k$ is defined componentwise by

$$u_0 \mapsto \int_0^T \langle w_i(t, u_0), g(t, u(t, u_0), 0)\rangle \, dt, \quad i = \overline{1, k}.$$

Here $[\,\cdot\,]^*$ denotes the transpose of some matrix, while $\langle \cdot\,, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^n$. We remark that there are other bifurcation functions encountered in the literature, that can eventually differ by a change of variable and/or a factor without zeros.

In the hypotheses and notations of this section we have that ([10], [11], [3]):

If there exists $u_0^* \in \mathcal{Z}$ *a simple zero of the Malkin bifurcation function* M *then from the* T*-periodic solution of* (2.2) *with initial value* u_0^* *emanates a branch of* T*-periodic solutions* $u(t, \varepsilon)$ *of* (2.1).

3. The analysis of the symmetric Euler top and its weak perturbation

In this section we will write explicitly the solutions of (1.1):

$$\dot{x} = -yz, \quad \dot{y} = xz, \quad \dot{z} = 0.$$

The next proposition is easy to be checked by straightforward calculation.

Proposition 3.1. *The solution* $(x(t), y(t), z(t))$ *of* (1.1) *satisfying the initial condition* $(x(0), y(0), z(0)) = (x_0, y_0, \eta) \in \mathbb{R}^3$ *is*

$$\begin{aligned} x(t, x_0, y_0, \eta) &= x_0 \cos \eta t - y_0 \sin \eta t, \\ y(t, x_0, y_0, \eta) &= x_0 \sin \eta t + y_0 \cos \eta t, \\ z(t, x_0, y_0, \eta) &= \eta. \end{aligned}$$

Consequently, we have:

(a) *The equilibria of* (1.1) *are* $(0, 0, \eta)$ *for each* $\eta \in \mathbb{R}$ *and* $(x_0, y_0, 0)$ *for each* $(x_0, y_0) \in \mathbb{R}^2$.
(b) *Each plane* $z = \eta$, *with* $\eta \neq 0$, *is invariant and foliated of periodic solutions of minimal period* $2\pi/|\eta|$.
(c) *The nontrivial* T*-periodic solutions of* (1.1) *have the initial values* $(x_0, y_0, 2k\pi/T)$ *for each* $(x_0, y_0) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ *and* $k \in \mathbb{Z} \setminus \{0\}$.

We proceed now to the analysis of the weak perturbation of the symmetric Euler top of the form

$$\dot{x} = -yz + p(t), \quad \dot{y} = xz + q(t), \quad \dot{z} = s(t). \tag{3.1}$$

Integrating with respect to t the last equation of (3.1) we obtain

$$z(t) = \eta + S(t), \quad \text{for all } t \in \mathbb{R}, \tag{3.2}$$

with $\eta \in \mathbb{R}$ an arbitrary constant and $S(t) = \int_0^t s(\sigma)\, d\sigma$.

In the case that $S(T) \neq 0$ the function z given by (3.2) is not periodic, hence (3.1) does not have any periodic solution. If not otherwise stated, from now on we assume that

$$\int_0^T s(\sigma)\, d\sigma = 0.$$

Introducing (3.2) into (3.1) we obtain

$$\dot{x} = -y(\eta + S(t)) + p(t), \quad \dot{y} = x(\eta + S(t)) + q(t),$$

that is a linear differential system with T-periodic coefficients, written equivalently using the complex variable $W = x + iy$ as

$$\dot{W} = i(\eta + S(t))W + p(t) + iq(t). \tag{3.3}$$

Denote $\tau(t,\eta) = \int_0^t (\eta + S(\sigma))\,d\sigma$. The linear homogeneous differential equation

$$\dot{W} = i(\eta + S(t))W \tag{3.4}$$

has the general solution $W(t) = Ce^{i\tau(t,\eta)}$, with $C \in \mathbb{C}$. We have that (3.4) has only the trivial T-periodic solution if and only if $e^{i\tau(T,\eta)} \neq 1$ that further is equivalent to $\eta \in \mathbb{R}\setminus\{2k\pi/T - \overline{S} : k \in \mathbb{Z}\}$. Here we denoted $\overline{S} = (1/T)\int_0^T S(t)\,dt$ the average of $S(t)$. Otherwise all its solutions are T-periodic. Using linear systems theory one can decide with respect to the existence of T-periodic solutions of (3.3) and finally obtain the following result.

Theorem 3.2. *Let $p, q, s \colon \mathbb{R} \to \mathbb{R}$ be continuous and T-periodic functions.*

(a) *If $\int_0^T s(\sigma)\,d\sigma \neq 0$ system* (3.1) *has no T-periodic solutions.*

From now on assume that $\int_0^T s(\sigma)\,d\sigma = 0$.

(b) *System* (3.1) *has a unique T-periodic solution that initiates in the plane $z = \eta \in \mathbb{R}\setminus\{2k\pi/T - \overline{S} : k \in \mathbb{Z}\}$.*

(c) *If for some $k \in \mathbb{Z}$, $\int_0^T e^{-i\tau(t,2k\pi/T - \overline{S})}(p(t) + iq(t))\,dt \neq 0$, then there is no T-periodic solution of* (3.1) *that initiates in the plane $z = 2k\pi/T - \overline{S}$. Otherwise all the points of the plane $z = 2k\pi/T - \overline{S}$ are initial values for the T-periodic solutions of* (3.1).

4. Analysis of the perturbed symmetric Euler top

The main result of this section follows and, mainly, gives the bifurcation functions corresponding to the problem of persistence of T-periodic solutions of (1.2):

$$\dot{x} = -yz + \varepsilon p(t,x,y,z), \quad \dot{y} = xz + \varepsilon q(t,x,y,z), \quad \dot{z} = \varepsilon s(t,x,y,z)$$

from the T-period manifolds $\mathcal{Z}_k^v$ and $\mathcal{Z}_k^h$ of the unperturbed system (1.1).

Theorem 4.1. *Let $p, q, s \colon \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ be C^2 and T-periodic in their first variable.*

(a) *Assume that there exists $\eta^* \in \mathbb{R}\setminus\{2k\pi/T : k \in \mathbb{Z}\}$ a simple zero of*

$$\eta \mapsto \int_0^T s(t,0,0,\eta)\,dt.$$

Then from the equilibrium point $(0,0,\eta^*) \in \mathcal{Z}_k^v$ (for any $k \in \mathbb{Z}$) of (1.1) *emanates a branch of T-periodic solutions of* (1.2).

(b) *Assume that, for some $k \in \mathbb{Z}$, there exists $(x^*, y^*) \in \mathbb{R}^2 \setminus \{(0,0)\}$ a simple zero of*

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \mapsto \begin{pmatrix} \displaystyle\int_0^T \varphi_k p(t, \varphi_k, \psi_k, \eta_k) + \psi_k q(t, \varphi_k, \psi_k, \eta_k)\, dt \\ \displaystyle\int_0^T s(t, \varphi_k, \psi_k, \eta_k)\, dt \end{pmatrix} \tag{4.1}$$

where $\eta_k = 2k\pi/T$ and we must replace

$$\begin{pmatrix} \varphi_k \\ \psi_k \end{pmatrix} = \begin{pmatrix} \cos(\eta_k t) & -\sin(\eta_k t) \\ \sin(\eta_k t) & \cos(\eta_k t) \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

Then from the T-periodic solution of (1.1) *with initial value (x^*, y^*, η_k) in $\mathcal{Z}_k^h$ emanates a branch of T-periodic solutions of* (1.2).

PROOF. Applying the general result from [10], [11] (see also [3]) presented in Section 2, we see that it is sufficient to prove that the T-periodic manifolds $\mathcal{Z}_k^v$ and $\mathcal{Z}_k^h$ are normally nondegenerate and, after, to construct the Malkin bifurcation function in each situation.

(a) We will prove now that *the T-periodic manifold $\mathcal{Z}_k^v$ of* (1.1) *is normally nondegenerate* for any $k \in \mathbb{Z}$.

For $f(x, y, z) = (-yz, xz, 0)$ we have

$$Df(x, y, z) = \begin{pmatrix} 0 & -z & -y \\ z & 0 & x \\ 0 & 0 & 0 \end{pmatrix}.$$

The first variational system of (1.1) around the equilibrium $(0, 0, \eta) \in \mathcal{Z}_k^v$ is

$$\dot{u} = -\eta v, \quad \dot{v} = \eta u, \quad \dot{w} = 0, \tag{4.2}$$

and, moreover, this has the principal fundamental matrix solution

$$\Phi_k^v(t, \eta) = \begin{pmatrix} \cos \eta t & -\sin \eta t & 0 \\ \sin \eta t & \cos \eta t & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It is easy to check that, since $\eta \in (2k\pi/T, 2(k+1)\pi/T) = (\eta_k, \eta_{k+1})$, the eigenvalue $+1$ of $\Phi_k^v(T, \eta)$ has geometric multiplicity 1. *Hence, the 1-dimensional period manifold $\mathcal{Z}_k^v$ of* (1.1) *is normally nondegenerate.* Note that $\Phi_k^v(T, 2k\pi/T)$ is the identity matrix, thus the eigenvalue $+1$ of this matrix has geometric multiplicity 3.

In order to construct the Malkin bifurcation function for (1.2) with respect to the 1-dimensional period manifold $\mathcal{Z}_k^v$ of (1.1), we need one non-null T-periodic solution for the adjoint system of (4.2), that, in fact, coincide to (4.2) and has the form

$$\dot{u} = -\eta v, \quad \dot{v} = \eta u, \quad \dot{w} = 0.$$

Since $\eta \in (2k\pi/T, 2(k+1)\pi/T)$, the only non-null T-periodic solution is $(0,0,1)$. Then the Malkin bifurcation function is

$$M(\eta) = \int_0^T \langle (0,0,1), (p,q,s) \rangle \, dt = \int_0^T s(t,0,0,\eta) \, dt.$$

(b) We will prove now that *the T-periodic manifold $\mathcal{Z}_k^v$ of* (1.1) *is normally nondegenerate* for any $k \in \mathbb{Z}$.
The first variational system of (1.1) around the T-periodic solution $(\varphi_k, \psi_k, \eta_k)$, see Proposition 3.1, with initial value $(x_0, y_0, \eta_k) \in \mathcal{Z}_k^h$ is

$$\begin{pmatrix} \dot u \\ \dot v \\ \dot w \end{pmatrix} = \begin{pmatrix} 0 & -\eta_k & -\psi_k \\ \eta_k & 0 & \varphi_k \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}, \tag{4.3}$$

and this has the principal fundamental matrix solution

$$\Phi_k^h(t, x_0, y_0) = \begin{pmatrix} \cos(\eta_k t) & -\sin(\eta_k t) & -\psi_k t \\ \sin(\eta_k t) & \cos(\eta_k t) & \varphi_k t \\ 0 & 0 & 1 \end{pmatrix}.$$

We have

$$\Phi_k^h(T, x_0, y_0) = \begin{pmatrix} 1 & 0 & -y_0 T \\ 0 & 1 & x_0 T \\ 0 & 0 & 1 \end{pmatrix}.$$

It is easy to check that, since $(x_0, y_0) \in \mathbb{R}^2 \setminus \{(0,0)\}$, the eigenvalue $+1$ of $\Phi_k^h(T, x_0, y_0)$ has geometric multiplicity 2. *Hence, the 2-dimensional period manifold $\mathcal{Z}_k^h$ of* (1.1) *is normally nondegenerate.* Note that $\Phi_k^h(T, 0, 0)$ is the identity matrix, thus the eigenvalue $+1$ of this matrix has geometric multiplicity 3.

In order to construct the Malkin bifurcation function for (1.2) with respect to the 2-dimensional period manifold $\mathcal{Z}_k^h$ of (1.1), we need two linearly independent T-periodic solutions for the adjoint system of (4.3), that has the form

$$\begin{pmatrix} \dot u \\ \dot v \\ \dot w \end{pmatrix} = \begin{pmatrix} 0 & -\eta_k & 0 \\ \eta_k & 0 & 0 \\ \psi_k & -\varphi_k & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}.$$

These solutions are $(\varphi_k, \psi_k, 0)$ and $(0,0,1)$. Then the Malkin bifurcation function is given by

$$M(x_0, y_0) = \begin{pmatrix} \displaystyle\int_0^T \langle (\varphi_k, \psi_k, 0), (p,q,s) \rangle \, dt \\ \displaystyle\int_0^T \langle (0,0,1), (p,q,s) \rangle \, dt \end{pmatrix}$$

with (p,q,s) evaluated at $(t, \varphi_k, \psi_k, \eta_k)$. Thus $M(x_0, y_0)$ is just (4.1). $\square$

5. Further problems

We propose new problems related to systems of the form (1.2). Consider the case that s does not depend on z, that is, consider the system

$$\dot{x} = -yz + \varepsilon p(t, x, y, z), \quad \dot{y} = xz + \varepsilon q(t, x, y, z), \quad \dot{z} = \varepsilon s(t, x, y),$$

and note that Theorem 4.1(a) fails for it because the bifurcation function is constant, thus can not have simple zeros. Also, in the case of the system

$$\dot{x} = -yz + \varepsilon p(t, x, y, z), \quad \dot{y} = xz + \varepsilon q(t, x, y, z), \quad \dot{z} = \varepsilon s(t, z),$$

Theorem 4.1(b) fails. Hence the problem of persistence of $\mathcal{Z}_k^v$ and, respectively, of $\mathcal{Z}_k^h$ has to be reconsidered in these cases.

Here we considered sufficiently smooth perturbations. But recently there is an increasing interest for nonsmooth systems (see for example [3], [4] and the references therein). Using the results in [3], [4] we consider that the conclusions of Theorem 4.1 remain valid in weaker hypothesis related to the smoothness of perturbations. Moreover, the problem of stability of the periodic solutions can also be treated in these weaker assumptions.

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