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EXISTENCE OF SIGN-CHANGING SOLUTIONS FOR ONE-DIMENSIONAL p-LAPLACIAN PROBLEMS WITH A SINGULAR INDEFINITE WEIGHT

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Abstract. In this paper, we establish a sequence $\{\nu_k^\infty\}$ of eigenvalues for the following eigenvalue problem

$$\begin{cases} \varphi_p(u'(t))' + \nu h(t)\varphi_p(u(t)) = 0 & \text{for } t \in (0,1), \\ u(0) = 0 = u(1), \end{cases}$$

where $\varphi_p(x) = |x|^{p-2}x$, $1 < p < 2$, ν a real parameter. In particular, $h \in C((0,1),(0,\infty))$ is singular at the boundaries which may not be of $L^1(0,1)$. Employing global bifurcation theory and approximation technique, we prove several existence results of sign-changing solutions for problems of the form

$$(\mathrm{QP}_\lambda) \qquad \begin{cases} \varphi_p(u'(t))' + \lambda h(t) f(u(t)) = 0 & \text{for } t \in (0,1), \\ u(0) = 0 = u(1), \end{cases}$$

when $f \in C(\mathbb{R},\mathbb{R})$ and $uf(u) > 0$, for all $u \neq 0$ and is odd with various combinations of growth conditions at 0 and ∞.

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1. Introduction

In this paper, we investigate existence and nonexistence of sign-changing solutions for singular one-dimensional p-Laplacian problem

$$(\mathrm{QP}_\lambda) \qquad \begin{cases} \varphi_p(u'(t))' + \lambda h(t) f(u(t)) = 0 & \text{for } t \in (0,1), \\ u(0) = 0 = u(1), \end{cases}$$

where $\varphi_p(x) = |x|^{p-2}x$, $p > 1$, λ a positive parameter, $f \in C(\mathbb{R},\mathbb{R})$ is odd and satisfies $uf(u) > 0$, for all $u \neq 0$, and $h \in C((0,1),(0,\infty))$.

Our concern is focused on the case that weight function h is singular at the boundary which may not be of $L^1(0,1)$ and nonlinear term f satisfies $0 \le f_0 < \infty$, where $f_0 \triangleq \lim_{u\to 0} f(u)/\varphi_p(u)$. In particular, for case $0 < f_0 < \infty$, it is known in continuous case (i.e. $h \in C[0,1]$) that the existence of solutions for problem (QP_λ) is closely related to the properties of eigenvalues for corresponding linearized problem such as

$$(\mathrm{QEP}_\mu) \qquad \begin{cases} \varphi_p(u'(t))' + \mu f_0 h(t) \varphi_p(u(t)) = 0 & \text{for } t \in (0,1), \\ u(0) = 0 = u(1). \end{cases}$$

There are few studies about the eigenvalues of problem (QEP_μ) for case $h \notin L^1(0,1)$. For precise description, let us introduce two classes of singularity for the weight functions:

$$\mathcal{A} \equiv \bigg\{ h \in C((0,1),(0,\infty)) : \int_0^1 s^{p-1}(1-s)^{p-1} h(s)\,ds < \infty \bigg\},$$

$$\mathcal{B} \equiv \bigg\{ h \in C((0,1),(0,\infty)) : \\ \int_0^{1/2} \varphi_p^{-1}\bigg(\int_s^{1/2} h(\tau)\,d\tau \bigg)\,ds + \int_{1/2}^1 \varphi_p^{-1}\bigg(\int_{1/2}^s h(\tau)\,d\tau \bigg)\,ds < \infty \bigg\}.$$

It is known that $L^1((0,1),(0,\infty)) \subsetneq \mathcal{A}\cap\mathcal{B}$, $\mathcal{A} = \mathcal{B} = \{h : \int_0^1 s(1-s)h(s)\,ds < \infty\}$ for $p = 2$, $\mathcal{A} \subsetneq \mathcal{B}$ for $1 < p < 2$ and $\mathcal{B} \subsetneq \mathcal{A}$ for $p > 2$ (see [3]).

For $h \in \mathcal{A}$ or $h \in \mathcal{B}$, by a solution to problem (QP_λ), we understand a function $u \in C[0,1] \cap C^1(0,1)$ and $\varphi_p(u')$ is absolutely continuous and u satisfies the first equation in (QP_λ) in $(0,1)$ and $u(0) = 0 = u(1)$.

Recently, Kajikiya–Lee–Sim [5] constructed a sequence $\{(\mu_k)\}$ of eigenvalues for problem (QEP_μ) when $h \in \mathcal{A}$. It diverges to ∞ and each eigenvalue is positive, simple and isolated. All corresponding eigenfunctions $\{(u_k)\}$ are of class $C^1[0,1]$ and u_k has exactly $k-1$ interior zeros. Furthermore, if there is an eigenvalue other than μ_k, then its corresponding eigenfunctions are not of $C^1[0,1]$. And the sequence becomes the set of all eigenvalues for problem (QEP_μ), when $h \in \mathcal{A}\cap\mathcal{B}$. They also proved that if $h \in \mathcal{A}$ and $0 \le f_0 < \infty$, then problem (QP_λ) has a C^1-solution. Thus setting up $C^1[0,1]$ as the solution space, they could define the corresponding integral operator for problem (QP_λ) which is completely continuous by the work of Manásevich–Mawhin [12] (or see Huidobro–Manásevich–Ward [4]). Well-definedness of the operator follows by showing $h(\cdot)f(u(\cdot)) \in L^1(0,1)$ with facts $u \in C^1[0,1]$ and $h \in \mathcal{A}$. Using bifurcation argument, Kajikiya–Lee–Sim [6] proved several existence and nonexistence results of sign-changing solutions for (QP_λ), specially when $h \in \mathcal{A}$.

In the sequel, it is natural to study eigenvalues of (QEP_μ) and existence results of (QP_λ) for case $1 < p < 2$ and $h \in \mathcal{B} \setminus \mathcal{A}$. It is, indeed, the aim of this paper. In this case, there are some difficulties mainly caused by the regularity of solutions. We easily check that there is no C^1-solution for (QEP_μ) (see Remark 2.1). Thus due to coarseness of topology in solution space and difference between classes $\mathcal{A}$ and $\mathcal{B}$, most approaches in [5] are not applied directly for the properties of eigenvalues. Further more, Manásevich–Mawhin type integral operator for problem (QP_λ) can not be applied either, since nonlinear term $h(\cdot)f(u(\cdot))$ may not be of $L^1(0,1)$ only with $u \in C[0,1]$ (see Remark 2.3). It is not successful to obtain properties of eigenvalues or well-definedness of corresponding integral operator, but we give some partial conclusions using approximation arguments.

In this work, we consider three different types of nonlinearities as follows.

(I) $0 < f_0 < \infty$ and $f_\infty \triangleq \lim_{u\to\infty} f(u)/\varphi_p(u) = 0$,
(II) $0 < f_0 < \infty$ and $f_\infty = \infty$,
(III) $f_0 = 0$ and $f_\infty = \infty$.

We expect that problems of cases (I) and (II) have certain bifurcation phenomena with respect to λ. For example, the existence of sign-changing solutions for (QP_λ) which belong to unbounded continuum of solutions bifurcating from $(\mu_k, 0)$ where μ_k is the k-th eigenvalue of (QEP_μ) is known in [8] when $h \in L^1(0,1)$ and in [5] when $h \in \mathcal{A}$.

In this paper, when $1 < p < 2$ and $h \in \mathcal{B} \setminus \mathcal{A}$, by using critical point theory of Lusternik–Schnirelmann and approximating argument, we construct a sequence $\{\mu_k^\infty\}$ of eigenvalues for (QEP_μ) which play a role as bifurcation points for (QP_λ). It is worth to note that we do not know whether the sequence $\{\mu_k^\infty\}$ is whole eigenvalues of (QEP_μ) or not. For each k, to obtain the existence of unbounded branch of sign-changing solutions for (QP_λ), we apply limit process in the sense of Whyburn [14] motivated by the work of Berestycki–Esteban [2]. We have main theorems for cases (I) and (II) as follows.

RESULT FOR CASE (I). *Assume* $1 < p < 2$, $h \in \mathcal{B} \setminus \mathcal{A}$. *Also assume* (I) *and*

$$\text{(L)} \qquad \lim_{t\to 0^+} t^p h(t) = 0 \quad \textit{and} \quad \lim_{t\to 1^-} (1-t)^p h(t) = 0.$$

Then for each $k \in \mathbb{N}$, *there exist* λ^* *and* λ^{**} *with* $0 < \lambda^{**} \le \lambda^* \le \mu_k^\infty$ *such that* (QP_λ) *has at least one solution for all* $\lambda \in (\lambda^*, \infty)$ *which has exactly* $k-1$ *many interior zeros in* $(0,1)$ *and no solution for* $\lambda \in (0, \lambda^{**})$.

Result for case (II). *Assume* $1 < p < 2$, $h \in \mathcal{B} \setminus \mathcal{A}$. *Also assume* (II) *and* (L). *Then for each* $k \in \mathbb{N}$, *there exist* λ_* *and* λ_{**} *with* $0 < \mu_k^\infty \le \lambda_* \le \lambda_{**}$ *such that* (QP_λ) *has at least one solution for* $\lambda \in (0, \lambda_*)$ *which has exactly* $k-1$ *many interior zeros in* $(0,1)$ *and no solution for* $\lambda \in (\lambda_{**}, \infty)$.

Condition (L) is required for technical reasons. However the condition does not make $\mathcal{B}$ and $\mathcal{A}$ coincide (see Example 2.4).

For the problems of case (III), on the other hand, we may not expect any bifurcation phenomena with respect to the parameter. So it is enough to set up the problem as follows:

$$(\mathrm{QP}) \qquad \begin{cases} \varphi_p(u'(t))' + h(t)f(u(t)) = 0 & \text{for } t \in (0,1), \\ u(0) = 0 = u(1). \end{cases}$$

To benefit from bifurcation arguments, we adopt an auxiliary equation

$$(\mathrm{A}_\lambda) \qquad \begin{cases} \varphi_p(u'(t))' + \lambda h(t)\varphi_p(u(t)) + h(t)f(u(t)) = 0 & \text{for } t \in (0,1), \\ u(0) = 0 = u(1). \end{cases}$$

Employing bifurcation arguments and approximation technique as in (I) and (II), we prove the existence of sign-changing solutions at $\lambda = 0$. The readers refer to Ma–Thompson [11] and Lee–Sim [9] for cases $h \in C[0,1]$ and $h \in L^1(0,1)$, respectively. We state the main theorem for case (III) as follows.

Result for case (III). *Assume* $1 < p < 2$, $h \in \mathcal{B} \setminus \mathcal{A}$. *Also assume* (III) *and* (L). *Then for each* $k \in \mathbb{N}$, (QP) *has at least one solution which has exactly* $k-1$ *many interior zeros in* $(0,1)$.

This paper is organized as follows: In Section 2, we introduce Lusternik–Schnirelmann theory, the Hardy inequality and get a sequence $\{\mu_k^\infty\}$ which will be eigenvalues for (QEP_μ). We also introduce Picone's identity which will be used in the subsequent sections. In Section 3, we prove some properties of interior zeros of solutions for (QP_λ). In Section 4, we prove that μ_k^∞ is an eigenvalue for (QEP_μ) and the existence results of sign-changing solutions for (QP_λ) with cases (I) and (II). Finally, in Section 5, we show the existence of sign-changing solutions for (QP) with case (III).

2. Preliminaries

In this section, we introduce the Hardy inequality, Lusternik–Schnirelmann theory and get a sequence $\{\mu_k^\infty\}$ which will be eigenvalues for (QEP_μ) and $(\mu_k^\infty, 0)$

will be a bifurcation point for problem (QP_λ). We also introduce Picone's identity which will be used in Section 4 and 5.

First of all, we give some properties of solutions for (QP_λ) in case of $1 < p < 2$ and $h \in \mathcal{B} \setminus \mathcal{A}$.

Proposition 2.1. *Assume* $0 < f_0 < \infty$. *If* (QP_λ) *has a nontrivial* C^1-*solution, then* $h \in \mathcal{A}$.

Proof. Let $u \in C^1[0,1]$ be a nontrivial solution of (QP_λ). Assume $u'(0) = 0$. Then we have

$$\varphi_p(u'(t)) = -\lambda \int_0^t h(\tau) f(u(\tau))\, d\tau.$$

If u is increasing on $[0,t)$ for small enough t, the above equation contradicts the positivity of u. If u is decreasing on $[0,t)$, the above equation contradicts the negativity of u. Therefore, u must have infinitely many zeros that converges to 0. By standard argument (see Lemma 3.1(a)), we have $u \equiv 0$. This is a contradiction. Thus, we only consider the case $u'(0) \neq 0$. Without loss of generality, we may assume $u'(0) > 0$. Since u is concave, there exists $\delta > 0$ such that $2u(t)/t \geq u'(0)$ for $t \in (0,\delta)$. Integrating (QP_λ) over (s,δ), we have

$$\varphi_p(u'(\delta)) - \varphi_p(u'(s)) + \lambda \int_s^\delta h(\tau) f(u(\tau))\, d\tau = 0.$$

Using the fact $0 < f_0 < \infty$, we get

$$\lambda C \int_s^\delta h(\tau) \left(\frac{\tau u'(0)}{2} \right)^{p-1} d\tau \leq \varphi_p(u'(s)) - \varphi_p(u'(\delta)).$$

As $s \to 0^+$, the right-hand side converges. Similarly, near $t = 1$, assuming $u'(1) > 0$, we have $\int_t^1 (1-\tau)^{p-1} h(\tau)\, d\tau < \infty$. Hence $h \in \mathcal{A}$. $\square$

Remark 2.2. Proposition 2.1 is valid for problem (QEP_μ) so that if $h \in \mathcal{B} \setminus \mathcal{A}$, then all solutions of (QEP_μ) are in $C[0,1] \cap C^1(0,1)$ but not in $C^1[0,1]$.

Remark 2.3. If u is a solution for (QP_λ) with $u \notin C^1[0,1]$ and $h \in \mathcal{B} \setminus \mathcal{A}$, then $h(\,\cdot\,) f(u(\,\cdot\,)) \notin L^1(0,1)$. Indeed, for $0 < s < \delta < 1$, we have

$$\lambda \int_s^\delta h(\tau) f(u(\tau))\, d\tau = \varphi_p(u'(s)) - \varphi_p(u'(\delta)).$$

As $s \to 0^+$, $h(\,\cdot\,) f(u(\,\cdot\,)) \notin L^1(0,\delta)$, since $u'(s)$ diverges.

The proof for $h(\,\cdot\,) f(u(\,\cdot\,)) \notin L^1(1-\delta,1)$ can be done by the same argument.

Let us recall the general Hardy inequality, for $p > 1$,

$$\int_0^1 \left| \frac{u(t)}{t} \right|^p dt \leq \left(\frac{p}{p-1} \right)^p \int_0^1 |u'(t)|^p\, dt, \quad \text{for all } u \in W_0^{1,p}(0,1). \tag{2.1}$$

It is easy to check the following inequality

$$\int_0^1 \left|\frac{u(t)}{1-t}\right|^p dt \le \left(\frac{p}{p-1}\right)^p \int_0^1 |u'(t)|^p\, dt, \quad \text{for all } u \in W_0^{1,p}(0,1). \tag{2.2}$$

For the technical reason, we will assume

$$\lim_{t\to 0^+} t^p h(t) = 0 \quad \text{and} \quad \lim_{t\to 1^-} (1-t)^p h(t) = 0. \tag{L}$$

However, this condition does not make $\mathcal{B} = \mathcal{A}$ for $1 < p < 2$. We shall give an example which holds condition (L) and is in $\mathcal{B}$ but not in $\mathcal{A}$.

EXAMPLE 2.4. Let a function $h \in C((0,1],(0,\infty))$ be singular only at $t = 0$ and satisfy $\int_0^1 \varphi_p^{-1}(\int_s^1 h(\tau)\,d\tau)\,ds < \infty$. Define a function v as $v(s) = \int_s^1 h(t)\,dt$. Then $v \in C^1(0,1]$ and v has a C^1 inverse function $w\colon [0,\infty) \to (0,1]$ with $w(0) = 1$. Putting $t = w(s)$, we have $h(w(s)) = -v'(w(s)) = -1/w'(s)$. Take $w(s) = 1/(s\log s)^{1/(p-1)}$, for $s \ge 2$ and patch w on the interval $[0,2]$ satisfying w is decreasing and $w(0) = 1$. Then $h \in \mathcal{B}$ and $h \notin \mathcal{A}$ (see [3]). Since $\lim_{t\to 0^+} t^p h(t) = 0$ is equivalent to $\lim_{s\to\infty}(w(s))^p h(w(s)) = 0$, we show the latter equality. Indeed,

$$\begin{aligned}
\lim_{s\to\infty}(w(s))^p h(w(s)) &= \lim_{s\to\infty} -\frac{(w(s))^p}{w'(s)} \\
&= \lim_{s\to\infty} -\frac{(1/(s\log s)^{1/(p-1)})^p}{(1/(p-1))s^{-p/(p-1)}(\log s)^{-1/(p-1)}[1-(\log s)^{-1}]} \\
&= -(p-1)\lim_{s\to\infty}\frac{(\log s)^{-p/(p-1)}}{(\log s)^{-1/(p-1)}} = 0.
\end{aligned}$$

REMARK 2.5. If h holds condition (L), then it holds the following condition: there exists $M > 0$ such that

$$h(t) \le M t^{-p}(1-t)^{-p}, \quad \text{for all } t \in (0,1). \tag{H$_1$}$$

Indeed, condition (L) implies that there exist $\delta_1, \delta_2 > 0$ such that

$$h(t) \le t^{-p}, \quad t \in (0,\delta_1] \quad \textit{and} \quad h(t) \le (1-t)^{-p}, \quad t \in [\delta_2, 1),$$

respectively. Therefore, $h(t) \le t^{-p}(1-t)^{-p}$, $t \in (0,\delta_1] \cup [\delta_2,1)$ and by the continuity of h in $[\delta_1,\delta_2]$, there exists $M_1 > 0$ such that $t^p(1-t)^p h(t) \le M_1$, $t \in [\delta_1,\delta_2]$. Hence, for $M = 2 + M_1 > 0$, we have, for all $t \in (0,1)$,

$$h(t) \le (2+M_1)t^{-p}(1-t)^{-p} = Mt^{-p}(1-t)^{-p}.$$

Now consider the eigenvalue problem for approximation argument

$$\begin{cases} \varphi_p(u'(t))' + \mu f_0 h_n(t)\varphi_p(u(t)) = 0 & \text{for } t \in (0,1), \\ u(0) = 0 = u(1), \end{cases} \tag{QEP$^n_\mu$}$$

where $h_n(t) = \min\{n, h(t)\}$, $n \in \mathbb{N}$. It is well-known that the critical point theory of Lusternik–Schnirelmann provides a sequence of eigenvalues for (QEP^n_μ) which are given by

$$\mu_k(h_n) \triangleq \inf_{K\in\mathcal{E}_k} \max_{u\in K} \frac{\int_0^1 |u'|^p\,dt}{\int_0^1 f_0 h_n(t)|u|^p\,dt},$$

where $\mathcal{E}_k = \{K \subset W_0^{1,p}(0,1) : K \text{ symmetrical compact, } 0 \notin K, \text{ and } \gamma(K) \geq k\}$, γ is the genus function, or equivalently

$$\frac{1}{\mu_k(h_n)} \triangleq \sup_{K\in\mathcal{F}_k} \min_{u\in K} \int_0^1 f_0 h_n(t)|u|^p\,dt,$$

where $\mathcal{F}_k = \{K \cap S : K \in \mathcal{E}_k\}$ and S is the unit sphere of $W_0^{1,p}(0,1)$. Furthermore, we have the variation form of the first eigenvalue as follows:

$$\frac{1}{\mu_1(h_n)} = \sup_{u\in S} \int_0^1 f_0 h_n(t)|u|^p\,dt = \int_0^1 f_0 h_n(t)|\phi|^p\,dt,$$

where for some $\phi \in S$. For convenience, we denote $\mu_k^n = \mu_k(h_n)$. The following properties are well-known (see, Theorem 1 in [1]):

(i) Every eigenfunction corresponding to the k-th eigenvalue μ_k^n has exactly $k-1$ many interior zeros in $(0,1)$.

(ii) For each k, μ_k^n is simple and verifies the strict monotonicity property with respect to indefinite weight h_n.

(iii) $\sigma_p(\varphi_p(u')', h_n) = \{\mu_k^n : k \in \mathbb{N}\}$. The eigenvalues are ordered as $0 < \mu_1^n < \ldots < \mu_k^n < \ldots \to \infty$ as $k \to \infty$.

The third property tells us the eigenvalues are only produced by Lusternik–Schnirelmann critical point theory. Hardy's inequality will be used to make a lower bound of a sequence $\{\mu_k^n\}$ of the strictly decreasing eigenvalues for (QEP^n_μ).

LEMMA 2.6. *Assume* $1 < p < 2$, $h \in \mathcal{B} \setminus \mathcal{A}$ *and* (L). *Then*

$$\mu_1^n \geq \frac{(1/2)^{p+1}}{f_0 M(p/(p-1))^p} > 0, \quad \textit{for all } n \in \mathbb{N}.$$

Moreover, we have

$$\mu_1^n \to \mu_1^\infty \geq \frac{(1/2)^{p+1}}{f_0 M(p/(p-1))^p}, \quad \textit{as } n \to \infty.$$

PROOF. Assume $1 < p < 2$, $h \in \mathcal{B} \setminus \mathcal{A}$ and (L). Then, since

$$t^{-p}\left(\frac{1}{2}\right)^{-p} \geq t^{-p}(1-t)^{-p}, \quad \text{for } t \in \left(0, \frac{1}{2}\right],$$
$$(1-t)^{-p}\left(\frac{1}{2}\right)^{-p} \geq t^{-p}(1-t)^{-p}, \quad \text{for } t \in \left[\frac{1}{2}, 1\right),$$

we have

$$t^{-p}(1-t)^{-p} \le \left(\frac{1}{2}\right)^{-p}(t^{-p}+(1-t)^{-p}), \quad \text{for } t \in (0,1).$$

Therefore, using (2.1) and (2.2), we get

$$\begin{aligned}\frac{\int_0^1 |u'(s)|^p\,ds}{\int_0^1 f_0 h_n(s)|u(s)|^p\,ds} &\ge \frac{\int_0^1 |u'(s)|^p\,ds}{\int_0^1 f_0 h(s)|u(s)|^p\,ds} \\ &\ge \frac{1}{f_0 M(1/2)^{-p}} \cdot \frac{\int_0^1 |u'(s)|^p\,ds}{\int_0^1 s^{-p}|u(s)|^p + (1-s)^{-p}|u(s)|^p\,ds} \\ &\ge \frac{(1/2)^p}{f_0 M} \cdot \frac{\int_0^1 |u'(t)|^p\,ds}{2(p/(p-1))^p \int_0^1 |u'(t)|^p\,ds} \\ &\ge \frac{(1/2)^{p+1}}{f_0 M (p/(p-1))^p} \triangleq c.\end{aligned}$$

Thus, we have

$$\sup_{u\in S}\int_0^1 f_0 h_n |u|^p\,ds \le \frac{1}{c}.$$

This implies $1/\mu_1^n \le 1/c$. Hence,

$$\mu_1^n \ge \frac{(1/2)^{p+1}}{f_0 M(p/(p-1))^p} > 0.$$

Since μ_1^n is decreasing, we obtain $\mu_1^n \to \mu_1^\infty$. □

Remark 2.7. We note that for each $k = 2, 3, \ldots$,

$$\mu_k^n \ge \mu_1^n \ge \frac{(1/2)^{p+1}}{f_0 M(p/(p-1))^p} > 0, \quad \text{for all } n \in \mathbb{N},$$

and μ_k^n is decreasing on n, thus we have

$$\mu_k^n \to \mu_k^\infty \ge \mu_1^\infty \ge \frac{(1/2)^{p+1}}{f_0 M(p/(p-1))^p}, \quad \text{as } n \to \infty.$$

We finally introduce the Picone's identity which will be used to figure out the shape of an unbounded subcontinuum. Let us consider the following two operators:

$$l_p[y] = (\varphi_p(y'))' + b_1(t)\varphi_p(y), \qquad L_p[z] = (\varphi_p(z'))' + b_2(t)\varphi_p(z).$$

Lemma 2.8 ([7, p. 382]). *Let $b_1, b_2 \in C(I)$, I an interval and if y and z are functions such that y, z, $\varphi_p(y')$, and $\varphi_p(z')$ are differentiable on I and $z(t) \ne 0$ for $t \in I$. Then we have the following identity:*

$$\begin{aligned}\frac{d}{dt}\left(\frac{|y|^p\varphi_p(z')}{\varphi_p(z)} - y\varphi_p(y')\right) &= (b_1 - b_2)|y|^p \\ &\quad - \left(|y'|^p + (p-1)\left|\frac{yz'}{z}\right|^p - p\varphi_p(y)y'\varphi_p\left(\frac{z'}{z}\right)\right) - yl_p(y) + \frac{|y|^p}{\varphi_p(z)}L_p(z).\end{aligned} \tag{2.3}$$

Remark 2.9. By Young's inequality, we get

$$|y'|^p + (p-1)\left|\frac{yz'}{z}\right|^p - p\varphi_p(y)y'\varphi_p\left(\frac{z'}{z}\right) \geq 0,$$

and the equality holds if and only if $\operatorname{sgn} y' = \operatorname{sgn} z'$ and $|y'/y|^p = |z'/z|^p$.

3. Properties of interior zeros of solutions

In this section, we investigate some properties of interior zeros of sign-changing solutions for (QP_λ).

Lemma 3.1. *Assume $h \in \mathcal{B}$. Also assume $0 \leq f_0 < \infty$. Let u be a solution of (QP_λ). Then the following assertions are valid.*

(a) *If $u(t_0) = u'(t_0) = 0$ for some $t_0 \in (0,1)$, then $u \equiv 0$ in $(0,1)$.*
(b) *If u has a sequence of zeros converging to 0 or 1, then $u \equiv 0$ in $(0,1)$.*
(c) *If $u \not\equiv 0$, then u has at most a finite number of zeros.*

Proof. (a) Let u be a solution of (QP_λ). Then by the fact $u \in C[0,1]$ and $0 \leq f_0 < \infty$, there exists a constant $C > 0$ such that

$$|f(u(t))| \leq C|u(t)|^{p-1} \quad \text{for } t \in [0,1]. \tag{3.1}$$

Define $A \equiv \{t \in (0,1) : u(t) = u'(t) = 0\}$. Then clearly A is relatively closed in $(0,1)$. We show that it is open too. Let $t_0 \in A$. We first find two sequences $\{t_n\}$ and $\{s_n\}$ satisfying

$$t_n > t_0 > s_n, \quad u(t_n) = u(s_n) = 0, \quad \lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = t_0.$$

We consider sequence $\{t_n\}$. Argument for sequence $\{s_n\}$ is exactly the same. Suppose on the contrary that $u(t) > 0$ for $t \in (t_0, t_0+\varepsilon)$ with small ε. In case of $u(t) < 0$ in this interval, our argument below remains the same. Then by assumption $sf(s) > 0$ for $s \neq 0$, $f(u(t)) > 0$ for $t \in (t_0, t_0+\varepsilon)$. However, since u is concave and $u'(t_0) = u(t_0) = 0$, $u(t) \leq 0$ in this interval. This contradiction provides us sequence $\{t_n\}$. If we show $u \equiv 0$ on (s_n, t_n) for some n, then set A is open. For this, let us fix n sufficiently large such that

$$\int_{t_{n-1}}^{t_n} \varphi_p^{-1}\left(\lambda C \int_{t_{n-1}}^{t_n} h(\tau)\,d\tau\right) ds < 1.$$

Suppose $u \not\equiv 0$ on (t_{n-1}, t_n) and without loss of generality, assume $u > 0$ on the interval. Then taking $u(\widetilde{t}_n) = \max_{t\in[t_{n-1},t_n]} u(t)$ and $\widetilde{t}_n \in (t_{n-1}, t_n)$, we have, from the equation in (QP_λ),

$$u(\widetilde{t}_n) = \int_{t_{n-1}}^{\widetilde{t}_n} \varphi_p^{-1}\left(\lambda \int_s^{\widetilde{t}_n} h(\tau) f(u(\tau))\,d\tau\right) ds.$$

Thus, we have

$$\begin{aligned}|u(\widetilde{t}_n)| &\le \int_{t_{n-1}}^{\widetilde{t}_n} \varphi_p^{-1}\bigg(\lambda \int_{t_{n-1}}^{\widetilde{t}_n} h(\tau)|f(u(\tau))|\,d\tau\bigg)\,ds \\ &\le \int_{t_{n-1}}^{\widetilde{t}_n} \varphi_p^{-1}\bigg(\lambda C \int_{t_{n-1}}^{\widetilde{t}_n} h(\tau)\,d\tau\bigg)\,ds|u(\widetilde{t}_n)|.\end{aligned}$$

This is a contradiction to the choice of n. Thus, we have $u \equiv 0$ on (t_0, t_n). By the similar way, we also have $u \equiv 0$ on (s_n, t_0). Thus A is open which implies $A \equiv (0,1)$. Therefore $u \equiv 0$ on $[0,1]$ and the proof of (a) is complete.

(b) Let $\{t_n\}$ be a sequence such that

$$t_n > t_{n+1} > 0, \quad \lim_{n\to\infty} t_n = 0, \quad u(t_n) = 0.$$

By the same argument as in the proof of (i), we have $u \equiv 0$ in (t_{n+1}, t_n) for sufficiently large n and eventually u vanishes in $(0,1)$. When $\{t_n\}$ converges to 1, the proof is the same.

(c) If u has a distinct sequence of zeros in $(0,1)$, then it has a convergent subsequence. Denote the limit by t_0. If $t_0 \in (0,1)$, then $u'(t_0) = u(t_0) = 0$, hence $u \equiv 0$ by (a). If $t_0 = 0$ or 1, then $u \equiv 0$ by (b). This completes the proof of (c). □

Let us consider

$$(\mathrm{QP}_\lambda^n) \qquad \begin{cases} \varphi_p(u'(t))' + \lambda h_n(t) f(u(t)) = 0 & \text{for } t \in (0,1), \\ u(0) = 0 = u(1), \end{cases}$$

where $h_n(t) = \min\{n, h(t)\}$, $n \in \mathbb{N}$.

Lemma 3.2. *Assume $h \in \mathcal{B}$. Also assume $0 \le f_0 < \infty$. Let u_n and u be nontrivial solutions of $(\mathrm{QP}_{\lambda_n}^n)$ and (QP_λ), respectively, and each u_n have exactly $k-1$ many interior zeros in $(0,1)$. If $u_n \to u$ and $\lambda_n \to \lambda > 0$, then u also has exactly $k-1$ many interior zeros in $(0,1)$.*

Proof. First, we claim that there exist $\delta_1, \delta_2 > 0$ such that

$$\bigcup_{n=1}^{\infty} \{t \in (0,1) : u_n(t) = 0\} \subset [\delta_1, \delta_2] \subset (0,1).$$

Let t_n be the first interior zero of u_n. Then it is enough to show that there exists $\delta_1 > 0$ such that $t_n > \delta_1$ for all n. We may prove this claim for the sequence of last interior zeros by similar fashion. Suppose on the contrary, $t_n \to 0$. We may assume, without loss of generality by Lemma 3.1(c), $u > 0$ near 0. Then since $u_n \to u$ in $C[0,1]$, considering a subsequence if necessary, we have an alternative, either $u_n > 0$ on $(0, t_n)$ for all n or $u_n < 0$ on $(0, t_n)$ for all n. If $u_n > 0$ on $(0, t_n)$ for all n, then u_n is a positive solution of the following problem

$$\begin{cases} \varphi_p(u_n'(t))' + \lambda_n h_n(t) f(u_n(t)) = 0, \\ u_n(0) = 0 = u_n(t_n). \end{cases} \tag{3.2}$$

For the other alternative, $v_n = -u_n$ is a positive solution of (3.2) by oddity of φ_p and f so that the proof follows exactly the same lines with v_n. Taking $u_n(\widetilde{t}_n) = \max_{t\in[0,t_n]} u_n(t)$, we get

$$\begin{aligned} &\int_0^{\widetilde{t}_n} \varphi_p^{-1}\left(\lambda_n \int_s^{\widetilde{t}_n} h_n(\tau) f(u_n(\tau))\, d\tau\right) ds \\ &\qquad\qquad = \int_{\widetilde{t}_n}^{t_n} \varphi_p^{-1}\left(\lambda_n \int_{\widetilde{t}_n}^{s} h_n(\tau) f(u_n(\tau))\, d\tau\right) ds. \end{aligned}$$

Since $u_n \to u$, we may assume $|u_n(t)| < ||u||_\infty + 1$, for all $t \in [0, t_n]$ and all n. On the interval $(0, t_n), u_n(t) > 0$ for all n and from condition $0 \le f_0 < \infty$, there exists $C_{\lambda,u} > 0$ such that

$$|\lambda_n h_n(s) f(u_n(s))| \le C_{\lambda,u} h(s) \varphi_p(u_n(s)).$$

Thus,

$$\begin{aligned} u_n(\widetilde{t}_n) &= \int_0^{\widetilde{t}_n} \varphi_p^{-1}\left(\int_s^{\widetilde{t}_n} \lambda_n h_n(\tau) f(u_n(\tau))\, d\tau\right) ds \\ &\le \int_0^{\widetilde{t}_n} \varphi_p^{-1}\left(\int_s^{\widetilde{t}_n} C_{\lambda,u} h(\tau) \varphi_p(u_n(\tau)) d\tau\right) ds \\ &\le u_n(\widetilde{t}_n) \int_0^{\widetilde{t}_n} \varphi_p^{-1}\left(\int_s^{\widetilde{t}_n} C_{\lambda,u} h(\tau)\, d\tau\right) ds. \end{aligned}$$

This implies

$$1 \le \int_0^{\widetilde{t}_n} \varphi_p^{-1}\left(\int_s^{\widetilde{t}_n} C_{\lambda,u} h(\tau)\, d\tau\right) ds.$$

Since $\varphi_p^{-1}(\int_s^{\widetilde{t}_n} C_{\lambda,u} h(\tau)\, d\tau) \in L^1(0, \gamma]$, for some $\gamma > 0$ and $\widetilde{t}_n \to 0$, the above inequality is not possible. This completes the claim.

Since u has at most a finite number of zeros, it is not hard to see that u has at most $k - 1$ interior zeros, all of them are in $[\delta_1, \delta_2]$. Suppose that the number of interior zeros of u is strictly less than $k - 1$. Then by the fact $u_n \to u$, there exist $t^* \in [\delta_1, \delta_2]$ a zero of u and $\delta > 0$ such that u_n (a subsequence if necessary) has at least two consecutive interior zeros, $t_{n,1}$, $t_{n,2}$ with $t_{n,1}, t_{n,2} \in (t^* - \delta, t^* + \delta)$ for all sufficiently large n. Also, we may assume $|u_n(t)| \le ||u||_\infty + 1$ and $u_n > 0$ on $(t_{n,1}, t_{n,2})$ for all n. It follows from condition $0 \le f_0 < \infty$ that there exists $\widetilde{C}_{\lambda,u} > 0$ such that $|\lambda_n h_n(s) f(u_n(s))| \le \widetilde{C}_{\lambda,u} h(s) u_n(s)^{p-1}$, on the interval.

Let us choose $\delta > 0$ satisfying

$$\delta < \frac{1}{2\varphi_p^{-1}(\int_{\delta_1}^{\delta_2} \widetilde{C}_{\lambda,u} h(\tau)\, d\tau)}.$$

Then for $u_n(\widetilde{t}_n) = \max_{t\in[t_{n,1},t_{n,2}]} u_n(t)$, we have

$$\begin{aligned} u_n(\widetilde{t}_n) &= \int_{t_{n,1}}^{\widetilde{t}_n} \varphi_p^{-1}\bigg(\int_s^{\widetilde{t}_n} \lambda_n h_n(\tau) f(u_n(\tau))\, d\tau\bigg)\, ds \\ &\leq \int_{t_{n,1}}^{\widetilde{t}_n} \varphi_p^{-1}\bigg(\int_s^{\widetilde{t}_n} \widetilde{C}_{\lambda,u} h(\tau) \varphi_p(u_n(\tau))\, d\tau\bigg)\, ds \\ &\leq u_n(\widetilde{t}_n) \int_{t_{n,1}}^{\widetilde{t}_n} \varphi_p^{-1}\bigg(\int_s^{\widetilde{t}_n} \widetilde{C}_{\lambda,u} h(\tau)\, d\tau\bigg)\, ds. \end{aligned}$$

This leads a contradiction as

$$1 \leq \varphi_p^{-1}\bigg(\int_{\delta_1}^{\delta_2} \widetilde{C}_{\lambda,u} h(\tau)\, d\tau\bigg) 2\delta < 1. \qquad \square$$

4. The case $0 < f_0 < \infty$

In this section, we show that μ_k^∞ is an eigenvalue for (QEP_μ) and the existence of unbounded continua of solutions for problem (QP_λ) which are bifurcating from $(\mu_k^\infty, 0)$, where μ_k^∞ is given in Lemma 2.6 and Remark 2.7. For general theory for global bifurcation argument, one may refer to [13]. We consider the following hypotheses.

- (A_1) $0 < f_0 < \infty$,
- (A_2) $f_\infty = 0$,
- (A_3) $f_\infty = \infty$.

Assumptions (A_1) and (A_2) imply that there exists $L_f > 0$ with $L_f \geq f_0$ such that

$$f(u) \leq L_f u^{p-1}, \quad \text{for all } u \geq 0.$$

Also assumptions (A_1) and (A_3) imply that there exists $\widetilde{L}_f > 0$ with $\widetilde{L}_f \leq f_0$ such that

$$f(u) \geq \widetilde{L}_f u^{p-1}, \quad \text{for all } u \geq 0.$$

Our main theorems in this section are as follows:

THEOREM 4.1. *Assume* $1 < p < 2$, $h \in \mathcal{B} \setminus \mathcal{A}$, (L), (A_1) *and* (A_2). *Then for each* $k \in \mathbb{N}$, *there exist* λ^* *and* λ^{**} *with* $0 < \lambda^{**} \leq \lambda^* \leq \mu_k^\infty$ *such that* (QP_λ) *has at least one solution for all* $\lambda \in (\lambda^*, \infty)$ *which has exactly* $k-1$ *many interior zeros in* $(0,1)$ *and no solution for* $\lambda \in (0, \lambda^{**})$. *Moreover, we have* $\lambda^* \geq (1/2)^{p+1}/(L_f M(p/(p-1))^p)$, *where* M *appeared in condition* (H_1).

Theorem 4.2. *Assume* $1 < p < 2$, $h \in \mathcal{B} \setminus \mathcal{A}$, (L), (A_1) *and* (A_3). *Then for each* $k \in \mathbb{N}$, *there exists* $\lambda_* \in [\mu_k^\infty, \mu_k^1 f_0/\widetilde{L}_f]$ *such that* (QP_λ) *has at least one solution for* $\lambda \in (0, \lambda_*)$ *which has exactly* $k-1$ *many interior zeros in* $(0,1)$ *and no solution for* $\lambda \in (\mu_k^1 f_0/\widetilde{L}_f, \infty)$.

To employ an approximation technique, let us consider

$$(\mathrm{QP}_\lambda^n) \qquad \begin{cases} \varphi_p(u'(t))' + \lambda h_n(t) f(u(t)) = 0 & \text{for } t \in (0,1), \\ u(0) = 0 = u(1), \end{cases}$$

where $h_n(t) = \min\{n, h(t)\}$, $n \in \mathbb{N}$. Since $h_n > 0$ and $h_n \in C[0,1]$, for all n, we obtain the following lemma from Theorem 2.1, Lemmas 3.3 and 4.1 in [8].

Lemma 4.3. *Assume* (A_1) *and* (A_2). *Then for each* $k \in \mathbb{N}$, *there exists an unbounded continuum* $\mathcal{B}_k^n$ *in* $\mathcal{S}^n$ *bifurcating from* $(\mu_k^n, 0)$, *where* $\mathcal{S}^n$ *is the closure of set of nontrivial solutions for* (QP_λ^n) *and* μ_k^n *is the* k*-th eigenvalue of problem* (QEP_μ^n). *If* $(\lambda, u_k^n) \in \mathcal{B}_k^n$, *then* u_k^n *has exactly* $k-1$ *many interior zeros in* $(0,1)$ *and* $\lambda \geq \mu_k^n f_0/L_f$.

Similarly, from Theorem 2.1, Lemmas 3.11 and 4.4 in [8], we have the following.

Lemma 4.4. *Assume* (A_1) *and* (A_3). *Then for each* $k \in \mathbb{N}$, *there exists an unbounded continuum* $\mathcal{C}_k^n$ *in* $\mathcal{S}^n$ *bifurcating from* $(\mu_k^n, 0)$. *If* $(\lambda, u_k^n) \in \mathcal{C}_k^n$, *then* u_k^n *has exactly* $k-1$ *many interior zeros in* $(0,1)$ *and* $\lambda \leq \mu_k^n f_0/\widetilde{L}_f$.

To show the existence of unbounded branch of sign-changing solutions of (QP_λ), we apply a limit process with components $\mathcal{B}_k^n$ (or $(\mathcal{C}_k^n)$) using some results in set topology. Let us start recalling the following definition.

Definition 4.5 ([14]). Let G be any infinite collection of point sets. The set of all points x such that every neighbourhood of x contains points of infinitely many sets of G is called the *superior limit of* G ($\limsup G$). The set of all points y such that every neighbourhood of y contains points of all but a finite number of sets of G is called the *inferior limit of* G ($\liminf G$).

Proposition 4.6 ([14]). *Let* $\{A_n\}_{n\in\mathbb{N}}$ *be a sequence of connected closed sets such that* $\liminf\{A_n\} \neq \emptyset$. *Then if set* $\cup_{n\in\mathbb{N}}\{A_n\}$ *is relatively compact, then* $\limsup\{A_n\}$ *is connected.*

Let $R > 0$ be given and let us denote $\mathcal{B}_k^n(R)$ (or $\mathcal{C}_k^n(R)$) the component containing $(\mu_k^n, 0)$ of $\mathcal{B}_k^n$ (or $\mathcal{C}_k^n$) $\cap ([0, \mu_k^\infty + R] \times \{u \in C[0,1] : ||u|| \leq R\})$. Clearly, $\{(\mu_k^\infty, 0)\} \subset \liminf \mathcal{B}_k^n(R)$ (or $\liminf \mathcal{C}_k^n(R)$). From now on, we denote $\mathcal{B}_k^\infty = \limsup \mathcal{B}_k^n$ (or $\mathcal{C}_k^\infty = \limsup \mathcal{C}_k^n$). Then, key step for proofs is to show that $\mathcal{B}_k^\infty$ (or $\mathcal{C}_k^\infty$) is connected, thus by Proposition 4.6, we only need to show that $\bigcup_{n\in\mathbb{N}}\{\mathcal{B}_k^n\}$ (or $\bigcup_{n\in\mathbb{N}}\{\mathcal{C}_k^n\}$) is relatively compact. We first show that $\mathcal{B}_k^n$ (or $\mathcal{C}_k^n$) will not be degenerated in λ-axis and $||\cdot||$-axis.

LEMMA 4.7. *Assume* (A_1). *Let I be any compact subinterval of* $[0,\mu_k^\infty)$. *Then for each $k\in\mathbb{N}$, there exists $a_I>0$ such that for all $(\lambda,u)\in\mathcal{B}_k^n$ (or $\mathcal{C}_k^n$) with $\lambda\in I$ and $n\in\mathbb{N}$, we have* $\|u\|\geq a_I$.

PROOF. Let I and k be given. If the conclusion is not true, then we have an alternative; either some $\mathcal{B}_k^n$ (or $\mathcal{C}_k^n$) contains a point on λ-axis other than $(\mu_k^n,0)$ or there exists a subsequence (if necessary) $\{(\lambda_k^n,u_k^n)\}\subset\mathbb{R}\times C[0,1]$ with $(\lambda_k^n,u_k^n)\in\mathcal{B}_k^n$ (or $\mathcal{C}_k^n$) and $\lambda_k^n\in I$ such that $\|u_k^n\|\to 0$, as $n\to\infty$. Since for each n, $\mathcal{B}_k^n$ (or $\mathcal{C}_k^n$) is unbounded by Lemma 4.3 (or Lemma 4.4), the first case does not happen. Thus it is enough to consider the second case of the alternative. We note that u_k^n is a solution of following problem

$$\begin{cases} \varphi_p({u_k^n}'(t))'+\lambda_k^n h_n(t)f(u_k^n(t))=0 & \text{for } t\in(0,1),\\ u_k^n(0)=0=u_k^n(1). \end{cases}$$

Let $\varepsilon=(1/2)(\mu_k^\infty/\lambda_M-1)f_0$, where $\lambda_M=\max\{\lambda:\lambda\in I\}$. Thus $\varepsilon>0$. It follows from (A_1) that there exists $\delta>0$ such that

$$f(u)<(f_0+\varepsilon)\varphi_p(u), \tag{4.1}$$

for $u\in(0,\delta)$. Since $\|u_k^n\|\to 0$, we can choose $N\in\mathbb{N}$ such that $\|u_k^N\|<\delta$.

For $k=1$, it follows from (4.1) that

$$\begin{aligned} \varphi_p({u_1^N}'(t))'+\lambda_1^N(f_0+\varepsilon)h_N(t)\varphi_p(u_1^N(t))&\\ \geq \varphi_p({u_1^N}'(t))'+\lambda_1^N h_N(t)f(u_1^N(t))&=0. \end{aligned}$$

Let ϕ_1^N be the first eigenfunction corresponding to μ_1^N with $\phi_1^N>0$;

$$\begin{cases} \varphi_p({\phi_1^N}'(t))'+\mu_1^N f_0 h_N(t)\varphi_p(\phi_1^N(t))=0 & \text{for } t\in(0,1),\\ \phi_1^N(0)=0=\phi_1^N(1). \end{cases}$$

Taking $y=u_1^N, b_1(t)=\lambda_1^N(f_0+\varepsilon)h_N(t)$ and $z=\phi_1^N$, $b_2(t)=\mu_1^N f_0 h_N(t)$ in Lemma 2.8 and integrating (2.3), we have

$$(\mu_1^N f_0-\lambda_1^N(f_0+\varepsilon))\int_0^1 h_N(t)|u_1^N(t)|^p\,dt\leq 0.$$

Thus, we get $\lambda_1^N(f_0+\varepsilon)-\mu_1^N f_0\geq 0$. This contradicts the choice of ε. Next, for $k\geq 2$, let t_1^* and t_1 be the first zeros of ϕ_k^N and u_k^N, with $\phi_k^N>0$ in $(0,t_1^*)$ and $u_k^N>0$ in $(0,t_1)$, and t_{k-1}^* and t_{k-1} be the last zeros of ϕ_k^N and u_k^N, respectively. First, for $k=2$, suppose $t_1\leq t_1^*$. Then it is easy to show the following equalities (Lemma 3.2 in [8])

$$\int_0^{t_1}\bigg\{\frac{|u_2^N|^p{\phi_2^N}'^{(p-1)}}{{\phi_2^N}^{(p-1)}}\bigg\}'\,dt=0 \quad\text{and}\quad \int_0^{t_1}-\{u_2^N{u_2^N}'^{(p-1)}\}'\,dt=0,$$

where $x^{(p-1)} = \varphi_p(x)$. Using (4.1), for $t \in (0, t_1)$, we get

$$\varphi_p({u_2^N}'(t))' + \lambda_2^N (f_0 + \varepsilon) h_N(t) \varphi_p(u_2^N(t)) \geq \varphi_p({u_2^N}'(t))' + \lambda_2^N h_N(t) f(u_2^N(t)).$$

Obviously, for $t \in (0, t_1)$, we have

$$0 = \varphi_p({\phi_2^N}'(t))' + \mu_2^N f_0 h_N(t) \varphi_p(\phi_2^N(t)).$$

If we take $y = u_2^N$, $b_1(t) = \lambda_2^N (f_0 + \varepsilon) h_N(t)$ and $z = \phi_2^N$, $b_2(t) = \mu_2^N f_0 h_N(t)$ and integrate (2.3) from 0 to t_1 in Lemma 2.8, then we have

$$(\mu_2^N f_0 - \lambda_2^N (f_0 + \varepsilon)) \int_0^{t_1} h_N(t) |u_2^N(t)|^p \, dt \leq 0.$$

Thus we get a contradiction as before. Suppose $t_1^* \leq t_1$. Then it is easy to show

$$\int_{t_1}^1 \left\{ \frac{|u_2^N|^p {\phi_2^N}'^{(p-1)}}{\phi_2^{N(p-1)}} \right\}' dt = 0 \quad \text{and} \quad \int_{t_1}^1 -\{u_2^N {u_2^N}'^{(p-1)}\}' \, dt = 0.$$

Since $u_2^N < 0$, for $t \in (t_1, 1)$, and

$$0 = \varphi_p({u_2^N}'(t))' + \lambda_2^N h_N(t) f(u_2^N(t)) \geq \varphi_p({u_2^N}'(t))' + \lambda_2^N (f_0 + \varepsilon) h_N(t) \varphi_p(u_2^N(t)),$$

we have

$$-u_2^N [\varphi_p({u_2^N}'(t))' + \lambda_2^N (f_0 + \varepsilon) h_N(t) \varphi_p(u_2^N(t))] \leq 0, \quad t \in (t_1, 1).$$

Clearly, for $t \in (t_1, 1)$, we have

$$0 = \varphi_p({\phi_2^N}'(t))' + \mu_2^N f_0 h_N(t) \varphi_p(\phi_2^N(t)).$$

Thus, if we take $y = u_2^N$, $b_1(t) = \lambda_2^N (f_0 + \varepsilon) h_N(t)$ and $z = \phi_2^N$, $b_2(t) = \mu_2^N f_0 h_N(t)$ and integrate (2.3) from t_1 to 1, then we obtain the same inequality as before and get a contradiction. Finally, suppose $k \geq 3$. If $t_1 \leq t_1^*$ or $t_{k-1}^* \leq t_{k-1}$, then we obtain a contradiction by the same process as in the case $k = 2$. If $t_1^* < t_1$ and $t_{k-1} < t_{k-1}^*$, then there exists an interval $(t_i, t_{i+1}) \subset (t_i^*, t_{i+1}^*)$ for some $1 < i < k$, and on the interval, we have

$$\int_{t_i}^{t_{i+1}} \left\{ \frac{|u_k^N|^p {\phi_k^N}'^{(p-1)}}{\phi_k^{N(p-1)}} - u_k^N {u_k^N}'^{(p-1)} \right\}' dt = 0.$$

Thus for either $u_k^N > 0$ in (t_i, t_{i+1}) or $u_k^N < 0$ in (t_i, t_{i+1}), we have

$$-u_k^N [\varphi_p({u_k^N}'(t))' + \lambda_k^N (f_0 + \varepsilon) h_N(t) \varphi_p(u_k^N(t))] \leq 0.$$

By the similar argument as in the case $k = 2$, we get a contradiction. $\square$

Lemma 4.8. *Assume* (A_1). *Let* I *be any compact subinterval of* (μ_k^∞, ∞). *Then for each* $k \in \mathbb{N}$, *there exists* $b_I > 0$ *such that for all* $(\lambda, u) \in \mathcal{B}_k^n$ (*or* $\mathcal{C}_k^n$) *with* $\lambda \in I$ *and* $n \in \mathbb{N}$, *we have* $||u|| \geq b_I$.

Proof. If the conclusion is not true, then without loss of generality, we assume that there exists a subsequence (if necessary) $\{(\lambda_k^n, u_k^n)\} \subset \mathbb{R} \times C[0,1]$ with $(\lambda_k^n, u_k^n) \in \mathcal{B}_k^n$ (or $\mathcal{C}_k^n$) and $\lambda_k^n \in I$ such that $||u_k^n|| \to 0$, as $n \to \infty$. We note that u_k^n is a solution of following problem

$$\begin{cases} \varphi_p({u_k^n}'(t))' + \lambda_k^n h_n(t) f(u_k^n(t)) = 0 & \text{for } t \in (0,1), \\ u_k^n(0) = 0 = u_k^n(1). \end{cases}$$

Assume $\lambda_k^n \to \widehat{\lambda} > \mu_k^\infty$ and let $\varepsilon = \frac{(}{1}/2) f_0(1 - \mu_k^\infty/\widehat{\lambda})$. Thus $\varepsilon > 0$. By condition (A_1), there exists $\delta > 0$ such that

$$f(u) > (f_0 - \varepsilon)\varphi_p(u), \tag{4.2}$$

for $u \in (0, \delta)$. Using the similar argument in the proof of Lemma 4.7 with (4.2) instead of (4.1), we get a contradiction. □

Lemma 4.9. *Assume* (A_1). *For given* $N > 0$ *and* $k \in \mathbb{N}$, *there exists* $\eta_N > 0$ *such that if* $(\lambda, u) \in \mathcal{B}_k^n$ (*or* $\mathcal{C}_k^n$) *and* $||u|| \leq N$, *then we have* $\lambda > \eta_N$.

Proof. If the conclusion is not true, then without loss of generality, we assume that there exists a subsequence (if necessary) $\{(\lambda_n, u_n)\} \subset \mathbb{R} \times C[0,1]$ with $(\lambda_n, u_n) \in \mathcal{B}_k^n$ (or $\mathcal{C}_k^n$) such that $||u_n|| \leq N$ and $\lambda_n \to 0$ as $n \to \infty$. By (A_1), there exists $A_N > 0$ such that $f(u) \leq A_N u^{p-1}$ for all $0 \leq u \leq N$. Using the similar argument in the proof Lemma 4.7, we get a contradiction and completes the proof. □

The next two lemmas are essential to apply Proposition 4.6.

Lemma 4.10. *Assume* (A_1). *For given* $R > 0$ *and* $k \in \mathbb{N}$, *let* $(\lambda_n, u_n) \in \mathcal{B}_k^n(R)$ (*or* $\mathcal{C}_k^n(R)$). *Then there exist subsequences* $\{u_m\}$, $\{\lambda_m\}$ *and* $u \in C[0,1]$ *such that* $u_m \to u$ *in* $C[0,1]$ *and* $\lambda_m \to \lambda > 0$. *Furthermore,* u *is a solution of* (QP_λ).

Proof. Let R and k be given and let $(\lambda_n, u_n) \in \mathcal{B}_k^n(R)$ (or $\mathcal{C}_k^n(R)$). Then since u_n is a solution for $(\mathrm{QP}_{\lambda_n}^n)$, we have

$$\begin{cases} \varphi_p(u_n'(t))' + \lambda_n h_n(t) f(u_n(t)) = 0 & \text{for } t \in (0,1), \\ u_n(0) = 0 = u_n(1). \end{cases}$$

Take $f_R := (\mu_k^\infty + R)\sup_{u \in [0,R]} f(u)$. Then by the similar arguments in Lü and O'Regan [10] with $c = 0$ (Lemmas 2.2–2.4), there exist unique solutions

$V, v \in C([0,1],\mathbb{R}) \cap C^1((0,1),\mathbb{R})$ for problems

$$\begin{cases} \varphi_p(V'(t))' + f_R h(t) = 0 & \text{for } t \in (0,1), \\ V(0) = 0 = V(1), \end{cases}$$

and

$$\begin{cases} \varphi_p(v'(t))' - f_R h(t) = 0 & \text{for } t \in (0,1), \\ v(0) = 0 = v(1), \end{cases}$$

respectively and V, v satisfy, for all n, the following relation:

$$v(t) \le u_n(t) \le V(t), \quad t \in [0,1].$$

Moreover, we may also prove that $\{u_n\}$ is equicontinuous on $[0,1]$ by the similar argument in the proof of Lemma 2.6 in [10]. Thus Ascoli–Arzela Theorem guarantees that there exist a subsequence $\{u_m\}$ and $u \in C[0,1]$ such that u_m converges to u in $C[0,1]$ with $u(0) = 0 = u(1)$. Since $\lambda_n \le \mu_k^\infty + R$, there exists a subsequence $\{\lambda_m\}$ such that $\lambda_m \to \lambda$ and by Lemma 4.9, $\lambda > 0$. Finally, we show that u is a solution for (QP_λ). Let $[a,b]$ be a compact interval in $(0,1)$. Then there exists n^* such that $h_m(t) = h(t)$ in $[a,b]$ for $m \ge n^*$. We know that u_m converges uniformly to u on $[a,b]$ and thus by Lebesgue Convergence Theorem, u satisfies

$$u(t) = u(a) + \int_a^t \varphi_p^{-1}\left(d + \int_\tau^b \lambda h(\tau) f(u(\tau))\,d\tau\right) ds, \quad a \le t \le b,$$

where d is a solution of the equation

$$\int_a^b \varphi_p^{-1}\left(d + \int_\tau^b \lambda h(\tau) f(u(\tau))\,d\tau\right) ds = u(b) - u(a).$$

This implies that u satisfies the equation in (QP_λ) on $[a,b]$. Since $[a,b]$ is arbitrary in $(0,1)$, $u \in C^1(0,1)$ and satisfies

$$\varphi_p(u')' + \lambda h(t) f(u(t)) = 0, \quad t \in (0,1).$$

This completes the proof. □

Once again, employing the above argument, we can show that μ_k^∞ is an eigenvalue for (QEP_μ), for each k.

Proposition 4.11. *For each $k \in \mathbb{N}$, μ_k^∞ is an eigenvalue of* (QEP_μ).

Proof. By the construction of μ_k^∞, there is a sequence $\{\mu_k^n\}$ such that converges to μ_k^∞ as $n \to \infty$ and μ_k^n is an eigenvalue of (QEP_μ^n). Let us denote its corresponding eigenfunction by u_k^n. Then we have

$$\begin{cases} \varphi_p({u_k^n}'(t))' + \mu_k^n f_0 h_n(t) \varphi_p(u_k^n(t)) = 0 & \text{for } t \in (0,1), \\ u_k^n(0) = 0 = u_k^n(1). \end{cases}$$

Denoting $v_n = u_k^n/||u_k^n||$, we have

$$\begin{cases} \varphi_p(v_n'(t))' + \mu_k^n f_0 h_n(t)\varphi_p(v_n(t)) = 0 & \text{for } t \in (0,1), \\ v_n(0) = 0 = v_n(1). \end{cases}$$

Since $\{\mu_k^n\}$ converges and $||v_n|| = 1$, we have

$$|\mu_k^n f_0 h_n(t)\varphi_p(v_n)| \le Mh(t), \tag{4.3}$$

for some $M > 0$. Once again, following the similar arguments in Lü and O'Regan [10], there exist unique solutions $W, w \in C([0,1],\mathbb{R}) \cap C^1((0,1),\mathbb{R})$ for problems

$$\begin{cases} \varphi_p(W'(t))' + Mh(t) = 0 & \text{for } t \in (0,1), \\ W(0) = 0 = W(1), \end{cases}$$

and

$$\begin{cases} \varphi_p(w'(t))' - Mh(t) = 0 & \text{for } t \in (0,1), \\ w(0) = 0 = w(1), \end{cases}$$

respectively, and W, w satisfy, for all n, the following relation:

$$w(t) \le v_n(t) \le W(t), \quad t \in [0,1].$$

Noting (4.3) and arguing the proof of Lemma 2.6 in [10], we may prove that $\{v_n\}$ is equicontinuous on $[0,1]$. Thus Ascoli–Arzela Theorem guarantees that there exist a subsequence $\{v_m\}$ and $v \in C[0,1]$ such that v_m converges to v in $C[0,1]$ with $v(0) = 0 = v(1)$. (Note that $||v|| = 1$, thus $v \not\equiv 0$). Finally, we can show that v is a nontrivial solution for (QEP_μ), following the similar argument in the last proof of Lemma 4.10. □

The following lemma is true for $\limsup \mathcal{C}_k^n$ by obvious modification.

LEMMA 4.12. *Assume* (A_1). *Then for each* $k \in \mathbb{N}$, $\limsup \mathcal{B}_k^n$ *is connected and if* $(\lambda, u) \in \limsup \mathcal{B}_k^n$, *then* u *is a solution of* (QP_λ).

PROOF. We know $(\mu_k^\infty, 0) \in \liminf \mathcal{B}_k^n(R)$. Thus $\liminf \mathcal{B}_k^n(R) \neq \emptyset$. For connectedness of $\limsup \mathcal{B}_k^n(R)$, we prove that $\bigcup \mathcal{B}_k^n(R)$ is relatively compact in $C[0,1]$. Let $\{(\lambda_n, u_n)\}$ be a sequence in $\bigcup \mathcal{B}_k^n(R)$. Without loss of generality, we may assume the following alternative; as a subsequence, either $(\lambda_m, u_m) \in \mathcal{B}_k^m(R)$ or there exists n_0 such that $(\lambda_m, u_m) \in \mathcal{B}_k^{n_0}(R)$ for all m. The first case can be easily verified by Lemma 4.10. In the second case, $\{(\lambda_m, u_m)\} \subset \mathcal{B}_k^{n_0}(R)$ is bounded and satisfies

$$\begin{cases} \varphi_p(u_m'(t))' + \lambda_m h_{n_0}(t) f(u_m(t)) = 0 & \text{for } t \in (0,1), \\ u_m(0) = 0 = u_m(1). \end{cases}$$

By the similar argument in the proof of Lemma 4.10, there exists a subsequence $\{(\lambda_{m_j}, u_{m_j})\}$ of $\{(\lambda_m, u_m)\}$ such that u_{m_j} converges to u in $C[0,1]$ and $\lambda_{m_j} \to \lambda > 0$. Thus the proof is done as $R \to \infty$. Finally, for given R, let $(\lambda, u) \in$

$\limsup \mathcal{B}_k^n(R)$, then by the definition of $\limsup \mathcal{B}_k^n(R)$, there exists $(\lambda_k^n, u_k^n) \in \mathcal{B}_k^n(R)$ such that $(\lambda_k^n, u_k^n) \to (\lambda, u)$ in $\mathbb{R} \times C[0,1]$. By Lemma 4.10, u is a solution of (QP_λ). Once again, the proof is done as $R \to \infty$. □

Lemma 4.13. *Assume* $1 < p < 2$, $h \in \mathcal{B} \backslash \mathcal{A}$, (L) *and* (A_1). *Let* $(\lambda, u_k) \in \mathcal{B}_k^\infty$ *(or* $\mathcal{C}_k^\infty$*) with* $u_k \not\equiv 0$. *Then* u_k *has exactly* $k-1$ *many interior zeros in* $(0,1)$.

Proof. The conclusion is obvious by the definition of $\mathcal{B}_k^\infty$ (or $\mathcal{C}_k^\infty$) and Lemma 3.2. □

As applications, we obtain the existence results of sign-changing solutions for (QP_λ) when f holds the case $f_\infty = 0$ or $f_\infty = \infty$. For the non-existence result, we have the following lemma.

Lemma 4.14. *Assume* $1 < p < 2$, $h \in \mathcal{B} \backslash \mathcal{A}$, (L), (A_1) *and* (A_2). *Then there exists* $\lambda^{**} > 0$ *such that if* (QP_λ) *has a solution at* λ, *then* $\lambda \geq \lambda^{**}$.

Proof. Let u be a solution for (QP_λ). Then, by Lemma 3.1(c), u has exactly $k-1$ many interior zeros in $(0,1)$, for some positive integer $k > 0$. Let $0 = t_0, t_1, \dots, t_k = 1$ be zeros of u. Then there exists $j \in \{1, \dots, k\}$ such that $t_j - t_{j-1} \geq 1/k$. Since f is odd, we may assume $u > 0$ on (t_{j-1}, t_j). Let

$$I = \left(\frac{3t_{j-1} + t_j}{4}, \frac{t_{j-1} + 3t_j}{4}\right) \quad \text{and} \quad M_I = \max_{t \in [(3t_{j-1}+t_j)/4, (t_{j-1}+3t_j)/4]} h(t),$$

then by (A_1) and (A_2), we have

$$\begin{aligned} 0 &= \varphi_p(u'(t))' + \lambda h(t) f(u(t)) \\ &\leq \varphi_p(u'(t))' + \lambda h(t) L_f \varphi_p(u(t)) \leq \varphi_p(u'(t))' + \lambda M_I L_f \varphi_p(u(t)), \end{aligned}$$

for $t \in I$. Consider the following eigenvalue problem

$$\begin{cases} \varphi_p(\phi'(t))' + \mu f_0 \varphi_p(\phi(t)) = 0 & \text{for } t \in I, \\ \phi\left(\dfrac{3t_{j-1} + t_j}{4}\right) = 0 = \phi\left(\dfrac{t_{j-1} + 3t_j}{4}\right). \end{cases}$$

Let us denote $\mu_1(I)$ the first eigenvalue and $\phi_1 > 0$ its corresponding eigenfunction for the above problem. Applying Picone's identity with $y = u$, $b_1(t) = \lambda M_I L_f$, $z = \phi_1$ and $b_2(t) = \mu_1 f_0$ and integrating over I, we have

$$\lambda M_I L_f - \mu_1 f_0 \geq 0.$$

Thus $\lambda \geq \mu_1(I) f_0 / M_I L_f$ and from $\mu_1(I) \geq \mu_1((0,1))$, we get

$$\lambda \geq \mu_1((0,1)) f_0 / M_I L_f \triangleq \lambda^{**}.$$ □

It is interesting to note that $\mu_1((0,1))$ depends only on p and M_I depends only on h. Now, we give an *a priori* estimate for case $f_\infty = 0$.

Lemma 4.15. *Assume* $1 < p < 2$, $h \in \mathcal{B} \setminus \mathcal{A}$, (L), (A_1) *and* (A_2). *Let* I *be a compact interval in* $(0, \infty)$ *and* $k \in \mathbb{N}$ *be given. Then there exists* $c_I > 0$ *such that for all possible solutions* u *of* (QP_λ) *having exactly* $k-1$ *interior zeros with* $\lambda \in I$, *one has* $||u|| \leq c_I$.

Proof. Suppose on the contrary that there exists a sequence $\{u_n\}$ of solutions for $(\mathrm{QP}_{\lambda_n})$ with $\{\lambda_n\} \subset I = [a, b]$ and $||u_n|| \to \infty$.

Let $\alpha \in (0, 1/(b\varphi_p(\gamma_p 2Q)))$, where $\gamma_p = \max\{1, 2^{(2-p)/(p-1)}\}$,

$$Q = \int_0^{1/2} \varphi_p^{-1}\bigg(\int_s^{1/2} h(\tau)\, d\tau\bigg)\, ds + \int_{1/2}^{1} \varphi_p^{-1}\bigg(\int_{1/2}^{s} h(\tau)\, d\tau\bigg)\, ds.$$

Then by (A_1) and (A_2), there exists $u_\alpha > 0$ such that $u > u_\alpha$ implies $f(u) < \alpha u^{p-1}$. Let $m_\alpha \triangleq \max_{u \in [0, u_\alpha]} f(u)$ and let $t_{1,n}, , \ldots , t_{k-1,n}$ be the zeros of u_n in $(0, 1)$. Put $u_n(\delta_n) = \max_{t \in [0,1]} u_n(t)$ (δ_n may not be unique). Then we can choose $[t_{j,n}, t_{j+1,n}] \ni \delta_n$, for some $j \in \{0, \ldots , k-1\}$ and $f(u([t_{j,n}, t_{j+1,n}])) \geq 0$. Let $A_{n,s} \triangleq \{t \in [s, \delta_n] : u_n(t) \leq u_\alpha,\ t_{j,n} \leq s \leq \delta_n\}$ and $B_{n,s} \triangleq \{t \in [s, \delta_n] : u_n(t) > u_\alpha,\ t_{j,n} \leq s \leq \delta_n\}$. Then, for $t_{j,n} \leq s \leq \delta_n$, we have

$$\begin{aligned}
u_n(\delta_n) &= \int_{t_{j,n}}^{\delta_n} \varphi_p^{-1}\bigg(\lambda_n \int_s^{\delta_n} h(\tau) f(u_n(\tau))\, d\tau\bigg)\, ds \\
&\leq \varphi_p^{-1}(\lambda_n) \int_{t_{j,n}}^{\delta_n} \varphi_p^{-1}\bigg(\int_{A_{n,s}} h(\tau) f(u_n(\tau))\, d\tau + \int_{B_{n,s}} h(\tau) f(u_n(\tau))\, d\tau\bigg)\, ds \\
&\leq \varphi_p^{-1}(\lambda_n) \int_{t_{j,n}}^{\delta_n} \varphi_p^{-1}\bigg(m_\alpha \int_s^{\delta_n} h(\tau)\, d\tau + \int_s^{\delta_n} h(\tau) f(u_n(\tau)) d\tau\bigg)\, ds.
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{1}{\varphi_p^{-1}(\lambda_n)} \leq \gamma_p \int_{t_{j,n}}^{\delta_n} \frac{\varphi_p^{-1}(m_\alpha)}{||u_n||} \varphi_p^{-1}\bigg(\int_s^{\delta_n} h(\tau)\, d\tau\bigg) \\
+ \varphi_p^{-1}\bigg(\int_s^{\delta_n} \frac{h(\tau) f(u_n(\tau))}{||u_n||^{p-1}}\, d\tau\bigg)\, ds.
\end{aligned}$$

On $B_{n,s}$, $u_n(s) > u_\alpha$ implies $f(u_n(s))/||u_n||^{p-1} \leq f(u_n(s))/u_n^{p-1}(s) \leq \alpha$. Thus

$$\frac{1}{\varphi_p^{-1}(\lambda_n)} \leq \gamma_p \bigg[\frac{\varphi_p^{-1}(m_\alpha) 2Q}{||u_n||} + \varphi_p^{-1}(\alpha) 2Q\bigg].$$

Since $0 < a \ \leq \lambda_n \leq b$, we have

$$\frac{1}{\varphi_p^{-1}(\lambda_n)} \geq \frac{1}{\varphi_p^{-1}(b)}$$

for all n and thus

$$\frac{1}{\varphi_p^{-1}(b)} \leq \gamma_p \bigg[\frac{\varphi_p^{-1}(m_\alpha) 2Q}{||u_n||} + \varphi_p^{-1}(\alpha) 2Q\bigg].$$

By the fact $||u_n|| \to \infty$ as $n \to \infty$, we get

$$\frac{1}{\varphi_p^{-1}(b)} \leq \gamma_p \varphi_p^{-1}(\alpha) 2Q < \gamma_p \varphi_p^{-1}\left(\frac{1}{b\varphi_p(\gamma_p 2Q)}\right) 2Q < \frac{1}{\varphi_p^{-1}(b)}.$$

This contradiction completes the proof. □

It is interesting to note that we do not use the fact that a solution belongs to $\mathcal{B}_k^\infty$ in the proof. Using the above results, we conclude the following.

PROOF OF THEOREM 4.1. From Lemma 4.7-4.13 and Lemma 4.15, we show the existence of an unbounded continuum $\mathcal{B}_k^\infty$ which is bifurcating from $(\mu_k^\infty, 0)$ and if (λ, u) is a solution in $\mathcal{B}_k^\infty$, then u has exactly $k-1$ many interior zeros in $(0,1)$. By Lemma 4.14, (QP_λ) has no solutions for $\lambda < \lambda^{**}$. Define $\lambda^* = \inf\{\lambda > 0 : (\lambda, u)$ is a solution of (QP_λ) in $\mathcal{B}_k^\infty\}$, then $\lambda^* \leq \mu_k^\infty$ and (QP_λ) has a solution which has exactly $k-1$ many interior zeros for all $\lambda > \lambda^*$. If $(\lambda, u) \in \mathcal{B}_k^\infty$, then u is a solution of (QP_λ) and by the definition of $\mathcal{B}_k^\infty$, there exists $(\lambda_n, u_k^n) \in \mathcal{B}_k^n$ such that $u_k^n \to u_k$ in $C[0,1]$ and $\lambda_n \to \lambda > 0$, as $n \to \infty$, respectively. By Lemma 4.3, $\lambda_n \geq \mu_k^n f_0/L_f$. Therefore, by Remark 2.7, $\lambda \geq \mu_k^\infty f_0/L_f \geq (1/2)^{p+1}/(L_f M(p/(p-1))^p)$. Thus, we have $\lambda^* \geq (1/2)^{p+1}/(L_f M(p/(p-1))^p)$ and this completes the proof. □

Now, let us consider the case $f_\infty = \infty$.

LEMMA 4.16. *Assume* $1 < p < 2$, $h \in \mathcal{B} \setminus \mathcal{A}$, (L), (A_1) *and* (A_3). *Let u be a solution for* (QP_λ) *which has exactly $k-1$ many interior zeros in* $(0,1)$, *then* $\lambda \leq \mu_k^1 f_0/\widetilde{L}_f$.

PROOF. Let u be a solution for (QP_λ) which has exactly $k-1$ many interior zeros in $(0,1)$ and ϕ be the k-th eigenfunction which is corresponding to the k-th eigenvalue μ_k^n for (QEP_μ^n). Denote the zeros of u and ϕ by $\{t_n\}$ and $\{s_n\}$, respectively. Then there are $i, j \in \{1, \dots, k\}$ such that $(s_{i-1}, s_i) \subset (t_{j-1}, t_j)$. We may assume that u and ϕ are positive in (t_{j-1}, t_j) and (s_{i-1}, s_i). From (A_1) and (A_3) and the definition of h_n, for $t \in (s_{i-1}, s_i)$, we have

$$\begin{aligned} 0 &= \varphi_p(u'(t))' + \lambda h(t) f(u(t)) \\ &\geq \varphi_p(u'(t))' + \lambda h(t) \widetilde{L}_f \varphi_p(u(t)) \geq \varphi_p(u'(t))' + \lambda h_n(t) \widetilde{L}_f \varphi_p(u(t)). \end{aligned}$$

Applying Picone's identity with $y = \phi$, $b_1(t) = \mu_k^n f_0 h_n(t)$, $z = u$ and $b_2(t) = \lambda h_n(t) \widetilde{L}_f$ and integrating over (s_{i-1}, s_i), we have $\lambda \leq \mu_k^n f_0/\widetilde{L}_f$. In fact, we need to mention that the case $t_{j-1} = s_{i-1}$ or $t_j = s_i$. We shall do it the case $t_{j-1} = s_{i-1}$ only. Other case can be done similarly. From the concavity of u, for sufficiently near t to t_{j-1}, we get $|u'(t)| \leq |u(t)/(t - t_{j-1})|$. Since

$\phi \in C^1[s_{i-1}, s_i]$, we have

$$\lim_{t\to t_{j-1}} \left|\frac{|\phi|^p\varphi_p(u')}{\varphi_p(u)}\right| = \lim_{t\to t_{j-1}} \left|\phi(t)\left(\frac{\phi(t)}{t-t_{j-1}}\right)^{p-1}\left[\frac{u'(t)}{u(t)/(t-t_{j-1})}\right]^{p-1}\right|$$
$$\leq \lim_{t\to t_{j-1}} \left|\phi(t)\left(\frac{\phi(t)}{t-t_{j-1}}\right)^{p-1}\right| = 0.$$

Since μ_k^n is decreasing on n, we have $\lambda \leq \mu_k^1 f_0/\widetilde{L}_f$. This completes the proof. □

We give an *a priori* estimate for case $f_\infty = \infty$.

Lemma 4.17. *Assume* $1 < p < 2$, $h \in \mathcal{B}\setminus\mathcal{A}$, (L), (A_1) *and* (A_3). *Let I be any compact interval in $(0,\infty)$ and $k \in \mathbb{N}$ be given. Then there exists $d_I > 0$ such that for all possible solutions u of* (QP_λ) *having exactly $k-1$ interior zeros with $\lambda \in I$, one has $||u|| \leq d_I$.*

Proof. Fix k and denote the zeros of u by $0 = t_0 < t_1 < \ldots < t_k = 1$. Then there exists an $n \in \{0,\ldots,k-1\}$ such that $t_n - t_{n-1} \geq 1/k$. By typical argument in the proof of Lemma 4.5 of [8], we conclude that there exists $C_1 > 0$ depending only on $I = [\alpha,\beta]$ such that $||u||_{L^\infty(t_{n-1},t_n)} \leq C_1$.

We claim that there exists $C_2 > 0$ depending only on I such that

$$t|u'(t)| \leq C_2 \quad \text{in } (t_{n-1}, s_n) \quad \text{and} \quad (1-t)|u'(t)| \leq C_2 \quad \text{in } (s_n, t_n),$$

where s_n is the unique critical point of u in (t_{n-1},t_n) (since $u(t) > 0$ in (t_{n-1},t_n), u is concave, and so u has a unique critical point in (t_{n-1},t_n)). We show that $\{s_n\}$ is bounded away from zero. As in (3.1) there exists $D > 0$ such that $|f(u(t)| \leq D|u(t)|^{p-1}$, for $t \in [0,1]$. If $s_n \geq 1/2$, we are done. Let $s_n \leq 1/2$. Since u is a solution for (QP_λ), we have

$$u(s_n) = \int_{t_{n-1}}^{s_n} \varphi_p^{-1}\left(\int_s^{s_n} \lambda h(\tau) f(u(\tau))\,d\tau\right) ds.$$

Hence, we get

$$|u(s_n)| \leq \varphi_p^{-1}(\lambda D)\int_{t_{n-1}}^{s_n} \varphi_p^{-1}\left(\int_s^{s_n} h(\tau)\,d\tau\right) ds|u(s_n)|.$$

This implies the existence of $\delta_1 > 0$ such that $\delta_1 < s_n - t_{n-1} \leq s_n \leq 1/2$. Similarly, there exists $\delta_2 > 0$ such that $s_n < 1-\delta_2$. Put $\delta_0 = \min\{\delta_1,\delta_2\}$. Then we have $\delta_0 < s_n < 1-\delta_0$. Because u is concave, $u'(t) > 0$ in (t_{n-1},s_n) and $u'(t) < 0$ in (s_n,t_n). Put $L = \|u\|_{L^\infty(t_{n-1},t_n)}$. Let $t \in (t_{n-1},s_n)$. Integrating (QP_λ) over (t,s_n), we get

$$u'(t)^{p-1} = \lambda\int_t^{s_n} h(s)f(u(s))\,ds \tag{4.4}$$
$$\leq \lambda\left(\sup_{|s|\leq L} f(s)\right)\int_t^{s_n} h(s)\,ds \leq \lambda\left(\sup_{|s|\leq L} f(s)\right)\int_t^{1-\delta_0} h(s)\,ds.$$

It follows from (4.4) that

$$t^{p-1}|u'(t)|^{p-1} \le \lambda\Big(\sup_{|s|\le L} f(s)\Big)t^{p-1}\int_t^{1-\delta_0} h(s)\,ds.$$

From (L), we have $\lim_{t\to 0^+} t^{p-1}\int_t^{1-\delta_0} h(s)\,ds = 0$. Since $\lambda \in I = [\alpha,\beta]$, we have

$$t^{p-1}|u'(t)|^{p-1} \le C_2' \quad \text{for } t\in(t_{n-1},s_n),$$

where

$$C_2' \equiv \beta \sup_{|s|\le L} f(s)t^{p-1}\int_t^{1-\delta_0} h(s)\,ds.$$

In the same fashion, we have

$$(1-t)^{p-1}|u'(t)|^{p-1} \le C_2'' \quad \text{for } t\in(s_n,t_n).$$

We denote $C_2 = \max\{C_2', C_2''\}$ and this completes the claim.

Finally, we show:

(i) If $t_{n-1} > 0$, then there exists $C_3 > 0$ depending only on I such that $|u(t)| \le C_3$ and $t|u'(t)| \le C_3$ in (t_{n-2},t_{n-1}).

(ii) If $t_n < 1$, then we have the same assertion (i) where we replace $t|u'(t)|$ and (t_{n-2},t_{n-1}) by $(1-t)|u'(t)|$ and (t_n,t_{n+1}), respectively.

To show (i), let $t_{n-1} > 0$. Then the interval $[t_{n-2},t_{n-1}]$ exists. By the claim, we have $0 < t_{n-1}u'(t_{n-1}) \le C_2$. Since $u(t) < 0$ in (t_{n-2},t_{n-1}), $u(t)$ is convex in this interval. Therefore the graph of $u(t)$ lies over the line $y = u'(t_{n-1})(x-t_{n-1})$. Hence we have

$$u'(t_{n-1})(t-t_{n-1}) \le u(t) < 0 \quad \text{in } (t_{n-2},t_{n-1}),$$

which means $|u(t)| \le u'(t_{n-1})(t_{n-1}-t) \le C_2$. We show that $t|u'(t)|$ has a priori bound. Let $t\in(t_{n-2},t_{n-1})$, Note that $t_{n-1} < s_n$. Integrating (QP_λ) over (t,s_n), we get

$$|u'(t)|^{p-1} \le \lambda\int_t^{s_n} h(s)|f(u(s))|\,ds.$$

In the same argument as in the claim, we obtain $C_3' > 0$ such that

$$t|u'(t)| \le C_3' \quad \text{in } [t_{n-2},t_{n-1}].$$

We denote $C_3 = \max\{C_2, C_3'\}$ and complete the assertion (i).

The assertion (ii) can be treated in the same way. By repeating the procedure of this, we have $d_I > 0$ depending only on I such that $||u||_{L^\infty(0,1)} \le d_I$. □

PROOF OF THEOREM 4.2. Once again, it follows from Lemmas 4.7-4.13 and Lemma 4.17 that the existence of an unbounded continuum $\mathcal{C}_k^\infty$ which is bifurcating from $(\mu_k^\infty, 0)$ and if (λ,u) is a solution in $\mathcal{C}_k^\infty$, then u has exactly $k-1$ many interior zeros in $(0,1)$. By Lemma 4.16, (QP_λ) has no solutions for $\lambda > \mu_k^1 f_0/\widetilde{L}_f$. Define $\lambda_* = \sup\{\lambda > 0 : (\lambda,u)$ is a solution of (QP_λ) in $\mathcal{C}_k^\infty\}$, then $\lambda_* \geq \mu_k^\infty$ and (QP$_\lambda$) has a solution which has exactly $k-1$ many interior zeros for all $\lambda < \lambda_*$. By the definition of λ_*, we get $\lambda_* \leq \mu_k^1 f_0/\widetilde{L}_f$. $\square$

5. The case $f_0 = 0$ and $f_\infty = \infty$

In this section, we show the existence of solutions for the following problem

$$\text{(QP)} \qquad \begin{cases} \varphi_p(u'(t))' + h(t)f(u(t)) = 0 & \text{for } t \in (0,1), \\ u(0) = 0 = u(1), \end{cases}$$

when f holds $f_0 = 0$ and $f_\infty = \infty$. We assume

(A$_4$) $f_0 = 0$,

and state the main result of this section.

THEOREM 5.1. *Assume $1 < p < 2$, $h \in \mathcal{B} \setminus \mathcal{A}$, (L), (A$_3$) and (A$_4$). Then for each $k \in \mathbb{N}$, (QP) has at least one solution which has exactly $k-1$ many interior zeros in $(0,1)$.*

Our method is the same to those of the previous sections, a bifurcation and an approximation. To apply a global bifurcation theorem, we need to modify (QP) as follows:

$$(\mathrm{A}_\lambda) \qquad \begin{cases} \varphi_p(u'(t))' + \lambda h(t)\varphi_p(u(t)) + h(t)f(u(t)) = 0 & \text{for } t \in (0,1), \\ u(0) = 0 = u(1), \end{cases}$$

where λ is a real parameter. Notice that if $(0,u)$ is a solution for (A$_\lambda$), then u is a solution for (QP). Since we do not know the operator equation for (A$_\lambda$), we shall employ an approximation method as before. Thus, let us consider

$$(\mathrm{A}_\lambda^n) \qquad \begin{cases} \varphi_p(u'(t))' + \lambda h_n(t)\varphi_p(u(t)) + h_n(t)f(u(t)) = 0 & \text{for } t \in (0,1), \\ u(0) = 0 = u(1), \end{cases}$$

where $h_n(t) = \min\{n, h(t)\}, n \in \mathbb{N}$. Using Theorem 3.4 and Lemma 3.5 in [9], we have the following result for problem (A$_\lambda^n$).

LEMMA 5.2. *Assume (A$_3$) and (A$_4$). Then for each $k \in \mathbb{N}$, there exists an unbounded continuum $\mathcal{D}_k^n$ in $\mathcal{T}^n$ bifurcating from $(\mu_k^n, 0)$ where $\mathcal{T}^n$ is the closure of set of nontrivial solutions for (A$_\lambda^n$) and μ_k^n is the k-th eigenvalue of problem*

$$\begin{cases} \varphi_p(u'(t))' + \mu h_n(t)\varphi_p(u(t)) = 0 & \textit{for } t \in (0,1), \\ u(0) = 0 = u(1). \end{cases}$$

If $(\lambda, u_k^n) \in \mathcal{D}_k^n$ then u_k^n has exactly $k-1$ many interior zeros in $(0,1)$ and $\lambda \leq \mu_k^n$.

As in Section 4, we shall apply Proposition 4.6. For given $R > 0$, let us denote $\mathcal{D}_k^n(R)$ the component containing $(\mu_k^n, 0)$ of

$$\mathcal{D}_k^n \cap ([-R, \mu_k^1] \times \{u \in C[0,1] : ||u|| \leq R\}).$$

Clearly, $\{(\mu_k^\infty, 0)\} \subset \liminf \mathcal{D}_k^n(R)$. The next lemma says that the continuum does not converge to zero when $\lambda = 0$.

Lemma 5.3. *Assume* (A$_3$) *and* (A$_4$). *For each* $k \in \mathbb{N}$, *let* $(0, u_k^n) \in \mathcal{D}_k^n$ *converge to* $(0, u)$ *in* $\mathbb{R} \times C[0,1]$. *Then* $u \not\equiv 0$.

Proof. Suppose on the contrary that there exists a sequence $\{(0, u_k^n)\} \subset \mathcal{D}_k^n$ such that $u_k^n \to 0$ in $C[0,1]$, as $n \to \infty$. Then u_k^n satisfies

$$\begin{cases} \varphi_p(u_k^{n\prime}(t))' + h_n(t) f(u_k^n(t)) = 0 & \text{for } t \in (0,1), \\ u_k^n(0) = 0 = u_k^n(1). \end{cases}$$

We first consider case $k = 1$. By condition (A$_4$), there exists $a > 0$ with $0 < a < \mu_1^\infty$ such that $f(u) \le a\varphi_p(u)$, for small $u > 0$. Hence for large n, we have

$$\varphi_p(u_1^{n\prime}(t))' + a\ h_n(t)\varphi_p(u_1^n(t)) \ge \varphi_p(u_1^{n\prime}(t))' + h_n(t) f(u_1^n(t)) = 0.$$

Let ϕ_1^n be the first eigenfunction corresponding to μ_1^n with $\phi_1^n > 0$ so it satisfies

$$\begin{cases} \varphi_p(\phi_1^{n\prime}(t))' + \mu_1^n h_n(t)\varphi_p(\phi_1^n(t)) = 0 & \text{for } t \in (0,1), \\ \phi_1^n(0) = 0 = \phi_1^n(1). \end{cases}$$

Taking $y = u_1^n$, $b_1(t) = a\ h_n(t)$ and $z = \phi_1^n$, $b_2(t) = \mu_1^n h_n(t)$ in Lemma 2.8 and integrating (2.3), we have

$$\int_0^1 (a - \mu_1^n) h_n(t) |u_1^n(t)|^p \, dt \ge 0.$$

Thus $a - \mu_1^n \ge 0$ and we get $a \ge \mu_1^n \ge \mu_1^\infty$. This is a contradiction to the choice of a. Next, consider case $k \ge 2$. Let t_1^*, t_1 be the first zeros of ϕ_k^n and u_k^n with $\phi_k^n, u_k^n > 0$ in $(0, t_1^*)$, $(0, t_1)$, respectively and t_{k-1}^*, t_{k-1} be the last zeros of ϕ_k^n and u_k^n. Then by the same argument in the proof of Lemma 4.7, replacing ϕ_1^N and u_k^N with ϕ_1^n and u_k^n, we get a contradiction. $\square$

Lemma 5.4. *Assume* (A$_3$) *and* (A$_4$). *For given* $R > 0$ *and* $k \in \mathbb{N}$, *let* $(\lambda_n, u_n) \in \mathcal{D}_k^n(R)$. *Then there exist subsequences* $\{u_m\}$, $\{\lambda_m\}$ *and* $u \in C[0,1]$ *such that* $u_m \to u$ *in* $C[0,1]$ *and* $\lambda_m \to \lambda$. *Furthermore, u is a solution for* (A$_\lambda$).

Proof. Fix k and let $(\lambda_n, u_n) \in \mathcal{D}_k^n(R)$. Then since u_n is a solution for (A$_{\lambda_n}^n$), we have

$$\begin{cases} \varphi_p(u_n'(t))' + \lambda_n h_n(t)\varphi_p(u_n(t)) + h_n(t) f(u_n(t)) = 0 & \text{for } t \in (0,1), \\ u_n(0) = 0 = u_n(1). \end{cases}$$

Put $g_R := (\mu_k^1 + 1)\sup_{u \in [0,R]}(\varphi_p(u) + f(u))$. By the same argument in the proof of Lemma 4.10, we get a conclusion. $\square$

LEMMA 5.5. *Assume* (A_3) *and* (A_4)*. Then for each* $k \in \mathbb{N}$, $\limsup\{\mathcal{D}_k^n\}$ *is connected and if* $(\lambda, u) \in \limsup\{\mathcal{D}_k^n\}$*, then* u *is a solution for* (A_λ).

PROOF. Clearly, $(\mu_k^\infty, 0) \in \liminf \mathcal{D}_k^n(R)$ so that $\liminf \mathcal{D}_k^n(R) \neq \emptyset$. To show $\bigcup \mathcal{D}_k^n(R)$ is relatively compact in $C[0,1]$, let $\{(\lambda_n, u_n)\}$ be a sequence in $\bigcup \mathcal{D}_k^n(R)$. Without loss of generality, it is enough to consider the following alternative as a subsequence if necessary; either $(\lambda_m, u_m) \in \mathcal{D}_k^m(R)$ or there exists n_0 such that $(\lambda_m, u_m) \in \mathcal{D}_k^{n_0}(R)$ for all m. The first case can be easily verified by Lemma 5.4. In the second case, $\{(\lambda_m, u_m)\} \subset \mathcal{D}_k^{n_0}(R)$ is bounded and satisfies

$$\begin{cases} \varphi_p(u_m'(t))' + \lambda_m h_{n_0}(t)\varphi_p(u_m(t)) + h_{n_0}(t) f(u_m(t)) = 0 & \text{for } t \in (0,1), \\ u_m(0) = 0 = u_m(1). \end{cases}$$

Following the same lines in the proof of Lemma 4.12, we get the conclusion. $\square$

We denote $\mathcal{D}_k^\infty = \limsup \mathcal{D}_k^n$.

LEMMA 5.6. *Assume* $1 < p < 2$, $h \in \mathcal{B} \setminus \mathcal{A}$, (L), (A_3) *and* (A_4)*. Let* $(\lambda, u_k) \in \mathcal{D}_k^\infty$ *with* $u_k \not\equiv 0$*. Then* u_k *has exactly* $k-1$ *many interior zeros in* $(0,1)$.

PROOF. Let $(\lambda, u_k) \in \mathcal{D}_k^\infty$. Then from the definition of $\mathcal{D}_k^\infty$, there exists $(\lambda_n, u_k^n) \in \mathcal{D}_k^n$ such that $u_k^n \to u_k$ in $C[0,1]$ and $\lambda_n \to \lambda$. We claim that there exists $\delta_1, \delta_2 > 0$ such that

$$\bigcup_{n=1}^{\infty} \{t \in (0,1) : u_k^n(t) = 0\} \subset [\delta_1, \delta_2] \subset (0,1).$$

To show the claim, let $t_{k,n}$ be the first interior zero of u_k^n. Then we show that there exists $\delta_1 > 0$ such that $t_{k,n} > \delta_1$ for all n. We can prove the result for the sequence of last interior zeros of u_n^k by similar fashion. Suppose on the contrary, $t_{k,n} \to 0$ as $n \to \infty$. Since $u_k > 0$ near 0 (it is possible because of Lemma 3.1(c)) and $u_n^k \to u_k$, considering a subsequence if necessary, we have an alternative, either $u_k^n > 0$ on $(0, t_{k,n})$ for all n or $u_k^n < 0$ on $(0, t_{k,n})$ for all n. If $u_k^n > 0$ on $(0, t_{k,n})$ for all n, then u_k^n is a positive solution of the following problem

$$\begin{cases} \varphi_p(u_n'(t))' + \lambda_n h_n(t)\varphi_p(u_n(t)) + h_n(t) f(u_n(t)) = 0, \\ u_n(0) = 0 = u_n(t_n). \end{cases} \tag{5.2}$$

For the other alternative, $v_k^n = -u_k^n$ is a positive solution of (5.1) because φ_p and f are odd and the proof follows exactly same lines with v_k^n. Let $u_k^n(\widetilde{t}_{k,n}) = \max_{t \in [0, t_{k,n}]} u_k^n(t)$, then we get

$$\begin{aligned} \int_0^{\widetilde{t}_{k,n}} \varphi_p^{-1}\bigg(\int_s^{\widetilde{t}_{k,n}} \lambda_n h_n(\tau)\varphi_p(u_k^n(\tau)) + h_n(\tau) f(u_k^n(\tau))\, d\tau \bigg)\, ds \\ = \int_{\widetilde{t}_{k,n}}^{t_{k,n}} \varphi_p^{-1}\bigg(\int_{\widetilde{t}_{k,n}}^{s} \lambda_n h_n(\tau)\varphi_p(u_k^n(\tau)) + h_n(\tau) f(u_k^n(\tau))\, d\tau \bigg)\, ds. \end{aligned}$$

Since $u_k^n \to u_k$, we may assume $|u_k^n(t)| < ||u_k||_\infty + 1$, for all $t \in [0, t_{k,n}]$, and sufficiently large n. On interval $(0, t_{k,n})$, $u_k^n(t) > 0$ for all n and by $f_0 = 0$, there exists $C_{\lambda,u_k} > 0$ such that

$$|\lambda_n h_n(\tau)\varphi_p(u_k^n(\tau)) + h_n(\tau)f(u_k^n(\tau))| \le C_{\lambda,u_k} h(\tau)\varphi_p(u_k^n(\tau)). \tag{5.2}$$

By the same computation in the proof of Lemma 3.2, we have

$$u_k^n(\widetilde{t}_{k,n}) \le u_k^n(\widetilde{t}_{k,n}) \int_0^{\widetilde{t}_{k,n}} \varphi_p^{-1}\biggl(\int_s^{\widetilde{t}_{k,n}} C_{\lambda,u_k} h(\tau)\,d\tau\biggr)\,ds.$$

This implies

$$1 \le \int_0^{\widetilde{t}_{k,n}} \varphi_p^{-1}\biggl(\int_s^{\widetilde{t}_{k,n}} C_{\lambda,u_k} h(\tau)\,d\tau\biggr)\,ds.$$

Since $\varphi_p^{-1}(\int_s^{\widetilde{t}_{k,n}} C_{\lambda,u_k} h(\tau)\,d\tau) \in L^1(0,\gamma]$, for some $\gamma > 0$ and $\widetilde{t}_{k,n} \to 0$, the above inequality is not possible and this completes the claim. The rest of proof follows on the lines of the proof of Lemma 3.2 making use of inequality (5.2). □

The following two lemmas make us to figure the shape of $\mathcal{D}_k^\infty$. First, we obtain λ-direction block.

Lemma 5.7. *Assume* $1 < p < 2$, $h \in \mathcal{B} \setminus \mathcal{A}$, (L), ($A_3$) *and* ($A_4$). *If* $(\lambda, u_k) \in \mathcal{D}_k^\infty$, *then* $\lambda \le \mu_k^1$.

Proof. From the definition of $\mathcal{D}_k^\infty$, there exists $(\lambda_n, u_k^n) \in \mathcal{D}_k^n$ such that $u_k^n \to u_k$ in $C[0,1]$ and $\lambda_n \to \lambda$. By Lemma 5.2, we have $\lambda_n \le \mu_k^n$. Therefore by decreasing monotonicity of μ_k^n, $\lambda_n \le \mu_k^1$ and $\lambda \le \mu_k^1$. □

Now we obtain an *a priori* estimate for (A_λ).

Lemma 5.8. *Assume* $1 < p < 2$, $h \in \mathcal{B} \setminus \mathcal{A}$, (L), ($A_3$) *and* ($A_4$). *Let* I *be any compact interval in* $[0,\infty)$ *and* $k \in \mathbb{N}$ *be given. Then there exists* $e_I > 0$ *such that for all solutions* (λ, u) *of* $\mathcal{D}_k^\infty$, *we have* $||u|| \le e_I$.

Proof. Fix k and denote the zeros of u by $0 = t_0 < t_1 < \ldots < t_k = 1$. Then there exists $n \in \{1, \ldots, k\}$ such that $t_n - t_{n-1} \ge 1/k$. Again, by typical argument in Lemma 4.5 of [8] with a little modification, we conclude that there exists $D_1 > 0$ depending only on $I \triangleq [\alpha,\beta]$ such that $||u||_{L^\infty(t_{n-1},t_n)} \le D_1$. We claim that there exists $D_2 > 0$ depending only on I such that

$$t|u'(t)| \le D_2 \quad \text{in } (t_{n-1}, s_n) \quad \text{and} \quad (1-t)|u'(t)| \le D_2 \quad \text{in } (s_n, t_n),$$

where s_n is the unique critical point of u in (t_{n-1}, t_n) (since $u(t) > 0$ in (t_{n-1}, t_n) and u is concave, so u has a unique critical point in (t_{n-1}, t_n)). We show that $\{s_n\}$ is bounded away from zero. As in (3.1), there exists $D > 0$ such that

$|f(u(t)| \le D|u(t)|^{p-1}$, for $t \in [0,1]$. If $s_n \ge 1/2$, then we are done. If $s_n \le 1/2$, then since u is a solution for (A_λ), we have

$$u(s_n) = \int_{t_{n-1}}^{s_n} \varphi_p^{-1}\bigg(\int_s^{s_n} [\lambda h(\tau)\varphi_p(u(\tau)) + h(\tau)f(u(\tau))]\,d\tau\bigg)\,ds.$$

Hence, we get

$$|u(s_n)| \le \varphi_p^{-1}(\lambda + D)\int_{t_{n-1}}^{s_n} \varphi_p^{-1}\bigg(\int_s^{s_n} h(\tau)\,d\tau\bigg)\,ds|u(s_n)|.$$

This implies that there exists $\delta_1 > 0$ such that $\delta_1 < s_n - t_{n-1} \le s_n \le 1/2$. Similarly, there exists $\delta_2 > 0$ such that $s_n < 1 - \delta_2$. Put $\delta_0 = \min\{\delta_1, \delta_2\}$. Then we have $\delta_0 < s_n < 1 - \delta_0$. Because u is concave, $u'(t) > 0$ in (t_{n-1}, s_n) and $u'(t) < 0$ in (s_n, t_n). Let $L = \|u\|_{L^\infty(t_{n-1},t_n)}$. Then for $t \in (t_{n-1}, s_n)$, integrating (A_λ) over (t, s_n), we get

$$\begin{aligned} u'(t)^{p-1} &= \int_t^{s_n} [\lambda h(s)\varphi_p(u(s)) + h(s)f(u(s))]\,ds \\ &\le (\lambda+1)\Big(\sup_{|s|\le L}(\varphi_p(s) + f(s))\Big)\int_t^{s_n} h(s)\,ds \\ &\le (\lambda+1)\Big(\sup_{|s|\le L}(\varphi_p(s) + f(s))\Big)\int_t^{1-\delta_0} h(s)\,ds. \end{aligned}$$

From (L), we have $\lim_{t\to 0^+} t^{p-1}\int_t^{1-\delta_0} h(s)\,ds = 0$. Since $\lambda \in I = [\alpha,\beta]$, we have

$$t^{p-1}|u'(t)|^{p-1} \le D_2' \quad \text{for } t \in (t_{n-1}, s_n),$$

where

$$D_2' \equiv (\beta+1)\sup_{|s|\le L}(\varphi_p(s) + f(s))t^{p-1}\int_t^{1-\delta_0} h(s)\,ds.$$

By the same fashion, we have

$$(1-t)^{p-1}|u'(t)|^{p-1} \le D_2'' \quad \text{for } t \in (s_n, t_n).$$

Taking $D_2 = \max\{D_2', D_2''\}$, we complete the claim. Finally, we show:

(i) If $t_{n-1} > 0$, then there exists $D_3 > 0$ depending only on I such that $|u(t)| \le D_3$ and $t|u'(t)| \le D_3$ in (t_{n-2}, t_{n-1}).

(ii) If $t_n < 1$, then we have the same assertion (i) where we replace $t|u'(t)|$ and (t_{n-2}, t_{n-1}) by $(1-t)|u'(t)|$ and (t_n, t_{n+1}), respectively.

To show (i), let $t_{n-1} > 0$. Then the interval $[t_{n-2}, t_{n-1}]$ exists. By the claim, we have $0 < t_{n-1}u'(t_{n-1}) \le D_2$. Since $u(t) < 0$ in (t_{n-2}, t_{n-1}), $u(t)$ is convex in this interval. Therefore the graph of $u(t)$ lies over the line $y = u'(t_{n-1})(x - t_{n-1})$. Hence we have

$$u'(t_{n-1})(t - t_{n-1}) \le u(t) < 0 \quad \text{in } (t_{n-2}, t_{n-1}),$$

which means $|u(t)| \le u'(t_{n-1})(t_{n-1} - t) \le D_2$. We show that $t|u'(t)|$ has a priori bound. Let $t \in (t_{n-2}, t_{n-1})$, Note that $t_{n-1} < s_n$. Integrating (A_λ) over (t, s_n), we get

$$|u'(t)|^{p-1} \le \int_t^{s_n} [\lambda h(s)|\varphi_p(u(s))| + h(s)|f(u(s))|]\, ds.$$

In the same argument as in the claim, we obtain $D_3' > 0$ such that

$$t|u'(t)| \le D_3' \quad \text{in } [t_{n-2}, t_{n-1}].$$

We denote $D_3 = \max\{D_2, D_3'\}$ and complete the assertion (i).

The assertion (ii) can be treated in the same way. By repeating the procedure of this, we have $e_I > 0$ depending only on I such that $||u||_{L^\infty(0,1)} \le e_I$. □

PROOF OF THEOREM 5.1. From Lemmas 5.2–5.6, we show that the existence of an unbounded continuum $\mathcal{D}_k^\infty$ which is bifurcating from $(\mu_k^\infty, 0)$ and if (λ, u) is a solution in $\mathcal{D}_k^\infty$, then u has exactly $k-1$ many interior zeros in $(0,1)$. Using Lemmas 5.7 and 5.8, the continuum must cross over $||\cdot||$-axis. Thus, we have a solution $(0, u) \in \mathcal{D}_k^\infty$ for (A_λ). Hence, u is a solution for (QP). □

REMARK 5.9. We leave a question to readers for the existence of sign-changing solutions of (QP) under assumptions $f_0 = \infty$ and $f_\infty = 0$.

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