HOMOCLINIC SOLUTIONS FOR A CLASS OF AUTONOMOUS SECOND ORDER HAMILTONIAN SYSTEMS WITH A SUPERQUADRATIC POTENTIAL

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Abstract. We will prove the existence of a nontrivial homoclinic solution for an autonomous second order Hamiltonian system \( \ddot{q} + \nabla V(q) = 0 \), where \( q \in \mathbb{R}^n \), a potential \( V: \mathbb{R}^n \to \mathbb{R} \) is of the form \( V(q) = -K(q) + W(q) \), \( K \) and \( W \) are \( C^1 \)-maps, \( K \) satisfies the pinching condition, \( W \) grows at a superquadratic rate, as \( |q| \to \infty \) and \( W(q) = o(|q|^2) \), as \( |q| \to 0 \). A homoclinic solution will be obtained as a weak limit in the Sobolev space \( W^{1,2}(\mathbb{R}, \mathbb{R}^n) \) of a sequence of almost critical points of the corresponding action functional. Before passing to a weak limit with a sequence of almost critical points each element of this sequence has to be appropriately shifted.

1. Introduction

This paper concerns the existence of homoclinic solutions for a certain class of autonomous second order Hamiltonian systems. Let us consider

\[ \ddot{q} + \nabla V(q) = 0, \]

where \( q \in \mathbb{R}^n \) and a potential \( V: \mathbb{R}^n \to \mathbb{R} \) satisfies the following conditions:

\[ (H_1) \quad V(q) = -K(q) + W(q), \] where \( K, W: \mathbb{R}^n \to \mathbb{R} \) are \( C^1 \)-maps,
(H2) there are constants $b_1, b_2 > 0$ such that for all $q \in \mathbb{R}^n$,
$$b_1|q|^2 \leq K(q) \leq b_2|q|^2,$$
\( (H_3) \) $(q, \nabla K(q)) \leq 2K(q)$ for all $q \in \mathbb{R}^n$,
\( (H_4) \) $2K(q) - (q, \nabla K(q)) = o(|q|^2)$, as $|q| \to 0$,
\( (H_5) \) $\nabla K$ is Lipschitzian in a neighbourhood of $0 \in \mathbb{R}^n$,
\( (H_6) \) $\nabla W(q) = o(|q|)$, as $|q| \to 0$,
\( (H_7) \) there is a constant $\mu > 2$ such that for every $q \in \mathbb{R}^n \setminus \{0\}$,
$$0 < \mu W(q) \leq (q, \nabla W(q)).$$

Here and subsequently, $(\cdot, \cdot) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ denotes the standard inner product in $\mathbb{R}^n$ and $|\cdot| : \mathbb{R}^n \to [0, \infty)$ is the induced norm.

Note that if $K : \mathbb{R}^n \to \mathbb{R}$ is a $C^2$-map satisfying (H2), then (H4) takes place. Let us also remark that (H6) and (H7) imply
\begin{equation}
(1.2) \quad W(q) = o(|q|^2), \quad \text{as } |q| \to 0.
\end{equation}

Moreover, from (H7) it follows that for $q \neq 0$ a map given by
$$W(q) = W(|q|^\mu) \zeta \quad (0, \infty) \ni \zeta \mapsto W(q) \zeta^{\mu}$$
is nonincreasing. Hence the following inequalities hold
\begin{equation}
(1.3) \quad W(q) \leq W\left(\frac{q}{|q|}\right)|q|^{\mu} \quad \text{if } 0 < |q| \leq 1,
\end{equation}
\begin{equation}
(1.4) \quad W(q) \geq W\left(\frac{q}{|q|}\right)|q|^{\mu} \quad \text{if } |q| \geq 1.
\end{equation}

By (H2) and (1.4) we get that a potential $V$ grows at a superquadratic rate, as $|q| \to \infty$, i.e.
$$\frac{V(q)}{|q|^2} \to \infty, \quad \text{as } |q| \to \infty.$$

Many mathematicians have written about Hamiltonian systems with a superquadratic potential, for example: V. Coti Zelati, I. Ekeland and E. Séré in [4], H. Hofer and K. Wysocki in [7], V. Coti Zelati and P. Rabinowitz in [5], P. Rabinowitz and K. Tanaka in [14], W. Omana and M. Willem in [11]. Our assumptions on the potential $V$ are natural, since one can immediately produce a lot of examples.

It is easily seen that $q \equiv 0$ is a solution of (1.1). In this work we are interested in the existence of nontrivial homoclinic solutions of (1.1) that emanate from 0 and terminate at 0, i.e. $\lim_{t \to \pm \infty} q(t) = q(\pm \infty) = 0$.

The existence of homoclinic orbits for first and second order Hamiltonian systems has been studied by many authors and the literature on this subject is vast (see [1], [2], [6], [8], [9], [12], [15]), but many questions are still open (see
the survey [13] by P. Rabinowitz). Finding homoclinic solutions in Hamiltonian systems can be quite difficult. In the last 20 years, a great progress was made by applying variational methods (see the survey [3] by T. Bartsch and A. Szulkin). For instance, the authors of [4] studied a class of first order Hamiltonian systems using a dual variational transformation and the Mountain Pass Theorem to prove the existence of two distinct homoclinic solutions. P. Rabinowitz in [12] examined a family of second order Hamiltonian systems applying the Mountain Pass Theorem to get a sequence of subharmonic solutions and suitable estimates to pass to a nontrivial limit which occurred to be a nontrivial homoclinic solution (see also [2], [8], [9]).

The theorem which we shall prove is as follows.

**Theorem 1.1.** If the assumptions \( (H_1) - (H_7) \) are satisfied then the Hamiltonian system \( (1.1) \) possesses a nontrivial homoclinic solution \( q_0 \in W^{1,2}(\mathbb{R}, \mathbb{R}^n) \) such that \( \dot{q}_0(\pm \infty) = 0 \).

This result is proved in Section 2 by studying the corresponding to \( (1.1) \) action functional \( I: W^{1,2}(\mathbb{R}, \mathbb{R}^n) \to \mathbb{R} \). Similarly to [14], by a general minimax principle (see Theorem 2.3) we obtain a sequence of almost critical points. However, its weak limit has not to be nontrivial. In order to get a nontrivial homoclinic orbit before passing to a weak limit with a sequence of almost critical points each element of this sequence has to be appropriately shifted.

**2. Proof of Theorem 1.1**

The proof of Theorem 1.1 will be divided into a sequence of lemmas. Let \( E \) be the Sobolev space \( W^{1,2}(\mathbb{R}, \mathbb{R}^n) \) with the standard norm

\[
\| q \|_E := \left( \int_{-\infty}^{\infty} (|q(t)|^2 + |\dot{q}(t)|^2) \, dt \right)^{1/2}.
\]

We first recall two elementary inequalities concerning functions in \( E \).

**Fact 2.1.** If \( q: \mathbb{R} \to \mathbb{R}^n \) is a continuous mapping such that \( \dot{q} \in L^2_{\text{loc}}(\mathbb{R}, \mathbb{R}^n) \), then for every \( t \in \mathbb{R} \),

\[
|q(t)| \leq \sqrt{2} \left( \int_{t-1/2}^{t+1/2} (|q(s)|^2 + |\dot{q}(s)|^2) \, ds \right)^{1/2}.
\]

The proof of Fact 2.1 can be found in [8]. (See Fact 2.8, p. 385.)

**Fact 2.2.** For each \( q \in E \),

\[
\| q \|_{L^\infty(\mathbb{R}, \mathbb{R}^n)} \leq \sqrt{2} \| q \|_E.
\]

Fact 2.2 is a direct consequence of the inequality (2.1).
Let $I: E \to \mathbb{R}$ be given by

$$I(q) := \int_{-\infty}^{\infty} \left[ \frac{1}{2} |\dot{q}(t)|^2 - V(q(t)) \right] dt.$$ 

By (H$_5$)-(H$_6$) it is obvious that $I \in C^1(E, \mathbb{R})$. Moreover,

$$I'(q)w = \int_{-\infty}^{\infty} \left[ (\dot{q}(t), \dot{w}(t)) - (\nabla V(q(t)), w(t)) \right] dt$$

for all $q, w \in E$ and any critical point of $I$ on $E$ is a classical solution of (1.1) with $q(\pm \infty) = 0$, as is easy to verify. In order to prove Theorem 1.1, we apply a general minimax principle. Let us remind it.

**Theorem 2.3** (see Theorem 4.3 in [10]). Let $K$ be a compact metric space, $K_0 \subset K$ a closed subset, $X$ a Banach space and $\chi \in C(K_0, X)$. Let $\mathcal{M}$ be a complete metric space given by

$$\mathcal{M} := \{ g \in C(K, X) : g(s) = \chi(s) \text{ if } s \in K_0 \}$$

with the usual distance. Let $\varphi \in C^1(X, \mathbb{R})$ and let us define

$$c = \inf_{g \in \mathcal{M}} \max_{s \in K} \varphi(g(s)), \quad c_1 = \max_{s \in K_0} \varphi.$$ 

If $c > c_1$ then for each $\varepsilon > 0$ and for each $h \in \mathcal{M}$ such that $\max_{s \in K} \varphi(h(s)) \leq c + \varepsilon$ there exists $v \in X$ such that $c - \varepsilon \leq \varphi(v) \leq \max_{s \in K} \varphi(h(s))$, $\text{dist}(v, h(K)) \leq \varepsilon^{1/2}$, $\|\varphi'(v)\|_{X^*} \leq \varepsilon^{1/2}$.

Set $\bar{b}_1 := \min\{1, 2b_1\}$, $\bar{b}_2 := \max\{1, 2b_2\}$, where $b_1$, $b_2$ are the constants of the pinching condition (H$_2$). By definition, $\bar{b}_1 \leq 1 \leq \bar{b}_2$. From (H$_2$) we have

$$I(q) \geq \frac{1}{2} \bar{b}_1 \|q\|_E^2 - \int_{-\infty}^{\infty} W(q(t)) dt$$

for every $q \in E$. By (1.2), (2.2) and (2.3), we conclude that there are constants $\alpha, \varrho > 0$ such that

$$I(q) \geq \alpha, \quad \text{if } \|q\|_E = \varrho.$$ 

Take $\nu \in C_0^\infty(\mathbb{R}, \mathbb{R}^n)$ such that $|\nu(t)| = 1$ for $|t| \leq 1$ and $\nu(t) = 0$ for $|t| \geq 2$. Set

$$m := \inf\{W(q) : |q| = 1\}.$$ 

From (1.4), for every $\xi \in \mathbb{R}$ such that $|\xi| \geq 1$, we have

$$\int_{-\infty}^{\infty} W(\xi \nu(t)) dt \geq \int_{-1}^{1} W(\xi \nu(t)) dt \geq \int_{-1}^{1} W\left( \frac{\xi \nu(t)}{|\xi \nu(t)|} \right) |\xi \nu(t)|^\mu dt \geq 2m|\xi|^\mu.$$
Combining this with (H2) we obtain
\[ I(\xi \nu) \leq \frac{1}{2} \bar{b}_2 \xi^2 \| \nu \|_E^2 - 2m|\xi|^{\mu}. \]
Since \( m > 0 \) and \( \mu > 2 \), for \(|\xi|\) sufficiently large, \( I(\xi \nu) < 0 \). Consequently, there exists \( Q \in E \) such that
\[ I(\xi \nu) \leq 1 \]
and \( I(Q) < 0 = I(0) \).
From now on, let
\[ M := \{ g \in C([0,1], E) : g(0) = 0 \text{ and } g(1) = Q \} \]
and
\[ c := \inf_{g \in M, s \in [0,1]} \max I(g(s)). \]
By (2.4)-(2.7), we get \( c \geq \alpha > 0 \).
Applying Theorem 2.3 we conclude that the following lemma holds.

**Lemma 2.4.** There exists a sequence \( \{q_k\}_{k \in \mathbb{N}} \) in \( E \) such that
\[ I(q_k) \to c \quad \text{and} \quad I'(q_k) \to 0, \quad \text{as } k \to \infty. \]

**Proof.** By (2.8), for large \( k \),
\[ \|I'(q_k)\|_E^* < 2 \quad \text{and} \quad |I(q_k) - c| < 1. \]
Applying (H3) and (H7) we obtain
\[ I(q_k) - \frac{1}{2} I'(q_k)q_k \geq \left( \frac{\mu}{2} - 1 \right) \int_{-\infty}^{\infty} W(q_k(t)) \, dt \]
for \( k \in \mathbb{N} \). Combining (2.10) with (2.9) we receive
\[ c + 1 + \|q_k\|_E \geq \left( \frac{\mu}{2} - 1 \right) \int_{-\infty}^{\infty} W(q_k(t)) \, dt \]
for large \( k \), and hence
\[ \int_{-\infty}^{\infty} W(q_k(t)) \, dt \leq \frac{2}{\mu - 2}(c + 1 + \|q_k\|_E). \]
By the use of (H2), (H3) and (H7), we get
\[ I'(q_k)q_k \leq \bar{b}_2 \|q_k\|_E^2 - \mu \int_{-\infty}^{\infty} W(q_k(t)) \, dt \]
for \( k \in \mathbb{N} \). From (2.3) and (2.12) it follows that
\[ \frac{1}{b_1} I(q_k) - \frac{1}{\mu \bar{b}_2} I'(q_k)q_k \geq \left( \frac{1}{2} - \frac{1}{\mu} \right) \|q_k\|_E^2 - \left( \frac{1}{b_1} - \frac{1}{\bar{b}_2} \right) \int_{-\infty}^{\infty} W(q_k(t)) \, dt \]
for \( k \in \mathbb{N} \). By (2.9) and (2.13), for large \( k \),

\[
\frac{1}{b_1}(c + 1) + \|q_k\|_E \geq \left( \frac{1}{2} - \frac{1}{\mu} \right) \|q_k\|_E^2 - \left( \frac{1}{b_1} - \frac{1}{b_2} \right) \int_{-\infty}^{\infty} W(q_k(t)) \, dt.
\]

Finally, from (2.11) and (2.14), for large \( k \),

\[
\left( \frac{1}{2} - \frac{1}{\mu} \right) \|q_k\|_E^2 \leq \frac{1}{b_1}(c + 1) + \|q_k\|_E + \frac{2}{\mu - 2} \left( \frac{1}{b_1} - \frac{1}{b_2} \right) (c + 1 + \|q_k\|_E).
\]

Since \( \mu > 2 \), (2.15) shows that \( \{q_k\}_{k \in \mathbb{N}} \) is bounded in \( E \).

For each \( k \in \mathbb{N} \) there is \( \tau_k \in \mathbb{R} \) such that a map \( q_{\tau_k} : \mathbb{R} \to \mathbb{R}^n \) given by

\[ q_{\tau_k}(t) := q_k(t + \tau_k), \]

where \( t \in \mathbb{R} \), achieves a maximum at \( 0 \in \mathbb{R} \), i.e.

\[
\max\{\|q_{\tau_k}(t)\| : t \in \mathbb{R}\} = |q_{\tau_k}(0)|.
\]

Then \( q_{\tau_k} \in E \) and it is easy to check that \( \|q_{\tau_k}\|_E = \|q_k\|_E, I(q_{\tau_k}) = I(q_k) \) and \( \|I'(q_{\tau_k})\|_{E^*} = \|I'(q_k)\|_{E^*} \). In consequence, by Lemma 2.4,

\[
I(q_{\tau_k}) \to c \quad \text{and} \quad I'(q_{\tau_k}) \to 0,
\]

as \( k \to \infty \), and by Lemma 2.5, the sequence \( \{q_{\tau_k}\}_{k \in \mathbb{N}} \) is bounded in \( E \). Since \( E \) is a reflexive Banach space, \( \{q_{\tau_k}\}_{k \in \mathbb{N}} \) possesses a weakly convergent subsequence in \( E \).

Let \( q_0 \) denote a weak limit of a weakly convergent subsequence of \( \{q_{\tau_k}\}_{k \in \mathbb{N}} \). Without loss of generality, we will write

\[
q_{\tau_k} \rightharpoonup q_0 \quad \text{in} \quad E,
\]

as \( k \to \infty \), which implies \( q_{\tau_k} \to q_0 \) in \( L^\infty_{\text{loc}}(\mathbb{R}, \mathbb{R}^n) \), as \( k \to \infty \).

**Lemma 2.6.** \( q_0 \) given by (2.18) is a homoclinic solution of (1.1).

**Proof.** Since \( q_0 \in E \), we see that \( q_0(t) \to 0 \), as \( t \to \pm \infty \), by Fact 2.1. Therefore, it is sufficient to show that \( I'(q_0) = 0 \). Fix \( w \in C^\infty_0(\mathbb{R}, \mathbb{R}^n) \) and assume that for some \( A > 0 \), \( \text{supp}(w) \subset [-A, A] \). We have

\[
I'(q_{\tau_k})w = \int_{-A}^{A} ([\dot{q}_{\tau_k}(t), \dot{w}(t)] - (\nabla V(q_{\tau_k}(t)), w(t))] \, dt
\]

for each \( k \in \mathbb{N} \). From (2.17) it follows that \( I'(q_{\tau_k})w \to 0 \), as \( k \to \infty \). On the other hand,

\[
\int_{-A}^{A} (\dot{q}_{\tau_k}(t), \dot{w}(t)) \, dt \to \int_{-A}^{A} (\dot{q}_0(t), \dot{w}(t)) \, dt,
\]

as \( k \to \infty \), by (2.18), and

\[
\int_{-A}^{A} (\nabla V(q_{\tau_k}(t)), w(t)) \, dt \to \int_{-A}^{A} (\nabla V(q_0(t)), w(t)) \, dt,
\]
as \( k \to \infty \), because \( q_{\tau_k} \to q_0 \) uniformly on \([-A, A]\). Thus \( I'(q_{\tau_k}) w \to I'(q_0) w \), as \( k \to \infty \), and, in consequence, \( I'(q_0) w = 0 \). Since \( C_0^\infty(\mathbb{R}, \mathbb{R}^n) \) is dense in \( E \), we get \( I'(q_0) = 0 \).

**Lemma 2.7.** Let \( q_0 \) be given by (2.18). Then \( \dot{q}_0(t) \to 0 \), as \( t \to \pm \infty \).

**Proof.** From Fact 2.1, we obtain

\[
|\dot{q}_0(t)|^2 \leq 2 \int_{t-1/2}^{t+1/2} |\ddot{q}_0(s)|^2 \, ds + 2 \int_{t-1/2}^{t+1/2} (|q_0(s)|^2 + |\dot{q}_0(s)|^2) \, ds.
\]

For this reason, it suffices to notice that

\[
\int_r^{r+1} |\ddot{q}_0(s)|^2 \, ds \to 0,
\]

as \( r \to \pm \infty \). Since \( q_0 \) satisfies (1.1), we have

\[
\int_r^{r+1} |\ddot{q}_0(s)|^2 \, ds = \int_r^{r+1} |\nabla V(q_0(s))|^2 \, ds.
\]

Take \( \varepsilon > 0 \). By (H2) and (H0), there is \( \eta > 0 \) such that for \( |q| < \eta \), \( |\nabla V(q)| < \varepsilon \).

Moreover, there is \( \delta > 0 \) such that, if \( |s| > \delta \), then \( |q_0(s)| < \eta \). Hence, if \( |r| > \delta + 1 \), then

\[
\int_r^{r+1} |\nabla V(q_0(s))|^2 \, ds < \varepsilon^2,
\]

which completes the proof. \( \square \)

To finish the proof of Theorem 1.1, we have to show that \( q_0 \neq 0 \).

On the contrary, suppose that \( q_0 \equiv 0 \). Consequently, we have \( q_{\tau_k}(0) \to 0 \), as \( k \to \infty \). From (2.16) it follows that \( q_{\tau_k} \to 0 \) uniformly on \( \mathbb{R} \), as \( k \to \infty \). By (2.17) and the boundedness of \( \{q_{\tau_k}\}_{k \in \mathbb{N}} \) in \( E \), we get \( 2I(q_{\tau_k}) - I'(q_{\tau_k}) q_{\tau_k} \to 2\varepsilon > 0 \), as \( k \to \infty \). On the other hand, by (H4), (H6) and (1.2),

\[
2I(q_{\tau_k}) - I'(q_{\tau_k}) q_{\tau_k} = \int_{-\infty}^\infty [((\nabla V(q_{\tau_k}(t)), q_{\tau_k}(t)) - 2V(q_{\tau_k}(t))] \, dt
\]

\[
= \int_{-\infty}^\infty [2K(q_{\tau_k}(t)) - (\nabla K(q_{\tau_k}(t)), q_{\tau_k}(t))] \, dt
\]

\[
+ \int_{-\infty}^\infty [(\nabla W(q_{\tau_k}(t)), q_{\tau_k}(t)) - 2W(q_{\tau_k}(t))] \, dt \to 0,
\]

as \( k \to \infty \). Indeed. Take \( \varepsilon > 0 \). From (H4), (H6) and (1.2), we deduce that there is \( \delta > 0 \) such that if \( |q| < \delta \), then \( 2K(q) - (\nabla K(q), q) \leq \varepsilon |q|^2 \), \( |\nabla W(q)| \leq \varepsilon |q| \) and \( |W(q)| \leq \varepsilon |q|^2 \). Since \( q_{\tau_k} \to 0 \) uniformly on \( \mathbb{R} \), there is \( k_0 \in \mathbb{N} \) such that for \( k > k_0 \) and for \( t \in \mathbb{R} \), \( |q_{\tau_k}(t)| < \delta \). Hence \( |2I(q_{\tau_k}) - I'(q_{\tau_k}) q_{\tau_k}| \leq 4\varepsilon \|q_{\tau_k}\|^2_E \) for \( k > k_0 \), which contradicts (2.17).
References


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