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NONTRIVIAL SOLUTIONS OF p-SUPERLINEAR ANISOTROPIC p-LAPLACIAN SYSTEMS VIA MORSE THEORY

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Abstract. We obtain nontrivial solutions of a class of p-superlinear anisotropic p-Laplacian systems using Morse theory.

1. Introduction

The purpose of this paper is to obtain nontrivial solutions of a class of p-superlinear anisotropic p-Laplacian systems using Morse theory.

As motivation, we begin by recalling a well-known result for the semilinear elliptic boundary value problem

$$\begin{cases} -\Delta u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n \geq 1$, $f \in C(\Omega \times \mathbb{R})$ satisfies the subcritical growth condition

$$|f(x,t)| \leq C(|t|^{r-1} + 1) \quad \text{for all } (x,t) \in \Omega \times \mathbb{R} \tag{1.2}$$

for some $r \in (1, 2^*)$,

$$2^* = \begin{cases} \dfrac{2n}{n-2} & \text{if } n > 2 \\ \infty & \text{if } n \leq 2, \end{cases}$$

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is the critical Sobolev exponent, and C denotes a generic positive constant. Assume

$$\lim_{t\to 0}\frac{f(x,t)}{t}=\lambda,\quad \text{uniformly in } x\in\Omega \tag{1.3}$$

and the Ambrosetti–Rabinowitz condition

$$0<F(x,t):=\int_0^t f(x,s)\,ds\le \frac{t}{\mu}\,f(x,t)\quad \text{for all } x\in\Omega,\ |t|\ge T, \tag{1.4}$$

for some $\mu>2$ and $T>0$. Note that (1.3) implies $f(x,0)\equiv 0$, so problem (1.1) has the trivial solution $u(x)\equiv 0$. Integrating (1.4) gives

$$F(x,t)\ge c(x)|t|^\mu - C\quad \text{for all } (x,t)\in\Omega\times\mathbb{R} \tag{1.5}$$

where $c(x)=\min F(x,\pm T)/T^\mu>0$, so f is superlinear. V. Benci [3] used a new approach to the Morse–Conley theory to obtain a nontrivial solution of this problem when $\lambda\notin\sigma(-\Delta)$, the Dirichlet spectrum of the negative Laplacian on Ω.

The idea of the proof may be restated in terms of critical groups as follows (see K. Chang [4] and Z. Q. Wang [14]). Weak solutions of (1.1) coincide with the critical points of the C^1-functional

$$\Phi(u)=\int_\Omega \frac{1}{2}\,|\nabla u|^2-F(x,u),\quad u\in H=H_0^1(\Omega),$$

and (1.4) ensures that Φ satisfies the (PS) condition. Suppose that Φ has no nontrivial critical points. Then the critical groups of Φ at zero are given by

$$C^q(\Phi,0)=H^q(\Phi^0,\Phi^0\setminus\{0\}),\quad q\ge 0$$

where Φ^0 is the sublevel set $\{u\in H:\Phi(u)\le 0\}$ and H^* denotes cohomology. By the second deformation lemma, Φ^0 is a deformation retract of H and $\Phi^0\setminus\{0\}$ deformation retracts to Φ^a for any $a<0$, so

$$C^q(\Phi,0)\approx H^q(H,\Phi^a).$$

By (1.4), if $|a|$ is sufficiently large, Φ^a is homotopic to the unit sphere in H and hence contractible, so

$$C^q(\Phi,0)=0\quad \text{for all } q. \tag{1.6}$$

On the other hand, if $\lambda_1<\lambda_2\le\ldots$ denote the Dirichlet eigenvalues of the Laplacian on Ω and $\lambda_k<\lambda<\lambda_{k+1}$ in (1.3), then

$$C^q(\Phi,0)\approx\delta_{qk}\mathcal{G}$$

where $\mathcal{G}$ is the coefficient group and $\delta_{\cdot,\cdot}$ denotes the Kronecker delta. This contradiction shows that Φ has a nontrivial critical point.

Remark 1.1. In the case $\lambda < \lambda_1$, A. Ambrosetti and P. H. Rabinowitz [1] obtained a positive solution and a negative solution using their mountain pass theorem, and Z. Q. Wang [14] obtained a third nontrivial solution using Morse theory. When f satisfies a global sign condition, P. H. Rabinowitz [13] used his linking theorem to obtain a nontrivial solution for all $\lambda \in \mathbb{R}$. S. J. Li and M. Willem [8] used a local linking to do the same when f satisfies only a local sign condition near zero.

K. Perera [11] extended the above result to the corresponding p-Laplacian problem

$$\begin{cases} -\Delta_p\, u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

where $\Delta_p\, u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian of u, $p \in (1,\infty)$, f now satisfies (1.2) with $r \in (1, p^*)$, and

$$p^* = \begin{cases} \dfrac{np}{n-p} & \text{if } n > p, \\ \infty & \text{if } n \le p. \end{cases}$$

Assume

$$\lim_{t\to 0} \frac{f(x,t)}{|t|^{p-2}t} = \lambda, \quad \text{uniformly in } x \in \Omega \tag{1.8}$$

and (1.4) with $\mu > p$ and $T > 0$, so $u(x) \equiv 0$ is a solution of (1.7) and f is p-superlinear by (1.5). K. Perera [11] obtained a nontrivial solution when λ is not an eigenvalue of the problem

$$\begin{cases} -\Delta_p\, u = \lambda |u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

This quasilinear eigenvalue problem is far more complicated. It is known that the first eigenvalue λ_1 is positive, simple, and has an associated eigenfunction φ_1 that is positive in Ω (see A. Anane [2] and P. Lindqvist [9], [10]). Moreover, λ_1 is isolated in the spectrum $\sigma(-\Delta_p)$, so the second eigenvalue $\lambda_2 = \inf \sigma(-\Delta_p) \cap (\lambda_1, \infty)$ is well-defined. In the ODE case $n = 1$, where Ω is an interval, the spectrum consists of a sequence of simple eigenvalues $\lambda_k \nearrow \infty$, and the eigenfunction φ_k associated with λ_k has exactly $k-1$ interior zeroes (see e.g. P. Drábek [6]). In the PDE case $n \ge 2$, an increasing and unbounded sequence of eigenvalues can be constructed using a standard minimax scheme involving the Krasnoselskiĭ's genus, but it is not known whether this gives a complete list of the eigenvalues.

The variational functional associated with problem (1.7) is

$$\Phi(u) = \int_\Omega \frac{1}{p}\, |\nabla u|^p - F(x,u), \quad u \in W = W_0^{1,p}(\Omega).$$

The argument of Wang [14] can easily be adapted to show that Φ^a is again contractible for $a < 0$ with $|a|$ sufficiently large, so (1.6) holds as before if zero is the only critical point of Φ. So the idea of K. Perera [11] was to use a minimax scheme involving the $\mathbb{Z}_2$-cohomological index of E. R. Fadell and P. H. Rabinowitz [7] to construct a new sequence of eigenvalues $\lambda_k \nearrow \infty$ such that if $\lambda_k < \lambda < \lambda_{k+1}$ in (1.8), then $C^k(\Phi, 0) \neq 0$, again contradicting (1.6).

Remark 1.2. When f satisfies a local sign condition near zero, M. Degiovanni, S. Lancelotti and K. Perera [5] used the notion of a cohomological local splitting introduced in K. Perera, R. P. Agarwal and D. O'Regan [12] to obtain a nontrivial solution for all $\lambda \in \mathbb{R}$.

Naturally we may ask whether there is an extension of these results to anisotropic p-Laplacian systems of the form

$$\begin{cases} -\Delta_{p_i} u_i = \dfrac{\partial F}{\partial u_i}(x, u) & \text{in } \Omega, \\ u_i = 0 & \text{on } \partial\Omega, \end{cases} \qquad i = 1, \ldots, m \tag{1.9}$$

where each $p_i \in (1, \infty)$, $u = (u_1, \ldots, u_m) \in \mathbb{R}^m$, $F \in C^1(\Omega \times \mathbb{R}^m)$ satisfies the subcritical growth conditions

$$\left| \frac{\partial F}{\partial u_i}(x, u) \right| \leq C \bigg(\sum_{j=1}^m |u_j|^{r_{ij} - 1} + 1 \bigg) \tag{1.10}$$

for all $(x, u) \in \Omega \times \mathbb{R}^m$, $i = 1, \ldots, m$, for some $r_{ij} \in (1, 1 + p_j^*/(p_i^*)')$, and $(p_i^*)' = p_i^*/(p_i^* - 1)$ is the Hölder conjugate of p_i^*. Here the associated functional is

$$\Phi(u) = I(u) - \int_\Omega F(x, u), \quad u \in W = W_0^{1,p_1}(\Omega) \times \ldots \times W_0^{1,p_m}(\Omega)$$

where

$$I(u) = \int_\Omega \sum_{i=1}^m \frac{1}{p_i} |\nabla u_i|^{p_i}.$$

Unlike in the scalar case, here I is not homogeneous except when $p_1 = \ldots = p_m$. However, it still has the following weaker property. Define continuous flows on both W and $\mathbb{R}^m$ by

$$(\alpha, u) \mapsto u_\alpha := (|\alpha|^{1/p_1 - 1} \alpha u_1, \ldots, |\alpha|^{1/p_m - 1} \alpha u_m), \quad \alpha \in \mathbb{R}.$$

Then

$$I(u_\alpha) = |\alpha|\, I(u) \quad \text{for all } \alpha \in \mathbb{R},\ u \in W. \tag{1.11}$$

This suggests that the appropriate class of eigenvalue problems to consider here are of the form

$$\begin{cases} -\Delta_{p_i} u_i = \lambda \dfrac{\partial J}{\partial u_i}(x, u) & \text{in } \Omega, \\ u_i = 0 & \text{on } \partial\Omega, \end{cases} \qquad i = 1, \ldots, m$$

where $J \in C^1(\Omega \times \mathbb{R}^m)$ satisfies

$$J(x, u_\alpha) = |\alpha|\, J(x, u) \quad \text{for all } \alpha \in \mathbb{R},\ (x, u) \in \Omega \times \mathbb{R}^m.$$

Differentiating this with respect to u_i gives

$$\frac{\partial J}{\partial u_i}(x, u_\alpha) = |\alpha|^{-1/p_i} \alpha\, \frac{\partial J}{\partial u_i}(x, u) \quad \text{and} \quad \Delta_{p_i} (u_\alpha)_i = |\alpha|^{-1/p_i} \alpha \Delta_{p_i} u_i,$$

so if u is an eigenvector associated with λ, then so is u_α for any $\alpha \neq 0$.

To fix ideas, let us take

$$J(x, u) = V(x)\, |u_1|^{r_1} \dots |u_m|^{r_m}$$

where $r_i \in (1, p_i)$ with $r_1/p_1 + \dots + r_m/p_m = 1$ and $V \in C^1(\Omega)$ is a (possibly indefinite) bounded weight function. Then

$$J(x, u_\alpha) = |\alpha|^{r_1/p_1 + \dots + r_m/p_m} V(x) |u_1|^{r_1} \dots |u_m|^{r_m} = |\alpha| J(x, u), \tag{1.12}$$

and the corresponding eigenvalue problem is

$$\begin{cases} -\Delta_{p_i} u_i = \lambda r_i V(x) |u_1|^{r_1} \dots |u_i|^{r_i - 2} u_i \dots |u_m|^{r_m} & \text{in } \Omega, \\ u_i = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.13}$$

for $i = 1, \dots, m$. Assume $u(x) \equiv 0$ is a solution of (1.9) and the behavior of F near zero is given by

$$F(x, u) = \lambda V(x) |u_1|^{r_1} \dots |u_m|^{r_m} + G(x, u), \tag{1.14}$$

where the higher-order term G satisfies

$$|G(x, u)| \le C \sum_{i=1}^m |u_i|^{s_i} \quad \text{for all } (x, u) \in \Omega \times \mathbb{R}^m \tag{1.15}$$

for some $s_i \in (p_i, p_i^*)$.

It is natural to replace (1.4) with

$$0 < F(x, u) \le \sum_{i=1}^m \frac{u_i}{\mu_i} \frac{\partial F}{\partial u_i}(x, u) \quad \text{for all } x \in \Omega,\ |u| \ge T \tag{1.16}$$

for some $\mu_i > p_i$, $i = 1, \dots, m$ and $T > 0$. We will also need to assume that

$$H(x, u) := \sum_{i=1}^m \frac{u_i}{p_i} \frac{\partial F}{\partial u_i}(x, u) - F(x, u) \ge -C \quad \text{for all } (x, u) \in \Omega \times \mathbb{R}^m \tag{1.17}$$

for some $C > 0$. Note that in the scalar case this follows from (1.2) and (1.4).

We shall prove

Theorem 1.3. *Assume* (1.10) *and* (1.14)–(1.17). *If* λ *is not an eigenvalue of* (1.13)*, then the system* (1.9) *has a nontrivial solution.*

Our proof will be based on an abstract framework for anisotropic systems introduced in Perera, Agarwal, and O'Regan [12], which we will recall in the next section, but first we show that (1.16) implies F is p_i-superquadratic in u_i, analogous to (1.5).

Let $\mu = (\mu_1, \dots, \mu_m)$ and set

$$r_\mu(u) = \sum_{i=1}^{m} |u_i|^{\mu_i}, \quad u \in \mathbb{R}^m.$$

There is an $R > 0$ such that $r_\mu(u) \geq R$ implies $|u| \geq T$. Then

$$r_\mu(u) \geq R \Rightarrow 0 < F(x,u) \leq \sum_{i=1}^{m} \frac{u_i}{\mu_i} \frac{\partial F}{\partial u_i}(x,u) \tag{1.18}$$

by (1.16).

Lemma 1.4. *If* (1.10) *and* (1.18) *hold, then*

$$F(x,u) \geq c(x) r_\mu(u) - C \quad \textit{for all } (x,u) \in \Omega \times \mathbb{R}^m \tag{1.19}$$

where

$$c(x) = \min_{r_\mu(u)=R} \frac{F(x,u)}{R} > 0$$

and $C > 0$.

Proof. Fix $u \in \mathbb{R}^m$ with $r_\mu(u) \geq R$. Let $\alpha_u = r_\mu(u)/R \geq 1$ and

$$\widetilde{u} = (\alpha_u^{-1/\mu_1} u_1, \dots, \alpha_u^{-1/\mu_m} u_m),$$

so that $r_\mu(\widetilde{u}) = \alpha_u^{-1} r_\mu(u) = R$, and consider the path

$$u(\alpha) = ((\alpha/\alpha_u)^{1/\mu_1} u_1, \dots, (\alpha/\alpha_u)^{1/\mu_m} u_m), \quad 1 \leq \alpha \leq \alpha_u$$

joining $\widetilde{u}$ to u. Noting that $r_\mu(u(\alpha)) = (\alpha/\alpha_u)\, r_\mu(u) = \alpha R \geq R$, we have

$$\frac{d}{d\alpha}\,(F(x,u(\alpha))) = \alpha^{-1} \sum_{i=1}^{m} \frac{u_i(\alpha)}{\mu_i} \frac{\partial F}{\partial u_i}(x,u(\alpha)) \geq \alpha^{-1} F(x,u(\alpha)) > 0$$

by (1.18), and integrating this from $\alpha = 1$ to α_u gives

$$F(x,u) \geq F(x,\widetilde{u})\alpha_u = \frac{F(x,\widetilde{u})}{R}\, r_\mu(u) \geq c(x) r_\mu(u). \qquad \square$$

2. Preliminaries

In this section we recall an abstract framework for anisotropic systems introduced in K. Perera, R. P. Agarwal and D. O'Regan [12].

For $i = 1, \dots, m$, let $(W_i, \| \cdot \|_i)$ be a real reflexive Banach space with the dual $(W_i^*, \| \cdot \|_i^*)$ and the duality pairing $(\cdot\, , \cdot)_i$. Then their product

$$W = W_1 \times \ldots \times W_m = \{u = (u_1, \dots, u_m) : u_i \in W_i\}$$

is also a reflexive Banach space with the norm

$$\|u\| = \left(\sum_{i=1}^m \|u_i\|_i^2 \right)^{1/2}$$

and has the dual

$$W^* = W_1^* \times \ldots \times W_m^* = \{L = (L_1, \dots, L_m) : L_i \in W_i^*\},$$

with the pairing

$$(L, u) = \sum_{i=1}^m (L_i, u_i)_i$$

and the dual norm

$$\|L\|^* = \left(\sum_{i=1}^m (\|L_i\|_i^*)^2 \right)^{1/2}.$$

We consider the system of operator equations

$$A_p\, u = F'(u) \tag{2.1}$$

in W^*, where $p = (p_1, \dots, p_m)$ with each $p_i \in (1, \infty)$,

$$A_p\, u = (A_{p_1} u_1, \dots, A_{p_m} u_m),$$

$A_{p_i} \in C(W_i, W_i^*)$ is

(A_{i1}) $(p_i - 1)$-homogeneous and odd:

$$A_{p_i}(\alpha u_i) = |\alpha|^{p_i - 2} \alpha A_{p_i} u_i \quad \text{for all } u_i \in W_i,\ \alpha \in \mathbb{R},$$

(A_{i2}) uniformly positive: there exists $c_i > 0$ such that

$$(A_{p_i} u_i, u_i)_i \ge c_i \|u_i\|_i^{p_i} \quad \text{for all } u_i \in W_i,$$

(A_{i3}) a potential operator: there is a functional $I_{p_i} \in C^1(W_i, \mathbb{R})$, called a potential for A_{p_i}, such that

$$I'_{p_i}(u_i) = A_{p_i} u_i \quad \text{for all } u_i \in W_i,$$

(A_4) A_p is of type (S): for any sequence $(u^j) \subset W$,

$$u^j \rightharpoonup u, \quad (A_p\, u^j, u^j - u) \to 0 \Rightarrow u^j \to u,$$

and $F \in C^1(W, \mathbb{R})$ with $F' = (F_{u_1}, \dots, F_{u_m}) : W \to W^*$ compact and $F(0) = 0$.

The following proposition is useful for verifying (A_4).

Proposition 2.1 ([12, Proposition 10.0.5]). *If each W_i is uniformly convex and*

$$(A_{p_i} u_i, v_i)_i \le r_i \|u_i\|_i^{p_i - 1} \|v_i\|_i, \quad (A_{p_i} u_i, u_i)_i = r_i \|u_i\|_i^{p_i} \quad \text{for all } u_i, v_i \in W_i$$

for some $r_i > 0$, then (A_4) holds.

By Proposition 1.0.2 of [12], A_p is also a potential operator and the potential I_p of A_p satisfying $I_p(0) = 0$ is given by

$$I_p(u) = \sum_{i=1}^{m} \frac{1}{p_i} (A_{p_i} u_i, u_i)_i.$$

Now the solutions of the system (2.1) coincide with the critical points of the C^1-functional

$$\Phi(u) = I_p(u) - F(u), \quad u \in W.$$

The following proposition is useful for verifying the (PS) condition for Φ.

Proposition 2.2 ([12, Lemma 3.1.3]). *Every bounded* (PS) *sequence of* Φ *has a convergent subsequence.*

Unlike in the scalar case, here the functional I_p is not homogeneous except when $p_1 = \ldots = p_m$. However, I_p still has the following weaker property. Define a continuous flow on W by

$$\mathbb{R} \times W \to W, \quad (\alpha, u) \mapsto u_\alpha := (|\alpha|^{1/p_1 - 1} \alpha u_1, \dots, |\alpha|^{1/p_m - 1} \alpha u_m).$$

Then $I_p(u_\alpha) = |\alpha| I_p(u)$ by (A_{i1}). This suggests that the appropriate class of eigenvalue problems to study for the operator A_p are of the form

$$A_p u = \lambda J'(u) \tag{2.2}$$

where the functional $J \in C^1(W, \mathbb{R})$ satisfies

$$J(u_\alpha) = |\alpha| J(u) \quad \text{for all } \alpha \in \mathbb{R},\ u \in W \tag{2.3}$$

and J' is compact. Taking $\alpha = 0$ shows that $J(0) = 0$, and taking $\alpha = -1$ shows that J is even, so J' is odd, in particular, $J'(0) = 0$. Moreover, if u is an eigenvector associated with λ, then so is u_α for any $\alpha \neq 0$ (see Proposition 10.1.2 of [12]).

Let

$$\mathcal{M} = \{u \in W : I_p(u) = 1\}, \qquad \mathcal{M}^\pm = \{u \in \mathcal{M} : J(u) \gtrless 0\}.$$

Then $\mathcal{M} \subset W \setminus \{0\}$ is a bounded complete symmetric C^1-Finsler manifold radially homeomorphic to the unit sphere in W, $\mathcal{M}^\pm$ are symmetric open submanifolds of $\mathcal{M}$, and the positive (resp. negative) eigenvalues of (2.2) coincide with the critical values of the even functionals

$$\Psi^\pm(u) = \frac{1}{J(u)}, \quad u \in \mathcal{M}^\pm$$

(see Lemmas 10.1.4 and 10.1.5 of [12]).

Denote by $\mathcal{F}^\pm$ the classes of symmetric subsets of $\mathcal{M}^\pm$ and by $i(M)$ the Fadell–Rabinowitz cohomological index of $M \in \mathcal{F}^\pm$. Then

$$\begin{aligned} \lambda_k^+ &:= \inf_{\substack{M \in \mathcal{F}^+ \\ i(M) \geq k}} \sup_{u \in M} \Psi^+(u), \quad 1 \leq k \leq i(\mathcal{M}^+), \\ \lambda_k^- &:= \sup_{\substack{M \in \mathcal{F}^- \\ i(M) \geq k}} \inf_{u \in M} \Psi^-(u), \quad 1 \leq k \leq i(\mathcal{M}^-) \end{aligned}$$

define nondecreasing (resp. nonincreasing) sequences of positive (resp. negative) eigenvalues of (2.2) that are unbounded when $i(\mathcal{M}^\pm) = \infty$ (see Theorems 10.1.8 and 10.1.9 of [12]). When $i(\mathcal{M}^\pm) = 0$ we set $\lambda_1^\pm = \pm\infty$ for convenience.

Returning to (2.1), suppose that $u = 0$ is a solution and the asymptotic behavior of F near zero is given by

$$F(u_\alpha) = \lambda J(u_\alpha) + \mathrm{o}(\alpha) \quad \text{as } \alpha \searrow 0, \text{ uniformly in } u \in \mathcal{M}. \tag{2.4}$$

PROPOSITION 2.3 ([12, Proposition 10.2.1]). *Assume that* (A_{i1})–(A_{i3}), (A_4), (2.3) *and* (2.4) *hold, F' and J' are compact, and zero is an isolated critical point of* Φ.

(a) *If $\lambda_1^- < \lambda < \lambda_1^+$, then $C^q(\Phi, 0) \approx \delta_{q0}\mathbb{Z}_2$.*
(b) *If $\lambda_{k+1}^- < \lambda < \lambda_k^-$ or $\lambda_k^+ < \lambda < \lambda_{k+1}^+$, then $C^k(\Phi, 0) \neq 0$.*

3. Proof of Theorem 1.3

First let us verify that our problem fits into the abstract framework of the previous section. Let $W_i = W_0^{1,p_i}(\Omega)$,

$$\begin{aligned} (A_{p_i} u_i, v_i)_i &= \int_\Omega |\nabla u_i|^{p_i - 2} \nabla u_i \cdot \nabla v_i, \qquad F(u) = \int_\Omega F(x, u), \\ J(u) = \int_\Omega J(x, u) &= \int_\Omega V(x) |u_1|^{r_1} \dots |u_m|^{r_m}, \quad G(u) = \int_\Omega G(x, u). \end{aligned}$$

Then (A_{i1}) is clear, $(A_{p_i} u_i, u_i)_i = \|u_i\|_i^{p_i}$ in (A_{i2}) and (A_{i3}) holds with

$$I_{p_i}(u_i) = \int_\Omega \frac{1}{p_i} |\nabla u_i|^{p_i}.$$

By the Hölder inequality,

$$(A_{p_i} u_i, v_i)_i \le \bigg(\int_\Omega |\nabla u_i|^{p_i}\bigg)^{1-1/p_i} \bigg(\int_\Omega |\nabla v_i|^{p_i}\bigg)^{1/p_i} = \|u_i\|_i^{p_i-1} \|v_i\|_i,$$

so (A_4) follows from Proposition 2.1. Integrating (1.12) gives (2.3). By (1.15),

$$|G(u_\alpha)| \le C \sum_{i=1}^m |\alpha|^{s_i/p_i} \|u_i\|_i^{s_i},$$

so (2.4) also holds.

By the growth condition (1.10),

$$|(F'(u), v)| = \bigg|\int_\Omega \sum_{i=1}^m \frac{\partial F}{\partial u_i}(x,u) v_i\bigg| \le C \sum_{i=1}^m \bigg(\sum_{j=1}^m \|u_j\|^{r_{ij}-1}_{L^{(r_{ij}-1)(p_i^*)'}(\Omega)} + 1\bigg) \|v_i\|_i.$$

Since $(r_{ij} - 1)(p_i^*)' < p_j^*$ and hence the imbedding $W_0^{1,p_j}(\Omega) \hookrightarrow L^{(r_{ij}-1)(p_i^*)'}(\Omega)$ is compact, the compactness of F' follows. We have

$$\bigg|\frac{\partial J}{\partial u_i}(x,u)\bigg| = r_i |V(x)|\, |u_1|^{r_1} \dots |u_i|^{r_i - 1} \dots |u_m|^{r_m} \le C \sum_{j=1}^m |u_j|^{p_j/p_i'}$$

since $r_1/p_1 + \dots + (r_i - 1)/p_i + \dots + r_m/p_m = 1 - 1/p_i = 1/p_i'$, and $p_j/p_i' < p_j^*/(p_i^*)'$, so the compactness of J' follows similarly.

Since λ is not an eigenvalue of (1.13), it now follows from Proposition 2.3 that $C^k(\Phi, 0) \ne 0$ for some $k \ge 0$. Now we show that Φ satisfies the (PS) condition and, for $a < 0$ with $|a|$ sufficiently large, Φ^a is homotopic to

$$\mathcal{M} = \{u \in W : I(u) = 1\}$$

and hence contractible. As in the scalar case, this then leads to a contradiction if Φ has no nontrivial critical points.

Lemma 3.1. *If* (1.10) *and* (1.16) *hold, then* Φ *satisfies the* (PS) *condition.*

Proof. By (1.10) and (1.16),

$$H_\mu(x,u) := \sum_{i=1}^m \frac{u_i}{\mu_i} \frac{\partial F}{\partial u_i}(x,u) - F(x,u) \ge -C \quad \text{for all } (x,u) \in \Omega \times \mathbb{R}^m \tag{3.1}$$

for some $C > 0$.

Let (u^j) be a (PS) sequence, i.e. $\Phi(u^j) = \mathrm{O}(1)$ and $\Phi'(u^j) = \mathrm{o}(1)$. By Proposition 2.2, it suffices to show that (u^j) is bounded. Writing $(u_1^j/\mu_1, \dots, u_m^j/\mu_m) = u^j/\mu$, we have

$$\Phi(u^j) - (\Phi'(u^j), u^j/\mu) = \int_\Omega \sum_{i=1}^m \bigg(\frac{1}{p_i} - \frac{1}{\mu_i}\bigg) |\nabla u_i^j|^{p_i} + H_\mu(x, u^j),$$

which together with (3.1) gives

$$\sum_{i=1}^m \left(\frac{1}{p_i} - \frac{1}{\mu_i}\right)\|u_i^j\|_i^{p_i} \le \mathrm{o}(1)\left(\sum_{i=1}^m \frac{1}{\mu_i^2}\|u_i^j\|_i^2\right)^{1/2} + \mathrm{O}(1).$$

Since $\mu_i > p_i > 1$, it follows from this that (u^j) is bounded. $\square$

Let $p = (p_1, \dots , p_m)$. Writing $(u_1/p_1, \dots , u_m/p_m) = u/p$, we have

$$\Phi(u) - (\Phi'(u), u/p) = \int_\Omega H(x, u) \ge -C|\Omega| =: a_0$$

where C is as in (1.17) and $|\Omega|$ is the volume of Ω, so all critical values of Φ are greater than or equal to a_0.

Lemma 3.2. *If $a < a_0$, then there is a C^1-map $A_a\colon \mathcal{M} \to (0, \infty)$ such that*

$$\Phi^a = \{u_\alpha : u \in \mathcal{M},\ \alpha \ge A_a(u)\} \simeq \mathcal{M}.$$

Proof. For $u \in \mathcal{M}$ and $\alpha > 0$,

$$\Phi(u_\alpha) = \alpha - \int_\Omega F(x, u_\alpha) \le \alpha - \sum_{i=1}^m \alpha^{\mu_i/p_i} \int_\Omega c(x)|u_i|^{\mu_i} + C|\Omega|$$

by (1.11) and (1.19), so $\Phi(u_\alpha) \le a$ for sufficiently large α. Moreover,

$$\begin{aligned}\frac{d}{d\alpha}(\Phi(u_\alpha)) &= 1 - \alpha^{-1} \int_\Omega \sum_{i=1}^m \frac{(u_\alpha)_i}{p_i}\, \frac{\partial F}{\partial u_i}(x, u_\alpha)\\ &= \alpha^{-1}\left(\Phi(u_\alpha) - \int_\Omega H(x, u_\alpha)\right) \le \alpha^{-1}(\Phi(u_\alpha) - a_0),\end{aligned}$$

so

$$\Phi(u_\alpha) \le a \Rightarrow \frac{d}{d\alpha}(\Phi(u_\alpha)) \le -\alpha^{-1}(a_0 - a) < 0.$$

Thus, there is a unique $A_a(u) > 0$ such that

$$\alpha < \text{resp. } =, > A_a(u) \Rightarrow \Phi(u_\alpha) > \text{resp. } =, < a$$

and the map A_a is C^1 by the implicit function theorem. Then $W \setminus \{0\}$, which is $\simeq \mathcal{M}$, deformation retracts to $\Phi^a = \{u_\alpha : u \in \mathcal{M},\ \alpha \ge A_a(u)\}$ via

$$\begin{gathered}(W \setminus \{0\}) \times [0, 1] \to W \setminus \{0\},\\ (u, t) \mapsto \begin{cases} u_{1-t+tA_a(u_{1/I(u)})/I(u)} & \text{for } u \in (W \setminus \{0\}) \setminus \Phi^a,\\ u & \text{for } u \in \Phi^a.\end{cases}\end{gathered}$$

$\square$

References

[1] A. Ambrosetti and P. H. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Funct. Anal. **14** (1973), 349–381.

[2] A. Anane, *Simplicité et isolation de la première valeur propre du p-Laplacien avec poids*, C. R. Acad. Sci. Paris Sér. I Math. **305** (1987), 725–728.

[3] V. Benci, *A new approach to the Morse–Conley theory and some applications*, Ann. Mat. Pura Appl. (4) **158** (1991), 231–305.

[4] K. Chang, *Infinite-dimensional Morse theory and multiple solution problems*, Progress in Nonlinear Differential Equations and their Applications, vol. **6**, Birkhäuser Boston Inc., Boston, MA, 1993.

[5] M. Degiovanni, S. Lancelotti and K. Perera, *Nontrivial solutions of p-superlinear p-Laplacian problems via a cohomological local splitting*, Comm. Contemp. Math. (to appear).

[6] P. Drábek, *Solvability and bifurcations of nonlinear equations*, Pitman Research Notes in Mathematics Series, vol. **264**, Longman Scientific & Technical, Harlow, 1992.

[7] E. R. Fadell and P. H. Rabinowitz, *Generalized cohomological index theories for Lie group actions with an application to bifurcation questions for Hamiltonian systems*, Invent. Math. **45** (1978), 139–174.

[8] S. J. Li and M. Willem, *Applications of local linking to critical point theory*, J. Math. Anal. Appl. **189** (1995), 6–32.

[9] P. Lindqvist, *On the equation* $\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u = 0.$, Proc. Amer. Math. Soc. **109** (1990), 157–164.

[10] ———, *Addendum: "On the equation* $\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u = 0$*"*, Proc. Amer. Math. Soc. **109** (1990), 157–164; Proc. Amer. Math. Soc. **116** (1992), 583–584.

[11] K. Perera, *Nontrivial critical groups in p-Laplacian problems via the Yang index*, Topol. Methods Nonlinear Anal. **21** (2003), 301–309.

[12] K. Perera, R. P. Agarwal and D. O'Regan, *Morse-Theoretic Aspects of p-Laplacian Type Operators*, Math. Surveys Monogr., vol. **161**, Amer. Math. Soc., Providence, RI, 2010.

[13] P. H. Rabinowitz, *Some critical point theorems and applications to semilinear elliptic partial differential equations*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **5** (1978), 215–223.

[14] Z. Q. Wang, *On a superlinear elliptic equation*, Ann. Inst. H. Poincaré Anal. Non Linéaire **8** (1991), 43–57.

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