NONTRIVIAL SOLUTIONS OF \(p\)-SUPERLINEAR ANISOTROPIC \(p\)-LAPLACIAN SYSTEMS VIA MORSE THEORY

KANISHKA PERERA — RAVI P. AGARWAL — DONAL O’REGAN

Abstract. We obtain nontrivial solutions of a class of \(p\)-superlinear anisotropic \(p\)-Laplacian systems using Morse theory.

1. Introduction

The purpose of this paper is to obtain nontrivial solutions of a class of \(p\)-superlinear anisotropic \(p\)-Laplacian systems using Morse theory.

As motivation, we begin by recalling a well-known result for the semilinear elliptic boundary value problem

\[
\begin{aligned}
-\Delta u &= f(x, u) \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

where \(\Omega\) is a bounded domain in \(\mathbb{R}^n\), \(n \geq 1\), \(f \in C(\Omega \times \mathbb{R})\) satisfies the subcritical growth condition

\[|f(x, t)| \leq C(|t|^{r-1} + 1) \quad \text{for all } (x, t) \in \Omega \times \mathbb{R}\]

for some \(r \in (1, 2^*)\),

\[2^* = \begin{cases} 
\frac{2n}{n-2} & \text{if } n > 2 \\
\infty & \text{if } n \leq 2,
\end{cases}
\]

2010 Mathematics Subject Classification. Primary 35J50; Secondary 47J10, 58E05.

Key words and phrases. \(p\)-Laplacian systems, anisotropic, \(p\)-superlinear, nontrivial solutions, Morse theory.

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is the critical Sobolev exponent, and $C$ denotes a generic positive constant. Assume

$$\lim_{t \to 0} \frac{f(x, t)}{t} = \lambda,$$

uniformly in $x \in \Omega$

and the Ambrosetti–Rabinowitz condition

$$0 < F(x, t) := \int_0^t f(x, s) \, ds \leq \frac{t}{\mu} f(x, t) \quad \text{for all } x \in \Omega, |t| \geq T,$$

for some $\mu > 2$ and $T > 0$. Note that (1.3) implies $f(x, 0) \equiv 0$, so problem (1.1) has the trivial solution $u(x) \equiv 0$. Integrating (1.4) gives

$$F(x, t) \geq c(x)|t|^\mu - C$$

for all $(x, t) \in \Omega \times \mathbb{R}$

where $c(x) = \min F(x, \pm T)/T^\mu > 0$, so $f$ is superlinear. V. Benci [3] used a new approach to the Morse–Conley theory to obtain a nontrivial solution of this problem when $\lambda \notin \sigma(-\Delta)$, the Dirichlet spectrum of the negative Laplacian on $\Omega$.

The idea of the proof may be restated in terms of critical groups as follows (see K. Chang [4] and Z. Q. Wang [14]). Weak solutions of (1.1) coincide with the critical points of the $C^1$-functional

$$\Phi(u) = \int_\Omega \frac{1}{2} |\nabla u|^2 - F(x, u), \quad u \in H = H^1_0(\Omega),$$

and (1.4) ensures that $\Phi$ satisfies the (PS) condition. Suppose that $\Phi$ has no nontrivial critical points. Then the critical groups of $\Phi$ at zero are given by

$$C^q(\Phi, 0) = H^q(\Phi^0, \Phi^0 \setminus \{0\}), \quad q \geq 0$$

where $\Phi^0$ is the sublevel set $\{u \in H : \Phi(u) \leq 0\}$ and $H^*$ denotes cohomology. By the second deformation lemma, $\Phi^0$ is a deformation retract of $H$ and $\Phi^0 \setminus \{0\}$ deformation retracts to $\Phi^a$ for any $a < 0$, so

$$C^q(\Phi, 0) \approx H^q(H, \Phi^a).$$

By (1.4), if $|a|$ is sufficiently large, $\Phi^a$ is homotopic to the unit sphere in $H$ and hence contractible, so

$$C^q(\Phi, 0) = 0 \quad \text{for all } q.$$ 

On the other hand, if $\lambda_1 < \lambda_2 \leq \ldots$ denote the Dirichlet eigenvalues of the Laplacian on $\Omega$ and $\lambda_k < \lambda < \lambda_{k+1}$ in (1.3), then

$$C^q(\Phi, 0) \approx \delta_{kq} \mathcal{G}$$

where $\mathcal{G}$ is the coefficient group and $\delta_{..}$ denotes the Kronecker delta. This contradiction shows that $\Phi$ has a nontrivial critical point.
Remark 1.1. In the case $\lambda < \lambda_1$, A. Ambrosetti and P. H. Rabinowitz [1] obtained a positive solution and a negative solution using their mountain pass theorem, and Z. Q. Wang [14] obtained a third nontrivial solution using Morse theory. When $f$ satisfies a global sign condition, P. H. Rabinowitz [13] used his linking theorem to obtain a nontrivial solution for all $\lambda \in \mathbb{R}$. S. J. Li and M. Willem [8] used a local linking to do the same when $f$ satisfies only a local sign condition near zero.

K. Perera [11] extended the above result to the corresponding $p$-Laplacian problem

$$\begin{cases} -\Delta_p u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial \Omega, \end{cases}$$

where $\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u)$ is the $p$-Laplacian of $u$, $p \in (1, \infty)$, $f$ now satisfies (1.2) with $r \in (1, p^*)$, and

$$p^* = \begin{cases} \frac{np}{n-p} & \text{if } n > p, \\ \infty & \text{if } n \leq p. \end{cases}$$

Assume

$$\lim_{t \to 0} \frac{f(x, t)}{|t|^{p-2}t} = \lambda, \quad \text{uniformly in } x \in \Omega$$

and (1.4) with $\mu > p$ and $T > 0$, so $u(x) \equiv 0$ is a solution of (1.7) and $f$ is $p$-superlinear by (1.5). K. Perera [11] obtained a nontrivial solution when $\lambda$ is not an eigenvalue of the problem

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial \Omega. \end{cases}$$

This quasilinear eigenvalue problem is far more complicated. It is known that the first eigenvalue $\lambda_1$ is positive, simple, and has an associated eigenfunction $\varphi_1$ that is positive in $\Omega$ (see A. Anane [2] and P. Lindqvist [9], [10]). Moreover, $\lambda_1$ is isolated in the spectrum $\sigma(-\Delta_p)$, so the second eigenvalue $\lambda_2 = \inf \sigma(-\Delta_p) \cap (\lambda_1, \infty)$ is well-defined. In the ODE case $n = 1$, where $\Omega$ is an interval, the spectrum consists of a sequence of simple eigenvalues $\lambda_k \nearrow \infty$, and the eigenfunction $\varphi_k$ associated with $\lambda_k$ has exactly $k - 1$ interior zeroes (see e.g. P. Drábek [6]). In the PDE case $n \geq 2$, an increasing and unbounded sequence of eigenvalues can be constructed using a standard minimax scheme involving the Krasnoselskiǐ’s genus, but it is not known whether this gives a complete list of the eigenvalues.

The variational functional associated with problem (1.7) is

$$\Phi(u) = \int_{\Omega} \frac{1}{p} |\nabla u|^p - F(x, u), \quad u \in W = W^{1,p}_0(\Omega).$$
The argument of Wang [14] can easily be adapted to show that $\Phi^a$ is again contractible for $a < 0$ with $|u|$ sufficiently large, so (1.6) holds as before if zero is the only critical point of $\Phi$. So the idea of K. Perera [11] was to use a minimax scheme involving the $\mathbb{Z}_2$-cohomological index of E. R. Fadell and P. H. Rabinowitz [7] to construct a new sequence of eigenvalues $\lambda_k \nearrow \infty$ such that $\lambda_k < \lambda < \lambda_{k+1}$ in (1.8), then $C^k(\Phi, 0) \neq 0$, again contradicting (1.6).

Remark 1.2. When $f$ satisfies a local sign condition near zero, M. Degiovanni, S. Lancelotti and K. Perera [5] used the notion of a cohomological local splitting introduced in K. Perera, R. P. Agarwal and D. O’Regan [12] to obtain a nontrivial solution for all $\lambda \in \mathbb{R}$.

Naturally we may ask whether there is an extension of these results to anisotropic $p$-Laplacian systems of the form

\begin{equation}
-\Delta_{p_i} u_i = \frac{\partial F}{\partial u_i}(x, u) \quad \text{in } \Omega, \\
u_i = 0 \quad \text{on } \partial \Omega, \quad i = 1, \ldots, m
\end{equation}

where each $p_i \in (1, \infty)$, $u = (u_1, \ldots, u_m) \in \mathbb{R}^m$, $F \in C^1(\Omega \times \mathbb{R}^m)$ satisfies the subcritical growth conditions

\begin{equation}
\left| \frac{\partial F}{\partial u_i}(x, u) \right| \leq C \left( \sum_{j=1}^m |u_j|^{r_{ij}-1} + 1 \right)
\end{equation}

for all $(x, u) \in \Omega \times \mathbb{R}^m$, $i = 1, \ldots, m$, for some $r_{ij} \in (1, 1 + p_j^*/(p_i^*)')$ and $(p_j^*)' = p_i^*/(p_i^* - 1)$ is the Hölder conjugate of $p_i^*$. Here the associated functional is

$$
\Phi(u) = I(u) - \int_{\Omega} F(x, u), \quad u \in W = W_0^{1,p_1}(\Omega) \times \ldots \times W_0^{1,p_m}(\Omega)
$$

where

$$
I(u) = \int_{\Omega} \sum_{i=1}^m \frac{1}{p_i} |\nabla u_i|^{p_i}.
$$

Unlike in the scalar case, here $I$ is not homogeneous except when $p_1 = \ldots = p_m$. However, it still has the following weaker property. Define continuous flows on both $W$ and $\mathbb{R}^m$ by

$$(\alpha, u) \mapsto u_\alpha := (|\alpha|^{1/p_1 - 1}\alpha u_1, \ldots, |\alpha|^{1/p_m - 1}\alpha u_m), \quad \alpha \in \mathbb{R}.
$$

Then

\begin{equation}
I(u_\alpha) = |\alpha| I(u) \quad \text{for all } \alpha \in \mathbb{R}, \quad u \in W.
\end{equation}

This suggests that the appropriate class of eigenvalue problems to consider here are of the form

\begin{equation}
-\Delta_{p_i} u_i = \lambda \frac{\partial J}{\partial u_i}(x, u) \quad \text{in } \Omega, \\
u_i = 0 \quad \text{on } \partial \Omega, \quad i = 1, \ldots, m
\end{equation}
where \( J \in C^1(\Omega \times \mathbb{R}^m) \) satisfies
\[
J(x, u_\alpha) = |\alpha| J(x, u) \quad \text{for all } \alpha \in \mathbb{R}, \ (x, u) \in \Omega \times \mathbb{R}^m.
\]
Differentiating this with respect to \( u_i \) gives
\[
\frac{\partial J}{\partial u_i}(x, u_\alpha) = |\alpha|^{-1/p_i} \alpha \frac{\partial J}{\partial u_i}(x, u) \quad \text{and} \quad \Delta_{p_i}(u_\alpha)_i = |\alpha|^{-1/p_i} \alpha \Delta_{p_i} u_i,
\]
so if \( u \) is an eigenvector associated with \( \lambda \), then so is \( u_\alpha \) for any \( \alpha \neq 0 \).

To fix ideas, let us take
\[
J(x, u) = V(x)|u_1|^{r_1} \cdots |u_m|^{r_m}
\]
where \( r_i \in (1, p_i) \) with \( r_1/p_1 + \ldots + r_m/p_m = 1 \) and \( V \in C^1(\Omega) \) is a (possibly indefinite) bounded weight function. Then
\[
J(x, u_\alpha) = |\alpha|^{r_1/p_1 + \ldots + r_m/p_m} V(x)|u_1|^{r_1} \cdots |u_m|^{r_m} = |\alpha| J(x, u),
\]
and the corresponding eigenvalue problem is
\[
\begin{aligned}
-\Delta_{p_i} u_i &= \lambda r_i V(x)|u_1|^{r_1} \cdots |u_i|^{r_i-2} u_i \cdots |u_m|^{r_m} \quad \text{in } \Omega, \\
 u_i &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]
for \( i = 1, \ldots, m \). Assume \( u(x) \equiv 0 \) is a solution of (1.9) and the behavior of \( F \) near zero is given by
\[
F(x, u) = \lambda V(x)|u_1|^{r_1} \cdots |u_m|^{r_m} + G(x, u),
\]
where the higher-order term \( G \) satisfies
\[
|G(x, u)| \leq C \sum_{i=1}^m |u_i|^{s_i} \quad \text{for all } (x, u) \in \Omega \times \mathbb{R}^m
\]
for some \( s_i \in (p_i, p_i^*) \).

It is natural to replace (1.4) with
\[
0 < F(x, u) \leq \sum_{i=1}^m \frac{u_i}{\mu_i} \frac{\partial F}{\partial u_i}(x, u) \quad \text{for all } x \in \Omega, \ |u| \geq T
\]
for some \( \mu_i > p_i, \ i = 1, \ldots, m \) and \( T > 0 \). We will also need to assume that
\[
H(x, u) := \sum_{i=1}^m \frac{u_i}{p_i} \frac{\partial F}{\partial u_i}(x, u) - F(x, u) \geq -C \quad \text{for all } (x, u) \in \Omega \times \mathbb{R}^m
\]
for some \( C > 0 \). Note that in the scalar case this follows from (1.2) and (1.4).

We shall prove
Theorem 1.3. Assume (1.10) and (1.14)–(1.17). If $\lambda$ is not an eigenvalue of (1.13), then the system (1.9) has a nontrivial solution.

Our proof will be based on an abstract framework for anisotropic systems introduced in Perera, Agarwal, and O’Regan [12], which we will recall in the next section, but first we show that (1.16) implies $F$ is $p_i$-superquadratic in $u_i$, analogous to (1.5).

Let $\mu = (\mu_1, \ldots, \mu_m)$ and set

$$r_{\mu}(u) = \sum_{i=1}^{m} |u_i|^\mu_i, \quad u \in \mathbb{R}^m.$$ 

There is an $R > 0$ such that $r_{\mu}(u) \geq R$ implies $|u| \geq T$. Then

$$r_{\mu}(u) \geq R \Rightarrow 0 < F(x,u) \leq \sum_{i=1}^{m} \frac{u_i}{\mu_i} \frac{\partial F}{\partial u_i}(x,u)$$

by (1.16).

Lemma 1.4. If (1.10) and (1.18) hold, then

$$F(x,u) \geq c(x)r_{\mu}(u) - C \quad \text{for all } (x,u) \in \Omega \times \mathbb{R}^m$$

where

$$c(x) = \min_{r_{\mu}(u) = R} \frac{F(x,u)}{R} > 0$$

and $C > 0$.

Proof. Fix $u \in \mathbb{R}^m$ with $r_{\mu}(u) \geq R$. Let $\alpha_u = r_{\mu}(u)/R \geq 1$ and

$$\tilde{u} = (\alpha_{u}^{-1/\mu_1} u_1, \ldots, \alpha_{u}^{-1/\mu_m} u_m),$$

so that $r_{\mu}(\tilde{u}) = \alpha_u^{-1} r_{\mu}(u) = R$, and consider the path

$$u(\alpha) = ((\alpha/\alpha_u)^{1/\mu_1} u_1, \ldots, (\alpha/\alpha_u)^{1/\mu_m} u_m), \quad 1 \leq \alpha \leq \alpha_u$$

joining $\tilde{u}$ to $u$. Noting that $r_{\mu}(u(\alpha)) = (\alpha/\alpha_u) r_{\mu}(u) = \alpha R \geq R$, we have

$$\frac{d}{d\alpha} (F(x,u(\alpha))) = \alpha^{-1} \sum_{i=1}^{m} \frac{u_i(\alpha)}{\mu_i} \frac{\partial F}{\partial u_i}(x,u(\alpha)) \geq \alpha^{-1} F(x,u(\alpha)) > 0$$

by (1.18), and integrating this from $\alpha = 1$ to $\alpha_u$ gives

$$F(x,u) \geq F(x,\tilde{u}) \alpha_u = \frac{F(x,\tilde{u})}{R} r_{\mu}(u) \geq c(x)r_{\mu}(u).$$
2. Preliminaries

In this section we recall an abstract framework for anisotropic systems introduced in K. Perera, R. P. Agarwal and D. O’Regan [12].

For \( i = 1, \ldots, m \), let \( (W_i, \| \cdot \|_i) \) be a real reflexive Banach space with the dual \( (W_i^*, \| \cdot \|_i^*) \) and the duality pairing \((\cdot, \cdot)_i\). Then their product

\[
W = W_1 \times \cdots \times W_m = \{ u = (u_1, \ldots, u_m) : u_i \in W_i \}
\]
is also a reflexive Banach space with the norm

\[
\| u \| = \left( \sum_{i=1}^{m} \| u_i \|_i^2 \right)^{1/2}
\]
and has the dual

\[
W^* = W_1^* \times \cdots \times W_m^* = \{ L = (L_1, \ldots, L_m) : L_i \in W_i^* \},
\]
with the pairing

\[
(L, u) = \sum_{i=1}^{m} (L_i, u_i)_i
\]
and the dual norm

\[
\| L \|_* = \left( \sum_{i=1}^{m} \| L_i \|_i^2 \right)^{1/2}.
\]

We consider the system of operator equations

\[
(2.1) \quad A_p u = F'(u)
\]
in \( W^* \), where \( p = (p_1, \ldots, p_m) \) with each \( p_i \in (1, \infty) \),

\[
A_p u = (A_{p_1} u_1, \ldots, A_{p_m} u_m),
\]
\( A_{p_i} \in C(W_i, W_i^*) \) is

(A1) \( (p_i - 1) \)-homogeneous and odd:

\[
A_{p_i}(\alpha u_i) = |\alpha|^{p_i - 2} \alpha A_{p_i}u_i \quad \text{for all } u_i \in W_i, \alpha \in \mathbb{R},
\]

(A2) uniformly positive: there exists \( c_i > 0 \) such that

\[
(A_{p_i} u_i, u_i)_i \geq c_i \| u_i \|_{i}^{p_i} \quad \text{for all } u_i \in W_i,
\]

(A3) a potential operator: there is a functional \( I_{p_i} \in C^1(W_i, \mathbb{R}) \), called a potential for \( A_{p_i} \), such that

\[
I'_{p_i}(u_i) = A_{p_i} u_i \quad \text{for all } u_i \in W_i,
\]

(A4) \( A_p \) is of type (S): for any sequence \( (u^j) \subset W \),

\[
u^j \rightharpoonup u, \quad (A_p u^j, u^j - u) \to 0 \Rightarrow u^j \to u,
\]
and $F \in C^1(W, \mathbb{R})$ with $F' = (F_{u_1}, \ldots , F_{u_m}) : W \to W^*$ compact and $F(0) = 0$.

The following proposition is useful for verifying $(A_4)$.

**Proposition 2.1** ([12, Proposition 10.0.5]). If each $W_i$ is uniformly convex and

$$(A_{p_i} u_i, v_i)_i \leq r_i \Vert u_i \Vert_{p_i}^{p_i - 1} \Vert v_i \Vert_i, \quad (A_{p_i} u_i, u_i)_i = r_i \Vert u_i \Vert_i^{p_i} \quad \text{for all } u_i, v_i \in W_i$$

for some $r_i > 0$, then $(A_4)$ holds.

By Proposition 1.0.2 of [12], $A_p$ is also a potential operator and the potential $I_p$ of $A_p$ satisfying $I_p(0) = 0$ is given by

$$I_p(u) = \sum_{i=1}^m \frac{1}{p_i} (A_{p_i} u_i, u_i).$$

Now the solutions of the system (2.1) coincide with the critical points of the $C^1$-functional

$$\Phi(u) = I_p(u) - F(u), \quad u \in W.$$  

The following proposition is useful for verifying the (PS) condition for $\Phi$.

**Proposition 2.2** ([12, Lemma 3.1.3]). Every bounded (PS) sequence of $\Phi$ has a convergent subsequence.

Unlike in the scalar case, here the functional $I_p$ is not homogeneous except when $p_1 = \ldots = p_m$. However, $I_p$ still has the following weaker property. Define a continuous flow on $W$ by

$$\mathbb{R} \times W \to W, \quad (\alpha, u) \mapsto u_\alpha := (|\alpha|^{1/p_1 - 1} \alpha u_1 , \ldots , |\alpha|^{1/p_m - 1} \alpha u_m).$$

Then $I_p(u_\alpha) = |\alpha| I_p(u)$ by $(A_{11})$. This suggests that the appropriate class of eigenvalue problems to study for the operator $A_p$ are of the form

$$(2.2) \quad A_p u = \lambda J'(u)$$

where the functional $J \in C^1(W, \mathbb{R})$ satisfies

$$(2.3) \quad J(u_\alpha) = |\alpha| J(u) \quad \text{for all } \alpha \in \mathbb{R}, \ u \in W$$

and $J'$ is compact. Taking $\alpha = 0$ shows that $J(0) = 0$, and taking $\alpha = -1$ shows that $J$ is even, so $J'$ is odd, in particular, $J'(0) = 0$. Moreover, if $u$ is an eigenvector associated with $\lambda$, then so is $u_\alpha$ for any $\alpha \neq 0$ (see Proposition 10.1.2 of [12]).

Let

$$\mathcal{M} = \{ u \in W : I_p(u) = 1 \}, \quad \mathcal{M}^\pm = \{ u \in \mathcal{M} : J(u) \geq 0 \}.$$
Then $\mathcal{M} \subset W \setminus \{0\}$ is a bounded complete symmetric $C^1$-Finsler manifold radially homeomorphic to the unit sphere in $W$, $\mathcal{M}^\pm$ are symmetric open submanifolds of $\mathcal{M}$, and the positive (resp. negative) eigenvalues of (2.2) coincide with the critical values of the even functionals

$$\Psi^\pm(u) = \frac{1}{J(u)}, \quad u \in \mathcal{M}^\pm$$

(see Lemmas 10.1.4 and 10.1.5 of [12]).

Denote by $\mathcal{F}^\pm$ the classes of symmetric subsets of $\mathcal{M}^\pm$ and by $i(M)$ the Fadell–Rabinowitz cohomological index of $M \in \mathcal{F}^\pm$.

Then

$$\lambda^+_k := \inf_{M \in \mathcal{F}^+, u \in M} \sup_{i(M) \geq k} \Psi^+(u), \quad 1 \leq k \leq i(\mathcal{M}^+),$$

$$\lambda^-_k := \sup_{M \in \mathcal{F}^-, u \in M} \inf_{i(M) \geq k} \Psi^-(u), \quad 1 \leq k \leq i(\mathcal{M}^-)$$

define nondecreasing (resp. nonincreasing) sequences of positive (resp. negative) eigenvalues of (2.2) that are unbounded when $i(\mathcal{M}^\pm) = \infty$ (see Theorems 10.1.8 and 10.1.9 of [12]). When $i(\mathcal{M}^\pm) = 0$ we set $\lambda^\pm_1 = \pm \infty$ for convenience.

Returning to (2.1), suppose that $u = 0$ is a solution and the asymptotic behavior of $F$ near zero is given by

$$F(\alpha) = \lambda J(\alpha) + o(\alpha) \quad \text{as} \quad \alpha \searrow 0, \quad \text{uniformly in} \quad u \in \mathcal{M}. \quad \text{(2.4)}$$

**Proposition 2.3** ([12, Proposition 10.2.1]). Assume that $(A_{i1})-(A_{i3})$, $(A_4)$, (2.3) and (2.4) hold, $F'$ and $J'$ are compact, and zero is an isolated critical point of $\Phi$.

(a) If $\lambda^-_1 < \lambda < \lambda^+_1$, then $C^q(\Phi, 0) \simeq \delta_0 \mathbb{Z}_2$.

(b) If $\lambda^-_{k+1} < \lambda < \lambda^+_k$ or $\lambda^+_k < \lambda < \lambda^-_{k+1}$, then $C^k(\Phi, 0) \neq 0$.

### 3. Proof of Theorem 1.3

First let us verify that our problem fits into the abstract framework of the previous section. Let $W_i = W_i^1, p_i, p_i^1(\Omega)$,

$$(A_p, u_i, v_i)_i = \int_\Omega |\nabla u_i|^{p_i-2} \nabla u_i \cdot \nabla v_i, \quad F(u) = \int_\Omega F(x, u),$$

$$J(u) = \int_\Omega J(x, u) = \int_\Omega V(x)|u_1|^r \cdots |u_m|^r, \quad G(u) = \int_\Omega G(x, u).$$

Then \((A_{i1})\) is clear, \((A_p, u_i, u_i)_i = \|u_i\|_{p_i}^{p_i}\) in \((A_{i2})\) and \((A_{i3})\) holds with

$$I_{p_i}(u_i) = \int_\Omega \frac{1}{p_i} |\nabla u_i|^{p_i}. $$
By the Hölder inequality,
\[ (A_p, u_i, v_i) \leq \left( \int_{\Omega} |\nabla u_i|^{p_i} \right)^{1-1/p_i} \left( \int_{\Omega} |\nabla v_i|^{p_i} \right)^{1/p_i} = ||u_i||^{-1}||v_i||, \]
so (A4) follows from Proposition 2.1. Integrating (1.12) gives (2.3). By (1.15),
\[ |G(u_n)| \leq C \sum_{i=1}^{m} |\alpha|^r_i ||u_i||^{r_i}, \]
so (2.4) also holds.

By the growth condition (1.10),
\[ |F'(u, v)| = \left| \int_{\Omega} \sum_{i=1}^{m} \frac{\partial F}{\partial u_i} (x, u)v_i \right| \leq C \sum_{i=1}^{m} \left( \sum_{j=1}^{m} ||u_j||^{r_j} ||\nabla^r_j(\Omega) + 1 \right)v_i. \]
Since \((r_j - 1)(p_j^*) < p_j^*\) and hence the imbedding \(W_{0}^{1,p_j}(\Omega) \hookrightarrow L^{(r_j - 1)(p_j^*)}(\Omega)\) is compact, the compactness of \(F\) follows. We have
\[ \left| \frac{\partial J}{\partial u_i}(x, u) \right| = r_i|V(x)| |u_1|^r_1 \ldots |u_i|^{r_i - 1} \ldots |u_m|^{r_m} \leq C \sum_{j=1}^{m} ||u_j||^{p_j/p_i}, \]
since \(r_1/p_1 + \ldots + r_i - 1)/p_i + \ldots + r_m/p_m = 1 - 1/p_i = 1/p_i,\) and \(p_j/p_i < p_j^*/(p_j^*),\) so the compactness of \(J\) follows similarly.

Since \(\lambda\) is not an eigenvalue of (1.13), it now follows from Proposition 2.3 that \(C^k(\Phi, 0) \neq 0\) for some \(k \geq 0.\) Now we show that \(\Phi\) satisfies the (PS) condition and, for \(a < 0\) with \(|a|\) sufficiently large, \(\Phi^a\) is homotopic to
\[ \mathcal{M} = \{ u \in W : I(u) = 1 \} \]
and hence contractible. As in the scalar case, this then leads to a contradiction if \(\Phi\) has no nontrivial critical points.

**Lemma 3.1.** If (1.10) and (1.16) hold, then \(\Phi\) satisfies the (PS) condition.

**Proof.** By (1.10) and (1.16),
\[ H_{\mu}(x, u) := \sum_{i=1}^{m} \frac{u_i}{\mu_i} \frac{\partial F}{\partial u_i}(x, u) - F(x, u) \geq -C \] for all \((x, u) \in \Omega \times \mathbb{R}^m\)
for some \(C > 0.\)

Let \((u^j)\) be a (PS) sequence, i.e., \(\Phi(u^j) = O(1)\) and \(\Phi'(u^j) = o(1).\) By Proposition 2.2, it suffices to show that \((u^j)\) is bounded. Writing \((u_i^j/\mu_1, \ldots, u_i^j/\mu_m) = u^j/\mu,\) we have
\[ \Phi(u^j) - (\Phi'(u^j), u^j/\mu) = \int_{\Omega} \sum_{i=1}^{m} \left( \frac{1}{p_i} - \frac{1}{\mu_i} \right) |\nabla u_i^j|^{p_i} + H_{\mu}(x, u^j), \]
which together with (3.1) gives

\[
\sum_{i=1}^{m} \left( \frac{1}{p_i} - \frac{1}{\mu_i} \right) \|u_i\|_{L^p_i}^{p_i} \leq \alpha(1) \left( \sum_{i=1}^{m} \frac{1}{\mu_i^2} \|u_i\|_{L^p_i}^2 \right)^{1/2} + O(1).
\]

Since \(\mu_i > p_i > 1\), it follows from this that \((u^j)\) is bounded. \(\square\)

Let \(p = (p_1, \ldots, p_m)\). Writing \((u_1^{p_1}, \ldots, u_m^{p_m}) = u/p\), we have

\[
\Phi(u) - (\Phi'(u), u/p) = \int_{\Omega} H(x, u) \geq -C|\Omega| := a_0
\]

where \(C\) is as in (1.17) and \(|\Omega|\) is the volume of \(\Omega\), so all critical values of \(\Phi\) are greater than or equal to \(a_0\).

**Lemma 3.2.** If \(a < a_0\), then there is a \(C^1\)-map \(A_a : M \to (0, \infty)\) such that

\[
\Phi^a = \{ u_\alpha : u \in M, \, \alpha \geq A_a(u) \} \simeq M.
\]

**Proof.** For \(u \in M\) and \(\alpha > 0\),

\[
\Phi(u_\alpha) = \alpha - \int_{\Omega} F(x, u_\alpha) \leq \alpha - \sum_{i=1}^{m} \frac{\alpha^{\mu_i/p_i}}{p_i} \int_{\Omega} c(x)|u_i|^\mu_i + C|\Omega|
\]

by (1.11) and (1.19), so \(\Phi(u_\alpha) \leq a\) for sufficiently large \(\alpha\). Moreover,

\[
\frac{d}{d\alpha}(\Phi(u_\alpha)) = 1 - \alpha^{-1} \int_{\Omega} \sum_{i=1}^{m} \frac{u_i}{p_i} \frac{\partial F}{\partial u_i}(x, u_\alpha)
\]

\[
= \alpha^{-1} \left( \Phi(u_\alpha) - \int_{\Omega} H(x, u_\alpha) \right) \leq \alpha^{-1}(\Phi(u_\alpha) - a_0),
\]

so

\[
\Phi(u_\alpha) \leq a \Rightarrow \frac{d}{d\alpha}(\Phi(u_\alpha)) \leq -\alpha^{-1}(a_0 - a) < 0.
\]

Thus, there is a unique \(A_a(u) > 0\) such that

\[
\alpha < \text{resp.} =, > A_a(u) \Rightarrow \Phi(u_\alpha) > \text{resp.} =, < a
\]

and the map \(A_a\) is \(C^1\) by the implicit function theorem. Then \(W \setminus \{0\}\), which is \(\simeq M\), deformation retracts to \(\Phi^a = \{ u_\alpha : u \in M, \, \alpha \geq A_a(u) \}\) via

\[
(W \setminus \{0\}) \times [0, 1] \to W \setminus \{0\},
\]

\[
(u, t) \mapsto \begin{cases} u_{1-t+tA_a(uz/t(u))}/I(u) & \text{for } u \in (W \setminus \{0\}) \setminus \Phi^a, \\ u & \text{for } u \in \Phi^a. \end{cases}
\]

\(\square\)
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Manuscript received May 8, 2009

Kanishka Perera and Ravi P. Agarwal
Department of Mathematical Sciences
Florida Institute of Technology
Melbourne, FL 32901, USA
E-mail address: kperera@fit.edu
agarwal@fit.edu

Donal O’Regan
Department of Mathematics
National University of Ireland
Galway, IRELAND
E-mail address: donal.oregan@nuigalway.ie

TMNA : Volume 35 – 2010 – No 2