

**EXISTENCE OF SOLUTIONS
FOR ANTI-PERIODIC BOUNDARY VALUE PROBLEMS
INVOLVING FRACTIONAL DIFFERENTIAL EQUATIONS
VIA LERAY–SCHAUDER DEGREE THEORY**

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ABSTRACT. In this paper, some existence results for a differential equation of fractional order with anti-periodic boundary conditions are presented. The main tool of study is Leray–Schauder degree theory.

1. Introduction

Fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes. This is the main advantage of fractional differential equations in comparison with classical integer-order models. Fractional differential equations arise in many engineering and scientific disciplines as the mathematical modelling of systems and processes in the fields of physics, chemistry, aerodynamics, electro dynamics of complex medium, polymer rheology, etc. involves derivatives of fractional order. In consequence, the subject of fractional differential equations is gaining much importance and attention. For details and examples, see [2]–[4], [6]–[7], [9], [10], [13], [18]–[23], [25], [28], [30], [36], [38] and the references therein. However, the theory of boundary value problems for nonlinear fractional differential equations is still in the initial stages and many aspects of this theory need to be explored.

2010 Mathematics Subject Classification. 34A12, 34A40.

Key words and phrases. Fractional differential equations, anti-periodic boundary conditions, existence, Leray–Schauder degree theory.

Anti-periodic problems have recently received considerable attention as anti-periodic boundary conditions appear in numerous situations. Examples include anti-periodic trigonometric polynomials in the study of interpolation problems [14], anti-periodic wavelets [11], difference equations [8], [34], ordinary, partial and abstract differential equations [16], [17], [24], [27], [33], [35], [37], and impulsive differential equations [1], [15], [26], etc. For some more application of anti-periodic boundary conditions in physics, see [12], [32] and the references therein.

In this paper, we apply Leray–Schauder degree theory to prove some existence results for the following anti-periodic fractional boundary value problem

$$(1.1) \quad \begin{cases} {}^c D^q u(t) = f(t, u(t)) & \text{for } t \in [0, T], \ 1 < q \leq 2, \\ u(0) = -u(T), \quad u'(0) = -u'(T), \end{cases}$$

where ${}^c D^q$ denotes the Caputo fractional derivative of order q , $f: [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ and T is a fixed positive constant.

2. Preliminaries

First of all, we recall some basic definitions [29], [31] on fractional calculus.

DEFINITION 2.1. For a function $g: [0, \infty) \rightarrow \mathbb{R}^1$, Caputo's derivative of fractional order $q > 0$ is defined as

$${}^c D^q g(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} g^{(n)}(s) ds, \quad n-1 < q < n, \quad n = [q] + 1,$$

where $[q]$ denotes the integer part of the real number q .

DEFINITION 2.2. The Riemann–Liouville fractional integral of order q is defined as

$$I^q g(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{g(s)}{(t-s)^{1-q}} ds, \quad q > 0,$$

provided the integral exists.

DEFINITION 2.3. The Riemann–Liouville fractional derivative of order q for a function $g(t)$ is defined by

$$D^q g(t) = \frac{1}{\Gamma(n-q)} \left(\frac{d}{dt} \right)^n \int_0^t \frac{g(s)}{(t-s)^{q-n+1}} ds, \quad n = [q] + 1,$$

provided the right hand side is pointwise defined on $(0, \infty)$.

LEMMA 2.4 ([38]). For $q > 0$, the general solution of the fractional differential equation ${}^c D^q u(t) = 0$ is given by

$$u(t) = b_0 + b_1 t + b_2 t^2 + \dots + b_{n-1} t^{n-1},$$

where $b_i \in \mathbb{R}$, $i = 0, \dots, n - 1$ ($n = [q] + 1$).

In view of Lemma 2.4, it follows that

$$(2.1) \quad I^q {}^c D^q u(t) = u(t) + b_0 + b_1 t + b_2 t^2 + \dots + b_{n-1} t^{n-1},$$

for some $b_i \in \mathbb{R}$, $i = 0, \dots, n - 1$ ($n = [q] + 1$).

For a given function σ , the most simple differential equation involving a fractional order $1 < q \leq 2$ is

$${}^c D^q u(t) = \sigma(t), \quad t \in [0, T].$$

Imposing the anti-periodic boundary conditions, we have the following new result for the problem:

$$(2.2) \quad \begin{cases} {}^c D^q u(t) = \sigma(t) & \text{for } 0 < t < T, \ 1 < q \leq 2, \\ u(0) = -u(T), \quad u'(0) = -u'(T). \end{cases}$$

LEMMA 2.5. *For any $\sigma \in C[0, T]$, there exists exactly one solution u of the problem (2.2). Moreover, a function u is a solution of the problem (2.2) if and only if*

$$u(t) = \int_0^T G(t, s) \sigma(s) ds,$$

where $G(t, s)$ is the Green's function given by

$$(2.3) \quad G(t, s) = \begin{cases} -\frac{(T-s)^{q-1}}{2\Gamma(q)} + \frac{(T-2t)(T-s)^{q-2}}{4\Gamma(q-1)} & \text{if } 0 \leq t < s \leq T, \\ \frac{(t-s)^{q-1} - (T-s)^{q-1}/2}{\Gamma(q)} + \frac{(T-2t)(T-s)^{q-2}}{4\Gamma(q-1)} & \text{if } 0 \leq s \leq t \leq T. \end{cases}$$

PROOF. Using (2.1), we have that

$$u(t) = I^q \sigma(t) - b_0 - b_1 t = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \sigma(s) ds - b_0 - b_1 t,$$

for arbitrary constants b_0 and b_1 . In view of the relations ${}^c D^q I^q u(t) = u(t)$ and $I^q I^p u(t) = I^{q+p} u(t)$ for $q, p > 0$, $u \in C[0, T]$, we obtain

$$u'(t) = \int_0^t \frac{(t-s)^{q-2}}{\Gamma(q-1)} \sigma(s) ds - b_1.$$

Applying the boundary conditions $u(0) = -u(T)$, $u'(0) = -u'(T)$, we find that

$$b_0 = \frac{1}{2\Gamma(q)} \int_0^T (T-s)^{q-1} \sigma(s) ds - \frac{T}{4\Gamma(q-1)} \int_0^T (T-s)^{q-2} \sigma(s) ds,$$

$$b_1 = \frac{1}{2\Gamma(q-1)} \int_0^T (T-s)^{q-2} \sigma(s) ds.$$

Thus, the unique solution of (2.2) is

$$u(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \sigma(s) ds - \frac{1}{2} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} \sigma(s) ds \\ + \frac{1}{4}(T-2t) \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} \sigma(s) ds = \int_0^T G(t,s) \sigma(s) ds,$$

where $G(t,s)$ is given by (2.3). \square

We remark that the Green's function $G(t,s)$ for $q=2$ takes the form:

$$G(t,s) = \begin{cases} \frac{1}{4}(-T-2t+2s) & \text{if } 0 \leq t < s \leq T, \\ \frac{1}{4}(-T+2t-2s) & \text{if } 0 \leq s \leq t \leq T, \end{cases}$$

which is the same as given in [35].

We will use the following fixed point theorem [5] to prove the existence of solutions for the nonlinear problem (1.1).

THEOREM 2.6. *Let Ω be an open bounded subset of a Banach space E with $0 \in \Omega$ and $\beta: \overline{\Omega} \rightarrow E$ be a compact operator. Then β has a fixed point in $\overline{\Omega}$ provided $\|\beta u - u\|^2 \geq \|\beta u\|^2 - \|u\|^2$, $u \in \partial\Omega$.*

3. Existence results

THEOREM 3.1. *Assume that there exist constants $0 \leq \kappa < 4\Gamma(q+1)/(6+q)$ and $M > 0$ such that*

$$|f(t,u)| \leq \frac{\kappa}{T^q} |u| + M \quad \text{for all } t \in [0, T], \quad u \in C[0, T].$$

Then the anti-periodic boundary value problem (1.1) has at least one solution.

PROOF. In view of Lemma 2.5, u is a solution of the problem (1.1) if and only if $\Gamma: C[0, T] \rightarrow C[0, T]$ satisfies the following condition

$$(3.1) \quad u = \Gamma(u),$$

where Γ is given by

$$(\Gamma u)(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s, u(s)) ds - \frac{1}{2} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(s, u(s)) ds \\ + \frac{1}{4}(T-2t) \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f(s, u(s)) ds,$$

for $t \in [0, T]$. Thus, we just need to prove the existence of at least one solution $u \in C[0, T]$ satisfying (3.1). Let us define a suitable ball $B_R \subset C[0, T]$ with radius $R > 0$ as

$$B_R = \left\{ u \in C[0, T] : \max_{t \in [0, T]} |u(t)| < R \right\},$$

where R will be fixed later. Then, it is sufficient to show that $\Gamma: \overline{B}_R \rightarrow C[0, T]$ satisfies

$$(3.2) \quad u \neq \lambda \Gamma u, \quad \text{for all } u \in \partial B_R \text{ and all } \lambda \in [0, 1].$$

Let us set

$$H(\lambda, u) = \lambda \Gamma u, \quad u \in C(\mathbb{R}), \lambda \in [0, 1].$$

Then, by Arzela–Ascoli theorem, $h_\lambda(u) = u - H(\lambda, u) = u - \lambda \Gamma u$ is completely continuous. If (3.2) is true, then the following Leray–Schauder degrees are well defined and by the homotopy invariance of topological degree, it follows that

$$\begin{aligned} \deg(h_\lambda, B_R, 0) &= \deg(I - \lambda \Gamma, B_R, 0) = \deg(h_1, B_R, 0) \\ &= \deg(h_0, B_R, 0) = \deg(I, B_R, 0) = 1 \neq 0, \end{aligned}$$

for $0 \in B_r$, where I denotes the unit operator. By the nonzero property of Leray–Schauder degree, $h_1(t) = u - \lambda \Gamma u = 0$ for at least one $u \in B_R$. In order to prove (3.2), we assume that $u = \lambda \Gamma u$ for some $\lambda \in [0, 1]$ and for all $t \in [0, T]$ so that

$$\begin{aligned} |u(t)| &= |\lambda \Gamma u(t)| \\ &\leq \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} |f(s, u(s))| ds + \frac{1}{2} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} |f(s, u(s))| ds \\ &\quad + \frac{1}{4} |T-2t| \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} |f(s, u(s))| ds \\ &\leq \left(\frac{\kappa}{T^q} \|u\| + M \right) \left[\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} ds + \frac{1}{2} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} ds \right. \\ &\quad \left. + \frac{1}{4} |T-2t| \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} ds \right] \\ &\leq \left(\frac{\kappa}{T^q} \|u\| + M \right) \left[\frac{t^q}{\Gamma(q+1)} + \frac{T^q}{2\Gamma(q+1)} + \frac{|T-2t|T^{q-1}}{4\Gamma(q)} \right] \\ &\leq \left(\frac{\kappa}{T^q} \|u\| + M \right) \left[\frac{3T^q}{2\Gamma(q+1)} + \frac{T^q}{4\Gamma(q)} \right] \\ &= \left(\frac{\kappa}{T^q} \|u\| + M \right) \frac{T^q(6+q)}{4\Gamma(q+1)}, \end{aligned}$$

which, on taking norm and solving for $\|u\|$, yields

$$\|u\| \leq \frac{MT^q(6+q)}{4\Gamma(q+1) - \kappa(6+q)}.$$

Letting $R = MT^q(6+q)/(4\Gamma(q+1) - \kappa(6+q) + 1)$, (3.2) holds. □

EXAMPLE 3.2. Consider the following anti-periodic boundary value problem

$$(3.3) \quad \begin{cases} {}^c D^q u(t) = \frac{1}{(4\pi)} \sin\left(\frac{2\pi u}{T^q}\right) + \frac{|u|}{1+|u|} & \text{for } t \in [0, T], \quad 1 < q \leq 2, \\ u(0) = -u(T), \quad u'(0) = -u'(T). \end{cases}$$

Clearly

$$|f(t, u)| = \left| \frac{1}{(4\pi)} \sin\left(\frac{2\pi u}{T^q}\right) + \frac{|u|}{1+|u|} \right| \leq \frac{1}{2T^q} \|u\| + 1,$$

with $\kappa = 1/2 < 4\Gamma(q+1)/(6+q)$ for $1 < q \leq 2$ and $M = 1$. Thus, the conclusion of Theorem 3.1 applies to the problem (3.3).

Now we modify the assumption on the nonlinear function $f(t, u)$ in (1.1) and develop the following existence results.

THEOREM 3.3. *If there exists a constant M_1 such that*

$$|f(t, u)| \leq \frac{4\Gamma(q+1)M_1}{T^q(6+q)}, \quad \text{for all } t \in [0, T], \quad u \in [-M_1, M_1],$$

Then the boundary value problem (1.1) has at least one solution.

PROOF. Let us define $\Lambda = \{u \in C[0, T] : \max_{t \in [0, T]} |u(t)| < M_1\}$ and $\Gamma: \bar{\Lambda} \rightarrow C[0, T]$. In view of Theorem 2.6, we just need to show that

$$(3.4) \quad \|\Gamma u\| \leq \|u\|, \quad \text{for all } u \in \partial\Lambda.$$

For all $t \in [0, T]$, $u \in \partial\Lambda$, we have

$$\begin{aligned} |\Gamma u(t)| &\leq \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} |f(s, u(s))| ds + \frac{1}{2} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} |f(s, u(s))| ds \\ &\quad + \frac{1}{4} |T-2t| \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} |f(s, u(s))| ds \\ &\leq \frac{4\Gamma(q+1)M_1}{T^q(6+q)} \left[\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} ds \right. \\ &\quad \left. + \frac{1}{2} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} ds + \frac{1}{4} |T-2t| \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} ds \right] \\ &\leq \frac{4\Gamma(q+1)M_1}{T^q(6+q)} \left[\frac{T^q(6+q)}{4\Gamma(q+1)} \right] = M_1 \end{aligned}$$

Since (3.4) holds, therefore, we obtain the result. \square

REMARK 3.4. In view of the assumption $|f(t, u)| \leq (\kappa/T^q)|u| + M$ of Theorem 3.3, we find that $M_1 = MT^q(6+q)/(4\Gamma(q+1) - \kappa(6+q))$.

EXAMPLE 3.5. Let us consider the following nonlinear function in (1.1):

$$f(t, u) = \frac{u^3}{3T^q} \sin\left(\frac{2\pi t}{T^q}\right).$$

Clearly

$$|f(t, u)| \leq \frac{|u|^3}{3T^q} < \frac{4\Gamma(q+1)M_1}{T^q(6+q)}M_1$$

with $M_1 = 1$. Thus, by Theorem 3.3, the nonlinear boundary value problem (1.1) has at least one solution.

THEOREM 3.6. Suppose that f is of class C^1 in the second variable and there exists a constant $0 \leq M_2 < 4\Gamma(q+1)/(T^q(6+q))$ such that $|f_u(t, u)| \leq M_2$ for all $t \in [0, T], u \in C[0, T]$. Then the boundary value problem (1.1) has at least one solution.

PROOF. For all $t \in [0, T]$, we find that

$$\begin{aligned} |\Gamma u(t)| &\leq \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} |f(s, u(s))| ds + \frac{1}{2} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} |f(s, u(s))| ds \\ &\quad + \frac{1}{4} |T-2t| \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} |f(s, u(s))| ds \\ &\leq \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} |(f_u(s, u(s))u(s) + \nu)| ds \\ &\quad + \frac{1}{2} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} |(f_u(s, u(s))u(s) + \nu)| ds \\ &\quad + \frac{1}{4} |T-2t| \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} |(f_u(s, u(s))u(s) + \nu)| ds \\ &\leq \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} (M_2\|u\| + \nu) ds + \frac{1}{2} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} (M_2\|u\| + \nu) ds \\ &\quad + \frac{1}{4} |T-2t| \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} (M_2\|u\| + \nu) ds \\ &\leq \frac{M_2 T^q (6+q)}{4\Gamma(q+1)} \|u\| + \nu_1, \end{aligned}$$

where $\nu_1 = T^q(6+q)\nu/(4\Gamma(q+1))$ (ν is a positive constant). For $R > 0$, we define

$$B_R = \left\{ u \in \mathbb{R}^1 : \max_{t \in [0, T]} |u(t)| < R \right\},$$

so that

$$\|\Gamma u\| \leq R \left(\frac{M_2 T^q (6+q)}{4\Gamma(q+1)} + \frac{\nu_1}{R} \right) \leq R,$$

for sufficiently large R . Therefore, by Schauder fixed point theorem, Γ has a fixed point. □

EXAMPLE 3.7. Consider

$$(3.5) \quad \begin{cases} {}^c D^q u(t) = \frac{1}{12T^q} \left(\frac{1-u^2}{1+u^2} \right) \sin \left(\frac{2\pi t}{T^q} \right) & \text{for } t \in [0, T], \quad 1 < q \leq 2, \\ u(0) = -u(T), \quad u'(0) = -u'(T). \end{cases}$$

Here

$$f(t, u) = \frac{1}{12T^q} \left(\frac{1-u^2}{1+u^2} \right) \sin \left(\frac{2\pi t}{T^q} \right).$$

Observe that

$$|f_u(t, u)| \leq \frac{1}{3T^q} \left(\frac{|u|}{(1+u^2)^2} \right) < \frac{4\Gamma(q+1)}{T^q(6+q)}.$$

Thus, the conclusion of Theorem 3.6 applies to the problem (3.5).

Acknowledgements. The authors are grateful to the anonymous referee for his/her valuable remarks that led to the improvement of the original manuscript.

The research of J. J. Nieto has been partially supported by Ministerio de Educacion y Ciencia and FEDER, project MTM2007-61724, and by Xunta de Galicia and FEDER, project PGIDIT06PXIB207023PR.

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Manuscript received March 18, 2009

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