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EXISTENCE AND MULTIPLICITY OF SOLUTIONS FOR RESONANT NONLINEAR NEUMANN PROBLEMS

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Abstract. We consider nonlinear Neumann problems driven by the p-Laplacian differential operator with a Caratheodory nonlinearity. Under hypotheses which allow resonance with respect to the principal eigenvalue $\lambda_0 = 0$ at $\pm\infty$, we prove existence and multiplicity results. Our approach is variational and uses critical point theory and Morse theory (critical groups).

1. Introduction

Let $Z \subseteq \mathbb{R}^N$ be a bounded domain with a C^2 boundary ∂Z. In this paper we study the following nonlinear Neumann problem:

$$\begin{cases} -\triangle_p x(z) = f(z, x(z)) & \text{a.e. on } Z, \\ \dfrac{\partial x}{\partial n} = 0 & \text{on } \partial Z. \end{cases} \tag{1.1}$$

Here $\triangle_p$ stands for the p-Laplacian differential operator defined by

$$\triangle_p x(z) = \operatorname{div}(\|Dx(z)\|_{\mathbb{R}^N}^{p-2} Dx(z)), \quad 1 < p < \infty,$$

$n(\,\cdot\,)$ stands for the outward unit normal on ∂Z and $f(z,x)$ is a Carathéodory nonlinearity. The goal of this paper is to prove existence and multiplicity results,

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when resonance with respect to the principal eigenvalue $\lambda_0 = 0$ is possible. Neumann problems driven by the p-Laplacian have not been studied as extensively as Dirichlet problems. Existence theorems were proved by G. Anello and G. Cordaro [2], D. Arcoya and L. Orsina [3], T. Godoy, J. P. Gosez and S. Paczka [14], S. Hu and N. S. Papageorgiou [15], D. Motreanu and N. S. Papageorgiou [21] and F. Papalini [22], [23]. In G. Anello and G. Cordaro [2], the authors assume $p > N$ (low dimensional problem) and exploit the fact that the Sobolev space $W^{1,p}(Z)$ is embedded compactly in $C(\overline{Z})$. D. Arcoya and L. Orsina [3] use Landesman–Lazer type conditions. T. Godoy, J. P. Gosez and S. Paczka [14] examine the antimaximum principle in the context of Neumann problems. Finally, S. Hu and N.S. Papageorgiou [15], D. Motreanu and N. S. Papageorgiou [21] and F. Papalini [22], [23], deal with problems having a nonsmooth potential function (hemivariational inequalities) and their methods of proof rely on the nonsmooth critical point theory.

Multiplicity results for the Neumann p-Laplacian can be found in the works of S. Aizicovici, N. S. Papageorgiou and V. Staicu [1], G. Barletta and N. S. Papageorgiou [4], G. Bonanno and P. Candito [6], M. Filippakis, L. Gasinski and N. S. Papageorgiou [10], S. Marano and D. Motreanu [18], D. Motreanu and N. S. Papageorgiou [21], B. Ricceri [26], and X. Wu and K. K. Tan [28]. In G. Bonanno and P. Candito [6], S. Marano and D. Motreanu [18], B. Ricceri [26], the authors deal with certain nonlinear eigenvalue problem and assume that $p > N$. As already mentioned, this condition implies that the Sobolev space $W^{1,p}(Z)$ is embedded compactly in $C(\overline{Z})$, and this fact plays a crucial role in their arguments. The approach in the aforementioned works is similar and is based on an abstract multiplicity result of B. Ricceri [25] or on variants of it. X. Wu and X. P. Tan [28] also assume $p > N$ (low dimensional problem) but their approach is variational, based on the critical point theory. M. Filippakis, L. Gasinski and N. S. Papageorgiou [10] use the second deformation theorem to prove a multiplicity theorem for a class of Neumann p-Laplacian problems. The papers of S. Aizicovici, N. S. Papageorgiou and V. Staicu [1], G. Barletta and N. S. Papageorgiou [4] and D. Motreanu and N. S. Papageorgiou [21] deal with hemivariational inequalities. In [4] and [21], the approach is degree theoretic and uses the degree map for certain multivalued perturbations of $(\mathrm{S})_+$−maps. In [21], the authors employ the nonsmooth local linking theorem (see L. Gasinski and N. S. Papageorgiou [12, p. 178]).

Our approach here combines variational methods with Morse theory (in particular, critical groups). More precisely, we use critical groups to distinguish between critical points and we also use a homological version of the local linking condition, the so called local (n,m)-linking, due to K. Perera [24], to generate two nontrivial solutions.

2. Preliminaries

In this section, we briefly present the basic mathematical tools that we are going to use in this work. We start with critical point theory. Let $(X, \|\cdot\|)$ be a Banach space and X^* its topological dual. By $\langle\cdot\,,\cdot\rangle$ we denote the duality brackets for the pair (X^*, X). Also $\xrightarrow{w}$ denotes weak convergence in X.

Let $\varphi \in C^1(X)$ and $c \in \mathbb{R}$. We say that φ satisfies the *Palais–Smale condition at level* c (the PS_c-condition, for short), if the following holds:

- every sequence $\{x_n\}_{n\geq 1} \subseteq X$ such that
$$\varphi(x_n) \to c \text{ and } \varphi'(x_n) \to 0 \text{ in } X^* \text{ as } n \to \infty,$$
has a strongly convergent subsequence.

If this condition is true at every level $c \in \mathbb{R}$, then we say that φ satisfies the PS-*condition*. Sometimes, it is necessary to use a more general compactness-type condition. So, we say that φ satisfies the *Cerami condition at the level* $c \in \mathbb{R}$ (the C_c-condition, for short), if the following holds:

- every sequence $\{x_n\}_{n\geq 1} \subseteq X$ such that
$$\varphi(x_n) \to c \text{ and } (1 + \|x_n\|)\varphi'(x_n) \to 0 \text{ in } X^* \text{ as } n \to \infty,$$
has a strongly convergent subsequence.

If this condition is true at every level $c \in \mathbb{R}$, then we say that φ satisfies the C-*condition*.

It was shown by P. Bartolo, V. Benci and D. Fortunato [5] that the deformation lemma, and consequently the minimax theory of the critical values of a function $\varphi \in C^1(X)$, remains valid if instead of the PS-condition, we employ the weaker C-condition.

The following topological notion plays a central role in critical point theory.

Definition 2.1. Let Y be a Hausdorff topological space and E_0, E and D be nonempty closed subsets of Y, with $E_0 \subseteq E$. We say that the pair $\{E, E_0\}$ is *linking with* D *in* Y, if

(a) $E_0 \cap D = \emptyset$;
(b) $\gamma(E) \cap D \neq \emptyset$ for all $\gamma \in C(E, Y)$, with $\gamma|_{E_0} = \mathrm{id}|_{E_0}$.

Using this notion, we have the following general minimax principle for the critical values of a C^1 function φ.

Theorem 2.2. *Suppose that* X *is a Banach space,* E_0*,* E *and* D *are nonempty, closed subsets of* X *such that* $\{E, E_0\}$ *is linking with* D *in* X*, and* $\varphi \in C^1(X)$*, with*
$$\sup_{E_0} \varphi \leq \inf_D \varphi.$$

Let $\Gamma = \{\gamma \in C(E,X) : \gamma|_{E_0} = \mathrm{id}|_{E_0}\}$ *and* $c = \inf_{\gamma\in\Gamma}\sup_{x\in E}\varphi(\gamma(x))$. *If also* φ *satisfies the* C_c*-condition, then* $c \geq \inf_D \varphi$ *and* c *is a critical value of* φ. *Moreover, if* $c = \inf_D \varphi$, *then there exists a critical point* x *of* φ, *such that* $\varphi(x) = c$ *and* $x \in D$.

Remark 2.3. With suitable choices of linking sets, from the above theorem we can produce as corollaries the mountain pass theorem, the saddle point theorem, and the generalized mountain pass theorem. For details we refer to L. Gasinski and N. S. Papageorgiou [13].

In our analysis of problem (1.1) we will use the following two spaces:

$$C^1_n(\overline{Z}) = \left\{x \in C^1(\overline{Z}) : \frac{\partial x}{\partial n} = 0 \text{ on } \partial Z\right\} \quad \text{and} \quad W^{1,p}_n(Z) = \overline{C^1_n(\overline{Z})}^{\|\cdot\|}$$

where $\|\cdot\|$ denotes the usual norm of $W^{1,p}(Z)$.

The next theorem links the variational and the Morse theoretic methods. The result was proved for $p \geq 2$ and a nonsmooth potential by G. Barletta and N. S. Papageorgiou [4], and for $1 < p < \infty$ and a smooth potential by D. Motreanu, V. Motreanu and N. S. Papageorgiou [20]. It extends to the Neumann setting the Dirichlet results of H. Brezis and L. Nirenberg [7] for $p = 2$, and of J. Garcia Azorero, J. Manfredi and I. Peral Alonso [11] for $p \neq 2$. So, assume the following:

(H_0) $f_0 \colon Z \times \mathbb{R} \to \mathbb{R}$ is a function such that:
- (a) for all $x \in \mathbb{R}$, $z \to f_0(z,x)$ is measurable;
- (b) for almost all $z \in Z$, $x \to f_0(z,x)$ is continuous;
- (c) for almost all $z \in Z$ and all $x \in \mathbb{R}$

$$|f_0(z,x)| \leq a_0(z) + c_0|x|^{r-1},$$

with $a_0 \in L^\infty(Z)_+$, $c_0 > 0$ and

$$1 < r < p^* = \begin{cases} Np/(N-p) & \text{if } p < N, \\ \infty & \text{if } p \geq N. \end{cases}$$

Let $F_0(z,x) = \int_0^x f_0(z,s)\,ds$ (the primitive of f_0) and consider the C^1-functional $\varphi_0 \colon W^{1,p}_n(Z) \to \mathbb{R}$ defined by

$$\varphi_0(x) = \frac{1}{p}\|Dx\|_p^p - \int_Z F_0(z,x(z))\,dz \quad \text{for all } x \in W^{1,p}_n(Z).$$

Throughout the paper, we use $\|\cdot\|_p$ to indicate the norm of $L^p(Z,\mathbb{R})$ or $L^p(Z,\mathbb{R}^N)$.

Theorem 2.4. *If $x_0 \in W^{1,p}_n(Z)$ is a local $C^1_n(\overline{Z})$-minimizer of φ_0, i.e. there exists $r_0 > 0$ such that*

$$\varphi_0(x_0) \leq \varphi_0(x_0 + u) \quad \text{for all } u \in C^1_n(\overline{Z}),\ \|u\|_{C^1_n(\overline{Z})} \leq r_0,$$

then $x_0 \in C^1_n(\overline{Z})$ and it is a local $W^{1,p}_n(Z)$-minimizer of φ_0, i.e. there exists $r_1 > 0$ such that

$$\varphi_0(x_0) \leq \varphi_0(x_0 + u) \quad \text{for all } u \in W^{1,p}_n(Z),\ \|u\| \leq r_1.$$

For $\varphi \in C^1(X)$ and $c \in \mathbb{R}$, we define the *sublevel set of φ at c* by

$$\varphi^c = \{x \in X : \varphi(x) \leq c\},$$

If $\{Y_1, Y_2\}$ is a topological pair with $Y_2 \subseteq Y_1 \subseteq X$, then for every integer $k \geq 0$, we denote by $H_k(Y_1, Y_2)$ the k^{th}-relative singular homology group of the pair $\{Y_1, Y_2\}$, with integer coefficients. We recall that the critical groups of φ at an isolated critical point $x_0 \in X$ with $\varphi(x_0) = c$, are defined by

$$C_k(\varphi, x_0) = H_k(\varphi^c \cap U, (\varphi^c \cap U) \setminus \{x_0\}), \quad \text{for all } k \geq 0,$$

where U is a neighborhood of x_0 such that $K \cap \varphi^c \cap U = \{x_0\}$ (see K. C. Chang [8] and J. Mawhin, M. Willem [19]). The excision property of the singular homology, implies that the above definition is independent of the particular neighborhood U we use.

Perera [24] introduced the following more general notion of local linking, which extends the one due to S. J. Li and J. Q. Liu [16].

Definition 2.5. Suppose $\varphi \in C^1(X)$, 0 is an isolated critical point of φ, $\varphi(0) = 0$ and m, $n \geq 0$ are integers. We say that φ has a local (n, m)-*linking* near the origin, if there exist a neighbourhood U of 0 and nonempty subsets E_0, E, D of U such that $E_0 \subseteq E$, $0 \notin E_0$, $E_0 \cap D = \emptyset$ and:

(a) if $K = \{x \in X : \varphi'(x) = 0\}$ (the *critical set of* φ), then

$$\varphi^0 \cap K \cap U = \{x_0\};$$

(b) if $i_1 : E_0 \to E$ and $i_2 : E_0 \to U \setminus D$ are the inclusion maps and

$$i_1^* : H_{n-1}(E_0) \to H_{n-1}(U \setminus D) \quad \text{and} \quad i_2^* : H_{n-1}(E_0) \to H_{n-1}(E)$$

are the corresponding group homomorphisms, then

$$\operatorname{rank}(i_1^*) - \operatorname{rank}(i_2^*) \geq m;$$

(c) $\varphi|_E \leq 0$;

(d) $\varphi|_{D \setminus \{0\}} > 0$.

Remark 2.6. An example in K. Perera [24] shows that this notion is weaker than the usual local linking condition of S. J. Li and J. Q. Liu [16] (see also L. Gasinski and N. S. Papageorgiou [13, p. 661]).

The next two theorems are due to K. Perera [24] and S. J. Li and J. Q. Liu [17], respectively.

Theorem 2.7. *If $\varphi \in C^1(X)$ has a local (n,m)-linking near the origin, then rank $C_n(\varphi,0) \geq m$.*

Theorem 2.8. *If $\varphi \in C^1(X)$, φ satisfies the* C*-condition, it is bounded below and has a critical point x which is homologically nontrivial (that is, $C_k(\varphi,x) \neq 0$ for some $k \geq 0$), and x is not a minimizer of φ, then φ has at least two nontrivial critical points.*

Finally, let us recall some basic facts about the spectrum of the negative Neumann p-Laplacian, denoted by $(-\triangle_p, W_n^{1,p}(Z))$. So, we consider the following nonlinear eigenvalue problem:

$$\begin{cases} -\triangle_p u(z) = \lambda |u(z)|^{p-2}u(z) & \text{on } Z, \\ \dfrac{\partial x}{\partial n_p} = 0 & \text{on } \partial Z. \end{cases} \tag{2.1}$$

By an *eigenvalue* of $(-\triangle_p, W_n^{1,p}(Z))$, we mean a $\lambda \in \mathbb{R}$ for which problem (2.1) has a nontrivial solution u, known as the *eigenfunction* corresponding to the eigenvalue λ. It is straightforward to check that a necessary condition for $\lambda \in \mathbb{R}$ to be an eigenvalue of (2.1) is that $\lambda \geq 0$. Note that 0 is an eigenvalue with corresponding eigenspace $\mathbb{R}$ (i.e. the space of constant functions). This first eigenvalue, denoted by λ_0, is isolated and admits the following variational characterization

$$\lambda_0 = \inf\left\{\frac{\|Du\|_p^p}{\|u\|_p^p} : u \in W^{1,p}(Z),\ u \neq 0\right\}. \tag{2.2}$$

Clearly, constant functions realize the infimum in (2.2).

The p-Laplacian is a $(p-1)$-homogeneous operator. So, by virtue of the Ljusternik–Schnirelmann theory, in addition to λ_0, $(-\triangle_p, W_n^{1,p}(Z))$ admits a whole strictly increasing sequence $\{\lambda_k\}_{k\geq 0}$ of eigenvalues such that $\lambda_k \to \infty$ as $k \to \infty$. These are known as the (LS)-eigenvalues of $(-\triangle_p, W_n^{1,p}(Z))$.

For $p = 2$ (linear eigenvalue problem), these are all the eigenvalues of the p-Laplacian $(-\triangle, W_n^{1,2}(Z))$. If $p \neq 2$ (nonlinear eigenvalue problem), then we do not know if this is the case. However, since λ_0 is isolated and the set $\sigma(p)$ of eigenvalues of $(-\triangle_p, W_n^{1,p}(Z))$ is closed, then

$$\widehat{\lambda}_1^* = \inf\{\lambda > 0 : \lambda \in \sigma(p)\} > 0.$$

This is the second (first nonzero) eigenvalue of $(-\triangle_p, W_n^{1,p}(Z))$. In fact we have

$$\lambda_1^* = \lambda_1,$$

i.e. the second eigenvalue of $(-\triangle_p, W_n^{1,p}(Z))$ and the second LS-eigenvalue coincide. The Ljusternik-Schnirelmann theory yields a minimax characterization of $\lambda_1 > 0$. However, for our purposes that characterization is not helpful. Instead, we will use an alternative one, essentially due to S. Aizicovici, N. S. Papageorgiou and V. Staicu ([1, Proposition 2]). (For the corresponding result for the Dirichlet p-Laplacian we refer to M. Cuesta, D. de Figueiredo and J. P. Gossez [9]). In what follows

$$u_0(z) = \frac{1}{|Z|_N^{1/p}} \quad \text{for all } z \in \overline{Z},$$

is the L^p-normalized principal eigenfunction corresponding to $\lambda_0 = 0$. (Here by $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$). Also let

$$\partial B_1^{L^p} = \{x \in L^p(Z) : \|x\|_p = 1\}.$$

Proposition 2.9. *If* $S = W_n^{1,p}(Z) \cap \partial B_1^{L^p}$ *and*

$$\Gamma_0 = \{\gamma \in C([-1,1], S) : \gamma(-1) = -u_0,\ \gamma(1) = u_0\},$$

then

$$\lambda_1 = \inf_{\gamma \in \Gamma_0} \max_{t \in [-1,1]} \|D\gamma(t)\|_p^p.$$

Moreover, it is well known (see, for example, L. Gasinski and N. S. Papageorgiou [13]) that, if

$$C(p) = \left\{u \in W_n^{1,p}(Z) : \int_Z |u|^{p-2} u \, dz = 0\right\}$$

then

$$\lambda_1 = \inf\left\{\frac{\|Du\|_p^p}{\|u\|_p^p} : u \in C(p)\right\} \tag{2.3}$$

and the infimum in (2.3) is realized.

Next we recall a notion which is useful in verifying the PS-condition and the C-condition.

Definition 2.10. We say that a map $A: X \to X^*$ is of type $(\mathrm{S})_+$, if for every sequence $\{x_n\}_{n\geq 1} \subseteq X$ such that $x_n \xrightarrow{w} x$ in X and

$$\limsup_{n\to\infty} \langle A(x_n), x_n - x\rangle \leq 0,$$

one has $x_n \to x$ in X.

Consider the nonlinear map $A\colon W_n^{1,p}(Z) \to W_n^{1,p}(Z)^*$ corresponding to the p-Laplacian, namely

$$\langle A(x), y\rangle = \int_Z \|Dx\|_{\mathbb{R}^N}^{p-2}(Dx, Dy)_{\mathbb{R}^N}\, dz \quad \text{for all } x, y \in W_n^{1,p}(Z). \tag{2.4}$$

We have the following result (see, e.g. G. Barletta and N. S. Papageorgiou [4]):

Proposition 2.11. *The map $A\colon W_n^{1,p}(Z) \to W_n^{1,p}(Z)^*$ defined by* (2.4) *is continuous, strictly monotone* (*hence maximal monotone, too*) *and of type* $(\mathrm{S})_+$.

3. An existence theorem

The hypotheses on the nonlinearity $f(z, x)$ are the following:

$(\mathrm{H(f)}_1)$ The function $f\colon Z \times \mathbb{R} \to \mathbb{R}$ is such that

(a) for every $x \in \mathbb{R}$, $z \to f(z, x)$ is measurable;

(b) for almost all $z \in Z$, $x \to f(z, x)$ is continuous and $f(z, 0) = 0$;

(c) for almost all $z \in Z$ and all $x \in \mathbb{R}$ we have

$$|f(z, x)| \le a(z) + c|x|^{r-1},$$

where $a \in L^\infty(Z)_+$, $c > 0$ and $p \le r < p^*$;

(d) if $F(z, x) = \int_0^x f(z, s)\, ds$, then

$$\limsup_{|x|\to\infty} \frac{pF(z, x)}{|x|^p} < \lambda_1, \quad \text{uniformly for a.a. } z \in Z;$$

(e) $\lim_{|x|\to\infty}[f(z, x)x - pF(z, x)] = \pm\infty$, uniformly for almost all $z \in Z$;

(f) $\lim_{|x|\to\infty} \int_Z F(z, x)\, dz = \infty$ and there exists $\delta > 0$ such that

$$\int_Z F(z, \xi)\, dz \le 0 \quad \text{for all } \xi \in \mathbb{R} \text{ with } |\xi| \le \delta.$$

Remark 3.1. Note that the above hypotheses incorporate in our framework of analysis problems where

$$\limsup_{|x|\to\infty} \frac{f(z, x)}{|x|^{p-2}x} = 0 \quad \text{uniformly for a.a. } z \in Z.$$

Hence, our analysis here covers problems which are resonant with respect to $\lambda_0 = 0$ at $\pm\infty$.

Example 3.2. We consider the following potential function (for the sake of simplicity, we drop the z-dependence):

$$F_1(x) := \frac{\lambda}{p}|x|^p - \frac{1}{\theta}\ln(1 + |x|^\theta)$$

with $1 < \theta \le p < p^*$, $0 \le \lambda < \lambda_1$. Then

$$f_1(x) := F_1'(x) = \lambda|x|^{p-2}x - \frac{|x|^{\theta-2}x}{1 + |x|^\theta}$$

satisfies hypotheses $(\mathrm{H(f)}_1)$. Similarly we may consider the function

$$F_2(x)=\frac{\theta}{p}|x|^p-\frac{1}{\tau}|x|^\tau \quad \text{with } 1<\tau<p, \text{ and } 0<\theta<\lambda_1.$$

In this case we have

$$f_2(x)=F_3'(x)=\theta|x|^{p-2}x-|x|^{\tau-2}x.$$

Let $\varphi\colon W^{1,p}_n(Z)\to\mathbb{R}$ be the Euler functional for problem (1.1), defined by

$$\varphi(x)=\frac{1}{p}\|Dx\|^p_p-\int_Z F(z,x(z))\,dz \quad \text{for all } x\in W^{1,p}_n(Z). \tag{3.1}$$

Hypotheses $(\mathrm{H(f)}_1)$(a)–(c) imply that $\varphi\in C^1(W^{1,p}_n(Z))$.

Proposition 3.3. *If hypotheses* $\mathrm{H}(f)_1$ *hold, then* φ *satisfies the* C-*condition.*

Proof. Let $(x_n)_{n\geq1}\subset W^{1,p}_n(Z)$ be a sequence such that

$$\varphi(x_n)\to c \tag{3.2}$$

and

$$(1+\|x_n\|)\varphi'(x_n)\to 0 \quad \text{in } W^{1,p}_n(Z)^* \text{ as } n\to\infty. \tag{3.3}$$

We know that

$$\varphi'(x_n)=A(x_n)-N(x_n) \quad \text{for all } n\geq 1, \tag{3.4}$$

where $N(u)(\,\cdot\,):=f(\,\cdot\,,u(\,\cdot\,))$ for all $u\in W^{1,p}_n(Z)$. From (3.3) and (3.4), we have

$$\left|\langle A(x_n),x_n\rangle-\int_Z f(z,x_n)x_n\,dz\right|\leq\varepsilon_n \quad \text{with } \varepsilon_n\downarrow 0,$$

hence

$$\left|\|Dx_n\|^p_p-\int_Z f(z,x_n)x_n\,dz\right|\leq\varepsilon_n \quad \text{with } \varepsilon_n\downarrow 0. \tag{3.5}$$

Also, from (3.2) we see that given $\varepsilon>0$, we can find $n_0=n_0(\varepsilon)\geq1$ such that

$$pc-\varepsilon\leq\|Dx_n\|^p_p-\int_Z pF(z,x_n)\,dz\leq pc+\varepsilon \quad \text{for all } n\geq n_0. \tag{3.6}$$

From (3.5) and (3.6), it follows that there exists $n_1=n_1(\varepsilon)\geq n_0\geq1$ such that

$$pc-2\varepsilon\leq\int_Z(f(z,x_n)x_n-pF(z,x_n))\,dz\leq pc+2\varepsilon \quad \text{for all } n\geq n_1,$$

hence

$$\int_Z(f(z,x_n)x_n-pF(z,x_n))\,dz\to pc \quad \text{as } n\to\infty. \tag{3.7}$$

Claim. The sequence $(x_n)_{n\geq1}\subset W^{1,p}_n(Z)$ is bounded.

We proceed by contradiction. So, suppose that the claim is not true. Then, by passing to a suitable subsequence if necessary, we may assume that $\|x_n\| \to \infty$. Set

$$y_n = \frac{x_n}{\|x_n\|}, \quad n \geq 1.$$

Then $\|y_n\| = 1$ for all $n \geq 1$, and so we may assume that

$$y_n \xrightarrow{w} y \quad \text{in } W_n^{1,p}(Z) \quad \text{and} \quad y_n \to y \quad \text{in } L^p(Z). \tag{3.8}$$

By virtue of hypotheses $(\mathrm{H(f)}_1)$(c) and (d), for almost all $z \in Z$ and all $x \in \mathbb{R}$, we have

$$F(z,x) \leq a_1(z) + \frac{\theta}{p}|x|^p \quad \text{with } a_1 \in L^\infty(Z)_+,\ 0 < \theta < \lambda_1. \tag{3.9}$$

From (3.1) and (3.2) it follows that we can find $M_1 > 0$ such that

$$\varphi(x_n) = \frac{1}{p}\|Dx_n\|_p^p - \int_Z F(z, x_n(z))\, dz \leq M_1 \quad \text{for all } n \geq 1,$$

hence, by (3.9),

$$\frac{1}{p}\|Dx_n\|_p^p - \frac{\theta}{p}\|x_n\|_p^p - c_1 \leq M_1 \quad \text{for all } n \geq 1,$$

with $c_1 = \|a\|_1$, therefore

$$\frac{1}{p}\|Dy_n\|_p^p - \frac{\theta}{p}\|y_n\|_p^p - \frac{c_1}{\|x_n\|^p} \leq \frac{M_1}{\|x_n\|^p} \quad \text{for all } n \geq 1. \tag{3.10}$$

If $y = 0$, by passing to the limit as $n \to \infty$ in (3.10), we obtain $Dy_n \to 0$ in $L^p(Z, \mathbb{R}^N)$ hence, by (3.8), $y_n \to 0$ in $W_n^{1,p}(Z)$, a contradiction to the fact that $\|y_n\| = 1$ for all $n \geq 1$. Therefore $y \neq 0$.

So, if $\widehat{Z} = \{z \in Z : y(z) \neq 0\}$, then $|\widehat{Z}|_N > 0$ and we have that $|x_n(z)| \to \infty$ for almost all $z \in \widehat{Z}$. Let

$$\beta(z,x) = f(z,x)x - pF(z,x).$$

We assume that in the hypothesis $(\mathrm{H(f)}_1)$(e) the ∞ limit is in effect (the argument is similar if the limit is $-\infty$). Then we have

$$\beta(z,x) \to \infty \quad \text{as } |x| \to \infty, \text{ uniformly for a.a. } z \in Z.$$

So, we can find $M_2 > 0$ such that

$$\beta(z,x) \geq -M_2 \quad \text{for a.a. } z \in Z, \text{ all } x \in \mathbb{R}. \tag{3.11}$$

Therefore

$$\begin{aligned} \int_Z \beta(z, x_n(z))\, dz &= \int_{\widehat{Z}} \beta(z, x_n(z))\, dz + \int_{Z \setminus \widehat{Z}} \beta(z, x_n(z))\, dz \\ &\geq \int_{\widehat{Z}} \beta(z, x_n(z))\, dz - M_2 |Z \setminus \widehat{Z}|_N > 0 \end{aligned} \tag{3.12}$$

(see (3.11)). We know that $\beta(z, x_n(z)) \to \infty$ as $n \to \infty$ for almost all $z \in \widehat{Z}$. So, from (3.12) and Fatou's lemma (it can be used due to (3.11)), we have

$$\int_Z \beta(z, x_n(z))\, dz = \int_Z (f(z, x_n(z))x_n(z) - pF(z, x_n(z)))\, dz \to \infty \quad \text{as } n \to \infty$$

which contradicts (3.7). This proves the claim.

Because of the claim and by passing to a suitable subsequence if necessary, we may assume that

$$x_n \xrightarrow{w} x \quad \text{in } W_n^{1,p}(Z) \quad \text{and} \quad x_n \to x \quad \text{in } L^r(Z) \text{ as } n \to \infty. \tag{3.13}$$

From (3.3), we have

$$\left| \langle A(x_n), x_n - x\rangle - \int_Z f(z, x_n)(x_n - x)\, dz \right| \leq \frac{\varepsilon_n}{1 + \|x_n\|} \|x_n - x\| \tag{3.14}$$

for all $n \geq 1$. Note that

$$\int_Z f(z, x_n)(x_n - x)\, dz \to 0$$

(see (3.13) and hypothesis $(\mathrm{H(f)}_1)$(c)), hence

$$\lim_{n\to\infty} \langle A(x_n), x_n - x\rangle = 0 \tag{3.15}$$

(see (3.14)), and by (3.13), (3.15) and Proposition 2.11 we obtain $x_n \to x$ in $W_n^{1,p}(Z)$. Therefore, we conclude that φ satisfies the C-condition. $\square$

From $(\mathrm{H(f)}_1)$(d), we get at once the following result.

PROPOSITION 3.4. *If hypotheses* $(\mathrm{H(f)}_1)$ *hold, then* $\varphi|_{\mathbb{R}}$ *is anticoercive, i.e.*

$$\varphi(\xi) \to -\infty \quad \textit{as } |\xi| \to \infty.$$

Let $D \subset W_n^{1,p}(Z)$ be the cone defined by

$$D = \{x \in W_n^{1,p}(Z) : \|Dx\|_p^p \geq \lambda_1 \|x\|_p^p\} \tag{3.16}$$

PROPOSITION 3.5. *If hypotheses* $(\mathrm{H(f)}_1)$ *hold, then* $m = \inf_D \varphi > -\infty$.

PROOF. Recall that for almost all $z \in Z$ and all $x \in \mathbb{R}$

$$F(z, x) \leq a_1(z) + \frac{\theta}{p}|x|^p \quad \text{with } a_1 \in L^\infty(Z)_+,\ 0 < \theta < \lambda_1$$

(see (3.9)). Hence, for all $x \in D$, we have

$$\begin{aligned}
\varphi(x) &= \frac{1}{p}\|Dx\|_p^p - \int_Z F(z,x(z))\,dz \\
&\geq \frac{1}{p}\|Dx\|_p^p - \frac{\theta}{p}\|x\|_p^p - c_1 && \text{with } c_1 = \|a_1\|_1 \\
&\geq \frac{1}{p}\left(1 - \frac{\theta}{\lambda_1}\right)\|Dx\|_p^p - c_1 && \text{(see (3.16))} \\
&\geq -c_1 && \text{(since } 0<\theta<\lambda_1\text{)},
\end{aligned}$$

therefore $m = \inf_D\ \varphi > -\infty$. □

By virtue of Propositions 3.4 and 3.5, we can find $\alpha > 0$ large such that

$$\varphi(\pm\alpha u_0) < \infty \tag{3.17}$$

Then we set $u = \alpha u_0$, $-u = -\alpha u_0$ and we consider the following sets

$$\begin{aligned}
E_0 &= \{-u, u\}, \\
E &= [-u,u] = \{(1-t)(-u) + tu : t \in [0,1]\}, \\
D &\text{ as in (3.16)}.
\end{aligned}$$

Proposition 3.6. *The pair $\{E_0, E\}$ is linking with D in $W_n^{1,p}(Z)$.*

Proof. Evidently, $E_0 \subset E$ and $E_0 \cap D = \emptyset$. Let

$$C = \{x \in W_n^{1,p}(Z) : \|Dx\|_p^p < \lambda_1\|x\|_p^p\}. \tag{3.18}$$

Claim. The set C is not path connected.

We argue indirectly. So, suppose that C is path connected. Then, since $-u_0$, $u_0 \in C$, we can find a continuous path $\widehat{\gamma}\colon [-1,1] \to W_n^{1,p}(Z)$ such that $\widehat{\gamma}(-1) = -u_0$, $\widehat{\gamma}(1) = u_0$ and $\widehat{\gamma}(t) \in C$ for all $t \in [-1,1]$. Since $0 \notin C$, we see that $\widehat{\gamma}(t) \neq 0$ for all $t \in [-1,1]$. So, we can define

$$\gamma_0(t) = \frac{\widehat{\gamma}(t)}{\|\widehat{\gamma}(t)\|_p} \quad \text{for all } t \in [-1,1].$$

Evidently, $\gamma_0\colon [-1,1] \to W_n^{1,p}(Z)$ is continuous and $\gamma_0(t) \in S \cap C$, where $S = W_n^{1,p}(Z) \cap \partial B_1^{L^p}$. Therefore $\gamma_0 \in \Gamma_0$ and by virtue of Proposition 2.9, we have

$$\lambda_1 \leq \max_{t\in[-1,1]} \|D\gamma_0(t)\|_p^p.$$

Let $t_0 \in [-1,1]$ be such that

$$\|D\gamma_0(t_0)\|_p^p = \max_{t\,\mathrm{inf}[-1,1]} \|D\gamma_0(t)\|_p^p$$

and set $x_0 = \gamma_0(t_0)$. Then

$$\lambda_1 \leq \|Dx_0\|_p^p, \quad \|x_0\|_p = 1,$$

which contradicts the fact that $\gamma_0(t_0) \in C$ (see (3.18)). This proves the claim.

From the above argument, it is clear that $-u_0$ and u_0 belong to different path components of C. Let C_+ be the path component of C containing u_0 and let $C_- = -C_+$ be the path component of C containing $-u_0$. We set

$$B_+ = \mathbb{R}_+ C_+, \quad B_- = \mathbb{R}_- C_- \quad \text{and} \quad B = B_+ \cup B_-.$$

Evidently, $u \in B_+$, $-u \in B_-$. Let $\gamma \in C(E, W_n^{1,p}(Z))$ be such that $\gamma|_{E_0} = \mathrm{id}|_{E_0}$. Then, clearly $\gamma(E) \cap \partial B \neq \emptyset$. But note that

$$\partial B = \{x \in W_n^{1,p}(Z) \colon \|Dx\|_p^p = \lambda_1 \|x\|_p^p\} \subseteq D,$$

hence $\gamma(E) \cap E \neq \emptyset$, and the pair $\{E_0, E\}$ is linking with D in $W_n^{1,p}(Z)$. $\square$

Now we are ready for the existence theorem for problem (1.1).

Theorem 3.7. *If hypotheses* $\mathrm{H}(f)_1$ *hold, then problem* (1.1) *has a nontrivial solution* $x_0 \in C_n^1(\overline{Z})$.

Proof. Propositions 3.3, 3.6 and (3.17) permit the use of Theorem 2.2, which gives $x_0 \in W_n^{1,p}(Z)$, a critical point of the function φ. We assume that x_0 is isolated, or otherwise we are done. Then

$$C_1(\varphi, x_0) \neq 0 \tag{3.19}$$

(see, for example, K. C. Chang [8, p. 88]). On the other hand, let $y \in C_n^1(\overline{Z})$ with $\|y\|_{C_n^1(\overline{Z})} \leq \delta$, where $\delta > 0$ is as in hypothesis $(\mathrm{H(f)}_1)$(f). Then

$$\int_Z F(z, y(z))\, dz \leq 0,$$

hence

$$\varphi(y) = \frac{1}{p}\|Dy\|_p^p - \int_Z F(z, y(z))\, dz \geq 0.$$

So, the origin is a local $C_n^1(\overline{Z})$-minimizer of φ. Invoking Theorem 2.4, we infer that it is also a $W_n^{1,p}(Z)$-minimizer of φ. Again, we assume without any loss of generality that $y = 0$ is an isolated critical point of φ. We know that

$$C_k(\varphi, 0) = \delta_{k,0}\mathbb{Z} \quad \text{for all } k \geq 0 \tag{3.20}$$

(see, for example, K. C. Chang [8, p. 33] and J. Mawhin and M. Willem [19, p. 173]). Comparing (3.19) and (3.20), we see that $x_0 \neq 0$. We have $\varphi'(x_0) = 0$, hence

$$A(x_0) = N(x_0). \tag{3.21}$$

From (3.21), reasoning as in D. Motreanu and N. S. Papageorgiou [21], using the nonlinear Green identity and nonlinear regularity theory (see, for example, L. Gasinski and N. S. Papageorgiou [13]) we conclude that $x_0 \in C_n^1(\overline{Z})$ and it solves problem (1.1). $\square$

4. A multiplicity theorem

In this section, we prove a multiplicity theorem for problem (1.1). The hypotheses on the nonlinearity $f(z,x)$ are:

(H(f)$_2$) The function $f\colon Z\times\mathbb{R}\to\mathbb{R}$ is such that

(a) for all $x\in\mathbb{R}$, $z\to f(z,x)$ is measurable;

(b) for almost all $z\in Z$, $x\to f(z,x)$ is continuous and $f(z,0)=0$;

(c) for almost all $z\in Z$ and all $x\in\mathbb{R}$ we have

$$|f(z,x)|\le a(z)+c|x|^{r-1},$$

with $a\in L^\infty(Z)_+$, $c>0$ and $p\le r<p^*$;

(d) if $F(z,x)=\int_0^x f(z,s)\,ds$, then $F(z,x)\to-\infty$, as $|x|\to\infty$, uniformly for almost all $z\in Z$;

(e) there exist $\delta>0$ and $\theta\in[0,\lambda_1]$ such that

$$0\le F(z,x)\le\frac{\theta}{p}|x|^p\quad\text{for a.a. } z\in Z \text{ and all } |x|\le\delta.$$

Remark 4.1. Hypotheses (H(f)$_2$)(d) and (e) permit resonance with respect to the principal eigenvalue $\lambda_0=0$, both at $\pm\infty$ and at zero (a double resonance situation).

Example 4.2. Consider the following potential function (as before, for the sake of simplicity, we drop the z-dependence):

$$F(x)=\begin{cases}\dfrac{\theta}{p}|x|^p & \text{if } |x|\le 1,\\[2mm] -\dfrac{1}{p}|x|^p+\dfrac{c}{x^2}+\dfrac{1+\theta}{p}-c & \text{if } |x|>1,\end{cases}$$

with $0<\theta<\lambda_1$ and $c=-(\theta+1)/2<0$. Then $f(x)=F'(x)$ satisfies (H(f)$_2$).

Again we consider the Euler functional $\varphi\colon W_n^{1,p}(Z)\to\mathbb{R}$ defined by (3.1).

Proposition 4.3. *If hypotheses* H$(f)_2$ *hold, then* φ *is coercive.*

Proof. Because of hypothesis H$(f)_2$(d) and Lemma 3 of C. L. Tang and X. P. Wu [27], we can find functions $g\in C(\mathbb{R})$, $g\ge 0$ and $h\in L^1(Z)_+$ such that

$$g \text{ is subadditive (i.e., } g(x+y)\le g(x)+g(y)\quad\text{for all } x,y\in\mathbb{R}); \tag{4.1}$$

$$g \text{ is coercive (i.e. } g(x)\to\infty \text{ as } |x|\to\infty); \tag{4.2}$$

$$g(x)\le 4+|x|\quad\text{for all } x\in\mathbb{R}; \tag{4.3}$$

$$F(z,x)\le h(z)-g(x)\quad\text{for a.a. } z\in Z \text{ and all } x\in\mathbb{R}. \tag{4.4}$$

We consider the following direct sum decomposition of the Sobolev space $W_n^{1,p}(Z)$:

$$W_n^{1,p}(Z)=\mathbb{R}\oplus V$$

with

$$V = \left\{ x \in W_n^{1,p}(Z) : \int_Z x(z)\,dz = 0 \right\}.$$

Hence, every $x \in W_n^{1,p}(Z)$ admits a unique decomposition of the form

$$x = \overline{x} + \widehat{x}, \quad \text{with } \overline{x} \in \mathbb{R} \text{ and } \widehat{x} \in V. \tag{4.5}$$

For every $x \in W_n^{1,p}(Z)$, we have

$$\varphi(x) = \frac{1}{p}\|Dx\|_p^p - \int_Z F(z, x(z))\,dz \ge \frac{1}{p}\|Dx\|_p^p + \int_Z g(x(z))\,dz - \|h\|_1 \tag{4.6}$$

(see (4.4)). Using the decomposition (4.5), we have

$$g(\overline{x}) = g(x(z) - \widehat{x}(z)) \le g(x(z)) + g(-\widehat{x}(z))$$

(see (4.1)), hence

$$g(\overline{x}) - g(-\widehat{x}(z)) \le g(x(z)) \quad \text{for all } z \in Z.$$

Therefore,

$$\int_Z g(x(z))\,dz \ge g(\overline{x})|Z|_N - \int_Z g(-\widehat{x}(z))\,dz \ge g(\overline{x})|Z|_N - c_2\|D\widehat{x}\|_p - c_3, \tag{4.7}$$

for some $c_2, c_3 > 0$ (see (4.3) and use the Poincaré–Wirtinger inequality). Using (4.7) in (4.6), we obtain

$$\varphi(x) \ge \frac{1}{p}\|D\widehat{x}\|_p^p + g(\overline{x})|Z|_N - c_2\|D\widehat{x}\|_p - c_4, \quad \text{with } c_4 = c_3 + \|h\|_1 > 0. \tag{4.8}$$

If $\|x\| \to \infty$, then

$$|\overline{x}| \to \infty \quad \text{and/or} \quad \|D\widehat{x}\|_p \to \infty \tag{4.9}$$

the last convergence being a consequence of the Poincaré–Wirtinger inequality. From (4.2), (4.8) and (4.9), and since $p > 1$, it follows that

$$\varphi(x) \to \infty \quad \text{as } \|x\| \to \infty,$$

hence φ is coercive. $\square$

Corollary 4.4. *If hypotheses* $\mathrm{H}(f)_2$ *hold, then* φ *is bounded below and satisfies the* PS-*condition*

Proof. Since φ is coercive (see Proposition 4.3), it is bounded below. Let $(x_n)_{n\ge1} \subset W_n^{1,p}(Z)$ be a PS-sequence. Then

$$|\varphi(x_n)| \le M_3 \quad \text{for some } M_3 > 0, \text{ all } n \ge 1, \tag{4.10}$$

and

$$\varphi'(x_n) \to 0 \quad \text{in } W_n^{1,p}(Z)^* \text{ as } n \to \infty. \tag{4.11}$$

From (4.10) and the coercivity of φ, it follows that $(x_n)_{n\geq 1} \subset W_n^{1,p}(Z)$ is bounded. Hence, we may assume that

$$x_n \xrightarrow{w} x \quad \text{in } W_n^{1,p}(Z) \quad \text{and} \quad x_n \to x \quad \text{in } L^r(Z) \text{ as } n \to \infty. \tag{4.12}$$

(recall that $r < p^*$). Then from (4.11) and (4.12) and Proposition 2.11, and reasoning as in the proof of Proposition 3.3, we obtain $x_n \to x$ in $W_n^{1,p}(Z)$, and so φ satisfies the PS-condition. $\square$

In what follows

$$C(p) = \left\{ x \in W_n^{1,p}(Z) : \int_Z |x(z)|^{p-2} x(z)\, dz = 0 \right\}.$$

Evidently, $C(p)$ is a cone of nodal functions.

The main result of this section is the following.

Theorem 4.5. *If hypotheses* $\mathrm{H}(f)_2$ *hold, then problem* (1.1) *has at least two nontrivial solutions* x_0, $y_0 \in C_n^1(\overline{Z})$.

Proof. Hypotheses $(\mathrm{H(f)}_2)$(c), (e) combined, imply

$$F(z,x) \leq \frac{\theta}{p}|x|^p + c_5|x|^\tau \quad \text{for a.a. } z \in Z, \text{ all } x \in \mathbb{R}$$

with $p < \tau < p^*$, $c_5 > 0$. Then, for all $x \in C(p)$, by using the variational characterization of $\lambda_1 > 0$ (see (2.3)), we have

$$\begin{aligned} \varphi(x) &= \frac{1}{p}\|Dx\|_p^p - \int_Z F(z,x(z))\, dz \\ &\geq \frac{1}{p}\|Dx\|_p^p - \frac{\theta}{p}\|x\|_p^p - c_6\|x\|^\tau \quad \text{for some } c_6 > 0 \\ &\geq \frac{1}{p}\left(1 - \frac{\theta}{\lambda_1}\right)\|Dx\|_p^p - c_6\|x\|^\tau \\ &\geq c_7\|x\|^p - c_6\|x\|^\tau \qquad \text{for some } c_7 > 0. \end{aligned} \tag{4.13}$$

Because $p < \tau$, from (4.13) it follows that we can find $\rho \in (0,1)$ small such that

$$\varphi\,|_{\overline{B}_\rho \cap C(p)\setminus\{0\}} > 0, \tag{4.14}$$

where $\overline{B}_\rho = \{x \in W_n^{1,p}(Z) : \|x\| \leq \rho\}$. On the other hand, by virtue of hypothesis $(\mathrm{H(f)}_2)$(e) and by choosing $\rho \in (0,1)$ even smaller if necessary, we will have

$$\varphi\,|_{\overline{B}_\rho \cap \mathbb{R}} \leq 0, \tag{4.15}$$

Let $U = \overline{B}_\rho$, $E_0 = \{-\rho, \rho\}$, $E = [-\rho, \rho]$ and $D = \overline{B}_\rho \cap C(p)$. Consider the inclusion maps $i_1\colon E_0 \to U \setminus D$ and $i_2\colon E_0 \to E$. If we consider the group homomorphisms $i_1^*\colon H_0(E_0) \to H_0(U \setminus D)$ and $i_2\colon H_0(E_0) \to H_0(E)$, then we have $\operatorname{rank} i_1^* = 2$ and $\operatorname{rank} i_2^* = 1$. Therefore

$$\operatorname{rank} i_1^* - \operatorname{rank} i_2^* = 1. \tag{4.16}$$

As before, we may assume that the origin is an isolated critical point of φ (otherwise we are done). From (4.14)–(4.16) it follows that φ has a local $(1,1)$-linking near the origin. So, by virtue of Theorem 2.7, we have

$$\operatorname{rank} C_1(\varphi, 0) \geq 1. \tag{4.17}$$

Therefore, the origin is a homologically nontrivial critical point of φ. Moreover, $x = 0$ is not a minimizer of φ, or otherwise $C_k(\varphi, 0) = \delta_{k,0}\mathbb{Z}$ for all $k \geq 0$, a contradiction to (4.17). Then Corollary 4.4 permits the use of Theorem 2.8, which implies that φ has at least two nontrivial critical points, x_0, $y_0 \in W_n^{1,p}(Z)$. As before, we infer that x_0, $y_0 \in C_n^1(\overline{Z})$ and they solve problem (1.1). $\square$

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