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INERTIAL MANIFOLDS FOR A SINGULAR PERTURBATION OF THE VISCOUS CAHN–HILLIARD–GURTIN EQUATION

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Abstract. We consider a singular perturbation of the generalized viscous Cahn-Hilliard equation based on constitutive equations introduced by M. E. Gurtin and we establish the existence of a family of inertial manifolds which is continuous with respect to the perturbation parameter $\varepsilon > 0$ as ε goes to 0. In a recent paper, we proved a similar result for the singular perturbation of the standard viscous Cahn–Hilliard equation, applying a construction due to X. Mora and J. Solà-Morales for equations involving linear self-adjoint operators only. Here we extend the result to the singularly perturbed Cahn–Hilliard–Gurtin equation which contains a non-self-adjoint operator. Our method can be applied to a larger class of nonlinear dynamical systems.

1. Introduction

Many infinite-dimensional dynamical systems possess a finite-dimensional global attractor. This is a compact set of the phase space which attracts uniformly the trajectories starting from bounded sets when time goes to infinity and thus appears as a suitable object in view of the study of the asymptotic behavior of such systems (see e.g. [46], cf. also [36]). The finite-dimensionality of the global attractor means that the *a priori* infinite-dimensional dynamical system has an asymptotic behavior which is determined by a finite number of degrees of freedom. This latter fact naturally leads one to consider the question of whether

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there is a finite-dimensional system which adequately captures the asymptotic nature of the original flow. The global attractor may have a complicated fractal structure, even for finite-dimensional dynamical systems, and a reasonably explicit description of the dynamics on the attractor might be out of reach. An answer to this question was given by C. Foias *et al.* who introduced the notion of inertial manifold (see [14]; cf. also [46]). An inertial manifold is a positively invariant smooth finite-dimensional manifold which contains the global attractor and which attracts the trajectories at a uniform exponential rate. These features entail that the *a priori* infinite-dimensional dynamical system reduces, on the inertial manifold, to a finite system of ordinary differential equations.

The Cahn–Hilliard equation plays a basic role in Materials Science. It is a conservation law (in the sense that the average of the order parameter is conserved) which describes very important qualitative features of phase separation processes, namely, the transport of atoms between unit cells (see [4], cf. also [41] and references therein). Several generalizations of this equation have been introduced by M. E. Gurtin in [28]; these are based on constitutive equations which take into account the work of internal microforces, the anisotropy and also the deformations of the material (see [33], [34] and references therein). When mechanical deformations are neglected and there are neither external mass supply nor external microforces, a typical example reads

$$\frac{\partial\rho}{\partial t}-\mathbf{d}\cdot\nabla\frac{\partial\rho}{\partial t}-\operatorname{div}\left(\widetilde{B}\nabla\frac{\partial\rho}{\partial t}-\alpha B\nabla\Delta\rho+B\nabla f'(\rho)\right)=0,$$

on a spatial domain $\Omega=\prod_{i=1}^{n}(0,L_i)$, $L_i>0$, $n\le 3$. Here $\alpha>0$ and $\mathbf{d}\in\mathbb{R}^n$ are given, while B and $\widetilde{B}$ are two symmetric positive definite $n\times n$ matrices with constant coefficients. More precisely, B is called the mobility tensor and $\widetilde{B}$ is a viscosity tensor representing some viscous effects (see [40]). Here the order parameter ρ, corresponding to a rescaled density of atoms, is supposed to be Ω-periodic, while f represents the coarse-grain free energy (a double-well potential) which accounts for the presence of two different phases.

We recall that the asymptotic behavior of the solutions to Cahn–Hilliard type equations was studied by many authors (see, e.g., [10]–[13], [27], [33], [39] and the references therein). Here our main goal is to compare the large time behavior of the above equation with the one of its singular perturbation

$$\varepsilon\frac{\partial^2\rho}{\partial t^2}+\frac{\partial\rho}{\partial t}-\mathbf{d}\cdot\nabla\frac{\partial\rho}{\partial t}-\operatorname{div}\left(\widetilde{B}\nabla\frac{\partial\rho}{\partial t}-\alpha B\nabla\Delta\rho+B\nabla f'(\rho)\right)=0,$$

for $\varepsilon\in(0,\varepsilon_0]$, $0<\varepsilon_0\le 1$, which has been proposed to model, for instance, rapid spinodal decompositions in certain glasses (see [16]–[20]). This equation presents different features depending on whether the viscosity term is present or not. In fact, in the former case, there is a regularization effect on the solutions as in the unperturbed equation, while, in the latter one, there is no regularization and the behavior of the equation is similar to the one of the damped semilinear wave equation, even though the equation is not hyperbolic (see [7], [8], [22]–[26], [47] and [48]).

The above generalized equation, subject to periodic boundary conditions on a n-rectangle ($n \leq 3$), was first studied in [1] (see also [3] for the unperturbed case). The author showed the existence of the global attractor and constructed a family of exponential attractors which is continuous (up to time shifts) as ε goes to zero. Then, in [2], the authors extended the results of [1] along several directions. More precisely, they proved the existence of a robust of family exponential attractors with respect to ε. In particular, they gave an explicit estimate of the symmetric distance between a proper exponential attractor of the perturbed problem and the natural lifting of the corresponding exponential attractor in the unperturbed one. They also established the upper semicontinuity of the global attractor at $\varepsilon = 0$, as well as its lower semicontinuity, provided that the stationary states are hyperbolic. Moreover, the authors proved the convergence of a given solution to a single equilibrium when f is real analytic. Finally, they constructed an inertial manifold in one and two space dimensions when $\mathbf{d} = \mathbf{0}$, showing also the (local) stability of the inertial manifold as ε goes to 0. This result was essentially based on a theory developed by X. Mora and J. Solà-Morales [37], [38] for equations of the form

$$\varepsilon \frac{\partial^2 \rho}{\partial t^2} + \frac{\partial L\rho}{\partial t} + A\rho + F(\rho) = 0,$$

where L and A are two linear positive self-adjoint operators with compact inverse.

The main goal of this paper is to compare the inertial manifolds of the above perturbed and unperturbed generalized equations in one and two space dimensions; in particular, $\mathbf{d}$ does not necessarily vanish. Since these equations contain a non-self-adjoint operator, the aforementioned theory of X. Mora and J. Solà-Morales is not applicable anymore. We thus suggest an alternative approach to the one followed in [2]. Up to Section 3, we introduce the functional setting of the problem and demonstrate some basic results. Then, in Sections 4 and 5, we prove the existence of inertial manifolds for the nonperturbed and perturbed problems, respectively. The final Section 6 is concerned with the convergence of the inertial manifolds of the perturbed problem as ε goes to 0.

2. Setting of the problem

We set $\Omega = \prod_{i=1}^{n}(0, L_i)$, $L_i > 0$, $n \leq 3$, and consider the following initial and boundary value problems:

$$
(2.1) \qquad \begin{cases} \dfrac{\partial \rho}{\partial t} - \mathbf{d} \cdot \nabla \dfrac{\partial \rho}{\partial t} - \operatorname{div}\left(\widetilde{B} \nabla \dfrac{\partial \rho}{\partial t} - \alpha B \nabla \Delta \rho + B \nabla f'(\rho) \right) = 0, \\ \rho|_{t=0} = \rho_0, \\ \rho \text{ is } \Omega\text{-periodic,} \end{cases}
$$

and

$$
(2.2) \qquad \begin{cases} \varepsilon \dfrac{\partial^2 \rho}{\partial t^2} + \dfrac{\partial \rho}{\partial t} - \mathbf{d} \cdot \nabla \dfrac{\partial \rho}{\partial t} - \operatorname{div}\left(\widetilde{B} \nabla \dfrac{\partial \rho}{\partial t} - \alpha B \nabla \Delta \rho + B \nabla f'(\rho) \right) = 0, \\ \rho|_{t=0} = \rho_0, \quad \left. \dfrac{\partial \rho}{\partial t} \right|_{t=0} = \rho_1, \\ \rho \text{ is } \Omega\text{-periodic.} \end{cases}
$$

Here we recall that $\varepsilon \in (0, \varepsilon_0]$, $0 < \varepsilon_0 \leq 1$, $\alpha > 0$ and $\mathbf{d} \in \mathbb{R}^n$ are given, while B and $\widetilde{B}$ are two symmetric positive definite $n \times n$ matrices with constant coefficients.

We assume that the potential f satisfies the following conditions:

$$
(2.3) \qquad f \in \mathcal{C}^4(\mathbb{R}), \quad f(s) \geq -C_0, \quad C_0 \geq 0, \quad \text{for all } s \in \mathbb{R},
$$

$$
(2.4) \qquad |f''(s)| \leq C_1(|s|^{2p} + 1), \quad C_1 > 0, \quad \text{for all } s \in \mathbb{R},
$$

where $p \geq 0$ is arbitrary when $n = 1, 2$ and $p \in [0, 2]$ when $n = 3$,

(2.5) for all $\gamma \in \mathbb{R}$, there exists $C_2 = C_2(\gamma) > 0$ and $C_3 = C_3(\gamma) \geq 0$ such that

$$
(s - \gamma) f'(s) \geq C_2 f(s) - C_3, \quad \text{for all } s \in \mathbb{R},
$$

where C_2 and C_3 are bounded when γ is bounded (with $\inf_{\gamma \in \mathbb{R}} C_2 \geq 0$),

$$
(2.6) \qquad f''(s) \geq -C_4, \quad C_4 \geq 0, \quad \text{for all } s \in \mathbb{R},
$$

(2.7) for all $\mu > 0$, there exists $C_5 = C_5(\mu) > 0$ such that

$$
|f'(s)| \leq \mu f(s) + C_5, \quad \text{for all } s \in \mathbb{R}.
$$

For instance, polynomials of degree $2p + 2$ with strictly positive leading coefficients (and with a double-well structure, e.g. $f(s) = (s^2 - 1)^{p+1}$) satisfy (2.3)–(2.7). However, we note that, in one space dimension, no growth assumption on f is needed, even in absence of viscous terms (see e.g. [21]).

From now on, the same letter c (and sometimes c_r, c'_r and c_j, $j = 0, 1, \ldots$) denotes positive constants which may change from line to line, but are always independent of ε. We denote by $\|\cdot\|$ and $(\,\cdot\,,\,\cdot\,)$ the usual norm and scalar product in the Hilbert space $L^2(\Omega)$ (and also in $L^2(\Omega)^n$). Moreover, for $u \in L^1(\Omega)$, $m(u)$ denotes the spatial average of u, that is,

$$
m(u) = \frac{1}{|\Omega|} \int_\Omega u(x)\, dx, \quad \overline{u} = u - m(u).
$$

For $u = (u_1, \dots , u_n)$, we set $m(u) = (m(u_1), \dots , m(u_n))$. If W is a Sobolev-type space, then we denote by W' its dual and we set

$$\dot{W} = \{q \in W,\ m(q) = 0\}.$$

We also define the linear operator $N = -\operatorname{div} B\nabla : \dot{H}^2_{\rm per}(\Omega) \to \dot{L}^2(\Omega)$ which is a self-adjoint and strictly positive operator with compact inverse N^{-1}. For every $r > 0$, we endow $(H^r_{\rm per}(\Omega))'$ with the norm $\|q\|_{-r} = (\|N^{-r/2}\overline{q}\|^2 + |m(q)|^2)^{1/2}$. We note that $\|q\|_r = (\|N^{r/2}\overline{q}\|^2 + |m(q)|^2)^{1/2}$ is a norm on $H^r_{\rm per}(\Omega)$ which is equivalent to the usual $H^r(\Omega)$-norm (see e.g. [46]). Furthermore, there exist two positive constants C_6 and C_7 such that

$$\|\overline{q}\|_{-1} \le C_6\|\overline{q}\| \le C_7\|\nabla q\|, \quad \text{for all } q \in H^1_{\rm per}(\Omega).$$

We endow the Hilbert spaces $\mathcal{H}^0_\varepsilon = H^1_{\rm per}(\Omega) \times (H^1_{\rm per}(\Omega))'$, $\mathcal{H}^1_\varepsilon = H^2_{\rm per}(\Omega) \times L^2(\Omega)$ and $\mathcal{H}^j_\varepsilon = H^{j+1}_{\rm per}(\Omega) \times H^{j-1}_{\rm per}(\Omega)$, $j = 2, 3$, with the norms (induced by the scalar products)

$$\begin{aligned}
\|(p, q)\|_{\mathcal{H}^0_\varepsilon} &= (\|p\|_1^2 + \varepsilon\|q\|_{-1}^2)^{1/2},\\
\|(p, q)\|_{\mathcal{H}^1_\varepsilon} &= (\|p\|_2^2 + \varepsilon\|q\|^2)^{1/2},\\
\|(p, q)\|_{\mathcal{H}^j_\varepsilon} &= (\|p\|_{(j+1)}^2 + \varepsilon\|q\|_{(j-1)}^2)^{1/2},
\end{aligned}$$

respectively. From Section 3 on, we will also work with spaces of complex-valued functions. However, we will keep the same notation since no confusion can arise. In particular, $x^\star$ and $\Re x$ denote the conjugate and the real part of x, respectively.

We also need to define the following complete metric spaces

$$\begin{aligned}
K_\delta &= \{u \in H^1_{\rm per}(\Omega),\ |m(u)| \le \delta\},\\
\widetilde{K}_\delta &= \{(u, v) \in \mathcal{H}^0_\varepsilon,\ |m(u)| + \varepsilon_0|m(v)| \le \delta\},\\
K^j_\delta &= K_\delta \cap H^{j+1}_{\rm per}(\Omega),\\
\widetilde{K}^j_\delta &= \widetilde{K}_\delta \cap \mathcal{H}^j_\varepsilon,
\end{aligned}$$

for $j = 1, 2, 3$ and some $\delta \ge 0$.

We now introduce a common weak formulation of problems (2.1) and (2.2).

(P$_\varepsilon$) *For any given $T > 0$, find $\rho_\varepsilon\colon [0, T] \to H^2_{\rm per}(\Omega)$ such that*

$$\rho_\varepsilon(0) = \rho_0, \qquad \varepsilon\frac{\partial\rho_\varepsilon}{\partial t}(0) = \varepsilon\rho_1$$

and, for almost every $t \in [0, T]$,

$$\frac{d}{dt}\left[\varepsilon\left(\frac{\partial\rho_\varepsilon}{\partial t}, q\right) + (\rho_\varepsilon, q) + (\rho_\varepsilon, \mathbf{d}\cdot\nabla q) + (\widetilde{B}\nabla\rho_\varepsilon, \nabla q)\right] + \alpha(\nabla B^{1/2}\nabla\rho_\varepsilon, \nabla B^{1/2}\nabla q) + (B\nabla f'(\rho_\varepsilon), \nabla q) = 0, \tag{2.8}$$

for all $q \in H^2_{\rm per}(\Omega)$.

We recall that

$$(\mathbf{d}\cdot\nabla p, q) = -(p, \mathbf{d}\cdot\nabla q), \quad \text{for all } p, q \in H^1_{\text{per}}(\Omega).$$

Moreover, taking $q = 1$ in (2.8), we obtain

$$\varepsilon\frac{d^2}{dt^2}m(\rho_\varepsilon) + \frac{d}{dt}m(\rho_\varepsilon) = 0,$$

so that

$$m(\rho_\varepsilon(t)) = m(\rho_0) + \varepsilon m(\rho_1)(1 - e^{-t/\varepsilon}), \quad \text{for all } t \geq 0, \tag{2.9}$$

and

$$m\bigg(\frac{\partial\rho_\varepsilon}{\partial t}(t)\bigg) = m(\rho_1)e^{-t/\varepsilon}, \quad \text{for all } t \geq 0. \tag{2.10}$$

Note that, when t goes to infinity, $m((\partial\rho_\varepsilon/\partial t)(t))$ goes to zero and $m(\rho_\varepsilon(t))$ goes to $m(\rho_0) + \varepsilon m(\rho_1)$.

We start by recalling the following result (see [3]).

Theorem 2.1. *We assume that* (2.3)–(2.6) *hold and that* $\rho_0 \in H^1_{\text{per}}(\Omega)$. *Then* (P_0) *possesses a unique solution* ρ *such that*

$$\rho \in \mathcal{C}([0,T]; H^1_{\text{per}}(\Omega)) \cap L^2(0,T; H^2_{\text{per}}(\Omega)), \quad \frac{\partial\rho}{\partial t} \in L^2(0,T; L^2(\Omega)),$$

for any $T > 0$. *Furthermore, if* $\rho_0 \in H^2_{\text{per}}(\Omega)$, *then*

$$\rho \in \mathcal{C}([0,T]; H^2_{\text{per}}(\Omega)), \quad \frac{\partial\rho}{\partial t} \in L^2(0,T; H^1_{\text{per}}(\Omega)).$$

Thanks to this result, we can define the semigroup

$$S(t)\colon H^1_{\text{per}}(\Omega) \to H^1_{\text{per}}(\Omega), \quad \rho_0 \mapsto \rho(t), \quad t \geq 0,$$

where $\rho(t)$ is the solution to (2.1) at time t. This semigroup possesses the global attractor $\mathcal{A}_\delta$ on K_δ which is bounded in K^3_δ (cf. [3]).

As far as problem (2.2) is concerned, we report the following (see [1]).

Theorem 2.2. *We assume that* (2.3)–(2.7) *hold and that* $(\rho_0, \rho_1) \in \mathcal{H}^0_\varepsilon$. *Then* (P_ε) *possesses a unique solution* ρ_ε *such that*

$$\bigg(\rho_\varepsilon, \frac{\partial\rho_\varepsilon}{\partial t}\bigg) \in \mathcal{C}([0,T]; \mathcal{H}^0_\varepsilon), \quad \rho_\varepsilon \in L^2(0,T; H^2_{\text{per}}(\Omega))$$

and

$$\frac{\partial\rho_\varepsilon}{\partial t} \in L^2(0,T; L^2(\Omega)), \quad \frac{\partial^2\rho_\varepsilon}{\partial t^2} \in L^2(0,T; (H^2_{\text{per}}(\Omega))'),$$

for any $T > 0$. *Moreover, if* $(\rho_0, \rho_1) \in \mathcal{H}^1_\varepsilon$, *then*

$$\left(\rho_\varepsilon, \frac{\partial \rho_\varepsilon}{\partial t}\right) \in \mathcal{C}([0,T]; \mathcal{H}^1_\varepsilon),$$

$$\frac{\partial \rho_\varepsilon}{\partial t} \in L^2(0,T; H^1_{\mathrm{per}}(\Omega)), \quad \frac{\partial^2 \rho_\varepsilon}{\partial t^2} \in L^2(0,T; (H^1_{\mathrm{per}}(\Omega))').$$

Thanks to Theorem 2.2, we can define the semigroup

$$S_\varepsilon(t): \mathcal{H}^0_\varepsilon \to \mathcal{H}^0_\varepsilon, \quad (\rho_0, \rho_1) \mapsto \left(\rho_\varepsilon(t), \frac{\partial \rho_\varepsilon}{\partial t}(t)\right), \quad t \ge 0,$$

where $\rho_\varepsilon(t)$ is the solution to (2.2) at time t and, for every $\varepsilon \in (0, \varepsilon_0]$, we have the existence of the global attractor $\mathcal{A}_{\varepsilon,\delta}$ for $S_\varepsilon(t)$ on $\widetilde{K}_\delta$ which is bounded in $\widetilde{K}^3_\delta$.

From now on we assume $n = 1$ or $n = 2$. In order to compare the long-time dynamics of problems (2.1) and (2.2), we introduce the natural lifting of an inertial manifold in the unperturbed case. Here the second component is reconstructed by means of the unperturbed equation, namely (see [29]; cf. also [2]), we define a mapping $\mathrm{L}: H^2_{\mathrm{per}}(\Omega) \to \dot{L}^2(\Omega)$ by setting

$$\mathrm{L}\rho = (I - \mathbf{d} \cdot \nabla - \operatorname{div} \widetilde{B} \nabla)^{-1} \operatorname{div} B \nabla(-\alpha \Delta \rho + f'(\rho)).$$

Note that L is well defined due to the regularity result given in Proposition 3.1 below, and $\mathrm{L}\rho(t) = (\partial \rho / \partial t)(t)$, for all $t \ge 0$, whenever $\rho(t)$ is solution to (2.1). Then we introduce the lifting of an inertial manifold $\mathfrak{M}_\delta$ for $S(t)$ on K^1_δ,

$$(\mathfrak{M}_\delta)_0 = \{(\rho, \mathrm{L}\rho) \in \mathcal{H}^1_\varepsilon,\ \rho \in \mathfrak{M}_\delta\}.$$

More generally, if B is a bounded set in $H^2_{\mathrm{per}}(\Omega)$, we indicate its lifting by $(B)_0$. Note that L can be also defined from $H^1_{\mathrm{per}}(\Omega)$ to its dual.

It was shown in [2], when $\mathbf{d} = \mathbf{0}$, the existence of a family of inertial manifolds $\mathfrak{M}^r_{\varepsilon,\delta}$ for the semigroup $S_\varepsilon(t)$ which converges in the $\mathcal{H}^1_1$-norm to a lifting of a corresponding inertial manifold $\mathfrak{M}^r_\delta$ for the unperturbed problem when ε goes to zero. The proof of this result was essentially based on the cited method devised by X. Mora and J. Solà-Morales. As we mentioned in the Introduction, here we extend the analysis to the case $\mathbf{d} \neq \mathbf{0}$, which requires a different approach.

3. Preliminaries

We recall that (see e.g. [39] and [46]) the family $\{e_k\}_{k \in \mathbb{Z}^n}$, where

$$e_k(x) = \sqrt{\frac{1}{|\Omega|}}\, e^{2\pi(kx/L)i},$$

with $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n$ and $kx/L = k_1 x_1/L_1 + \ldots + k_n x_n/L_n$, which consists of all the eigenvectors associated with the operator $-\Delta: H^2_{\mathrm{per}}(\Omega) \to L^2(\Omega)$, is an orthonormal basis of $L^2(\Omega)$. The corresponding eigenvalues have the form

$$\lambda_k = \begin{cases} 4\pi^2 \dfrac{k^2}{L^2} & \text{if } n = 1, \\ 4\pi^2 \left(\dfrac{k_1^2}{L_1^2} + \dfrac{k_2^2}{L_2^2} \right) & \text{if } n = 2. \end{cases}$$

We write $B = (b_{ij})_{i,j}$ and $\widetilde{B} = (\widetilde{b}_{ij})_{i,j}$ (it is understood that $B = b$ and $\widetilde{B} = \widetilde{b}$ when $n = 1$). We can note that, for any $k \in \mathbb{Z}^n$, we have $-\operatorname{div} B \nabla e_k = \beta_k e_k$ and $-\operatorname{div} \widetilde{B} \nabla e_k = \widetilde{\beta}_k e_k$, where

$$\beta_k = \begin{cases} 4b\pi^2 \dfrac{k^2}{L^2} & \text{if } n = 1, \\ 4\pi^2 \left(b_{11} \dfrac{k_1^2}{L_1^2} + 2b_{12} \dfrac{k_1}{L_1} \dfrac{k_2}{L_2} + b_{22} \dfrac{k_2^2}{L_2^2} \right) & \text{if } n = 2, \end{cases}$$

and

$$\widetilde{\beta}_k = \begin{cases} 4\widetilde{b}\pi^2 \dfrac{k^2}{L^2} & \text{if } n = 1, \\ 4\pi^2 \left(\widetilde{b}_{11} \dfrac{k_1^2}{L_1^2} + 2\widetilde{b}_{12} \dfrac{k_1}{L_1} \dfrac{k_2}{L_2} + \widetilde{b}_{22} \dfrac{k_2^2}{L_2^2} \right) & \text{if } n = 2. \end{cases}$$

Since B and $\widetilde{B}$ are two symmetric positive definite matrices, β_k and $\widetilde{\beta}_k$ are strictly positive numbers. Moreover, we set

$$\mathbf{d} = \begin{cases} d & \text{if } n = 1, \\ (d_1, d_2) & \text{if } n = 2, \end{cases}$$

and

$$\widetilde{\lambda}_k = \begin{cases} 2\pi d \dfrac{k}{L} & \text{if } n = 1, \\ 2\pi \left(d_1 \dfrac{k_1}{L_1} + d_2 \dfrac{k_2}{L_2} \right) & \text{if } n = 2. \end{cases}$$

We also recall that the functions of $H^m_{\mathrm{per}}(\Omega)$ are easily characterized by their Fourier series expansions (see e.g. [39] and [46])

$$H^m_{\mathrm{per}}(\Omega) = \left\{ u,\ u = \sum_{k \in \mathbb{Z}^n} e^{2\pi(kx/L)i} u_k,\ u_{-k} = u_k^*,\ \sum_{k \in \mathbb{Z}^n} |k|^{2m} |u_k|^2 < \infty \right\}$$

and the norm $\|u\|_m$ is equivalent to the norm $\{\sum_{k \in \mathbb{Z}^n} (1 + |k|^{2m}) |u_k|^2\}^{1/2}$.

We first show the following regularity result.

Proposition 3.1. *Let $f \in L^2(\Omega)$. Then the solution $u \in H^1_{\mathrm{per}}(\Omega)$ to the probem*

$$\begin{cases} u - \mathbf{d} \cdot \nabla u - \operatorname{div} \widetilde{B} \nabla u = f, \\ u \text{ is } \Omega\text{-periodic}, \end{cases}$$

satisfies $u \in H^2_{\mathrm{per}}(\Omega)$ and there exists a constant $C > 0$ such that $\|u\|_2 \le C\|f\|$.

Proof. For a given $f \in L^2(\Omega)$, the existence and the uniqueness of the solution u in $H^1_{\mathrm{per}}(\Omega)$ is immediate since $(\mathbf{d} \cdot \nabla u, u) = 0$. Let us show the regularity of u. To do so, we use the Fourier series expansions of f and u (cf. e.g. [43] and [46]). We write

$$f = \sum_{k \in \mathbb{Z}^n} e^{2\pi(kx/L)i} f_k, \quad f_{-k} = f_k^\star,$$

where $\sum_{k \in \mathbb{Z}^n} |f_k|^2 < \infty$. Now, if we expand u similarly,

$$u = \sum_{k \in \mathbb{Z}^n} e^{2\pi(kx/L)i} u_k, \quad u_{-k} = u_k^\star,$$

we have

$$u - \mathbf{d} \cdot \nabla u - \operatorname{div} \widetilde{B} \nabla u = \sum_{k \in \mathbb{Z}^n} (1 - i\widetilde{\lambda}_k + \widetilde{\beta}_k) e^{2\pi(kx/L)i} u_k$$

and, comparing the coefficients in $u - \mathbf{d} \cdot \nabla u - \operatorname{div} \widetilde{B} \nabla u = f$, we obtain

$$(1 - i\widetilde{\lambda}_k + \widetilde{\beta}_k) u_k = f_k,$$

so that we have

$$u_k = \frac{f_k}{1 - i\widetilde{\lambda}_k + \widetilde{\beta}_k}.$$

Since the mapping $q \mapsto \| \operatorname{div} \widetilde{B} \nabla q \| + \|q\|$ defines a norm on $H^2_{\mathrm{per}}(\Omega)$ which is equivalent to the usual H^2-norm (see e.g. [35, Lemma 2.1]), there exists $C_8 > 0$ such that

$$\|u\|_2^2 \le C_8 \sum_{k \in \mathbb{Z}^n} (1 + \widetilde{\beta}_k^2) |u_k|^2,$$

that is,

$$\|u\|_2^2 \le C_8 \sum_{k \in \mathbb{Z}^n} \frac{(1 + \widetilde{\beta}_k^2)}{(1 + \widetilde{\beta}_k)^2 + \widetilde{\lambda}_k^2} |f_k|^2.$$

It follows that

$$\|u\|_2^2 \le C_9 \sum_{k \in \mathbb{Z}^n} |f_k|^2 \le C \|f\|^2$$

which yields that, if $f \in L^2(\Omega)$, then $u \in H^2_{\mathrm{per}}(\Omega)$. □

We now introduce the (nonlinear) manifold $\mathbb{M} = \{(u, v) \in \mathcal{H}^0_\varepsilon : v = \mathrm{L}u\}$ and we define the (strongly) continuous semigroup $S_0(t)\colon \mathbb{M} \to \mathbb{M}$ by setting

$$S_0(t)(\rho_0, \mathrm{L}\rho_0) = (S(t)\rho_0, \mathrm{L}S(t)\rho_0), \quad \text{for all } t \ge 0.$$

It is known (see [1]) that there exists a bounded absorbing set for the semigroup $S_\varepsilon(t)$ in $\widetilde{K}_\delta$ of the form

$$B_{2,\delta} = \{(u, v) \in \widetilde{K}^1_\delta,\ \|(u, v)\|_{\mathcal{H}^1_\varepsilon} \le r\},$$

with r independent of ε. We assume that

$$B_\delta = \{u \in K^1_\delta,\ \|u\|_2 \le r\}$$

is a bounded absorbing set for $\{S(t)\}_{t\geq 0}$ in K_δ. From now on, we set

$$\widetilde{B}_{2,\delta} = S_\varepsilon(1)B_{2,\delta}, \quad \widetilde{B}_\delta = S(1)B_\delta.$$

We will always assume that

$$\widetilde{B}_{2,\delta} \subset \{(u,v) \in \widetilde{K}^1_\delta,\ \|(u,v)\|_{\mathcal{H}^1_\varepsilon} \leq \widetilde{r}\} \tag{3.1}$$

and

$$\widetilde{B}_\delta \subset \{u \in K^1_\delta,\ \|u\|_2 \leq \widetilde{r}\},$$

where $\widetilde{r}$ only depends on r. Note that $\widetilde{B}_{2,\delta}$ and $\widetilde{B}_\delta$ are bounded absorbing sets for $S_\varepsilon(t)$ and $S(t)$, respectively.

The following result was proven in [2, Proposition 4.1].

Proposition 3.2. *Let* (2.3)–(2.7) *hold. Then there exists* $t_\star > 0$ *such that*

$$\|S_\varepsilon(t)(\rho_0,\rho_1) - S_0(t)(\rho_0, \mathrm{L}\rho_0)\|^2_{\mathcal{H}^1_1} \leq \sqrt{\varepsilon}c(r,t), \quad \textit{for all } t \geq t_\star, \tag{3.2}$$

for any $(\rho_0,\rho_1) \in \widetilde{B}_{2,\delta}$ *and any* $\varepsilon \in (0,\varepsilon_0]$.

4. Inertial manifolds for the unperturbed problem

Let E be a metric space and $\{S(t)\}_{t\geq 0}$ be a continuous semigroup on E. We recall the definition of an inertial manifold (see e.g. [31] and [46]).

Definition 4.1. A set $\mathfrak{M}$ is called an inertial manifold for $S(t)$ if:

(a) $\mathfrak{M}$ is a finite-dimensional Lipschitz manifold in E;
(b) $\mathfrak{M}$ is smooth, that is, $\mathfrak{M}$ is of class $\mathcal{C}^1$;
(c) $\mathfrak{M}$ is positively invariant under the flow, that is, $S(t)\mathfrak{M} \subset \mathfrak{M}$, for all $t \geq 0$;
(d) $\mathfrak{M}$ is exponentially attracting, that is, there exists a constant c_0 such that, for every $u_0 \in E$, there exists a constant $c_1(u_0) > 0$ such that

$$\mathrm{dist}_E(S(t)u_0, \mathfrak{M}) \leq c_1 e^{-c_0 t}, \quad \text{for all } t \geq 0,$$

where dist_E is the Hausdorff semi-distance with respect to the metric of E:

$$\mathrm{dist}_E(A,B) = \sup_{a\in A} \inf_{b \in B} \|a-b\|_E.$$

The symmetric Hausdorff distance between A and B is given by

$$\mathrm{dist}^{\mathrm{sym}}_E(A,B) = \max\{\mathrm{dist}_E(A,B), \mathrm{dist}_E(B,A)\}.$$

Since problem (2.1) possesses a bounded absorbing set $\widetilde{B}_\delta$ in $H^2_{\mathrm{per}}(\Omega)$, we can truncate the nonlinear term for large $\|\rho\|_2$ and consider the so-called prepared equation. Then we construct an inertial manifold which is globally realized as a graph for the prepared equation. This equation coincides with the original one on $\widetilde{B}_\delta$ so that the intersection of the graph with $\widetilde{B}_\delta$ defines an inertial manifold for the original equation (see e.g. [15] and [30]).

We define a truncated function by setting

$$g(\rho) = \theta\left(\frac{\|\rho\|_2}{\widetilde{r}}\right) f'(\rho), \tag{4.1}$$

where $\widetilde{r}$ is related with the absorbing set $\widetilde{B}_\delta$ and is defined in Section 3 and $\theta\colon \mathbb{R}^+ \to [0,1]$ is a $\mathcal{C}^\infty$ function such that

$$\theta(s) = \begin{cases} 1 & \text{if } 0 \le s \le 1, \\ 0 & \text{if } s > 2, \end{cases}$$

and

$$|\theta'(s)| \le 2, \quad \text{for all } s \ge 0.$$

We now consider the following prepared equation which is "equivalent" to the original equation for t large:

$$\frac{\partial \rho}{\partial t} - \mathbf{d}\cdot\nabla\frac{\partial \rho}{\partial t} - \operatorname{div}\left(\widetilde{B}\nabla\frac{\partial \rho}{\partial t} - \alpha B\nabla\Delta\rho + B\nabla g(\rho)\right) = 0. \tag{4.2}$$

For a given integer n_0 (whose value will be chosen in the proof of Proposition 4.3 below), we consider the orthogonal projector P in $L^2(\Omega)$ onto the space spanned by $\{e_k\}_{|k|\le n_0}$ and we introduce the corresponding projections of ρ, $p = P\rho$ and $q = (I-P)\rho = Q\rho$. We have the orthogonal decomposition $H^j_{\rm per}(\Omega) = PH^j_{\rm per}(\Omega) \oplus QH^j_{\rm per}(\Omega)$ and we set $E^j_1 = PH^{j+1}_{\rm per}(\Omega) \cap K_\delta$, $E^j_2 = QH^{j+1}_{\rm per}(\Omega)$, $j = 0,1,2$. We can note that the spaces E^0_1, E^1_1 and E^2_1 consist of the same finite-dimensional subspace, endowed with different, though equivalent, scalar products.

If ρ is a solution to (4.2), then p and q satisfy the system:

$$\frac{\partial p}{\partial t} - \mathbf{d}\cdot\nabla\frac{\partial p}{\partial t} - \operatorname{div}\left(\widetilde{B}\nabla\frac{\partial p}{\partial t} - \alpha B\nabla\Delta p + B\nabla P g(p+q)\right) = 0, \tag{4.3}$$

$$\frac{\partial q}{\partial t} - \mathbf{d}\cdot\nabla\frac{\partial q}{\partial t} - \operatorname{div}\left(\widetilde{B}\nabla\frac{\partial q}{\partial t} - \alpha B\nabla\Delta q + B\nabla Q g(p+q)\right) = 0. \tag{4.4}$$

Our aim is to find a mapping Φ such that $q = \Phi(p)$ and $p + \Phi(p)$ is solution to (4.2) for (p,q) solution to (4.3)–(4.4).

We denote by $\mathcal{F}_{\mathfrak{b},l}$ the functions Φ which satisfy the following conditions:

$$\begin{cases} \Phi\colon E^1_1 \to E^1_2, \\ \|\Phi(p)\|_2 \le \mathfrak{b}, & \text{for all } p \in E^1_1, \\ \|\Phi(p_1) - \Phi(p_2)\|_2 \le l\|p_1 - p_2\|_2, & \text{for all } p_1, p_2 \in E^1_1, \\ \operatorname{supp}\Phi \subset \{p \in E^1_1,\ \|p\|_2 \le 4\widetilde{r} + \delta\}, \end{cases} \tag{4.5}$$

and we introduce an explicit nonlinear transformation of Φ through the following system:

$$\frac{\partial p}{\partial t} + Ap + PG(p + \Phi(p)) = 0, \tag{4.6}$$

$$\frac{\partial q}{\partial t} + Aq + QG(p + \Phi(p)) = 0, \tag{4.7}$$

where

$$A = (I - \mathbf{d}\cdot\nabla - \operatorname{div}\widetilde{B}\nabla)^{-1}(\alpha \operatorname{div} B\nabla\Delta)\colon H^3_{\mathrm{per}}(\Omega) \to H^1_{\mathrm{per}}(\Omega),$$
$$G(\rho) = -(I - \mathbf{d}\cdot\nabla - \operatorname{div}\widetilde{B}\nabla)^{-1}[\operatorname{div} B\nabla g(\rho)].$$

For $p(0) = p_0$ given in E_1^1, equation (4.6) possesses a unique solution $p(t)$ defined for all $t \in \mathbb{R}$ and, therefore, equation (4.7) admits a unique solution $q(t)$ which remains bounded as $t \to -\infty$ and $q(0)$ is given by the formula (see [45]; cf. also [15] and [46])

$$q(0) = -\int_{-\infty}^{0} e^{sAQ} QG(p(s) + \Phi(p(s)))\, ds. \tag{4.8}$$

The right-hand side of (4.8) defines the value of the image of Φ at p_0 by a nonlinear functional operator T, that is, $q(0) = T\Phi(p_0)$.

An inertial manifold $\mathfrak{M}_\delta$ for the semigroup $S(t)$ generated by the modified equation (4.2), acting on K_δ^1, will be sought as the graph of a mapping Φ belonging to $\mathcal{F}_{\mathfrak{b},l}$ and solution to the following fixed point equation:

$$q(0) = \Phi(p_0) = T\Phi(p_0), \quad \text{for all } p_0 \in E_1^1. \tag{4.9}$$

This theory of existence is based on the Lyapunov–Perron method. Other approaches, such as the Hadamard method, or the graph transform method, are available. However, we can note that all these different methods use the same type of spectral gap condition (see e.g. [30], [32], [43]).

We have the following result (see e.g. [46]).

Proposition 4.2. *Let* (2.3)–(2.7) *hold. Then there exist two constants* Θ_1 *and* Θ_2 *such that, for any* u *and* v *in* K_δ^1,

$$\|G(u) - G(v)\|_2 \le \Theta_1 \|u - v\|_2 \tag{4.10}$$

and

$$\|G(u)\|_2 \le \Theta_2. \tag{4.11}$$

We now state the

PROPOSITION 4.3. *Let* (2.3)–(2.7) *hold and let* L_1/L_2 *be a rational number when* $n = 2$. *Then:*

(a) $T\Phi(p)$ *belongs to* E_2^1 *and there exists* $\mathfrak{b}$ *such that*

$$\|T\Phi(p)\|_2 \leq \mathfrak{b}, \quad \textit{for all } p \in E_1^1. \tag{4.12}$$

(b) *Let* $l \in (0, 1/8)$. *For* Φ *in* $\mathcal{F}_{\mathfrak{b},l}$, *we have*

$$\|T\Phi(p_1) - T\Phi(p_2)\|_2 \leq l\|p_1 - p_2\|_2, \quad \textit{for all } p_1, p_2 \in E_1^1. \tag{4.13}$$

(c) *For every* $\Phi \in \mathcal{F}_{\mathfrak{b},l}$, *the support of* $T\Phi$ *satisfies*

$$\operatorname{supp} T\Phi \subset \{p \in E_1^1,\ \|p\|_2 \leq 4\widetilde{r} + \delta\}. \tag{4.14}$$

PROOF. The proof of this proposition for the classical Cahn–Hilliard equation was given in [39] (cf. also [15] and [46]). Thus, we only give the main points. We have, for every $k \in \mathbb{Z}^n$, $Ae_k = \gamma_k e_k$, where

$$\gamma_k = \frac{\alpha\lambda_k\beta_k(1 + \widetilde{\beta}_k)}{(1 + \widetilde{\beta}_k)^2 + \widetilde{\lambda}_k^2} + i\frac{\alpha\lambda_k\beta_k\widetilde{\lambda}_k}{(1 + \widetilde{\beta}_k)^2 + \widetilde{\lambda}_k^2}.$$

We assume that the $\Re\gamma_k$ are rearranged in an increasing sequence as $|k| = 0, 1, \dots$. For a given $0 < l \leq 1/8$, there exists n_0 such that $\Lambda_2 \geq 4\Theta_1$ and

$$\Lambda_2 - \Lambda_1 \geq \Theta_1(1 + l)(1 + l^{-1}), \tag{4.15}$$

where $\Lambda_1 = \Re\gamma_{\mathfrak{k}_0}$ and $\Lambda_2 = \Re\gamma_{\mathfrak{k}_1}$, with $|\mathfrak{k}_0| = n_0$ and $|\mathfrak{k}_1| = n_0 + 1$. Indeed, we have

$$\Re\gamma_k = \frac{\alpha\lambda_k\beta_k}{\widetilde{\beta}_k} - \frac{\alpha\lambda_k\beta_k}{\widetilde{\beta}_k^2}\left(1 + \frac{\widetilde{\lambda}_k^2}{\widetilde{\beta}_k}\right) + O\left(\frac{1}{|k|^2}\right),$$

where $O(1/|k|^2) \to 0$ when $|k| \to \infty$. In one space dimension, we find

$$\Re\gamma_j = \frac{4\alpha\pi^2 b}{L^2\widetilde{b}}j^2 - \frac{\alpha b}{\widetilde{b}^3}(b + d^2) + O\left(\frac{1}{j^2}\right).$$

In the two-dimensional case, we have

$$\Re\gamma_k \sim c_1\lambda_k + c_2 \quad \text{when } |k| \to \infty,$$

for some positive constants c_1 and c_2. Since L_1/L_2 is rational, a result from number theory (see [32] and [42]) implies that

$$\limsup_{|k|\to\infty}(\lambda_{k'} - \lambda_k) = \infty, \quad |k'| = |k| + 1$$

and, therefore, (4.15) holds.

Next, we can note that $\|e^{sAQ}\|_{\mathcal{L}(E_2^0, E_2^0)}$ is bounded by $e^{s\Lambda_2}$, for any $s \leq 0$. We take the norm $\|\cdot\|_2$ of (4.8). For any $p_0 \in E_1^1$, we obtain

$$\|T\Phi(p_0)\|_2 \leq \int_{-\infty}^0 e^{s\Lambda_2}\|G(p(s) + \Phi(p(s)))\|_2\, ds$$

and estimate (4.12) follows, owing to (4.11). On the other hand, for any p_{01} and p_{02} in E_1^1 such that $m(p_{01}) = m(p_{02})$, there holds

$$\|T\Phi(p_{01}) - T\Phi(p_{02})\|_2 \leq \int_{-\infty}^{0} e^{s\Lambda_2} \|G(p_1(s) + \Phi(p_1(s))) - G(p_2(s) + \Phi(p_2(s)))\|_2 \, ds.$$

Now, we have, due to (4.10),

$$\|G(p_1(s) + \Phi(p_1(s))) - G(p_2(s) + \Phi(p_2(s)))\|_2 \leq \Theta_1(1 + l)\|p_1(s) - p_2(s)\|_2.$$

Using the Poincaré-type inequality

$$\Re(Ap, N^2\overline{p}) \leq \Lambda_1 \|\overline{p}\|_2^2, \quad \text{for all } p \in E_1^1,$$

we also deduce from (4.6) that

$$\|p_1(s) - p_2(s)\|_2 \leq \|p_{01} - p_{02}\|_2 e^{-s[\Lambda_1 + \Theta_1(1+l)]}, \quad \text{for all } s \leq 0.$$

Using the spectral gap condition given by (4.15), estimate (4.13) follows.

Finally, let us prove (4.14). Let $p_0 \in E_1^1$ be such that $\|p_0\|_2 > 4\widetilde{r} + \delta$. By continuity, there exists a neighbourhood $\mathcal{O}$ of 0 such that $p(t)$ satisfies $\|p(t)\|_2 > 2\widetilde{r} + \delta$, for all $t \in \mathcal{O}$. Since

$$\|p(t) + \Phi(p(t))\|_2 = \|p(t)\|_2 + \|\Phi(p(t))\|_2,$$

we find

$$\|p(t) + \Phi(p(t))\|_2 \geq \|p(t)\|_2 > 2\widetilde{r} + \delta,$$

so that $g(p(t) + \Phi(p(t))) = 0$. In $\mathcal{O}$, (4.6) reduces to

$$\frac{\partial p}{\partial t} + Ap = 0. \tag{4.16}$$

We take the L^2-scalar product of (4.16) with $N^2\overline{p}$ and obtain

$$\frac{d}{dt}\|\overline{p}\|_2^2 + 2\,\Re(Ap, N^2\overline{p}) = 0, \quad \text{for all } t \in \mathcal{O},$$

and, therefore,

$$\frac{d}{dt}\|\overline{p}\|_2^2 + 2\Lambda_0\|\overline{p}\|_2^2 \leq 0, \quad \text{for all } t \in \mathcal{O}, \tag{4.17}$$

where $\Lambda_0 = \Re\gamma_{\mathfrak{k}}$, $|\mathfrak{k}| = 1$. Now, observe that $|m(p_0)| \leq |m(\rho_0)| \leq \delta$ and $\|p_0\|_2^2 = \|\overline{p}_0\|_2^2 + |m(p_0)|^2 > (2\widetilde{r} + \delta)^2$. Therefore, $\|\overline{p}_0\|_2^2 > (2\widetilde{r} + \delta)^2 - \delta^2 > 4\widetilde{r}^2$. It follows from (4.17), for $t < 0$ and $t \in \mathcal{O}$, that

$$2\widetilde{r} < \|\overline{p}_0\|_2 \leq e^{2\Lambda_0 t}\|\overline{p}(t)\|_2 \leq \|\overline{p}(t)\|_2 \leq \|p(t)\|_2,$$

which implies that $\|p(t)\|_2 > 2\widetilde{r}$, for all $t < 0$ (see e.g. [15, p. 321]). Therefore, $\|p(t) + \Phi(p(t))\|_2 > 2\widetilde{r}$ and $g(p(t) + \Phi(p(t))) = 0$, for all $t < 0$. Equation (4.7) reduces to

$$\frac{\partial q}{\partial t} + Aq = 0\, quad \text{for all } t < 0. \tag{4.18}$$

The unique solution to (4.18) which remains bounded as $t \to -\infty$ vanishes on $(-\infty, 0]$. We deduce from (4.8) and (4.9) that $q(0) = T\Phi(p_0) = 0$, for any $\Phi \in \mathcal{F}_{\mathfrak{b},l}$. This completes the proof of (4.14). □

REMARK 4.4. We take the opportunity to recall that in [2, Proposition 7.2] we forgot to specify the assumption on L_1/L_2 which ensures the validity of the gap condition.

A consequence of Proposition 4.3 is the following result (cf. [15], [39], [45] and [46]).

THEOREM 4.5. *Let* $l \in (0, 1/8)$ *and let the assumptions of Proposition* 4.3 *hold. Then there exists* $\mathfrak{b} > 0$ *such that:*

(a) T *is a strict contraction from* $\mathcal{F}_{\mathfrak{b},l}$ *into itself: by the Contraction Principle, it possesses a unique fixed point* Φ *in* $\mathcal{F}_{\mathfrak{b},l}$*;*

(b) *the graph of* Φ,

$$\mathfrak{M}_\delta = \{(p, \Phi(p)),\ p \in E_1^1\},$$

is an inertial manifold for equation (4.2) *on* K_δ^1 *of dimension* $\mathfrak{n}_0$.

REMARK 4.6. In fact, we have

$$\mathfrak{M}_\delta = \bigcup_{|\mu| \le \delta} \{\rho = p + \Phi(p),\ (\rho) = \mu\}.$$

5. Inertial manifolds for the perturbed problem

We now consider the following prepared equation which is "equivalent" to the original one for t large:

$$\varepsilon \frac{\partial^2 \rho}{\partial t^2} + \frac{\partial \rho}{\partial t} - \mathbf{d} \cdot \nabla \frac{\partial \rho}{\partial t} - \operatorname{div}\left(\widetilde{B} \nabla \frac{\partial \rho}{\partial t} - \alpha B \nabla \Delta \rho + B \nabla \mathfrak{g}(\rho)\right) = 0, \tag{5.1}$$

where $\mathfrak{g}$ is defined like g (cf. (4.1)) and where $\widetilde{r}$ is related with the absorbing set $\widetilde{B}_{2,\delta}$ (see (3.1)). We introduce the following change of variables:

$$\frac{\partial \rho}{\partial t} = -\frac{1}{2\varepsilon}\rho + \frac{1}{\sqrt{\varepsilon}} v.$$

We can write (5.1) in the following form:

$$\frac{\partial U}{\partial t} + \mathcal{A}U + \mathcal{G}(U) = 0, \tag{5.2}$$

where

$$U = \begin{pmatrix} \rho \\ v \end{pmatrix},$$

$$\mathcal{A} = \begin{pmatrix} \frac{1}{2\varepsilon} & -\frac{1}{\sqrt{\varepsilon}} \\ \frac{\alpha}{\sqrt{\varepsilon}} \operatorname{div} B\nabla\Delta + \frac{1}{2\varepsilon\sqrt{\varepsilon}}\left(-\frac{1}{2}I+\mathbf{d}\cdot\nabla+\operatorname{div}\widetilde{B}\nabla\right) & -\frac{1}{\varepsilon}\left(-\frac{1}{2}I+\mathbf{d}\cdot\nabla+\operatorname{div}\widetilde{B}\nabla\right) \end{pmatrix}$$

and

$$\mathcal{G}(U) = \begin{pmatrix} 0 \\ -\dfrac{1}{\sqrt{\varepsilon}} \operatorname{div} B\nabla\mathfrak{g}(\rho) \end{pmatrix}.$$

For every $0 < \varepsilon \le \varepsilon_0$, we have the following result.

Proposition 5.1. *Let* (2.3)–(2.7) *hold. Then there exist two constants* $\widetilde{\Theta}_1$ *and* $\widetilde{\Theta}_2$*, independent of* ε*, such that, for any* U *and* V *in* $\widetilde{K}^1_\delta$*,*

$$\|\mathcal{G}(U) - \mathcal{G}(V)\|_{\mathcal{H}^1_\varepsilon} \le \widetilde{\Theta}_1 \|U - V\|_{\mathcal{H}^1_\varepsilon} \tag{5.3}$$

and

$$\|\mathcal{G}(U)\|_{\mathcal{H}^1_\varepsilon} \le \widetilde{\Theta}_2. \tag{5.4}$$

Proof. Let $U = \binom{u_1}{v_1}$ and $V = \binom{u_2}{v_2}$. By definition, we have

$$\|\mathcal{G}(U) - \mathcal{G}(V)\|_{\mathcal{H}^1_\varepsilon} = \|\operatorname{div} B\nabla(\mathfrak{g}(u_1) - \mathfrak{g}(u_2))\|$$

and there exists a constant $k_1(r) > 0$, depending only on r, such that

$$\|\operatorname{div} B\nabla(\mathfrak{g}(u_1) - \mathfrak{g}(u_2))\| \le k_1(r)\|\Delta(u_1 - u_2)\|.$$

On the other hand, there exists a constant $\Theta > 0$, independent of ε, such that

$$\|\Delta(u_1 - u_2)\| \le \Theta\|U - V\|_{\mathcal{H}^1_\varepsilon},$$

hence the result. Similarly, we have (see e.g. [46])

$$\|\operatorname{div} B\nabla\mathfrak{g}(u)\| \le k_2(r),$$

hence (5.4). □

Now, we denote by $\{\mu_k\}_{k\in\mathbb{Z}^n}$ the eigenvalues of the operator $\mathcal{A}\colon \mathcal{H}^2_\varepsilon \to \mathcal{H}^0_\varepsilon$. These read

$$\begin{cases} \mu_k^+ = \dfrac{1}{2\varepsilon}\left[\widetilde{\beta}_k + 1 + \sqrt{\dfrac{C_k + \sqrt{C_k^2 + D_k^2}}{2}}\right] + \dfrac{i}{2\varepsilon}\left[\widetilde{\lambda}_k + \sqrt{\dfrac{-C_k + \sqrt{C_k^2 + D_k^2}}{2}}\right], \\ \mu_k^- = \dfrac{1}{2\varepsilon}\left[\widetilde{\beta}_k + 1 - \sqrt{\dfrac{C_k + \sqrt{C_k^2 + D_k^2}}{2}}\right] + \dfrac{i}{2\varepsilon}\left[\widetilde{\lambda}_k - \sqrt{\dfrac{-C_k + \sqrt{C_k^2 + D_k^2}}{2}}\right], \end{cases} \tag{5.5}$$

when $\widetilde{\lambda}_k \geq 0$ and

$$(5.6)\quad \begin{cases} \mu_k^+ = \dfrac{1}{2\varepsilon}\left[\widetilde{\beta}_k + 1 + \sqrt{\dfrac{C_k + \sqrt{C_k^2 + D_k^2}}{2}}\right] + \dfrac{i}{2\varepsilon}\left[\widetilde{\lambda}_k - \sqrt{\dfrac{-C_k + \sqrt{C_k^2 + D_k^2}}{2}}\right], \\ \mu_k^- = \dfrac{1}{2\varepsilon}\left[\widetilde{\beta}_k + 1 - \sqrt{\dfrac{C_k + \sqrt{C_k^2 + D_k^2}}{2}}\right] + \dfrac{i}{2\varepsilon}\left[\widetilde{\lambda}_k + \sqrt{\dfrac{-C_k + \sqrt{C_k^2 + D_k^2}}{2}}\right], \end{cases}$$

when $\widetilde{\lambda}_k \leq 0$, where

$$C_k = (\widetilde{\beta}_k + 1)^2 - \widetilde{\lambda}_k^2 - 4\varepsilon\alpha\lambda_k\beta_k, \quad D_k = 2\widetilde{\lambda}_k(\widetilde{\beta}_k + 1).$$

Let n_0 be some given integer. Set

$$\begin{aligned} X_{n_0} &= \operatorname{span}\left\{ \begin{pmatrix} e_k(x) \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ e_k(x) \end{pmatrix},\ |k| = 0, \dots, n_0 \right\}, \\ Y_{n_0} &= \operatorname{span}\left\{ \begin{pmatrix} e_k(x) \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ e_k(x) \end{pmatrix},\ |k| = n_0 + 1, n_0 + 2, \dots \right\}. \end{aligned}$$

The orthogonal decomposition $\mathcal{H}_\varepsilon^0 = X_{n_0} \oplus Y_{n_0}$ holds. By restricting the operator $\mathcal{A}$ to X_{n_0}, we find the eigenvalues $\mu_k^\pm$ and corresponding eigenvectors $U_k^\pm = (e_k, -\mu_k^\pm e_k)$, with $|k| \leq n_0$. For all ε, we can see that the $\Re\mu_k^\pm$ are all positive real numbers. The family $\Re\mu_k^\pm$ can be rearranged such that $\Re\mu_{\mathfrak{k}_0}^-$ and $\Re\mu_{\mathfrak{k}_1}^-$ (with $|\mathfrak{k}_0| = n_0$ and $|\mathfrak{k}_1| = n_0 + 1$) are consecutive numbers (see [48, Proposition 4.1]).

We now prove the following result.

PROPOSITION 5.2. *Let L_1/L_2, $\widetilde{b}_{11}$, $\widetilde{b}_{12}$, $\widetilde{b}_{22}$ be rational numbers when $n = 2$. Then the following properties hold:*

(a) *the spectrum of $\mathcal{A}$ can be divided into two parts σ_1 and σ_2, σ_1 being finite;*

(b) *if $\widetilde{\Lambda}_1 = \sup\{\Re\mu,\ \mu \in \sigma_1\}$ and $\widetilde{\Lambda}_2 = \inf\{\Re\mu,\ \mu \in \sigma_2\}$, then*

$$(5.7)\quad \widetilde{\Lambda}_2 - \widetilde{\Lambda}_1 \geq \widetilde{\Theta}_1(1 + l)(1 + l^{-1}),$$

for a given $l \in (0, 1/8]$ and for any $\varepsilon \in (0, \varepsilon_0]$;

(c) *there exists an orthogonal decomposition $\widetilde{K}_\delta = \mathcal{E}_1 \oplus \mathcal{E}_2$, with $\mathcal{E}_1 = \mathcal{P}\mathcal{H}_\varepsilon^0 \cap \widetilde{K}_\delta$ and $\mathcal{E}_2 = \mathcal{Q}\mathcal{H}_\varepsilon^0$, where $\mathcal{P}$ and $\mathcal{Q}$ are orthogonal projectors onto $\mathcal{E}_1$ and $\mathcal{E}_2$, respectively.*

PROOF. Let $(\cdot\,,\cdot)_{\mathcal{H}_\varepsilon^0}$ be the scalar product defined by

$$(U, V)_{\mathcal{H}_\varepsilon^0} = (N^{1/2}\overline{u}, N^{1/2}\overline{y}^*) + m(u)m(y^*) + \varepsilon(N^{-1/2}\overline{z}^*, N^{-1/2}\overline{v}) + \varepsilon m(z^*)m(v),$$

where $U = (u, v)$, $V = (y, z) \in \mathcal{H}_\varepsilon^0$. Note that this is the extension of the scalar product defined on the real Hilbert space $\mathcal{H}_\varepsilon^0$ (see Section 2) to the corresponding complex Hilbert space.

We then set

$$\begin{aligned}\sigma_1 &= \{\mu_j^-, \mu_m^+,\ \max\{\Re\mu_j^-, \Re\mu_m^+\} \le \Re\mu_{\mathfrak{k}_0}^-\},\\ \sigma_2 &= \{\mu_j^+, \mu_m^\pm,\ \Re\mu_j^- \le \Re\mu_{\mathfrak{k}_0}^- < \min\{\Re\mu_j^+, \Re\mu_m^\pm\}\}.\end{aligned}$$

Obviously, we have $X_{n_0} = X_{n_0}^1 \oplus X_{n_0}^2$, where

$$\begin{aligned}X_{n_0}^1 &= \operatorname{span}\{U_j^-, U_m^+, \mu_j^-, \mu_m^+ \in \sigma_1\},\\ X_{n_0}^2 &= \operatorname{span}\{U_j^+, \Re\mu_j^- \le \Re\mu_{\mathfrak{k}_0}^- < \Re\mu_j^+\}\end{aligned}$$

and $\mathcal{H}_\varepsilon^0 = X_{n_0}^1 \oplus X_{n_0}^2 \oplus Y_{n_0}$. Observe that $\mu_{\mathfrak{k}_0}^- \in X_{n_0}^1$ and $\mu_{\mathfrak{k}_1}^- \in X_{n_0}^2$. We also note that $X_{n_0}^1$ is orthogonal to Y_{n_0}, but it is not orthogonal to $X_{n_0}^2$, with respect to the scalar product $(\,\cdot\,,\,\cdot\,)_{\mathcal{H}_\varepsilon^0}$. We then introduce the following equivalent scalar product in $\mathcal{H}_\varepsilon^0$:

$$\langle U, V\rangle = \Re\Psi_1(\mathrm{P}_{X_{n_0}}U, \mathrm{P}_{X_{n_0}}V) + \Re\Psi_2(\mathrm{P}_{Y_{n_0}}U, \mathrm{P}_{Y_{n_0}}V),$$

where $\mathrm{P}_{X_{n_0}}$ and $\mathrm{P}_{Y_{n_0}}$ are, respectively, the projections from $\mathcal{H}_\varepsilon^0$ onto X_{n_0} and Y_{n_0} and the functions $\Psi_1\colon X_{n_0} \to \mathbb{C}$ and $\Psi_2\colon Y_{n_0} \to \mathbb{C}$ are defined by

$$\begin{aligned}\Psi_1(U,V) &= (u, y^\star) + ((-\operatorname{div}\widetilde{B}\nabla)^{1/2}\overline{u}, (-\operatorname{div}\widetilde{B}\nabla)^{1/2}\overline{y}^\star)\\ &\quad - \varepsilon(R^{1/2}\overline{u}, R^{1/2}\overline{y}^\star) + \varepsilon(N^{-1/2}\overline{z}, N^{1/2}\overline{u}^\star)\\ &\quad + \varepsilon(N^{-1/2}\overline{v}, N^{1/2}\overline{y}^\star) + \varepsilon(N^{-1/2}\overline{z}^\star, N^{-1/2}\overline{v}) + \varepsilon m(z^\star)m(v),\\ \Psi_2(U,V) &= (N^{1/2}\overline{u}, N^{1/2}\overline{y}^\star) + m(u)m(y^\star)\\ &\quad + \varepsilon(N^{-1/2}\overline{z}^\star, N^{1/2}\overline{u}) + \varepsilon(N^{-1/2}\overline{v}^\star, N^{1/2}\overline{y})\\ &\quad + \varepsilon(N^{-1/2}\overline{z}^\star, N^{-1/2}\overline{v}) + \varepsilon m(z^\star)m(v),\end{aligned}$$

with $U = (u, v)$, $V = (y, z)$ in X_{n_0} and Y_{n_0}, respectively.

The operator $R\colon \dot{H}^2_{\mathrm{per}}(\Omega) \to \dot{L}^2(\Omega)$ is such that $Re_k = \zeta_k e_k$, $\zeta_0 = 0$, $\zeta_k = \Re(\mu_k^{+*}\mu_k^-/\beta_k)$ for $k \neq 0$, and there exist $c_1, c_2 > 0$, independent of ε, $\varepsilon \in (0, \varepsilon_0]$, such that $c_1\|q\|_1^2 \le \|q\|^2 + \|(-\operatorname{div}\widetilde{B}\nabla)^{1/2}q\|^2 - \varepsilon\|R^{1/2}q\|^2 \le c_2\|q\|_1^2$, for all $q \in \dot{H}^2_{\mathrm{per}}(\Omega)$. We can note that $\Re\Psi_1(U,U) \ge 0$, for all $U \in X_{n_0}$, and $\Re\Psi_2(U,U) \ge 0$, for all $U \in Y_{n_0}$, so that $\langle\,\cdot\,,\,\cdot\,\rangle$ indeed defines a scalar product on $\mathcal{H}_\varepsilon^0$. We have $\mu_j^+ + \mu_j^- = (1 + \widetilde{\beta}_j + i\widetilde{\lambda}_j)/\varepsilon$ and $(e_j, e_j^*) = 1$. Thus, for $U_j^- \in X_{n_0}^1$ and $U_j^+ \in X_{n_0}^2$, we have

$$\langle U_j^-, U_j^+\rangle = \Re\Psi_1(U_j^-, U_j^+) = 1 + \widetilde{\beta}_j - \varepsilon\zeta_j - \varepsilon\Re(\mu_j^+ + \mu_j^-) + \varepsilon\zeta_j = 0.$$

As a consequence, the decomposition $\mathcal{H}_\varepsilon^0 = X_{n_0} \oplus Y_{n_0}$ is orthogonal with respect to the equivalent scalar product $\langle\,\cdot\,,\,\cdot\,\rangle$ and we set $\mathcal{E}_1 = X_{n_0}^1 \cap K_\delta$ and $\mathcal{E}_2 = X_{n_0}^2 \oplus Y_{n_0}$, where n_0 is chosen to satisfy the spectral gap condition (5.7). Indeed, it follows from (5.5)–(5.7) that

$$\begin{aligned}\Re\mu_k^- &= \frac{1}{2\varepsilon}\left(1-\sqrt{\frac{1+\sqrt{\mathfrak{C}}}{2}-2\alpha\varepsilon\frac{\lambda_k\beta_k}{\widetilde{\beta}_k^2}}\right)\widetilde{\beta}_k \\ &+\frac{1}{2\varepsilon}\left(1-\frac{1}{4\sqrt{(1+\sqrt{\mathfrak{C}})/2-2\alpha\varepsilon\lambda_k\beta_k/\widetilde{\beta}_k^2}}\left(2-\frac{\widetilde{\lambda}_k^2}{\widetilde{\beta}_k}+\frac{\mathfrak{D}}{\sqrt{\mathfrak{C}}}\right)\right)+O\left(\frac{1}{|k|^2}\right),\end{aligned}$$

where $O(1/|k|^2)\to 0$ when $|k|\to\infty$ and

$$\mathfrak{C}=1-8\alpha\varepsilon\frac{\lambda_k\beta_k}{\widetilde{\beta}_k^2}+16\alpha^2\varepsilon^2\frac{\lambda_k^2\beta_k^2}{\widetilde{\beta}_k^4},\qquad \mathfrak{D}=2+\frac{\widetilde{\lambda}_k^2}{\widetilde{\beta}_k}-8\alpha\varepsilon\frac{\lambda_k\beta_k}{\widetilde{\beta}_k^2}+4\alpha\varepsilon\frac{\lambda_k\beta_k\widetilde{\lambda}_k^2}{\widetilde{\beta}_k^3}.$$

In the one-dimensional case, we find

$$\begin{aligned}\Re\mu_j^- &= \frac{2\pi^2}{\varepsilon L^2}\left(\widetilde{b}-\sqrt{\frac{\widetilde{b}^2-4\varepsilon\alpha b+\sqrt{C}}{2}}\right)j^2 \\ &\quad+\frac{1}{2\varepsilon}\left(1-\frac{1}{4\sqrt{(\widetilde{b}^2-4\varepsilon\alpha b+\sqrt{C})/2}}\left(2\widetilde{b}-d^2+\frac{D}{\sqrt{C}}\right)\right)+O\left(\frac{1}{j^2}\right),\end{aligned}$$

where

$$C=\widetilde{b}^4+16\varepsilon^2\alpha^2b^2-8\varepsilon\alpha b\widetilde{b}^2,\quad D=2\widetilde{b}^3+\widetilde{b}^2d^2-8\varepsilon\alpha b\widetilde{b}+4\varepsilon\alpha bd^2.$$

We can also note that

$$\begin{aligned}&\frac{1}{\varepsilon}\left(\widetilde{b}-\sqrt{\frac{\widetilde{b}^2-4\varepsilon\alpha b+\sqrt{C}}{2}}\right) \\ &\qquad=\frac{8\alpha b\widetilde{b}^2}{\widetilde{b}+\sqrt{\left(\widetilde{b}^2-4\varepsilon\alpha b+\sqrt{\widetilde{b}^4+16\varepsilon^2\alpha^2b^2-8\varepsilon\alpha b\widetilde{b}^2}\right)/2}} \\ &\qquad\qquad\times\frac{1}{\widetilde{b}^2+4\varepsilon\alpha b+\sqrt{\widetilde{b}^4+16\varepsilon^2\alpha^2b^2-8\varepsilon\alpha b\widetilde{b}^2}}.\end{aligned}$$

In the two-dimensional case, we have

$$\begin{aligned}\Re\mu_k^- &\sim \frac{1}{2\varepsilon}\left(1-\sqrt{\frac{1+\sqrt{\widetilde{C}}}{2}-2\alpha\varepsilon\widetilde{c}_1}\right)\widetilde{\beta}_k \\ &\qquad+\frac{1}{2\varepsilon}\left(1-\frac{1}{4\sqrt{(1+\sqrt{\widetilde{C}})/2-2\alpha\varepsilon\widetilde{c}_1}}\left(2-\widetilde{c}_3+\frac{\widetilde{D}}{\sqrt{\widetilde{C}}}\right)\right)\end{aligned}$$

when $|k|\to\infty$ and where

$$\widetilde{C}=1-8\alpha\varepsilon\widetilde{c}_1+16\alpha^2\varepsilon^2\widetilde{c}_1^2,\qquad \widetilde{D}=2+\widetilde{c}_3-8\alpha\varepsilon\widetilde{c}_1+4\alpha\varepsilon\widetilde{c}_1\widetilde{c}_3,$$

for some positive constants $\widetilde{c}_1, \widetilde{c}_2$ and $\widetilde{c}_3$. Since L_1/L_2, $\widetilde{b}_{11}$, $\widetilde{b}_{12}$ and $\widetilde{b}_{22}$ are rational numbers, a result from number theory (see [32] and [42]) implies that

$$\limsup_{|k|\to\infty} (\widetilde{\beta}_{k'} - \widetilde{\beta}_k) = \infty, \quad |k'| = |k| + 1.$$

Thus, in both cases, there exists n_0 which can be chosen independently of ε, $0 < \varepsilon \le \varepsilon_0$, such that $\widetilde{\Lambda}_1 = \Re\mu^-_{\mathfrak{k}_0}$ and $\widetilde{\Lambda}_2 = \Re\mu^-_{\mathfrak{k}_1}$ satisfy the spectral gap condition (5.7). Thus (a) and (b) are satisfied.

Let $\mathcal{P}$ and $\mathcal{Q}$ be the unique orthogonal projections onto $X^1_{n_0}$ and $\mathcal{E}_2$, respectively, and we also see that (c) holds.

We conclude by observing that, denoting by $|||\cdot|||_{\mathcal{H}^0_\varepsilon}$ the norm induced by the scalar product $\langle\,\cdot\,,\,\cdot\,\rangle$, then it is easy to see that there exist two positive constants c_1 and c_2, both independent of ε, such that

$$c_1\|U\|_{\mathcal{H}^0_\varepsilon} \le |||U|||_{\mathcal{H}^0_\varepsilon} \le c_2\|U\|_{\mathcal{H}^0_\varepsilon}, \quad \text{for all } U \in \mathcal{H}^0_\varepsilon.$$

Hence, we can use $\|\cdot\|_{\mathcal{H}^0_\varepsilon}$ in place of $|||\cdot|||_{\mathcal{H}^0_\varepsilon}$ in what follows. □

Remark 5.3. Note that, if $\mathcal{E}^1_1 = \mathcal{P}\mathcal{H}^0_\varepsilon \cap \widetilde{K}^1_\delta$ and $\mathcal{E}^1_2 = \mathcal{Q}\mathcal{H}^1_\varepsilon$, then we have $\widetilde{K}^1_\delta = \mathcal{E}^1_1 \oplus \mathcal{E}^1_2$ and the spaces $\mathcal{E}_1$ and $\mathcal{E}^1_1$ consist of the same finite-dimensional subspace, endowed with different, though equivalent, scalar products.

If U is solution to (5.2), then $\mathrm{p}(t) = \mathcal{P}U$ and $\mathrm{q}(t) = \mathcal{Q}U$ satisfy the system

$$\frac{\partial \mathrm{p}}{\partial t} + \mathcal{A}_\mathrm{p}\mathrm{p} + \mathcal{P}\mathcal{G}(\mathrm{p}+\mathrm{q}) = 0, \tag{5.8}$$

$$\frac{\partial \mathrm{q}}{\partial t} + \mathcal{A}_\mathrm{q}\mathrm{q} + \mathcal{Q}\mathcal{G}(\mathrm{p}+\mathrm{q}) = 0, \tag{5.9}$$

where $\mathcal{A}_\mathrm{p} = \mathcal{A}|_{\mathcal{E}_1}$ and $\mathcal{A}_\mathrm{q} = \mathcal{A}|_{\mathcal{E}_2}$. Our aim is to find a mapping Φ_ε such that $\mathrm{q} = \Phi_\varepsilon(\mathrm{p})$ and $\mathrm{p} + \Phi_\varepsilon(\mathrm{p})$ is solution to (5.2) for (p,q) solution to (5.8)–(5.9).

We denote by $\mathcal{F}^\varepsilon_{\mathfrak{b},l}$ the functions Φ_ε which satisfy the following conditions:

$$\begin{cases} \Phi_\varepsilon: \mathcal{E}^1_1 \to \mathcal{E}_2, \\ \|\Phi_\varepsilon(\mathrm{p})\|_{\mathcal{H}^1_\varepsilon} \le \mathfrak{b}, & \text{for all } \mathrm{p} \in \mathcal{E}^1_1, \\ \|\Phi_\varepsilon(\mathrm{p}^1) - \Phi_\varepsilon(\mathrm{p}^2)\|_{\mathcal{H}^1_\varepsilon} \le l\|\mathrm{p}^1 - \mathrm{p}^2\|_{\mathcal{H}^1_\varepsilon}, & \text{for all } \mathrm{p}^1, \mathrm{p}^2 \in \mathcal{E}^1_1, \\ \operatorname{supp}\Phi_\varepsilon \subset \{\mathrm{p} \in \mathcal{E}^1_1,\ \mathrm{p} = (p_1,p_2), \|p_1\|_2 \le 4\widetilde{r} + \delta\}, \end{cases}$$

and we introduce an explicit nonlinear transformation of Φ_ε through the following system:

$$\frac{\partial \mathrm{p}}{\partial t} + \mathcal{A}_\mathrm{p}\mathrm{p} + \mathcal{P}\mathcal{G}(\mathrm{p}+\Phi_\varepsilon(\mathrm{p})) = 0, \tag{5.10}$$

$$\frac{\partial \mathrm{q}}{\partial t} + \mathcal{A}_\mathrm{q}\mathrm{q} + \mathcal{Q}\mathcal{G}(\mathrm{p}+\Phi_\varepsilon(\mathrm{p})) = 0. \tag{5.11}$$

For $\mathrm{p}(0) = \mathrm{p}_0$ given in $\mathcal{E}_1^1$, equation (5.10) possesses a unique solution $\mathrm{p}(t)$ defined for all $t \in \mathbb{R}$ and, therefore, equation (5.11) admits a unique solution $\mathrm{q}(t)$ which remains bounded as $t \to -\infty$ and $\mathrm{q}(0)$ is given by

$$\mathrm{q}(0) = -\int_{-\infty}^{0} e^{s\mathcal{A}_{\mathrm{q}}\mathcal{Q}} \mathcal{Q}\mathcal{G}(\mathrm{p}(s) + \Phi_\varepsilon(\mathrm{p}(s)))\, ds. \tag{5.12}$$

The right-hand side of (5.12) defines the value at p_0 of the image of Φ_ε by a nonlinear functional operator T_ε, that is, $\mathrm{q}(0) = T_\varepsilon\Phi_\varepsilon(\mathrm{p}_0)$.

As for the unperturbed problem, an inertial manifold $\mathfrak{M}_{\varepsilon,\delta}$ for the semigroup $S_\varepsilon(t)$ generated by equation (5.2), acting in $\widetilde{K}_\delta^1$, will be sought as the graph of a mapping Φ_ε belonging to $\mathcal{F}_{b,l}^\varepsilon$ and solution to the fixed point equation

$$\mathrm{q}(0) = \Phi_\varepsilon(\mathrm{p}_0) = T_\varepsilon\Phi_\varepsilon(\mathrm{p}_0), \quad \text{for all } \mathrm{p}_0 \in \mathcal{E}_1^1. \tag{5.13}$$

We now state and prove the

PROPOSITION 5.4. *Let* (2.3)–(2.7) *and the assumptions of Proposition* 5.2 *hold. Then we have:*

(a) $T_\varepsilon\Phi_\varepsilon(\mathrm{p})$ *belongs to* $\mathcal{E}_2$ *and there exists* $\mathfrak{b} > 0$ *such that*

$$\|T_\varepsilon\Phi_\varepsilon(\mathrm{p})\|_{\mathcal{H}_\varepsilon^1} \le \mathfrak{b}, \quad \textit{for all } \mathrm{p} \in \mathcal{E}_1^1. \tag{5.14}$$

(b) *Let* $l \in (0, 1/8]$. *Then, for* Φ_ε *in* $\mathcal{F}_{\mathfrak{b},l}^\varepsilon$, *we have*

$$\|T_\varepsilon\Phi_\varepsilon(\mathrm{p}^1) - T_\varepsilon\Phi_\varepsilon(\mathrm{p}^2)\|_{\mathcal{H}_\varepsilon^1} \le l\|\mathrm{p}^1 - \mathrm{p}^2\|_{\mathcal{H}_\varepsilon^1}, \quad \textit{for all } \mathrm{p}^1, \mathrm{p}^2 \in \mathcal{E}_1^1. \tag{5.15}$$

(c) *For every* $\Phi_\varepsilon \in \mathcal{F}_{\mathrm{b},l}^\varepsilon$, *the support of* $T_\varepsilon\Phi_\varepsilon$ *satisfies*

$$\operatorname{supp} T_\varepsilon\Phi_\varepsilon \subset \{\mathrm{p} \in \mathcal{E}_1^1,\ \mathrm{p} = (p_1, p_2),\ \|p_1\|_2 \le 4\widetilde{r} + \delta\}. \tag{5.16}$$

PROOF. We first note that $\|e^{s\mathcal{A}_{\mathrm{q}}\mathcal{Q}}\|_{\mathcal{L}(\mathcal{E}_2,\mathcal{E}_2)}$ is bounded by $e^{s\widetilde{\Lambda}_2}$, for any $s \le 0$. Let us take the $\mathcal{H}_\varepsilon^1$-norm of (5.12). For any $\mathrm{p}_0 \in \mathcal{E}_1^1$, we obtain

$$\|T_\varepsilon\Phi_\varepsilon(\mathrm{p}_0)\|_{\mathcal{H}_\varepsilon^1} \le \int_{-\infty}^{0} e^{s\widetilde{\Lambda}_2} \|\mathcal{G}(\mathrm{p}(s) + \Phi_\varepsilon(\mathrm{p}(s)))\|_{\mathcal{H}_\varepsilon^1}\, ds$$

and estimate (5.14) follows, owing to (5.4). On the other hand, for any p_0^1 and p_0^2 in $\mathcal{E}_1^1$ such that $m(\mathrm{p}_0^1) = m(\mathrm{p}_0^2)$, there holds

$$\begin{aligned}&\|T_\varepsilon\Phi_\varepsilon(\mathrm{p}_0^1) - T_\varepsilon\Phi_\varepsilon(\mathrm{p}_0^2)\|_{\mathcal{H}_\varepsilon^1}\\ &\qquad \le \int_{-\infty}^{0} e^{s\widetilde{\Lambda}_2} \|\mathcal{G}(\mathrm{p}^1(s) + \Phi_\varepsilon(\mathrm{p}^1(s))) - \mathcal{G}(\mathrm{p}^2(s) + \Phi_\varepsilon(\mathrm{p}^2(s)))\|_{\mathcal{H}_\varepsilon^1}\, ds.\end{aligned}$$

Now, due to (5.3), we have

$$\|\mathcal{G}(\mathrm{p}^1(s) + \Phi_\varepsilon(\mathrm{p}^1(s))) - \mathcal{G}(\mathrm{p}^2(s) + \Phi_\varepsilon(\mathrm{p}^2(s)))\|_{\mathcal{H}_\varepsilon^1} \le \widetilde{\Theta}_1(1 + l)\|\mathrm{p}^1(s) - \mathrm{p}^2(s)\|_{\mathcal{H}_\varepsilon^1}.$$

Using the Poincaré-type inequality

$$\Re(\mathcal{A}_{\mathrm{p}}\mathrm{p}, \mathcal{N}\overline{\mathrm{p}}) \le \widetilde{\Lambda}_1 \|\overline{\mathrm{p}}\|^2_{\mathcal{H}^1_\varepsilon}, \quad \text{for all } \mathrm{p} \in \mathcal{E}^1_1,$$

where $\mathcal{N}\mathrm{p} = (N^2 p_1, \varepsilon p_2)$ for $\mathrm{p} = (p_1, p_2)$, we also deduce from (5.10) that

$$\|\mathrm{p}^1(s) - \mathrm{p}^2(s)\|_{\mathcal{H}^1_\varepsilon} \le \|\mathrm{p}^1_0 - \mathrm{p}^2_0\|_{\mathcal{H}^1_\varepsilon} e^{-s[\widetilde{\Lambda}_1 + \widetilde{\Theta}_1(1+l)]}, \quad \text{for all } s \le 0.$$

Using the spectral gap condition given by Proposition 5.2, estimate (5.15) follows.

We now prove (5.16). Let $\mathrm{p}_0 = (p_{01}, p_{02}) \in \mathcal{E}^1_1$ be such that $\|p_{01}\|_2 > 4\widetilde{r} + \delta$. There exists a neighbourhood $\mathfrak{O}$ of 0 such that $\|p_1(t)\|_2 > 2\widetilde{r} + \delta$, for all $t \in \mathfrak{O}$. This implies that $\|\rho(t)\|_2 = \|p_1(t) + q_1(t))\|_2 > 2\widetilde{r}$ and then $\mathfrak{g}(\rho(t))) = 0$. Therefore, we have

$$\frac{\partial \mathrm{p}}{\partial t} + \mathcal{A}_{\mathrm{p}}\mathrm{p} = 0, \quad \text{for all } t \in \mathfrak{O}. \tag{5.17}$$

The first component of p in (5.17) then satisfies the equation

$$\frac{\partial p_1}{\partial t} + \mathcal{A}^1_{\mathrm{p}}\mathrm{p} = 0, \quad \text{for all } t \in \mathfrak{O}, \tag{5.18}$$

where $\mathcal{A}_{\mathrm{p}}\mathrm{p} = (\mathcal{A}^1_{\mathrm{p}}\mathrm{p}, \mathcal{A}^2_{\mathrm{p}}\mathrm{p})$. Taking the L^2-scalar product of (5.18) with $N^2\overline{p}_1$, we obtain

$$\frac{d}{dt}\|\overline{p}_1\|^2_2 + 2\Re(\mathcal{A}^1_{\mathrm{p}}\mathrm{p}, N^2\overline{p}_1) = 0, \quad \text{for all } t \in \mathfrak{O},$$

and, therefore,

$$\frac{d}{dt}\|\overline{p}_1\|^2_2 + 2\widetilde{\Lambda}_0\|\overline{p}_1\|^2_2 \le 0, \quad \text{for all } t \in \mathfrak{O}, \tag{5.19}$$

where $\widetilde{\Lambda}_0 = \Re\gamma_{\widetilde{\mathfrak{k}}}$, $|\widetilde{\mathfrak{k}}| = 1$. We have $|m(p_{01})| \le |m(\rho_0)| \le \delta$ and $\|p_{01}\|^2_2 = \|\overline{p}_{01}\|^2_2 + |m(p_{01})|^2 > (2\widetilde{r} + \delta)^2$. Therefore, $\|\overline{p}_{01}\|^2_2 > (2\widetilde{r} + \delta)^2 - \delta^2 > 4\widetilde{r}^2$.

From (5.19), we find, for $t < 0$ and $t \in \mathfrak{O}$,

$$2\widetilde{r} < \|\overline{p}_{01}\|_2 \le e^{2\widetilde{\Lambda}_0 t}\|\overline{p}_1(t)\|_2 \le \|\overline{p}_1(t)\|_2.$$

Consequently, $\|p_1(t)\|_2 > 2\widetilde{r}$, for all $t < 0$, and therefore $\mathfrak{g}(\rho(t))) = 0$, for all $t < 0$. Then equation (5.9) reduces to

$$\frac{\partial \mathrm{q}}{\partial t} + \mathcal{A}_{\mathrm{q}}\mathrm{q} = 0, \quad \text{for all } t < 0. \tag{5.20}$$

The unique solution to (5.20) which remains bounded as $t \to -\infty$ vanishes on $(-\infty, 0]$. From (5.12) and (5.13), it follows that $\mathrm{q}(0) = T_\varepsilon\Phi_\varepsilon(\mathrm{p}_0) = 0$, for any $\Phi_\varepsilon \in \mathcal{F}^\varepsilon_{\mathfrak{b},l}$. This completes the proof of (5.16). □

As for Theorem 4.5, Proposition 5.4 implies the following result.

THEOREM 5.5. *Let* $l \in (0, 1/8]$ *and let the assumptions of Proposition* 5.4 *hold. Then there exists* $\mathfrak{b} > 0$*, independent of* ε*, such that, for any* $\varepsilon \in (0, \varepsilon_0]$,

(a) T_ε *is a strict contraction from* $\mathcal{F}^\varepsilon_{\mathfrak{b},l}$ *into itself; by the Contraction Principle, it possesses a unique fixed point* Φ_ε *in* $\mathcal{F}^\varepsilon_{\mathfrak{b},l}$*;*

(b) *the graph* $\mathfrak{N}_{\varepsilon,\delta} = \{(\mathrm{p}, \Phi_\varepsilon(\mathrm{p})),\ \mathrm{p} \in \mathcal{E}^1_1\}$ *of* Φ_ε *is an inertial manifold for equation* (5.2) *on* $\widetilde{K}^1_\delta$ *of dimension* $\mathfrak{n}_0$ ($\mathfrak{n}_0$ *is the same as in Theorem* 4.5).

REMARK 5.6. Actually, we have

$$\mathfrak{N}_{\varepsilon,\delta} = \bigcup_{|\mu|+\varepsilon_0|\sigma|\le\delta} \{U(t) = \mathrm{p}(t) + \Phi_\varepsilon(\mathrm{p}(t)),\ m(U(t)) = (f_1(\mu,\sigma,t), f_2(\mu,\sigma,t)\},$$

where

$$\begin{aligned} f_1(\mu,\sigma,t) &= \frac{\mu}{2} + \sqrt{\varepsilon}\sigma + \left(\frac{\mu}{2} - \sqrt{\varepsilon}\sigma\right)e^{-t/\varepsilon}, \\ f_2(\mu,\sigma,t) &= \frac{\mu}{4\sqrt{\varepsilon}} + \frac{\sigma}{2} - \left(\frac{\mu}{4\sqrt{\varepsilon}} - \frac{\sigma}{2}\right)e^{-t/\varepsilon}. \end{aligned}$$

6. Continuity of the inertial manifolds at $\varepsilon = 0$

Here we want to compare the inertial manifolds of problems (5.1) with their unperturbed counterpart. For all $\varepsilon \in (0, \varepsilon_0]$, we introduce the auxiliary semigroups $\widetilde{T}_\varepsilon(t)\colon \mathcal{H}^0_\varepsilon \to \mathcal{H}^0_\varepsilon$ defined by

$$\widetilde{T}_\varepsilon(t)(\rho_0, \rho_1) = \left(\rho_\varepsilon(t), \sqrt{\varepsilon}\frac{\partial\rho_\varepsilon}{\partial t}(t) + \frac{1}{2\sqrt{\varepsilon}}\rho_\varepsilon(t)\right), \quad \text{for all } t \ge 0,$$

where $\rho_\varepsilon(t)$ is the solution to problem (5.1) with the initial conditions

$$\rho_\varepsilon|_{t=0} = \rho_0, \qquad \left.\frac{\partial\rho_\varepsilon}{\partial t}\right|_{t=0} = -\frac{1}{2\varepsilon}\rho_0 + \frac{1}{\sqrt{\varepsilon}}\rho_1.$$

Introducing the matrix

$$C_\varepsilon = \begin{pmatrix} 1 & 0 \\ -1/(2\varepsilon) & 1/\sqrt{\varepsilon} \end{pmatrix},$$

we can also write

$$\widetilde{S}_\varepsilon(t) = C_\varepsilon \circ \widetilde{T}_\varepsilon(t) \circ C_\varepsilon^{-1}, \quad \text{for all } \varepsilon > 0.$$

Clearly, $\mathfrak{N}_{\varepsilon,\delta}$ is an inertial manifold for $\widetilde{T}_\varepsilon(t)$, for every $0 < \varepsilon \le \varepsilon_0$. Let us set

$$\mathfrak{M}_{\varepsilon,\delta} = C_\varepsilon \mathfrak{N}_{\varepsilon,\delta}, \quad \text{for all } \varepsilon \in (0, \varepsilon_0].$$

For every $\varepsilon \in (0, \varepsilon_0]$, $\mathfrak{M}_{\varepsilon,\delta}$ is an inertial manifold for the semigroup $\widetilde{S}_\varepsilon(t)$ generated by (5.1) with respect to the metric induced by the norm $|||C_\varepsilon^{-1} \cdot |||_{\mathcal{H}^0_\varepsilon}$ and $\dim \mathfrak{M}_{\varepsilon,\delta} = \dim \mathfrak{N}_{\varepsilon,\delta}$. We observe that the inertial manifolds $\mathfrak{M}_\delta$ and $\mathfrak{N}_{\varepsilon,\delta}$, given by Theorems 4.5 and 5.5, respectively, are not only positively invariant, but also satisfy $\widetilde{S}(t)\mathfrak{M}_\delta = \mathfrak{M}_\delta$ and $\widetilde{T}_\varepsilon(t)\mathfrak{N}_{\varepsilon,\delta} = \mathfrak{N}_{\varepsilon,\delta}$, for all $t \in \mathbb{R}$, where $\widetilde{S}(t)$ and $\widetilde{T}_\varepsilon(t)$ are the semigroups generated by the modified equations (5.1) and (5.2), respectively (see e.g. [9, p. 399]; cf. also [30, p. 611–613]). We also note that the lifting $(\mathfrak{M}_\delta)_0$ is an inertial manifold for the semigroup $\{\widetilde{S}_0(t)\}_{t\geq 0}$ on $\widetilde{K}^1_\delta \cap \mathbb{M}$. We now set

$$\mathfrak{M}^{\widetilde{r}}_{\varepsilon,\delta} = \mathfrak{M}_{\varepsilon,\delta} \cap \widetilde{B}_{2,\delta}, \quad \mathfrak{M}^{\widetilde{r}}_\delta = \mathfrak{M}_\delta \cap \widetilde{B}_\delta.$$

As noted above, $(\mathfrak{M}^{\widetilde{r}}_\delta)_0$ and $\mathfrak{M}^{\widetilde{r}}_{\varepsilon,\delta}$ are inertial manifolds for the semigroups generated by the original equations (see e.g. [48, p. 875]) and are only positively invariant under the flow, that is, $S_0(t)(\mathfrak{M}^{\widetilde{r}}_\delta)_0 \subset (\mathfrak{M}^{\widetilde{r}}_\delta)_0$ and $S_\varepsilon(t)\mathfrak{M}^{\widetilde{r}}_{\varepsilon,\delta} \subset \mathfrak{M}^{\widetilde{r}}_{\varepsilon,\delta}$, for all $t \geq 0$.

We now show the following result for trajectories lying on the inertial manifold $\mathfrak{M}_\delta$.

Proposition 6.1. *For any solution ρ of problem (4.2) such that the trajectory $(\rho(t))_{t\in\mathbb{R}}$ lies on $\mathfrak{M}_\delta$ and $\rho_0 \in \mathfrak{M}^{\widetilde{r}}_\delta$, we have*

$$\left\|\frac{\partial^2\rho}{\partial t^2}(t)\right\|^2 \leq M_1 e^{-(\Lambda_1+\widetilde{\Theta}_1)t}, \quad \text{for all } t \leq 0, \tag{6.1}$$

where Λ_1 and $\widetilde{\Theta}_1$ are the same as in Propositions 4.3 and 5.1, respectively.

Proof. Let the complete orbit $(\rho(t))_{\in\mathbb{R}}$ lie in $\mathfrak{M}_\delta$. Any function ρ of $\mathfrak{M}_\delta$ has the form $\rho(t) = p(t) + \Phi(p(t))$, where $p(t)$ satisfies $p \in \mathcal{C}(\mathbb{R}, E^2_1)$ and is the solution to (4.6) with initial datum p_0 in $P\widetilde{B}_\delta$. We now observe that $\partial\rho/\partial t$ is solution to the linearized problem:

$$\frac{\partial s}{\partial t} - \mathbf{d}\cdot\nabla\frac{\partial s}{\partial t} - \operatorname{div}\left(\widetilde{B}\nabla\frac{\partial s}{\partial t} - \alpha B\nabla\Delta s + B\nabla(sg'(\rho))\right) = 0, \tag{6.2}$$

$$s(t)|_{t=0} = \mathrm{L}\rho_0. \tag{6.3}$$

On account of (4.15), there exists a time varying $\mathcal{C}^1$-finite-dimensional invariant manifold $\mathcal{M}_t$, $t \in \mathbb{R}$, for problem (6.2)–(6.3) of the form

$$\mathcal{M}_t = \{(p, \Phi_t(p)),\ p \in E^1_1\},$$

where

$$\begin{cases} \Phi_t\colon E^1_1 \to E^1_2, & \\ \|\Phi_t(p)\|_2 \leq \mathfrak{b}, & \text{for all } p \in E^1_1, \\ \|\Phi_t(p_1) - \Phi_t(p_2)\|_2 \leq l\|p_1 - p_2\|_2, & \text{for all } p_1, p_2 \in E^1_1, \end{cases} \tag{6.4}$$

$q(t) = \Phi_t(p(t))$, $p(t) + q(t)$ is solution to (6.2)–(6.3) for $(p(t),\, q(t))$ solution to the system

$$\frac{\partial p}{\partial t} + Ap + P\widetilde{G}(t, p + \Phi_t(p)) = 0, \tag{6.5}$$

$$\frac{\partial q}{\partial t} + Aq + Q\widetilde{G}(t, p + \Phi_t(p)) = 0, \tag{6.6}$$

where

$$\widetilde{G}(t,s(t)) = -(I - \mathbf{d}\cdot\nabla - \operatorname{div}\widetilde{B}\nabla)^{-1}[\operatorname{div} B\nabla(g'(\rho(t))s(t))].$$

Similarly to (4.11) (see also [38, Lemma 3.1]), we can show that

$$\|g'(\rho(t))\|_2 \le c, \quad \text{for all } t \in \mathbb{R}.$$

For practical reasons, we can then assume that

$$\|\widetilde{G}(t,s(t))\|_2 \le \widetilde{\Theta}_1 \|s(t)\|_2, \quad \text{for all } t \in \mathbb{R}. \tag{6.7}$$

From (6.4), (6.6) and (6.7), we deduce that

$$\left\|\frac{\partial}{\partial t}\Phi_t(p(t))\right\| \le c(\|p(t)\|_2 + 1), \quad \text{for all } t \in \mathbb{R}. \tag{6.8}$$

We take the L^2-scalar product of (6.5) with $N^2 p$ (note that $m(p) = 0$) and obtain

$$\frac{1}{2}\frac{d}{dt}\|p\|_2 + \Re(Ap, N^2p) + (P\widetilde{G}(t, p + \Phi_t(p)), N^2 p) = 0,$$

hence

$$\left|\frac{1}{2}\frac{d}{dt}\|p\|_2^2 + \Re(Ap, N^2p)\right| \le \widetilde{\Theta}_1\|p\|_2^2 + c\|p\|_2.$$

We then deduce that

$$-\|\overline{p}\|_2 \frac{d}{dt}\|p\|_2 \le (\Lambda_1 + \widetilde{\Theta}_1)\|p\|_2^2 + c\|p\|_2$$

and, therefore,

$$-\frac{d}{dt}\|p\|_2 \le (\Lambda_1 + \widetilde{\Theta}_1)\|p\|_2 + c. \tag{6.9}$$

We now apply the Gronwall lemma to (6.9) between t and 0, $t \le 0$, and we find

$$\|p(t)\|_2 \le (\|p_0\|_2 + c)e^{-(\Lambda_1+\widetilde{\Theta}_1)t}, \quad \text{for all } t \le 0.$$

Since $\rho_0 \in \widetilde{B}_\delta$, we have $\|\mathrm{L}\rho_0\| \le c_r$ and then $\|p_0\| \le c_r$. We note that

$$\|\overline{p}_0\|_2^2 \le c\lambda_{\mathfrak{k}_0}^2\|\overline{p}_0\|^2,$$

where $\mathfrak{k}_0$ is the same as the one defining Λ_1. Therefore,

$$\|p(t)\|_2 \le c_r e^{-(\Lambda_1+\widetilde{\Theta}_1)t}, \quad \text{for all } t \le 0.$$

From (6.5), it also follows that there exists c_r such that

$$\left\|\frac{\partial p}{\partial t}(t)\right\| \le c_r\left(\|p(t)\|_2 + 1\right) \le c_r e^{-(\Lambda_1+\widetilde{\Theta}_1)t}, \quad \text{for all } t \le 0. \tag{6.10}$$

Consequently, it follows, on account of (6.8), (6.10) and the fact that $s(t) = p(t) + \Phi_t(p(t))$, that

$$\left\|\frac{\partial s}{\partial t}(t)\right\| \le Ce^{-(\Lambda_1+\widetilde{\Theta}_1)t}, \quad \text{for all } t \le 0,$$

and, in particular, (6.1) holds. □

Remark 6.2. It follows from Proposition 6.1 that

$$\left\|\frac{\partial^2\rho}{\partial t^2}(s)\right\| \le M_1 e^{-s(\widetilde\Lambda_1+\widetilde\Theta_1)}, \quad \text{for all } s\le 0, \tag{6.11}$$

since $\Lambda_1 < \widetilde\Lambda_1$, for any $\varepsilon\in(0,\varepsilon_0]$, $\widetilde\Lambda_1$ being the same as in the proof of Proposition 5.2.

We now show the following result.

Proposition 6.3. *Let the assumptions of Theorem* 5.5 *hold. Then the family of inertial manifolds* $\mathfrak{N}_{\varepsilon,\delta}\cup(\mathfrak{M}^{\widetilde r}_\delta)_0$ *are lower semicontinuous at* $\varepsilon=0$ *in the* $\mathcal{H}^1_1-$ *norm and, for every* $0<\varepsilon\le\varepsilon_0$*, there holds*

$$\operatorname{dist}_{\mathcal{H}^1_1}((\mathfrak{M}^{\widetilde r}_\delta)_0,\mathfrak{N}_{\varepsilon,\delta}) \le M_2\sqrt{\varepsilon}, \tag{6.12}$$

where the constant $M_2>0$ *is independent of* ε.

Proof. We consider an element $W_0=(\rho_0,\mathrm{L}\rho_0)$ of $(\mathfrak{M}^{\widetilde r}_\delta)_0$. Set

$$W(t)=(\rho(t),\sqrt{\varepsilon}\mathrm{L}\rho(t)+2^{-1}\varepsilon^{-1/2}\rho(t)),$$

where $\rho(t)=S(t)\rho_0$. Thus, $W(t)=(\rho(t),v(t))$ satisfies the following problem:

$$\frac{\partial W}{\partial t}+\mathcal{A}W+\mathcal{G}(W)=\sqrt{\varepsilon}\begin{pmatrix}0\\ \frac{\partial^2\rho}{\partial t^2}\end{pmatrix},$$
$$W(0)=(\rho_0,\mathrm{L}\rho_0).$$

Let $w_\varepsilon(t)\in\mathfrak{N}_{\varepsilon,\delta}$ be a solution to (5.2). The function $z(t)=W(t)-w_\varepsilon(t)$ satisfies the equation

$$\frac{\partial z}{\partial t}+\mathcal{A}z+\mathcal{G}(W)-\mathcal{G}(w_\varepsilon)=\sqrt{\varepsilon}\begin{pmatrix}0\\ \frac{\partial^2\rho}{\partial t^2}\end{pmatrix}. \tag{6.13}$$

We write $z(t)=p(t)+q(t)$, where $p=\mathcal{P}z$ and $q=\mathcal{Q}z$. In particular, $z(0)=p(0)+q(0)$ and we have

$$z(0)=p(0)-\int_{-\infty}^0 e^{s\mathcal{A}_q\mathcal{Q}}\mathcal{Q}\Big[\mathcal{G}(W(s))-\mathcal{G}(w_\varepsilon(s))+\sqrt{\varepsilon}\begin{pmatrix}0\\ \frac{\partial^2\rho}{\partial t^2}(s)\end{pmatrix}\Big]\,ds. \tag{6.14}$$

Since $\mathcal{E}^1_1$ is a finite-dimensional subspace of $\mathcal{H}^1_\varepsilon$, we can choose $w_\varepsilon(0)$ such that $p(0)=0$. Thus, on account of (6.11), we deduce from (6.14) that

$$\begin{aligned}\|z(0)\|_{\mathcal{H}^1_\varepsilon} &\le \int_{-\infty}^0 e^{s\widetilde\Lambda_2}\left[\|\mathcal{G}(W(s))-\mathcal{G}(w_\varepsilon(s))\|_{\mathcal{H}^1_\varepsilon}+\sqrt{\varepsilon}\left\|\begin{pmatrix}0\\ \frac{\partial^2\rho}{\partial t^2}(s)\end{pmatrix}\right\|_{\mathcal{H}^1_\varepsilon}\right]ds \\ &\le \widetilde\Theta_1\int_{-\infty}^0 e^{s\widetilde\Lambda_2}\|z(s)\|_{\mathcal{H}^1_\varepsilon}\,ds+\varepsilon\int_{-\infty}^0 e^{s\widetilde\Lambda_2}\left\|\frac{\partial^2\rho}{\partial t^2}(s)\right\|ds, \\ &\le \widetilde\Theta_1\int_{-\infty}^0 e^{s\widetilde\Lambda_2}\|z(s)\|_{\mathcal{H}^1_\varepsilon}\,ds+c\varepsilon,\end{aligned} \tag{6.15}$$

due to the fact that (5.7) holds. We now observe that $m(z(t)) = 0$, since $m(p(0)) = 0$. We take the L^2-scalar product of (6.13) with $\mathcal{N}z$. We then deduce

$$\frac{1}{2}\frac{d}{dt}\|z\|^2_{\mathcal{H}^1_\varepsilon} + \Re(\mathcal{A}z, \mathcal{N}z) + (\mathcal{G}(W) - \mathcal{G}(w_\varepsilon), \mathcal{N}z) = \sqrt{\varepsilon}\left(\begin{pmatrix} 0 \\ \frac{\partial^2\rho}{\partial t^2}\end{pmatrix}, \mathcal{N}z\right)$$

and, therefore,

$$\left|\frac{1}{2}\frac{d}{dt}\|z\|^2_{\mathcal{H}^1_\varepsilon} + \Re(\mathcal{A}z, \mathcal{N}z)\right| \leq \widetilde{\Theta}_1\|z\|^2_{\mathcal{H}^1_\varepsilon} + \varepsilon\left\|\frac{\partial^2\rho}{\partial t^2}\right\|\|z\|_{\mathcal{H}^1_\varepsilon}.$$

From this latter equation, we deduce, owing to (6.11) and the fact that $m(z) = 0$,

$$-\|z\|_{\mathcal{H}^1_\varepsilon}\frac{d}{dt}\|z\|_{\mathcal{H}^1_\varepsilon} \leq (\widetilde{\Lambda}_1 + \widetilde{\Theta}_1)\|z\|^2_{\mathcal{H}^1_\varepsilon} + \varepsilon e^{-(\widetilde{\Lambda}_1+\widetilde{\Theta}_1)t}\|z\|_{\mathcal{H}^1_\varepsilon},$$

hence

$$-\frac{d}{dt}\|z\|_{\mathcal{H}^1_\varepsilon} \leq (\widetilde{\Lambda}_1 + \widetilde{\Theta}_1)\|z\|_{\mathcal{H}^1_\varepsilon} + \varepsilon e^{-(\widetilde{\Lambda}_1+\widetilde{\Theta}_1)t}. \tag{6.16}$$

Applying the Gronwall lemma to (6.16) between s and 0, $s \leq 0$, and using (6.1), we find

$$\|z(s)\|_{\mathcal{H}^1_\varepsilon} \leq (\|z(0)\|_{\mathcal{H}^1_\varepsilon} - c\varepsilon s)e^{-(\widetilde{\Lambda}_1+\widetilde{\Theta}_1)s}, \quad \text{for all } s \leq 0.$$

Noting that $\widetilde{\Lambda}_2 - \widetilde{\Lambda}_1 \geq 3\widetilde{\Theta}_1$, it follows from (6.15) that there exists $M_3 > 0$ such that

$$\|z(0)\|_{\mathcal{H}^1_\varepsilon} \leq \frac{1}{2}\|z(0)\|_{\mathcal{H}^1_\varepsilon} + \varepsilon M_3. \tag{6.17}$$

For any $0 < \varepsilon \leq \varepsilon_0$, we have

$$\|z(0)\|_{\mathcal{H}^1_1} \leq \frac{1}{\sqrt{\varepsilon}}\|z(0)\|_{\mathcal{H}^1_\varepsilon}. \tag{6.18}$$

Estimates (6.17) and (6.18) entail $\|z(0)\|_{\mathcal{H}^1_1} \leq M_4\sqrt{\varepsilon}$, that is,

$$\|(\rho_0, \mathrm{L}\rho_0) - w_\varepsilon(0)\|_{\mathcal{H}^1_1} \leq M_4\sqrt{\varepsilon}.$$

This implies the lower semicontinuity estimate (6.12). □

We now state and prove two propositions about the upper and lower semicontinuity.

Proposition 6.4. *Let the assumptions of Theorem* 5.5 *hold. Then the inertial manifolds* $\mathfrak{M}^{\widetilde{r}}_{\varepsilon,\delta} \cup (\mathfrak{M}^{\widetilde{r}}_{\delta})_0$ *are upper semicontinuous at* $\varepsilon = 0$ *with respect to the metric induced by the* $\mathcal{H}^1_1$*-norm, that is, for any* $\eta > 0$*, there exist* t_η *and* ε_η *such that, for all* $\varepsilon \leq \varepsilon_\eta$*,*

$$\mathrm{dist}_{\mathcal{H}^1_1}(S_\varepsilon(t_\eta)\mathfrak{M}^{\widetilde{r}}_{\varepsilon,\delta}, (\mathfrak{M}^{\widetilde{r}}_{\delta})_0) \leq \eta.$$

Proof. Let $(\rho_0,\rho_1)\in\mathfrak{M}^{\widetilde{r}}_{\varepsilon,\delta}$. Since $\mathfrak{M}^{\widetilde{r}}_{\varepsilon,\delta}\subset\widetilde{B}_{2,\delta}$, the following estimate holds (see (3.2)):

$$\|S_\varepsilon(t)(\rho_0,\rho_1)-S_0(t)(\rho_0,\mathrm{L}\rho_0)\|^2_{\mathcal{H}^1_1}\le c(r,t)\sqrt{\varepsilon},\quad\text{for all } t\ge t_\star.$$

As noticed above, $(\mathfrak{M}^{\widetilde{r}}_\delta)_0$ is an inertial manifold for $\{S_0(t)\}_{t\ge0}$ on $\widetilde{K}^1_\delta\cap\mathbb{M}$ and, therefore,

$$\mathrm{dist}_{\mathcal{H}^1_1}(S_0(t)(\rho_0,\mathrm{L}\rho_0),(\mathfrak{M}^{\widetilde{r}}_\delta)_0)\to0\quad\text{when } t\to\infty.$$

This shows that, if $\eta>0$, then there exist (φ,ϕ) belonging to $(\mathfrak{M}^{\widetilde{r}}_\delta)_0$ and $t_\eta\ge t_\star$ depending only on η such that

$$\|S_0(t_\eta)(\rho_0,\mathrm{L}\rho_0)-(\varphi,\phi)\|^2_{\mathcal{H}^1_1}\le\frac{\eta^2}{2}.\tag{6.19}$$

We now choose ε_η (which only depends on η) such that $c(r,t_\eta)\sqrt{\varepsilon_\eta}\le\eta^2/2$. For any $0<\varepsilon\le\varepsilon_\eta$, we have

$$\|S_\varepsilon(t_\eta)(\rho_0,\rho_1)-S_0(t_\eta)(\rho_0,\mathrm{L}\rho_0)\|^2_{\mathcal{H}^1_1}\le\frac{\eta^2}{2}.\tag{6.20}$$

We deduce from (6.19) and (6.20) that

$$\|S_\varepsilon(t_\eta)(\rho_0,\rho_1)-(\varphi,\phi)\|^2_{\mathcal{H}^1_1}\le\eta^2.$$

This result is uniform in (ρ_0,ρ_1) and we then obtain, for all $\varepsilon\le\varepsilon_\eta$,

$$\mathrm{dist}_{\mathcal{H}^1_1}(S_\varepsilon(t_\eta)\mathfrak{M}^{\widetilde{r}}_{\varepsilon,\delta},(\mathfrak{M}^{\widetilde{r}}_\delta)_0)\le\eta.$$

This completes the proof of the upper semicontinuity. □

Proposition 6.5. *Let the assumptions of Theorem* 5.5 *hold. Then the inertial manifolds* $\mathfrak{M}^{\widetilde{r}}_{\varepsilon,\delta}\cup(\mathfrak{M}^{\widetilde{r}}_\delta)_0$ *are lower semicontinuous at* $\varepsilon=0$ *with respect to the metric induced by the* $\mathcal{H}^1_1$*-norm, that is, for any* $\eta>0$*, there exist* t_η *and* ε_η *such that, for all* $\varepsilon\le\varepsilon_\eta$*,*

$$\mathrm{dist}_{\mathcal{H}^1_1}(S_0(t_\eta)(\mathfrak{M}^{\widetilde{r}}_\delta)_0,\mathfrak{M}^{\widetilde{r}}_{\varepsilon,\delta})\le\eta.$$

Proof. Let $(\rho_0,\mathrm{L}\rho_0)\in(\mathfrak{M}^{\widetilde{r}}_\delta)_0$. There exists a sequence $\{(\rho_\varepsilon,v_\varepsilon)\}_{\varepsilon>0}$ of solutions to problem (5.2) such that $(\rho_\varepsilon,v_\varepsilon)$ belongs to $\mathfrak{N}_{\varepsilon,\delta}\cap C_\varepsilon^{-1}\widetilde{B}_{2,\delta}$ and converges to $(\rho_0,\mathrm{L}\rho_0)$ in the $\mathcal{H}^1_1$-norm as ε goes to zero, due to Proposition 6.3. Set $u_\varepsilon=-2\varepsilon^{-1}\rho_\varepsilon+2\varepsilon^{-1/2}v_\varepsilon$. Since $(\rho_\varepsilon,v_\varepsilon)\in\mathfrak{N}_{\varepsilon,\delta}\cap C_\varepsilon^{-1}\widetilde{B}_{2,\delta}$, we have, by definition, $(\rho_\varepsilon,u_\varepsilon)\in\mathfrak{M}^{\widetilde{r}}_{\varepsilon,\delta}$. There exists a subsequence, still denoted by $\{(\rho_\varepsilon,u_\varepsilon)\}_{\varepsilon>0}$, which converges to some $(\varphi,u)\in(\mathfrak{M}^{\widetilde{r}}_\delta)_0$, due to Proposition 6.4. Clearly, $\varphi=\rho_0$ and $u=\mathrm{L}\rho_0$ and this limit is independent of the subsequence chosen. Consequently, the whole sequence $(\rho_\varepsilon,u_\varepsilon)_{\varepsilon>0}$ converges to $(\rho_0,\mathrm{L}\rho_0)$, hence the lower semicontinuity result. □

To conclude, we can state the following

Theorem 6.6. *Let the assumptions of Theorem* 5.5 *hold. Then the inertial manifolds* $\mathfrak{M}^{\widetilde{r}}_{\varepsilon,\delta}$ *converge to* $(\mathfrak{M}^{\widetilde{r}}_{\delta})_0$*, with respect to the metric induced by the* $\mathcal{H}^1_1$*-norm, when* ε *goes to zero in the sense of Propositions* 6.4 *and* 6.5.

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