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ON THE EXISTENCE OF PERIODIC SOLUTIONS FOR A CLASS OF NON-AUTONOMOUS DIFFERENTIAL DELAY EQUATIONS

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Abstract. This paper considers the existence of periodic solutions for a class of non-autonomous differential delay equations

$$x'(t) = -\sum_{i=1}^{n-1} f(t, x(t-i\tau)), \tag{$*$}$$

where $\tau > 0$ is a given constant. It is shown that under some conditions on f and by using symplectic transformations, Floquet theory and some results in critical point theory, the existence of single periodic solution of the differential delay equation $(*)$ is obtained. These results generalize previous results on the cases that the equations are autonomous.

1. Introduction and main results

In this paper, we study the existence of nontrivial 2π-periodic solutions for a class of non-autonomous differential delay equations of the following form:

$$x'(t) = -[f(t, x(t-\tau)) + f(t, x(t-2\tau)) + \ldots + f(t, x(t-(n-1)\tau))], \tag{1.1}$$

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where $f(t,x) \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ is odd with respect to x, $\tau > 0$ is a given constant and n is a positive integer.

Before introducing our assumptions on f and stating the main results, we first recall some earlier work on (1.1) with f independent of t, that is the following autonomous differential delay equation

$$x'(t) = -[f(x(t-\tau)) + f(x(t-2\tau)) + \ldots + f(x(t-(n-1)\tau))]. \tag{1.2}$$

The equation (1.2) with $n = 2$ and $\tau = 1$ arises from a variety of practical problems such as communication systems [9], population growth models [5], the operation of a control system working with potentially explosive chemical reactions [13], and economic studies of business cycles [2]. Thus many authors were attracted to consider various questions on (1.2) and there has been a great deal of research. To the best of our knowledge, the equation (1.2) was first considered by G. S. Jones in [13] on the existence of periodic solutions. Following the Jones's work, J. Kaplan, J. Yorke, R. D. Nussbaum, H. O. Walther, S. N. Chow, etc., studied the existence of periodic solutions, bifurcations, stability of periodic solutions, slowly oscillating, homoclinic solutions and a lot of remarkable results have been contributed in 1970s and 1980s of the last century (see [4], [10], [11], [14], [15], [23]–[27]). In 1990s of the last century, some authors [16], [17] made use of the original ideas in [14] to study multiple periodic solutions of (1.2). Specifically, they reduced the existence of periodic solutions of (1.2) to the existence of periodic solutions of an associated ordinary differential system. When $f(x) \in C(\mathbb{R}, \mathbb{R})$ is odd, $xf(x) > 0$ for $x \neq 0$ and $f(x)$ satisfies suitable conditions at 0 and ∞, they proved the existence of multiple periodic periodic solutions of the equation (1.2). Later, G. Fei [7], [8] continued the work done by X. He and J. Li in [16] and [17] at the beginning of this century. The author relaxed some conditions on $f(x)$ which were often employed in previous papers, such as $xf(x) > 0$ for $x \neq 0$ and by applying the pseudo index theory constructed in [6], Galerkin approximation method and S^1 index theory in [21], the author obtained the multiple periodic solutions of the equation (1.2). In common, the tools employed in [7], [8], [16], [17] are variational methods. For some other methods to study periodic solutions of (1.2), please see [12], [19]. Since the functionals used in [7], [8] are not S^1-invariant anymore for the non-autonomous equation (1.1), the methods used in [7], [8] can not be applied to study periodic solutions of (1.1). Some other methods are needed.

Motivated by the lack of results on the existence of periodic solutions for non-autonomous differential delay equations, we study in this paper the existence of periodic solutions of the non-autonomous equation (1.1). Now we give the following assumptions:

(H_1) $f(t,x) \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ is odd with respect to x and τ-periodic in t.

(H_2) $f(t,x) \in C^1(\mathbb{R}\times\mathbb{R},\mathbb{R})$ is odd with respect to x and τ-periodic in t.

(H_3) The following limits hold uniformly in $t \in [0,\tau]$,

$$\lim_{x\to 0}\frac{f(t,x)}{x} = \beta_0(t), \qquad \lim_{x\to\infty}\frac{f(t,x)}{x} = \beta_\infty(t).$$

It is obvious that $\beta_0(t)$ and $\beta_\infty(t)$ are two τ-periodic functions.

Since there is much difference between n being even and odd, in this paper we always assume that $n = 2N+1 \in \mathbb{Z}^+$ is odd and the constant $\tau = \pi/n$. In order to state our main results, we need to introduce a $4N\times 4N$ matrix defined below. For $k \in \mathbb{Z}^+$ and $\alpha \in \mathbb{R}$, we define

$$T_k(\alpha) = \begin{pmatrix} -\alpha M & -kA_{2N}^{-1} \\ kA_{2N}^{-1} & -\alpha M \end{pmatrix},$$

where M and A_{2N} are two $2N\times 2N$ matrices defined as follows:

$$M = \begin{pmatrix} 2 & -1 & 1 & -1 & \ldots & 1 & -1 \\ -1 & 2 & -1 & 1 & \ldots & -1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & -1 & 1 & -1 & \ldots & 2 & -1 \\ -1 & 1 & -1 & 1 & \ldots & -1 & 2 \end{pmatrix},$$

$$A_{2N} = \begin{pmatrix} 0 & -1 & \ldots & -1 & -1 \\ 1 & 0 & \ldots & -1 & -1 \\ 1 & 1 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & 0 & -1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

It is easy to check that M is positive definite and symmetric and A_{2N} is skew symmetric. Moreover, A_{2N} is a Hamiltonian matrix, i.e. it satisfies $JA_{2N} + A_{2N}^{\top}J = 0$, where $J = \left(\begin{smallmatrix} 0 & -I_{2N} \\ I_{2N} & 0 \end{smallmatrix}\right)$ is the standard symplectic matrix and I_{2N} is the identity matrix in $\mathbb{R}^{2N}$. Let $M^-(\,\cdot\,)$, $M^+(\,\cdot\,)$ and $M^0(\,\cdot\,)$ denote the negative, the positive and the zero Morse indices of the symmetric matrix define it, respectively. For the symmetric matrix αM, we define our index as

$$i^-(\alpha) = \sum_{k=1}^{\infty}(M^-(T_k(\alpha M)) - 2N) \quad \text{and} \quad i^0(\alpha) = \sum_{k=1}^{\infty} M^0(T_k(\alpha M)).$$

Noting that $T_k(\alpha M)$ is symmetric, its eigenvalues are all real. Since for k large enough, one has $M^-(T_k(\alpha M)) = 2N$ and $M^0(T_k(\alpha M)) = 0$. Hence, $i^-(\alpha M)$ and $i^0(\alpha M)$ are well defined.

Write $\alpha_0 = (1/\tau)\int_0^\tau \beta_0(t)\,dt$, $\alpha_\infty = (1/\tau)\int_0^\tau \beta_\infty(t)\,dt$. Then our main results read as follows.

Theorem 1.1. *Suppose that* (H_1) *and* (H_3) *hold. If* $i^0(\alpha_0 M) = i^0(\alpha_\infty M) = 0$ *and* $i^-(\alpha_0 M) \neq i^-(\alpha_\infty M)$*, then the equation* (1.1) *possesses at least one nontrivial* 2π*-periodic solution* $x(t)$ *satisfying* $x(t) = -x(t-\pi)$.

Theorem 1.2. *Suppose that* (H_2) *and* (H_3) *hold. If* $i^0(\alpha_\infty M) = 0$ *and* $i^-(\alpha_\infty M) \notin [i^-(\alpha_0 M), i^-(\alpha_0 M) + i^0(\alpha_0 M)]$*, then the equation* (1.1) *possesses at least one nontrivial* 2π*-periodic solution* $x(t)$ *satisfying* $x(t) = -x(t-\pi)$.

Remark 1.3. As we pointed out before, Theorems 1.1 and 1.2 are concerned with the existence of periodic solutions for the non-autonomous differential delay equation (1.1). Therefore, our results generalize the results gotten in the references.

Remark 1.4. In the sequel, the equation (1.1) was first changed to a form of Hamiltonian system. Thus, finding periodic solutions of (1.1) is equivalent to seeking periodic solutions of the Hamiltonian system. Periodic solutions of the Hamiltonian system are obtained as critical points of a functional ϕ defined on a Hilbert space E. We shall apply Galerkin approximation method and two well known critical point results to obtain periodic solutions of the non-autonomous differential delay equation (1.1).

The present paper is organized as follows. In Section 2, we transform (1.1) to a form of Hamiltonian system which is asymptotically linear both at 0 and ∞. In Section 3, we construct a symplectic transformation with respect to "A_{2N}" which reduces the linear parts of the Hamiltonian system to constant coefficients. Subsequently, in Section 4, we recall two critical point results while some useful lemmas are also given. Finally, the proofs of Theorems 1.1 and 1.2 will be carried out in Section 5.

2. An equivalent Hamiltonian system

In this section, we change (1.1) to an equivalent Hamiltonian system. We show that the main idea in [14], [17], [8] can be applied to seek for periodic solutions of the equation (1.1) for $n = 2N$. Precisely speaking, if a 2π-periodic solution $X(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ of the following system:

$$\frac{d}{dt}X(t) = A_n F(t, X(t)) \tag{2.1}$$

satisfies the following symmetric structure

$$x_1(t) = -x_n(t-\tau), x_2(t) = x_1(t-\tau), \ldots, x_n(t) = x_{n-1}(t-\tau), \tag{2.2}$$

then $x(t) = x_1(t)$ is a 2π-periodic solution of the equation (1.1) and satisfies $x(t - n\tau) = -x(t)$. Here $F(t, X) = (f(t, x_1), \ldots, f(t, x_n))^\top$ and $A_n = A_{2N}$ is defined in Section 1.

Since $n = 2N + 1$, the matrix A_{2N+1} is not a Hamiltonian matrix anymore. For each $y = (y_1, \dots, y_{2N})^\top \in \mathbb{R}^{2N}$, we define a Hamiltonian function as follows:

$$H(t, y) = \int_0^{y_1} f(t, x)\, dx + \dots + \int_0^{y_{2N}} f(t, x)\, dx + \int_0^{\sum_{j=1}^{N}(y_{2j} - y_{2j-1})} f(t, x)\, dx.$$

Then following the ideas in [17], the system (2.1) can be written as the following Hamiltonian system

$$y'(t) = A_{2N} \nabla_y H(t, y), \tag{2.3}$$

where $\nabla_y H(t, y)$ denotes the gradient of $H(t, y)$ with respect to y.

REMARK 2.3. Here we point out that the matrix A_{2N} is the symplectic structure associated to the Hamiltonian system (2.3), since according to [22] there is a non-degenerate matrix S such that $SA_{2N}S^\top = J$, where J is called the standard symplectic structure of a Hamiltonian system. In the following of the present paper, we only need to study the Hamiltonian system (2.3).

3. A symplectic transformation with respect to "A_{2N}"

In this section, we construct a symplectic transformation with respect to the symplectic structure "A_{2N}" which reduces coefficients of the linear parts of the Hamiltonian system (2.3) to constants. Since the coefficients of the linear parts at 0 and ∞ may be different, the Floquet theory can not be applied directly. We want to use Hamiltonian flow mapping to construct such global symplectic transformation. For this purpose, it is enough for us to construct the corresponding Hamiltonian function.

By the condition (H_3), one has

$$f(t, x) = \begin{cases} \beta_0(t)x + o(|x|) & \text{as } |x| \to 0, \\ \beta_\infty(t)x + o(|x|) & \text{as } |x| \to \infty. \end{cases}$$

Then $H(t, y)$ is even with respect to y and satisfies

$$\nabla_y H(t, y) = \begin{cases} \beta_0(t)My + o(|y|) & \text{as } |y| \to 0, \\ \beta_\infty(t)My + o(|y|) & \text{as } |y| \to \infty. \end{cases}$$

Where M is the symmetric definite matrix defined in Section 1. Thus, the corresponding Hamiltonian system (2.3) satisfies

$$y'(t) = A_{2N}M\beta_0(t)y + o(|y|) \quad \text{as } |y| \to 0, \tag{3.1}$$

$$y'(t) = A_{2N}M\beta_\infty(t)y + o(|y|) \quad \text{as } |y| \to \infty. \tag{3.2}$$

For the system (3.1), let $y = P_1(t, z) = e^{Q_1(t)}z$, where $Q_1(t) = A_{2N}M \int_0^t \gamma_0(\xi)\, d\xi$ and $\gamma_0(t) = \beta_0(t) - \alpha_0$. Then the transformation $y = P_1(t, z)$ is symplectic.

Then with the transformation $y = P_1(t,z)$, the system (3.1) is changed to the following system

$$z'(t) = \alpha_0 A_{2N} M z + o(|z|) \quad \text{as } |z| \to 0. \tag{3.3}$$

Similarly, let $y = P_2(t,z)$, where $P_2(t,z) = e^{Q_2(t)}z$, $Q_2(t) = A_{2N}M\int_0^t \gamma_\infty(\xi)d\xi$ and $\gamma_\infty(t) = \beta_\infty(t) - \alpha_\infty$. In the same way we have that the transformation $y = P_2(t,z)$ is also symplectic. Then the equation (3.2) can be transformed to the following system

$$z'(t) = \alpha_\infty A_{2N} M z + o(|z|) \quad \text{as } |z| \to \infty. \tag{3.4}$$

Noting that $Q_i(t) (i = 1,2)$ are τ-periodic, the functions $e^{-Q_i(t)} (i = 1,2)$ are bounded. Therefore, there are two positive constants $\overline{r}$ and $\overline{R}$ with $\overline{r} < \overline{R}$ such that

$$\overline{r}|y| \le |e^{-Q_i(t)}y| \le \overline{R}|y| \quad (i = 1,2).$$

We now manage to construct a global symplectic transformation $\Psi(t,z)$ such that

$$y = \Psi(t,z) = \begin{cases} P_1(t,z) & \text{if } |z| < \overline{r}/\overline{R}, \\ P_2(t,z) & \text{if } |z| > \overline{R}/\overline{r}. \end{cases}$$

Let $\Gamma_0(t) = \int_0^t \gamma_0(\xi)\,d\xi$ and $\Gamma_\infty(t) = \int_0^t \gamma_\infty(\xi)\,d\xi$. Then $\Gamma_0(t)$ and $\Gamma_\infty(t)$ are two τ-periodic functions. Let $\rho_1(w)$ and $\rho_2(w)$ be two smooth functions satisfying

$$\rho_1(|w|^2) = \begin{cases} 1 & \text{as } |w| < \overline{r}, \\ 0 & \text{as } |w| > \overline{R}, \end{cases} \qquad \rho_2(|w|^2) = \begin{cases} 0 & \text{as } |w| < \overline{r}, \\ 1 & \text{as } |w| > \overline{R}. \end{cases}$$

Set

$$\widetilde{H}_t(w) = \frac{1}{2}\Gamma_0(t)(Mw,w)\rho_1(|w|^2) + \frac{1}{2}\Gamma_\infty(t)(Mw,w)\rho_2(|w|^2),$$

where t is regarded as a parameter and $(\,\cdot\,,\,\cdot\,)$ denotes the inner product in $\mathbb{R}^{2N}$. It is easy to see that $\widetilde{H}_t(w)$ is a τ-periodic function with respect to the parameter t.

Now we consider the following Hamiltonian system

$$\frac{dw}{ds} = A_{2N}\nabla_w \widetilde{H}_t(w). \tag{3.5}$$

Note that the above Hamiltonian system (3.5) is autonomous with respect to s and t is regarded as a parameter. Let $\Psi_t(s,w)$ be the general solution of (3.5). By the uniqueness of solutions of ordinary differential equations, $\Psi_t(s,w)$ is τ-periodic with respect to the parameter. Now we prove that the flow mapping $\Psi_t(s,w)$ is symplectic with respect to the symplectic structure "A_{2N}", that is,

$$\frac{\partial \Psi_t(s,w)}{\partial w} A_{2N} \left(\frac{\partial \Psi_t(s,w)}{\partial w}\right)^{\top} = A_{2N}.$$

Since A_{2N} is skew symmetric and non-degenerate, according to [22], there is a non-degenerate matrix S such that

$$SA_{2N}S^{\top} = J,$$

where J is the standard symplectic matrix. Let $w = S^{-1}\widetilde{z}$. Then the system (3.5) is transformed to the following standard Hamiltonian system

$$\frac{d\widetilde{z}}{ds} = J\nabla_{\widetilde{z}}\widetilde{\widetilde{H}}_t(\widetilde{z}), \tag{3.6}$$

where $\widetilde{\widetilde{H}}_t(\widetilde{z}) = \widetilde{H}_t(S^{-1}\widetilde{z})$. Let $\widetilde{\Psi}_t(s,\widetilde{z}) = S\Psi_t(s, S^{-1}\widetilde{z})$. Then $\widetilde{\Psi}_t(s,\widetilde{z})$ is the general solution of the Hamiltonian system (3.6). Since the Jacobian of a flow mapping of a Hamiltonian system is symplectic, we have that

$$\frac{\partial\widetilde{\Psi}_t(s,\widetilde{z})}{\partial\widetilde{z}} J\left(\frac{\partial\widetilde{\Psi}_t(s,\widetilde{z})}{\partial\widetilde{z}}\right)^{\top} = J,$$

i.e.

$$\begin{aligned} S\frac{\partial\Psi_t}{\partial w}S^{-1}J\left(S\frac{\partial\Psi_t}{\partial w}S^{-1}\right)^{\top} = J &\Rightarrow S\frac{\partial\Psi_t}{\partial w}(S^{-1}J(S^{-1})^{\top})\left(\frac{\partial\Psi_t}{\partial w}\right)^{\top}S^{\top} = J \\ &\Rightarrow \frac{\partial\Psi_t}{\partial w}A_{2N}\left(\frac{\partial\Psi_t}{\partial w}\right)^{\top} = A_{2N}. \end{aligned}$$

Let $y = \Psi(t,z) = \Psi_t(s,w)|_{s=1,w=z} = \Psi_t(1,z)$. Then one has

$$\frac{\partial\Psi}{\partial z}A_{2N}\left(\frac{\partial\Psi}{\partial z}\right)^{\top} = A_{2N}.$$

For $|w| < \overline{r}$, the system (3.5) becomes $dw/ds = \Gamma_0(t)A_{2N}Mw$. So $w(s) = e^{\Gamma_0(t)A_{2N}Ms}z$. If $|s| < 1$, $|z| < \overline{r}/\overline{R}$, then $|w(s)| \le |e^{\Gamma_0(t)A_{2N}Ms}||z| \le \overline{R}|z| < \overline{r}$. Thus for $|s| < 1, |z| < \overline{r}/\overline{R}$, we have $w(s) = e^{\Gamma_0(t)A_{2N}Ms}z$. Therefore $\Psi(t,z) = P_1(t,z)$ as $|z| < \overline{r}/\overline{R}$. In the same way we can prove that $\Psi(t,z) = P_2(t,z)$ as $|z| > \overline{R}/\overline{r}$.

Let $z = \Phi(t,y)$ be the inverse mapping of the transformation $y = \Psi(t,z)$. Then the system (2.3) is changed to the following system

$$\frac{dz}{dt} = A_{2N}\nabla_z\widehat{H}(t,z) + \frac{\partial\Phi}{\partial t}, \tag{3.7}$$

where $\widehat{H}(t,z) = H(t,\Psi(t,z))$. By a direct computation, $A_{2N}^{-1}(\partial^2\Phi/\partial t\partial y)$ is a symmetric matrix. According to [22], there is a smooth function R such that

$$A_{2N}\nabla_z R = \frac{\partial\Phi}{\partial t}.$$

Then $H^*(t,z) = \widehat{H}(t,z) + R(t,z)$ is the Hamiltonian function of the system (3.7) and the Hamiltonian system (3.7) can be written as the following form:

$$\frac{dz}{dt} = A_{2N}\nabla_z H^*(t,z). \tag{3.8}$$

By (3.3) and (3.4), $H^*(t,z)$ satisfies the following asymptotically linear properties.

$$\nabla_z H^*(t,z) = \alpha_0 Mz + o(|z|) \quad as|z| \to 0, \tag{3.9}$$

$$\nabla_z H^*(t,z) = \alpha_\infty Mz + o(|z|) \quad \text{as } |z| \to \infty. \tag{3.10}$$

Now in order to prove the main results of this paper, we only need to consider the Hamiltonian system (3.8). It is enough for us to prove the following two theorems.

THEOREM 3.1. *Under the conditions of Theorem* 1.1, *the system* (3.8) *possesses at least one nontrivial* 2π*-periodic solution* x *with* $x(t) = -x(t-\pi)$.

THEOREM 3.2. *Under the conditions of Theorem* 1.2, *the system* (3.8) *possesses at least one nontrivial* 2π*-periodic solution* x *with* $x(t) = -x(t-\pi)$.

4. Two critical point theorems and some lemmas

In this section, let $E = W^{1/2,2}(S^1,\mathbb{R}^{2N})$. Then E is a Hilbert space. Denote the inner product and the norm in E by $\langle\,\cdot\,,\,\cdot\,\rangle$ and $\|\cdot\|$, respectively. A better way to understand this space seems as follows. The space E consists of all $z(t)$ in $L^2(S^1,\mathbb{R}^{2N})$ whose Fourier series

$$z(t) = a_0 + \sum_{k=1}^{\infty}(a_k\cos kt + b_k\sin kt)$$

satisfies

$$|a_0|^2 + \sum_{k=1}^{\infty}k(|a_k|^2 + |b_k|^2) < \infty,$$

where $a_0, a_k, b_k \in \mathbb{R}^{2N}$. The inner product on E is defined by

$$\langle z_1,z_2\rangle = (a_0^1,a_0^2) + \sum_{k=1}^{\infty}k[(a_k^1,a_k^2) + (b_k^1,b_k^2)],$$

where

$$z_i = a_0^i + \sum_{k=1}^{\infty}(a_k^i\cos kt + b_k^i\sin kt) \quad (i=1,2).$$

Observe that A_{2N}^{-1} is also a nonsingular skew symmetric. For each $z,y\in E$, we define an operator A on E by extending the bilinear form

$$\langle Az,y\rangle = \int_0^{2\pi}(A_{2N}^{-1}z'(t),y(t))\,dt.$$

It is not difficult to check that the operator A is a bounded self-adjoint linear operator on E. For any $z \in E$, a direct computation yields that

$$Az = \sum_{k=1}^{\infty} (A_{2N}^{-1} b_k \cos kt - A_{2N}^{-1} a_k \sin kt).$$

For each $z \in E$ we define a functional ϕ on E by

$$\phi(z) = \frac{1}{2} \langle Az(t), z(t) \rangle - \int_0^{2\pi} H^*(t, z(t))\, dt.$$

It is well known that critical points of ϕ are solutions of the system (3.8). Hence, finding periodic solutions of the system (3.8) is equivalent to seeking critical points of ϕ.

For any $\alpha \in \mathbb{R}$ and $z, y \in E$, we define another operator B_α on E by

$$\langle B_\alpha z, y \rangle = - \int_0^{2\pi} (\alpha M z(t), y(t))\, dt. \tag{4.1}$$

Then B_α is also a bounded self-adjoint linear operator on E. Moreover, B_α is compact and from a direct check, we have for any $z \in E$,

$$B_\alpha z = -\alpha M a_0 + \sum_{k=1}^{\infty} \frac{1}{k} (-\alpha M a_k \cos kt - \alpha M b_k \sin kt). \tag{4.2}$$

Combining (4.1) and (4.2), for any $z(t) \in E$, one has

$$\begin{aligned} (A + B_\alpha) z(t) &= -\alpha M a_0 \\ &+ \sum_{k=1}^{\infty} \left(\left(A_{2N}^{-1} b_k - \frac{1}{k} \alpha M a_k \right) \cos kt + \left(- A_{2N}^{-1} a_k - \frac{1}{k} \alpha M b_k \right) \sin kt \right). \end{aligned} \tag{4.3}$$

Lemma 4.1. *Suppose that* (H_1) *and* (H_3) *hold. Then the functional* ϕ *satisfies*

$$\|\phi'(z) - (A + B_{\alpha_0}) z\| = o(\|z\|) \qquad \text{as } \|z\| \to 0, \tag{4.4}$$

$$\|\phi'(z) - (A + B_{\alpha_\infty}) z\| = o(\|z\|) \qquad \text{as } \|z\| \to \infty. \tag{4.5}$$

Proof. From (3.9), for any $\widetilde{r} > 0$, there exists a constant $C(\widetilde{r})$ (here and in the following C denotes various constants) such that

$$|\nabla_x H^*(t, x) - \alpha_0 M x| \le \widetilde{r} |x| + C(\widetilde{r}) |x|^2, \quad \text{for each } x \in \mathbb{R}^{2N}.$$

Note that

$$\phi(z) = \frac{1}{2} \langle (A + B_{\alpha_0}) z(t), z(t) \rangle - \int_0^{2\pi} \Big(H^*(t, z(t)) - \frac{1}{2} (\alpha_0 M z(t), z(t))\Big)\, dt.$$

We have

$$\|\phi'(z) - (A + B_{\alpha_0}) z\| \le C \|\nabla_z H^*(t, z) - \alpha_0 M z\| \le C(\widetilde{r} \|z\| + C(\widetilde{r}) \|z\|^2),$$

which means (4.4). By (3.10), we have

$$|\nabla_x H^*(t,x) - \alpha_\infty Mx| \le \widetilde{r}|x| + C(\widetilde{r}), \quad \text{for each } x \in \mathbb{R}^{2N}.$$

Observing that

$$\phi(z) = \frac{1}{2}\langle (A + B_{\alpha_\infty})z(t), z(t)\rangle - \int_0^{2\pi} \Big(H^*(t,z(t)) - \frac{1}{2}(\alpha_\infty Mz(t), z(t))\Big)\,dt,$$

one has

$$\|\phi'(z) - (A+B_{\alpha_\infty})z\| \le C\|\nabla_z H^*(t,z) - \alpha_\infty Mz\| \le C(\widetilde{r}\|z\| + C(\widetilde{r})).$$

Thus, we get (4.5). □

In order to obtain solutions of (3.8) with the symmetric structure (2.2), we define the following $2N \times 2N$ matrix T_{2N} by

$$T_{2N} = \begin{pmatrix} 1 & -1 & \dots & 1 & -1 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

For any $z(t) \in E$, define an action on z by

$$\delta z(t) = T_{2N} z(t-\tau).$$

Then by a direct computation we have that $\delta^{2N+1}z(t) = -z(t-(2N+1)\tau)$, $\delta^{4N+2}z(t) = z(t)$ and $G = \{\delta, \delta^2, \dots, \delta^{4N+2}\}$ is a compact group action over E. It is not difficult to see that if $\delta z(t) = z(t)$ holds, then $z(t)$ has the symmetric structure (2.2).

We claim that a solution y of (2.3) also has the symmetric structure (2.2) when $\delta z(t) = z(t)$ holds, where z is a solution of (3.8). Hence, the solution y gives a solution to (1.1). By $y = \Psi(t,z)$, we only need to show that $\Psi(t,\delta z) = \delta\Psi(t,z)$, i.e. Ψ is G-invariant with respect to z. In fact, note that $A_{2N}M$ can commute with T_{2N} and T_{2N} is isometric, i.e. $|T_{2N}w|^2 = |w|^2$. We set $\widetilde{z}(s) = \Psi_t(s, T_{2N}w)$, $\widetilde{y}(s) = \Psi_t(s,w)$. Then

$$\begin{aligned} \frac{d\widetilde{z}(s)}{ds} &= A_{2N}\{\Gamma_0(t)MT_{2N}w\rho_1(|w|^2) + \Gamma_0(t)\langle Mw,w\rangle\rho_1'(|w|^2)T_{2N}w \\ &\quad + \Gamma_\infty(t)MT_{2N}w\rho_2(|w|^2) + \Gamma_\infty(t)\langle Mw,w\rangle\rho_2'(|w|^2)T_{2N}w\} \\ &= T_{2N}A_{2N}\{\Gamma_0(t)Mw\rho_1(|w|^2) + \Gamma_0(t)\langle Mw,w\rangle\rho_1'(|w|^2)w \\ &\quad + \Gamma_\infty(t)Mw\rho_2(|w|^2) + \Gamma_\infty(t)\langle Mw,w\rangle\rho_2'(|w|^2)w\} = T_{2N}\frac{d\widetilde{y}(s)}{ds}, \end{aligned}$$

this, jointly with $\Psi_t(0, T_{2N}w) = T_{2N}w = T_{2N}\Psi_t(0,w)$ shows that $T_{2N}\Psi_t(s,w) = \Psi_t(s, T_{2N}w)$, i.e. $\Psi(t,\delta z) = \delta\Psi(t,z)$.

Set $SE = \{z \in E : \delta z(t) = z(t)\}$ and write $\beta_k = 2k - 1$. Then by a similar proof with Lemma 2.1 of [8], one has:

$$SE = \bigg\{z(t) = \sum_{k=1}(a_k \cos(2k-1)t + b_k \sin(2k-1)t) : \\ \begin{pmatrix} a_k \\ b_k \end{pmatrix} \in \operatorname{span}\bigg\{\begin{pmatrix} u_k \\ w_k \end{pmatrix}, \begin{pmatrix} -w_k \\ u_k \end{pmatrix}\bigg\}\bigg\},$$

where

$$\begin{aligned} u_k &= (1, \cos\beta_k, \cos(2\beta_k), \dots, \cos((2N-1)\beta_k))^T, \\ w_k &= (0, \sin\beta_k, \sin(2\beta_k), \dots, \sin((2N-1)\beta_k))^T. \end{aligned}$$

Let $\Omega = \{P_m : m = 1, 2, \dots\}$ be a sequence of orthogonal projections. Ω is called a Galerkin approximation scheme with respect to the operator $A + B_\alpha$, if it satisfies the following properties:

(1) The image of P_m, as a subspace of E, has finite dimension;
(2) $P_m z \to z$ as $m \to \infty$ for any $z \in E$;
(3) P_m commutes with $A + B_\alpha$.

We now define a subspace E_m of E by

$$E_m = \bigg\{z(t) : z(t) = a_0 + \sum_{k=1}^{m}(a_k \cos kt + b_k \sin kt)\bigg\}.$$

Let $SE_m = E_m \cap SE$ and $P_m\colon E \to SE_m$ be the orthogonal projection. One may check easily that P_m satisfies the above properties (1)–(3). Therefore, $\Omega = \{P_m : m = 1, 2, \dots\}$ is a Galerkin approximation method with respect to $A + B_\alpha$.

Let A_m and $\phi_m$be the restrictions of A and ϕ on SE_m, respectively. We have the following important lemma.

Lemma 4.2. *Let* (H_1) *and* (H_2) *hold. If* $i^0(\alpha_\infty M) = 0$, *then*

(a) ϕ *satisfies* $(\mathrm{PS})^*$ *condition over* E, *i.e. every sequence* $\{z_j\} \subset E$ *with* $z_j \in SE_j$, $\phi_j(z_j) \to 0$ *and* $\phi_j(z_j)$ *being bounded, possesses a convergent subsequence.*
(b) ϕ_j *satisfies* (PS) *condition, i.e. every sequence* $\{z_i\} \subset SE_j$, $\phi_j(z_i) \to 0$ *and* $\phi_j(z_i)$ *being bounded, possesses a convergent subsequence.*

Proof. Note that $i^0(\alpha_\infty M) = 0$ yields that $A + B_{\alpha_\infty}$ has a bounded inverse. Then the proof is standard. □

Observe that our functional ϕ_m acts on the finite dimensional space SE_m. The following two theorems were well known results in critical point theory(see [1], [3], [18]) and will be used in our arguments.

Theorem 4.3. *Let ϕ_m be a C^1 function satisfying* (4.4)–(4.5). *If*

$$M^0(A_m + P_m B_{\alpha_0} P_m) = M^0(A_m + P_m B_{\alpha_\infty} P_m) = 0$$

and $M^-(A_m + P_m B_{\alpha_0} P_m) \neq M^-(A_m + P_m B_{\alpha_\infty} P_m)$, then ϕ_m has at least one nontrivial critical point.

Theorem 4.4. *Let ϕ_m be a C^2 function satisfying* (4.4)–(4.5). *If*

$$M^0(A_m + P_m B_{\alpha_\infty} P_m) = 0$$

and $M^-(A_m + P_m B_{\alpha_\infty} P_m) \notin [M^-(A_m + P_m B_{\alpha_0} P_m), M^-(A_m + P_m B_{\alpha_0} P_m) + M^0(A_m + P_m B_{\alpha_0} P_m)]$, then ϕ_m has at least one nontrivial critical point.

Notice that $i^0(\alpha M) = 0$ means that $A + B_\alpha$ is invertible, A and B are both linear self-adjoint operators. Moreover, B is compact. Then with the same proof as Lemma 2.3 of [18], we have the following lemma.

Lemma 4.5. *If $i^0(\alpha M) = 0$, then*

$$M^-(A_m + P_m B_\alpha P_m) - M^-(A_m) = M^-(A_k + P_k B_\alpha P_k) - M^-(A_k)$$

for m, k large enough.

For the linear compact operator B_α, we define

$$\begin{aligned} I^-(B_\alpha) &= \{k = M^-(A_m + P_m B_\alpha P_m) - M^-(A_m), \text{ for infinitely many } m\}, \\ I^0(B_\alpha) &= \{k = M^0(A_m + P_m B_\alpha P_m), \text{ for infinitely many } m\}. \end{aligned}$$

Let

$$\begin{aligned} I^-(\phi, \infty) &= I^-(B_{\alpha_\infty}), \quad I^-(\phi, 0) = I^-(B_{\alpha_0}); \\ I^0(\phi, \infty) &= I^0(B_{\alpha_\infty}), \quad I^0(\phi, 0) = I^0(B_{\alpha_0}). \end{aligned}$$

By Lemma 4.5, the indices above are well defined. Now we prove the following lemma.

Lemma 4.6. *Let $\phi|_{SE}$ be the restriction of ϕ over SE. Then with respect to the approximation scheme Ω, one has*

$$\begin{aligned} I^-(\phi|_{SE}, 0) &= \sum_{k=1}^{\infty} (M^-(T_k(\alpha_0 M)) - 2N) = i^-(\alpha_0 M), \\ I^-(\phi|_{SE}, \infty) &= \sum_{k=1}^{\infty} (M^-(T_k(\alpha_\infty M)) - 2N) = i^-(\alpha_\infty M), \\ I^0(\phi|_{SE}, 0) &= \sum_{k=1}^{\infty} M^0(T_k(\alpha_0 M)) = i^0(\alpha_0 M). \end{aligned}$$

Proof. Note that

$$SE_m = SE \cap E_m = \left\{ z(t) : z(t) = \sum_{k=1}^{m} (a_k \cos(2k-1)t + b_k \sin(2k-1)t) \right\}.$$

For $k \geq 1$, set

$$SE(k) = \{z(t) = a_k \cos(2k-1)t + b_k \sin(2k-1)t\}.$$

Then $SE_m = SE(1) \oplus SE(2) \ldots \oplus SE(m)$ and $SE = \bigoplus_{k=1}^{\infty} SE_k$. It follows from the definition of the negative Morse index of $T_k(\alpha_0 M)$ and (4.3) that the dimension of the negative eigenspace of the operator $A_m + P_m B_{\alpha_0} P_m$ on $SE_m = \bigoplus_{j=1}^{m} SE(j)$ is equal to $\sum_{k=1}^{m}(M^-(T_k(\alpha_0 M))$. For $\alpha_0 = 0$, $M^-(T_k(\alpha_0 M) = 2N$. Therefore, the formula

$$I^-(\phi|_{SE}, 0) = \sum_{k=1}^{\infty} (M^-(T_k(\alpha_0 M)) - 2N) = i^-(\alpha_0 M)$$

holds. The other formulas hold similarly. □

5. Proof of the main results

We are now ready to give the proofs of our results. We first prove Theorem 3.1.

Proof of Theorem 3.1. As we already pointed above, critical points of ϕ in SE are indeed nonconstant classic 2π-periodic solutions of (2.3) with the symmetric structure (2.2), and hence they give solutions of (1.1) with the property $x(t-(2N+1)\tau) = x(t-\pi) = -x(t)$. Therefore, we will seek critical points of ϕ in SE, i.e. critical points of $\phi|_{SE}$.

Set $g_\infty(z) = \phi|_{SE}(z) - (1/2)\langle (A + B_{\alpha_\infty})z, z\rangle$. Under the assumptions of Theorem 3.1 and by (4.5), for $\varepsilon = (1/2)\|(A+B_{\alpha_\infty})^{-1}\|^{-1}$, there is a large $R > 0$ such that

$$\|g'_\infty(z)\| < \varepsilon\|z\| \quad \text{for } \|z\| > R.$$

That yields

$$\|\phi'|_{SE}(z)\| \geq \|(A + B_{\alpha_\infty})^{-1}\|^{-1}\|z\| - \|g_\infty(z)\| > \varepsilon\|z\| \quad \text{for } \|z\| > R.$$

This means $\phi|_{SE}$ has no critical points outside the ball

$$B_R = \{z \in SE : \|z\| < R\}.$$

Let $g_0(z) = \phi|_{SE}(z) - (1/2)\langle (A+B_{\alpha_0})z, z\rangle$. Since $A + B_{\alpha_0}$ has bounded inverse and $\|(I - P_m)B_{\alpha_0}\| \to 0$ as $m \to 0$, there exists a constant C such that $\|(A_m + P_m B_{\alpha_0} P_m)^{-1}\| < C$. By (4.4), we can take r small enough with $r < R$ such that

$$\|g'_0(z)\| < \frac{1}{2C}\|z\| \quad \text{for } \|z\| < r.$$

Thus, for m large enough and for $z \in SE_m$ with $\|z\| < r$, one has

$$\begin{aligned}\|\phi'|_{SE}(z) - (A_m + P_m B_{\alpha_0} P_m)(z)\| &\leq \|g_0'(z)\| < \frac{1}{2C}\|z\| \\ &< \frac{1}{2}\|(A_m + P_m B_{\alpha_0} P_m)^{-1}\|^{-1}\|z\|.\end{aligned}$$

From the above inequality, we get

$$\begin{aligned}\|\phi'|_{SE}(z)\| &> \|(A_m + P_m B_{\alpha_0} P_m)(z)\| - \frac{1}{2}\|(A_m + P_m B_{\alpha_0} P_m)^{-1}\|^{-1}\|z\| \\ &> \|(A_m + P_m B_{\alpha_0} P_m)^{-1}\|^{-1}\|z\| - \frac{1}{2}\|(A_m + P_m B_{\alpha_0} P_m)^{-1}\|^{-1}\|z\| \\ &= \frac{1}{2}\|(A_m + P_m B_{\alpha_0} P_m)^{-1}\|^{-1}\|z\|.\end{aligned}$$

Therefore, 0 is the unique critical point $\phi|_{SE}$ inside the ball $B_r = \{z \in SE : \|z\| < r\}$.

By Lemma 4.6, $M^-(A_m + P_m B_{\alpha_0} P_m) \neq M^-(A_m + P_m B_{\alpha_\infty} P_m)$. Then Lemmas 4.1 and 4.3 yield ϕ_m has a nontrivial critical point z_m inside the annular area $\Theta = \{z : r < \|z_m\| < R\}$. By Lemma 4.2, $\phi|_{SE}$ satisfies the (PS)* condition. Hence, $\{z_m\}$ has a subsequence convergent to a point z, which is just a critical point of $\phi|_{SE}$ inside $\Theta = \{z : r < \|z\| < R\}$. □

Proof of Theorem 3.2. Let $(A+B_{\alpha_0})|_{SE}$ denote the restriction of $A+B_{\alpha_0}$ over SE. Noting that $M^0(T_m(\alpha_0)) = 0$ for m large enough, the null space of $(A + B_{\alpha_0})|_{SE}$ can be included in SE_m. Thus, there is a constant C such that

$$\|(A_m + P_m B_{\alpha_0} P_m)^\sharp\| < C \quad \text{for } m \text{ large enough},$$

where $(A_m + P_m B_{\alpha_0} P_m)^\sharp$ denotes the inverse of $A_m + P_m B_{\alpha_0} P_m$ restricted in the range of $A_m + P_m B_{\alpha_0} P_m$. By (4.4) we can take a small r such that

$$\|\phi_m''(z) - (A + B_{\alpha_0})\| \leq \frac{1}{2C} \quad \text{as } \|z\| < r.$$

As $\|z\| < 2r$ one has

$$\begin{aligned}\|\phi_m''(z) - (A_m + P_m B_{\alpha_0} P_m)\| &\leq \|\phi_m''(z) - (A + B_{\alpha_0})\| \\ &\leq \frac{1}{2C} < \frac{1}{2}\|(A_m + P_m B_{\alpha_0} P_m)^\sharp\|^{-1}.\end{aligned}$$

Then by a similar argument with [4, Theorem 1.3], we have the fact that if ϕ_m has critical points, then at least one of them is outside B_r. By Lemma 4.6, we have for m large enough

$$\begin{aligned}&M^-(A_m + P_m B_{\alpha_\infty} P_m) \\ &\notin [M^-(A_m + P_m B_{\alpha_0} P_m), M^-(A_m + P_m B_{\alpha_0} P_m) + M^0(A_m + P_m B_{\alpha_0} P_m)].\end{aligned}$$

Hence, Lemma 4.4, jointly with Lemma 4.1 and the proof of Theorem 3.1 yields that ϕ_m has a nontrivial critical point z_m inside the annular area $\Theta = \{z : r <$

$\|z_m\| < R\}$. By Lemma 4.2, $\phi|_{SE}$ satisfies the (PS)* condition. Therefore, $\{z_m\}$ has a subsequence convergent to a point z, which is just a critical point of $\phi|_{SE}$ inside $\Theta = \{z : r < \|z\| < R\}$. □

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