Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 35, 2010, 99–126

INDEX AT INFINITY AND BIFURCATIONS OF TWICE DEGENERATE VECTOR FIELDS

Alexander Krasnosel'skiĭ

(*Submitted by J. Mawhin*)

Abstract. We present a method to study twice degenerate at infinity asymptotically linear vector fields, i.e. the fields with degenerate principal linear parts and next order bounded terms. The main features of the method are sharp asymptotic expansions for projections of nonlinearities onto the kernel of the linear part. The method includes theorems in abstract Banach spaces, the expansions which are the main assumptions of these abstract theorems, and lemmas on the exact form of the expansions for generic functional nonlinearities with saturation. The method leads to several new results on solvability and bifurcations for various classic BVPs.

If the leading terms in the expansions are of order 0, then solvability conditions (and conditions for the index at infinity to be non-zero) coincide with Landesman–Lazer conditions, traditional for the BVP theory. If the terms of order 0 vanish (the Landesman–Lazer conditions fail), then it is necessary to determine and to take into account nonlinearities that are smaller at infinity. The presented method uses such nonlinearities and makes it possible to obtain the expansions with the leading terms of arbitrary possible orders.

The method is applicable if the linear part has simple degeneration, if the corresponding eigenfunction vanishes, and if the small nonlinearities decrease at infinity sufficiently fast. The Dirichlet BVP for a second order ODE is the main model example, scalar and vector cases being considered separately. Other applications (the Dirichlet problem for the Laplace PDE and the Neumann problem for the second order ODE) are given rather schematically.

2000 *Mathematics Subject Classification.* Primary: 34B15, 34C23; Secondary: 47H11.

Key words and phrases. Operator equations, index at infinity, bifurcations at infinity, degenerate vector fields, degree theory, asymptotic expansions, Dirichlet problem.

The author was supported by the Russian Foundation for Basic Researches, Grants 06-01-00256 and 06-01-72552.

1. Introduction

We study various equations and vector fields that contain nonlinear superposition operators $x(t) \mapsto \mathfrak{f}x = f(t, x(t))$ acting in functional spaces of scalar-valued functions $x: \Omega \to \mathbb{R}$, the main case being $\Omega = [0, T]$. The functions f are always bounded and continuous w.r.t. all the variables and always are of the form

$$f(t, x) = b(t) + a(t)\,\mathrm{sign}(x) + g(t, x), \quad t \in \Omega,\ x \in \mathbb{R}. \tag{1.1}$$

The nonlinearity g in (1.1) is asymptotically small:

$$\lim_{|x|\to\infty} \sup_{t\in\Omega} |g(t, x)| = 0$$

and has a jump at zero to compensate the jump of the function $\mathrm{sign}(\,\cdot\,)$. Any function f satisfying the saturation condition ([1])

$$\lim_{x\to\pm\infty} \sup_{t\in\Omega} |f(t, x) - f^{\pm}(t)| = 0 \tag{1.2}$$

admits representation (1.1) for

$$b(t) = \frac{f^+(t) + f^-(t)}{2}, \qquad a(t) = \frac{f^+(t) - f^-(t)}{2},$$
$$g(t, x) = f(t, x) - b(t) - a(t)\,\mathrm{sign}(x).$$

Every function (1.1) obviously satisfies (1.2) for $f^{\pm}(t) = b(t) \pm a(t)$.

Consider in $L^2 = L^2(\Omega, \mathbb{R})$ a completely continuous linear operator A, it generates the Hammerstein vector field

$$\Phi x = x - A(x + \mathfrak{f}x). \tag{1.3}$$

Since $\mathfrak{f}$ is continuous, we see that Φ is completely continuous.

In the non-degenerate case ([2]) $1 \notin \sigma(A)$, the principal linear part $I - A$ of the field Φ is continuously invertible, the set $\{x \in L^2 : \Phi x = 0\}$ is bounded, therefore the index at infinity ([3]) $\mathrm{ind}_\infty \Phi$ of the field Φ is well defined. It may be calculated by the formula $\mathrm{ind}_\infty \Phi = (-1)^\beta$, where β is the sum of multiplicities of all real eigenvalues of the operator A that are greater than 1.

Throughout the paper we investigate the degenerate case $1 \in \sigma(A)$.

([1]) To eliminate at least some cumbersome formulas we use the symbols $\pm$ and $\mp$. If we do not specify the opposite, the presence of these symbols (usually in several places in the same formula) means the correctness of two formulas that appear if we choose either all upper or all lower symbols at all places. In particular, one formula (1.2) means two relations, one for the upper symbol "+" and another for the lower symbol "−".

([2]) $\sigma(A)$ denotes the spectrum of the operator A.

([3]) The index at infinity [11] is the common (independent from r) rotation of the field Φ on spheres $\{\|x\| = r\}$ for sufficiently large r. The index is well defined if and only if the set of all singular points of the field Φ is bounded.

Let the eigenvalue 1 be simple, $e = Ae$, $\|e\| = 1$, and $\mathrm{mes}\{t \in \Omega : e(t) = 0\} = 0$. Let $A^*e^* = e^*$, $\langle e, e^* \rangle = 1$, where A^* is the adjoint operator. Consider the values

$$\mu^{\pm} = \int_{\Omega} e^*(t)(b(t) \pm a(t)\,\mathrm{sign}(e(t)))\,dt, \tag{1.4}$$

they appear in the projections $\mu^{\pm}e(t)$ of the terms $b(t) \pm a(t)\,\mathrm{sign}(e(t))$ onto the one dimensional kernel of $I - A$. If $\mu^{\pm} \neq 0$, then $\mathrm{ind}_{\infty}\Phi$ is well defined and

$$\mathrm{ind}_{\infty}\Phi = \frac{\mathrm{sign}(\mu^+) - \mathrm{sign}(\mu^-)}{2}(-1)^{\beta} = \begin{cases} 0 & \text{if } \mu^-\mu^+ > 0, \\ (-1)^{\beta}\mathrm{sign}(\mu^+) & \text{if } \mu^-\mu^+ < 0. \end{cases} \tag{1.5}$$

This statement was used by various authors in more or less explicit form (to study solvability of degenerate BVPs, see, e.g. [2], [3], [13]; in [6] the problem on the index at infinity computation is directly investigated), starting from the famous paper [12]. The assumption $\mu^-\mu^+ < 0$ for concrete problems is often written in the form $\mu^- < 0 < \mu^+$ (or $\mu^+ < 0 < \mu^-$) and is called the Landesman–Laser condition.

Under assumption $\mu^{\pm} \neq 0$ the field Φ is once degenerate: its principal linear part is degenerate (i.e. $1 \in \sigma(A)$), the remaining part $A\mathfrak{f}x$ contains nondegenerate leading term of order 0. This paper is devoted to the case where at least one of the numbers $\mu^{\pm}$ equals zero. Such cases are twice degenerate: the linear part is degenerate as well as the projections $\mu^{\pm}e(t)$ of the leading homogeneous term $b(t) \pm a(t)\,\mathrm{sign}(e(t))$.

The study of twice degenerate problems for $a(t) \equiv 0$ was started in [2], [5], see also [6] and references therein; it was continued for the case $a(t) \not\equiv 0$ in [9]. There the main restriction on the function g is of the form

$$\text{either } \mathrm{sign}(x)g(t,x) \geq \varphi(|x|) \text{ or } \mathrm{sign}(x)g(t,x) \leq -\varphi(|x|), \ |x| \geq x_0; \tag{1.6}$$

it is supplemented with the estimate from below of the rate of decreasing at infinity of the positive monotone function φ:

$$\int^{\infty} u^{\theta}\varphi(u)\,du = \infty. \tag{1.7}$$

The number θ is defined by the asymptotics at zero of the permutations [4] of the function $|e|$, for the Dirichlet BVP, analyzed below, $\theta = 1$.

In [6], [9] the estimates at 0 of the function $\chi(\delta) = \mathrm{mes}\{t \in \Omega : |e(t)| \leq \delta\}$ are used. If $\Omega = [0,T]$, then $\chi(\delta)$ is inverse to the equimeasurable permutation of $|e|$. In the present paper we use the behavior of e in the vicinity of any its zero separately instead of the total estimates of $\chi(\delta)$. The main assumption has the opposite to (1.7) form (see (3.3), (3.4)), any analogous to (1.6) assumptions are not used. As a result we obtain sharp expansions for projections of nonlinearities instead of estimates. Formulas for the index at infinity have an explicit form and are defined by some integrals; the formulas are sharp: the signs of these integrals define the value of the index. In a certain sense the results of this paper cover the most of unknown cases.

Integrals of the form

$$G_r = \int_0^T q(t) g(t, re(t) + h^*(t) + \widetilde{h}(t))\, dt$$

and

$$S_r = \int_0^T q(t)(\operatorname{sign}(re(t) + h^*(t) + \widetilde{h}(t)) - \operatorname{sign}(e(t)))\, dt$$

and their expansions play a determinative role in several applications. The main new technical observation is the possibility to compute exact uniform w.r.t. $\widetilde{h} \to 0$ asymptotics of G_r and S_r as $r \to \infty$. It turns out that the asymptotics of G_r is independent of the rate of decreasing of g to zero; both quantities G_r and S_r at infinity generically have the form $Kr^{-s} + o(r^{-s})$, the coefficients K may be explicitly computed. The number s is the same for G_r and S_r, it is defined by the functions e and q, in almost all our applications $e = q$ and $s = 2$.

All results of this paper are substantial if $\{t \in [0, T] : e(t) = 0\} \neq \emptyset$. In the opposite case ([4]) $e(t) \geq \varepsilon_0 > 0$, the asymptotics of G_r depends on the asymptotics of g: if $g(t, x) \sim |x|^{-\delta}$ for some $\delta > 0$, then also $G_r \sim r^{-\delta}$. The reason of such incongruity with the main case $\{t \in [0, T] : e(t) = 0\} \neq \emptyset$ is as follows. For the case $e(t) \geq \varepsilon_0 > 0$, the integral G_r depends on the values $g(t, x)$ for large $|x| \sim r$ only; for the main case G_r depends also on $g(t, x)$ with small $|x|$, moreover the values of $g(t, x)$ for small $|x|$ define constants in the asymptotics of G_r.

The paper is organized as follows.

Results on vector fields in abstract Banach spaces are in Section 2. Theorem 2.1 concerns the index at infinity computation, its main assumptions are expansions (2.6) for projections of the nonlinearity, these expansions explain the role of the asymptotics of the expressions G_r and S_r for Hammerstein vector fields. The possibility to obtain such expansions for concrete BVPs defines the applicability of Theorem 2.1.

Theorem 2.4 on asymptotic bifurcation points concludes Section 2. Theorem 2.4 does not use the classical Changing Index Principle [11], it is based on asymptotic expansions (2.9) (similar to (2.6)), and follows from direct constructions.

The results (Lemmas 3.1 and 3.2) and their corollaries) presented in Section 3 give expansions (2.6) for concrete BVPs. Sections 4–7 contain various new results

([4]) This case appears in the applications, e.g. the function $e \equiv 1$ is an eigenfunction for the operator $-x''$ with the Neumann ($x'(0) = x'(\pi) = 0$) or periodic ($x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0$) boundary conditions. We do not consider such cases in the paper.

(Theorems 4.1–7.2) on solvability and bifurcations, they are mainly based on the possibility to obtain expansions (2.6) and (2.9) and to compute the index at infinity for corresponding vector fields. All these results follow from Theorems 2.1 and 2.4 and from the statements of Section 3. The most part of applications and examples are formulated for the solvability of the Dirichlet BVP, the scalar case is in Section 4, the vector case is in Section 5. In Section 6 we discuss applications of Theorems 2.1 and 2.4 to bifurcations at infinity. Various examples of BVPs with degenerate linear part independent of the parameter are considered, bifurcation diagrams are defined by bounded nonlinearities. The possibilities to apply the presented methods for some other BVPs are discussed in Section 7.

Theorems on BVPs have an illustrative character to demonstrate facilities of the presented method, their formulations are rather variable and simple. Other applications are also possible, the corresponding results using more cumbersome formulas.

2. Abstract operator equations and vector fields

2.1. Problem formulation. Consider a Hilbert space H with a norm $\|\cdot\|_H$ and a scalar product $\langle\,\cdot\,,\,\cdot\,\rangle$. Consider in H a vector field

$$\Phi x = x - Ax - AF(x), \tag{2.1}$$

here A is a linear completely continuous operator, F is a nonlinear continuous uniformly bounded ($\|F(x)\|_H \le c_F$) operator. Let 1 be a simple eigenvalue of A, $Ae = e$, $\|e\|_H = 1$; then 1 is also a simple eigenvalue of the adjoint operator A^*, suppose $A^*e^* = e^*$ and $\langle e, e^*\rangle = 1$. If A is self-adjoint, then $e = e^*$. Put

$$Px = \langle e^*, x\rangle e, \quad Q = I - P.$$

The projectors P and Q commute with A, the subspaces PH and QH ($\operatorname{codim} QH = \dim PH = 1$) are invariant for A. The operator $I - A$ is continuously invertible in QH.

Suppose A maps continuously H to some Banach space $B \subset H$ with a stronger norm $\|\cdot\|_B$, i.e. for some $c_B > 0$ the estimate $\|x\|_H \le c_B\|x\|_B$ holds for any $x \in B$ and $AH \subset B$. In applications below $H = L^2$, $B = C^1$, $AL^2 \subset C^1$, $F = \mathfrak{f}$.

Let the nonlinearity F be asymptotically homogeneous: there exist values α_0^+ and α_0^- such that for every $c > 0$ the limits

$$\lim_{r\to\infty} \sup_{\|h\|_B \le c} |\langle e^*, F(\pm re + h)\rangle - \alpha_0^\pm| = 0 \tag{2.2}$$

exist (the definition from [7] is given for the case of the simple eigenvalue 1 only). If $\alpha_0^\pm \neq 0$, then $\mathrm{ind}_\infty \Phi$ is well defined and

$$\mathrm{ind}_\infty \Phi = \frac{1}{2}(-1)^\beta(\mathrm{sign}(\alpha_0^+) - \mathrm{sign}(\alpha_0^-));$$

the number β again is the sum of the multiplicities of all the real eigenvalues of the operator A that are greater than 1. In addition to (2.2), assume that there exist $F^\pm \in H$ such that for any $c > 0$ the stronger relations

$$\lim_{r\to\infty} \sup_{\|h\|_B \le c} \|F(\pm re + h) - F^\pm\|_H = 0 \tag{2.3}$$

are valid. Obviously, relation (2.2), where $\alpha_0^\pm = \langle e^*, F^\pm\rangle$, follows from (2.3).

2.2. Relations (2.2) and (2.3) in functional spaces. Suppose the function f, generating the nonlinearity $F = \mathfrak{f}$, satisfies (1.2) (i.e. has form (1.1)).

If $\{t : e(t) = 0\} = 0$, then (2.2), where $\alpha_0^\pm = \mu^\pm = \langle e^*, b \pm a\,\mathrm{sign}(e)\rangle$, follows from

$$\begin{aligned} &\lim_{\xi\to\pm\infty} \sup_{\|h\|_B \le c} \left| \int_\Omega e^*(t) g(t, \xi e(t) + h(t))\, dt \right| \\ &\quad = \lim_{\xi\to\pm\infty} \sup_{\|h\|_B \le c} \left| \int_\Omega e^*(t) a(t) (\mathrm{sign}(\xi e(t) + h(t)) - \mathrm{sign}(\xi e(t)))\, dt \right| = 0, \end{aligned} \tag{2.4}$$

and (2.3), where $F^\pm = b(t) \pm a\,\mathrm{sign}(e(t))$, follows from analogous equalities

$$\begin{aligned} &\lim_{\xi\to\pm\infty} \sup_{\|h\|_B \le c} \|g(t, \xi e(t) + h(t))\|_{L^2} \\ &\qquad = \lim_{\xi\to\pm\infty} \sup_{\|h\|_B \le c} \|\mathrm{sign}(\xi e(t) + h(t)) - \mathrm{sign}(\xi e(t))\|_{L^2} = 0. \end{aligned}$$

Such relations hold for $B = L^1$ (and hence for $B = L^p$ and $B = C^k$), they were used in various forms, starting from papers by A. C. Lazer (e.g. [12], the bibliography see in [2]). If $\Omega = [0, T]$, then similar to (2.4) relations

$$\lim_{\xi\to\pm\infty} \sup_{\|h\|_{C^1} \le c} \left| \int_0^T e^*(t) p(t, \xi e(t) + h(t))\, dt \right| = 0$$

are valid for several bounded functions p, e.g. for almost periodic p with zero average (see [10], the basic idea follows [1], here C^1 cannot be replaced by C).

2.3. Theorem on the index computation. Let for some $F^\pm \in H$ relations (2.3) hold for any $c > 0$. Denote by $h^\pm \subset QH$ unique solutions of the equations $h = AQh + AQF^\pm$, obviously, $h^\pm = (I - AQ)^{-1} AQF^\pm \in B$.

For an $\varepsilon = \varepsilon(r) > 0$ such that $\varepsilon \to 0$ as $r \to \infty$, consider the balls $\mathcal{H}_r = \mathcal{H}_r(\varepsilon) = \{\|\widetilde{h}\|_B \le \varepsilon\}$. Below we use assumptions of the form

$$\lim_{r\to\infty} \sup_{\widetilde{h}\in\mathcal{H}_r} |r^{k_\pm} \langle e^*, F(\pm re + h^\pm + \widetilde{h})\rangle - \alpha^\pm| = 0. \tag{2.5}$$

Relations (2.5) mean that the expansions

$$PF(\pm re + h^{\pm} + \widetilde{h}) = \alpha^{\pm} r^{-k_{\pm}} e + o(r^{-k_{\pm}}) \tag{2.6}$$

are valid uniformly w.r.t. $\widetilde{h} \in \mathcal{H}_r$. The convergence rate in (2.5) depends on $\varepsilon(r)$.

If at least one of the values $\alpha_0^{\pm}$ in (2.2) is different from 0, then the corresponding leading term $\alpha^{\pm} r^{-k_{\pm}} e$ in the right-hand side of (2.6) has the form $\alpha_0^{\pm} e$ with $k_{\pm} = 0$. In the case $k_+ = k_- = 0$ relations (2.5) follow from (2.2), $\alpha^{\pm} = \alpha_0^{\pm}$.

THEOREM 2.1. *Let for some $\alpha^{\pm} \neq 0$ and $k_{\pm} \geq 0$ relations* (2.5) *be valid for any $\varepsilon(r)$. Then* $\mathrm{ind}_\infty \Phi$ *is well defined and*

$$\mathrm{ind}_\infty \Phi = \frac{1}{2}(-1)^{\beta}(\mathrm{sign}(\alpha^+) - \mathrm{sign}(\alpha^-)).$$

In other words, if $\alpha^-\alpha^+ > 0$, then $\mathrm{ind}_\infty \Phi = 0$, *if $\alpha^-\alpha^+ < 0$, then* $\mathrm{ind}_\infty \Phi = (-1)^{\beta}\mathrm{sign}(\alpha^+)$.

If for vector field (1.3) both the inequalities $\mu^+ \neq 0$ and $\mu^- \neq 0$ hold, then $k_{\pm} = 0$, $\alpha^{\pm} = \mu^{\pm}$, and the condition $\alpha^+\alpha^- < 0$ corresponds to the Landesman–Lazer condition. The possibility to obtain expansions (2.6) for Hammerstein type nonlinearities gives the possibility to compute the index at infinity for the vector fields, corresponding concrete BVPs. In the next section we explain how to obtain such asymptotic expansions. In fact, the projections of the Hammerstein operator $A\mathfrak{f}$ (if $A = A^*$ and $f\colon [0,T] \times \mathbb{R} \to \mathbb{R}$) generically admit the stronger expansions

$$PF(\pm re + h^{\pm} + \widetilde{h}) = (\alpha_0^{\pm} + \alpha^{\pm} r^{-2} + o(r^{-2}))e.$$

A variant of Theorem 2.1 for $k_{\pm} = 0$ and without assumption (2.3), follows from the principal result of the paper [7].

2.4. Proof of Theorem 2.1. Consider the cylinder $Z_r = \{\|Px\|_H \leq r, \|Qx\|_H \leq \rho\} \subset H$ and compute for sufficiently large r the rotation $\gamma(\Phi, \partial Z_r)$ of the field Φ on the boundary ∂Z_r of Z_r.

The value $\rho = 1 + c_F\|(I - AQ)^{-1}AQ\|_{QH \to QH}$ (the quantity c_F comes from the assumed estimate $\|F(x)\|_H \leq c_F$) is fixed throughout the proof; all the solutions $h \in QH$ of the equations $h = Ah + AQF(x)$, $x \in H$ satisfy $\|h\|_H < \rho$. For large r the rotation of Φ on the spheres $\{\|x\|_H = r\}$ coincides with $\gamma(\Phi, \partial Z_r)$ and with the index at infinity.

To compute the rotation $\gamma(\Phi, \partial Z_r)$ it is convenient to use the following proposition from [8], it generalizes the usual Rotation Product Formula (see, e.g. [11]).

Let Y be a Banach space, consider in the space $\mathbb{Y} = \{(x, y) : x \in \mathbb{R}, y \in Y\}$ the cylinder $Z = [a_1, a_2] \times B_Y$, where $B_Y \subset Y$ is a fixed closed ball. A pair of completely continuous operators $\phi\colon [a_1, a_2] \times B_Y \to \mathbb{R}$ and $\Psi\colon [a_1, a_2] \times B_Y \to Y$ defines the completely continuous vector field $\mathbb{F} = (\phi(x, y), y - \Psi(x, y))$ on $Z \subset \mathbb{Y}$.

Let for any $x \in [a_1, a_2]$ the rotation γ of the field $y - \Psi(x,y)$ on the boundary ∂B_Y of the ball B_Y be well defined (i.e. the completely continuous in Y field $y - \Psi(x,y)$ is non-degenerate on ∂B_Y) and let $\gamma \neq 0$. Then for each $x \in [a_1, a_2]$ the set $U_x = \{y : y = \Psi(x,y)\} \subset B_Y$ is non-empty.

Proposition 2.2. *Let $\phi(a_j, y) \neq 0$ and $s_j\phi(a_j, y) > 0$ for $y \in U_{a_j}$, $|s_j| = 1$, $j = 1,2$. Then the rotation γ_1 of the field $\mathbb{F}$ on the boundary of Z equals $(s_2 - s_1)\gamma/2$.*

The assumption of this proposition means that the values $\phi(a_1, y)$ have the common sign s_1 for all $y \in U_{a_1}$ and the values $\phi(a_2, y)$ have the common sign s_2 for all $y \in U_{a_2}$. This proposition makes it possible to use properties (such as localization, smoothness, asymptotics, *a priori* estimates) of the infinite dimensional component y to analyze the scalar component ϕ at the points a_j.

Let us come back to the proof of Theorem 2.1. If h satisfies $Q\Phi = 0$, then $\|h\|_B \leq \rho_1 = c_F \, \|AQ(I - AQ)^{-1}\|_{H\to B}$; therefore by (2.3),

$$\begin{aligned}\|h - h^{\pm}\|_B &= \|AQ(I - AQ)^{-1}(F(\pm re + h) - F^{\pm})\|_B \\ &\leq \|AQ(I - AQ)^{-1}\|_{H\to B} \sup_{\|h\|_B \leq \rho_1} \|F(\pm re + h) - F^{\pm}\|_H = \varepsilon(r) \to 0\end{aligned}$$

for such h. Define $\mathcal{H}_r$ in (2.5) by this $\varepsilon(r)$. Now for sufficiently large r the relations $\operatorname{sign}\langle e^*, F(\pm re + h)\rangle = \operatorname{sign}\alpha^{\pm}$ follow from $h = AQh + AQF(\pm re + h)$ and it is possible to apply Proposition 2.2 to the case $[a_1, a_2] = [-r, r]$, $\mathbb{Y} = H$, $Z = Z_\rho$, $B_Y = \{\|h\|_H \leq \rho\} \subset Y = QH$, $\phi = \langle e^*, \ \Phi\rangle$, $\Psi = Q\Phi$, $\mathbb{F} = \Phi$, $s_2 = \operatorname{sign}(\alpha^+)$, $s_1 = \operatorname{sign}(\alpha^-)$. By Proposition 2.2, $\gamma(\Phi, \partial Z_r) = \gamma_P \times \gamma_Q$, where $\gamma_Q = (-1)^\beta$ is the common rotation of the fields (5) $h - AQh - AQF(\pm re + h)$ and $\gamma_P = (\operatorname{sign}(\alpha^+) - \operatorname{sign}(\alpha^-))/2$, obviously, $\gamma_P = 0$ if $\alpha^+\alpha^- > 0$ and $\gamma_P = \operatorname{sign}(\alpha^+)$ if $\alpha^+\alpha^- < 0$. This concludes the proof. □

2.5. Theorem on asymptotic bifurcation points. In this subsection we apply assumptions of the type (2.5) to study asymptotic bifurcation points of the field $\Phi_\lambda x = x - Ax - AF(x;\lambda)$ depending on the parameter $\lambda \in \Lambda \subset \mathbb{R}$, where Λ is a given closed interval, and $F\colon H \times \Lambda$ is continuous w.r.t. the both variables. Let us stress, that the linear operator A is independent of λ.

Definition 2.3. The value λ_0 is called an asymptotic bifurcation point of the field $\Phi_\lambda\colon H \to H$ if for any $N > 0$ there exist λ_N and x_N such that $|\lambda_0 - \lambda_N| < N^{-1}$, $\|x_N\|_H > N$, $\Phi_{\lambda_N}(x_N) = 0$.

In other words, the value λ_0 is an asymptotic bifurcation point if and only if the set $\{x : \Phi_\lambda x = 0, \ |\lambda - \lambda_0| \leq \varepsilon\}$ is unbounded in H for any $\varepsilon > 0$.

(5) The rotation γ_Q is defined by the non-degenerate linear part $h - AQh$, it is independent of the term $-QF(\pm re + h)$, and equals $(-1)^\beta$ on the spheres $\{\|h\|_H = \rho\} \subset QH$.

Theorem 2.4 is valid for the choice either of the upper symbol "+" or of the lower symbol "−" everywhere in the assumptions and in the assertion.

THEOREM 2.4. *Let the following hypotheses be valid.*

(a) *For any $\lambda \in \Lambda$ there exist $F^{\pm}(\lambda) \in H$ satisfying*

$$\lim_{r\to\infty} \sup_{\|h\|_B \le c, \lambda\in\Lambda} \|F(\pm re + h;\lambda) - F^{\pm}(\lambda)\|_H = 0. \tag{2.7}$$

(b) *For some $k_{\pm} \ge 0$ and $\alpha^{\pm}(\lambda)$ the relation*

$$\lim_{r\to\infty} \sup_{\widetilde{h}\in\mathcal{H}_r:\lambda\in\Lambda} |r^{k_{\pm}}\langle e^*, F(\pm re + h^{\pm}(\lambda) + \widetilde{h};\lambda)\rangle - \alpha^{\pm}(\lambda)| = 0 \tag{2.8}$$

holds for any $\varepsilon(r) \to 0$, where $h^{\pm}(\lambda) = (I - AQ)^{-1}AQF^{\pm}(\lambda) \in B$.

(c) *The value $\lambda_0 \in \operatorname{Int}\Lambda$ is a robust* (6) *zero of the function $\alpha^{\pm}$.*

Then λ_0 is an asymptotic bifurcation point of the field Φ_λ in H; more precisely, for any vicinity $\mathcal{O} \subset \Lambda$ of the point λ_0 the set of all solutions of $\Phi_\lambda x = 0$ for $\lambda \in \mathcal{O}$ satisfying $\pm\langle e^*, x\rangle > 0$, is unbounded in H.

Some applications of Theorem 2.4 are presented in Subsection 6.2. Let us conclude this subsection by small remarks, concerning Theorem 2.4.

(1) The vectors $h^{\pm}(\lambda)$ are unique solutions of the linear nonhomogeneous equations $h - AQh - AQF^{\pm}(\lambda) = 0$. The functions $F^{\pm}\colon \Lambda \to H$ are continuous in λ by (2.7) and by the continuity of F; therefore h^+ and h^- are also continuous.

(2) Conditions (2.8) are the uniform w.r.t. $\widetilde{h}$ and λ expansions

$$PF(\pm re + h^{\pm}(\lambda) + \widetilde{h};\lambda) = \alpha^{\pm}(\lambda)r^{-k_{\pm}}e + o(r^{-k_{\pm}}). \tag{2.9}$$

(3) The set $\{(x,\lambda) : \Phi_\lambda x = 0\}$ is an unbounded continuous branch [11] in $H \times \Lambda$.

(4) In addition, suppose

$$PF(re + h;\lambda) = \alpha_0^+(\lambda)e + \alpha^+(\lambda)r^{-k_+}e + o(r^{-k_+}), \quad k_+ > 0;$$

(α_0^+ is continuous in λ by (2.8)) let λ_0 be a robust zero of the function α_0^+ and let $\alpha^+(\lambda_0)(\lambda - \lambda_0)\alpha_0^+(\lambda) > 0$ for $\lambda \ne \lambda_0$. Then for all $\lambda < \lambda_0$ sufficiently close to λ_0 the field Φ_λ has a singular point

$$x = (-\alpha^+(\lambda_0)/\alpha_0^+(\lambda))^{1/k_+}e + o(|\alpha_0^+(\lambda)|^{-1/k_+}).$$

(6) A zero is *robust* if it is isolated and the function is monotone in a vicinity of the zero.

2.6. Proof of Theorem 2.4. The proof is given for the choice "+" of the sign in the formulation of Theorem 2.4. Let us rewrite the equation $\Phi_\lambda x = 0$ as

$$P\Phi_\lambda x = 0, \qquad Q\Phi_\lambda x = 0 \tag{2.10}$$

and prove that for each sufficiently large r there exist $\widetilde{h} = \widetilde{h}(r) \in QH \bigcap B$ and $\lambda = \lambda(r)$ such that $x = re + h^+(\lambda) + \widetilde{h}$ is a solution of (2.10) for $\lambda = \lambda(r)$ and such that $\|\widetilde{h}\|_B \to 0, \lambda \to \lambda_0$ as $r \to \infty$. This means we swap the roles of r and λ in (2.10): the unknown r becomes a new parameter, the parameter λ becomes a new unknown.

Solutions $(\lambda, \widetilde{h})$ of system (2.10) are singular points of the vector field

$$\begin{aligned}\Theta_r(\lambda, \widetilde{h}) &= (\alpha^+(\lambda)e + \nu_1, \widetilde{h} - AQ\widetilde{h} - \nu_2) = r^{k_+} P\Phi_\lambda x + Q\Phi_\lambda x,\\ \nu_1 &= \nu_1(\lambda, \widetilde{h}, r) = r^{k_+} PF(re + h^+ + \widetilde{h}; \lambda) - \alpha^+(\lambda)e,\\ \nu_2 &= \nu_2(\lambda, \widetilde{h}, r) = AQF(re + h^+ + \widetilde{h}; \lambda) - AQF^+(\lambda).\end{aligned}$$

Let $\delta > 0$ be so small that the function α^+ is monotone on $[\lambda_0 - \delta, \lambda_0 + \delta]$, by assumption $\alpha^+(\lambda_0 - \delta)\alpha^+(\lambda_0 + \delta) < 0$. The relation $\|\nu_2\|_B \to 0$ as $r \to \infty$ holds uniformly w.r.t. λ and $\widetilde{h}$, therefore the rotation of the completely continuous vector field $\widetilde{h} - AQ\widetilde{h} - \nu_2$ in QH on the sphere $\{\|\widetilde{h}\|_B = 1\}$ equals $(-1)^\beta$ for any $\lambda \in [\lambda_0 - \delta, \lambda_0 + \delta]$.

For sufficiently large r the values $\alpha^+(\lambda) + \langle e^*, \nu_1 \rangle$ and $\alpha^+(\lambda)$ for $\lambda = \lambda_0 \pm \delta$ are different from zero and have the same common signs $\mathrm{sign}(\alpha^+(\lambda_0 + \delta))$ and $\mathrm{sign}(\alpha^+(\lambda_0 - \delta))$, the rotation of the field Θ_r on the boundary of the set $D_\delta = \{(\lambda, \widetilde{h}) : |\lambda - \lambda_0| \le \delta, \|\widetilde{h}\|_B \le 1\}$ equals $(-1)^\beta \mathrm{sign}(\alpha^+(\lambda_0 + \delta))$ by Proposition 2.2, i.e. it is different from 0. Therefore for any $\delta > 0$ for sufficiently large r there exists $(\lambda, \widetilde{h}) \in D_\delta$ such that $\Theta_r(\lambda, \widetilde{h}) = 0$, i.e. $\Phi_\lambda(er + h^+(\lambda) + \widetilde{h}) = 0$. ☐

3. Expansions of projections of functional nonlinearities

Expansions (2.6) for nonlinearities of the Hammerstein type and for superposition operators can be proved in many cases, the exact form of (2.6) follows from the results of this section.

3.1. General approach. Consider the integral

$$\int_0^T e^*(t) f(t, x(t))\, dt, \qquad x(t) = \pm re(t) + h(t)$$

(the interval of integration $\Omega = [0, T]$ is usually defined by boundary conditions) with the function f satisfying (1.1). All the terms in the right-hand side of

$$\begin{aligned}\int_0^T e^*(t) f(t, x(t))\, dt &= \int_0^T e^*(t)(b(t) + a(t)\mathrm{sign}(x(t)) + g(t, x(t)))\, dt\\ = \mu^\pm + \int_0^T e^*(t) g(t, x(t))\, dt &\pm \int_0^T e^*(t) a(t)(\mathrm{sign}(re(t) \pm h(t)) - \mathrm{sign}(e(t)))\, dt\end{aligned}$$

($\mu^\pm$ are defined by (1.4) except $\mu^\pm$ are small as $r \to \infty$. If $\mu^+ \neq 0$ or $\mu^- \neq 0$, then the corresponding expansion is ready: $\alpha^\pm = \mu^\pm$, $k_\pm = 0$. If either one or both values $\mu^\pm$ equals zero, then it is necessary to study the expressions

$$\int_0^T e^*(t)g(t, \pm re(t) + h(t))\, dt, \qquad \int_0^T e^*(t)a(t)(\operatorname{sign}(\pm re(t) + h(t)) \mp \operatorname{sign}(e(t)))\, dt$$

and to find their leading terms. The leading terms depend on the behavior of e, e^*, and $h^\pm$ at the vicinities of zeros of the function e; the integrals over the sets, where e is separated from zero, are smaller under the assumptions of results below. Therefore we split the interval $[0, T]$ into subintervals Δ_j such that each Δ_j contains exactly one zero ([7]) t_0 of the function e, then we separately analyze the integrals

$$\int_{\Delta_j} e^*(t)g(t, \pm re(t) + h(t))\, dt, \qquad \int_{\Delta_j} e^*(t)a(t)(\operatorname{sign}(\pm re(t) + h(t)) \mp \operatorname{sign}(e(t)))\, dt.$$

Below we present expansions for these integrals, the formulas for leading terms are different if either t_0 is the end (left or right) of Δ_j, or $t_0 \in \operatorname{Int} \Delta_j$.

3.2. Assumptions. Fix an interval $\Delta = \Delta_j \subset [0, T]$, let a scalar function $e \in C^1(0, T)$ have a unique zero $t_0 \in \Delta$. Let

$$\begin{gathered} e(t_0) = 0, \quad e(t) = (t - t_0)(e' + \beta(t)), \quad e' \neq 0, \\ |\beta'(t)| \leq c, \quad |\beta(t)| \to 0, \quad t \to t_0. \end{gathered} \tag{3.1}$$

Let another fixed function $q(t)$ satisfy

$$q(t) = (t - t_0)^k (q^{(k)} + \alpha(t)), \quad q^{(k)} \neq 0,\ |\alpha(t)| \to 0,\ t \to t_0. \tag{3.2}$$

Here $k \geq 0$ is a fixed integer (the number of the first nonzero derivative of q at the point t_0), for applications the main cases are $k = 0$ and $k = 1$, the case $k = 1$ naturally arises if $q = e$. Relations (3.1)–(3.2) follow from the smoothness of e and q.

The main assumptions on the function g are of the form

$$|g(t, x)| \leq G(|x|), \qquad |g(t, x) - g(s, x)| \leq |t - s| G(|x|). \tag{3.3}$$

Everywhere below $G\colon [0, \infty) \to [0, \infty)$ is a monotone decreasing function satisfying

$$\int^\infty u^k G(u)\, du < \infty \tag{3.4}$$

(k is the number from (3.2)).

([7]) All the cases $t_0 = 0$, $t_0 = T$, and $t_0 \in \operatorname{Int} \Delta_j$ appear later in the applications.

Finally, fix $c_0 > 0, \varepsilon = \varepsilon(r) \to 0$ as $r \to \infty$, and $h^*(t) \in C^1$ and consider the family $\mathfrak{H} = \mathfrak{H}_{\varepsilon,c_0}(h^*, r)$ of functions $h(t) \in C^1$ such that

$$h(t) = h^*(t) + \widetilde{h}(t), \quad |\widetilde{h}(t)| \le \varepsilon, \quad |\widetilde{h}'(t)| \le c_0, \quad t \in \Delta.$$

Obviously,

$$c_{\mathfrak{H}} \stackrel{\text{def}}{=} \sup_{h \in \mathfrak{H}} \|h\|_C < \infty. \tag{3.5}$$

3.3. The main lemma. Let $e(t_0) = 0$, and let t_0 be one of the ends of Δ. Put

$$I_0^{\pm} = I_0^{\pm}(r; q, e, h) = \pm \int_{t_0}^{t_1} q(t) g(t, re(t) + h(t))\, dt,$$

the sign "+" corresponds to the case $t_0 < t_1$ and $\Delta = [t_0, t_1]$, the sign "−" corresponds to the case $t_0 > t_1$ and $\Delta = [t_1, t_0]$.

Since t_0 is the end of Δ and t_0 is a unique on Δ zero of the function $e \in C$, we see that for $t \in \Delta$, $t \ne t_0$ either $e(t) > 0$ or $e(t) < 0$. Put $\sigma = +$ if $e(t) > 0$ and $\sigma = -$ if $e(t) < 0$ (Δ is a symbol, not a number). Obviously, if $\operatorname{sign}((t_1 - t_0)e') = \pm 1$, then $\sigma = \pm$; denote $-\sigma = \pm$ if $\sigma = \mp$. To simplify and to unify notations put

$$\int_{\mathbb{R}^+} \text{ instead of } \int_0^{\infty} \quad \text{and} \quad \int_{\mathbb{R}^-} \text{ instead of } \int_{-\infty}^0 .$$

Lemma 3.1. *Let all the assumptions of Subsection* 3.2 *hold and let t_0 be an end of the interval Δ. Then*

$$\lim_{r\to\infty} \sup_{h\in\mathfrak{H}} \left| r^{k+1} I_0^{\pm} - \operatorname{sign}(e')(e')^{-k-1} q^{(k)} \int_{\mathbb{R}^\sigma} u^k g(t_0, u + h^*(t_0))\, du \right| = 0. \tag{3.6}$$

The case $t_0 \in \operatorname{Int}\Delta = (t_1, t_2)$ follows from this lemma. Denote

$$I_0 = I_0(r; q, e, h) = \int_{t_1}^{t_2} q(t) g(t, re(t) + h(t))\, dt.$$

By Lemma 3.1,

$$\lim_{r\to\infty} \sup_{h\in\mathfrak{H}} \left| r^{k+1} I_0 - \operatorname{sign}(e')(e')^{-k-1} q^{(k)} \int_{-\infty}^{+\infty} u^k g(t_0, u + h^*(t_0))\, du \right| = 0.$$

3.4. Proof of Lemma 3.1. We prove (3.6) for I_0^+ and $e' > 0$ (i.e. $t_1 > t_0$, $\sigma = +$), other cases can be proved in analogous way. To simplify formulas, suppose $t_0 = 0$.

Put $\delta = \delta(r) = (\ln(\ln r))^{-1}$, the variable δ is chosen to satisfy $\delta \to 0$, $r\delta \to \infty$, and $r^{k+1} G(\kappa r \delta) \to 0$ for any κ. Consider for sufficiently large r the integrals

$$J_r = \int_0^{\delta} t^k G(|re(t) + h(t)|)\, dt, \qquad J_r^* = \int_0^{\delta} t^k g(0, re(t) + h(t))\, dt$$

and estimate them. The value δ is small, hence for $t \in [0,\delta]$ the relation $e(t) > e't/2$ holds. Therefore, $|re(t)+h(t)| > re't/2 - c_{\mathfrak{H}}$ for $t > 2c_{\mathfrak{H}}(re')^{-1}$ and

$$\begin{aligned} J_r &= \int_0^{2c_{\mathfrak{H}}/(re')} + \int_{2c_{\mathfrak{H}}/(re')}^{\delta} \\ &\le \frac{1}{k+1}\left(\frac{2c_{\mathfrak{H}}}{re'}\right)^{k+1} G(0) + \int_{2c_{\mathfrak{H}}/(re')}^{\delta} t^k G\left(\frac{re'}{2}t - c_{\mathfrak{H}}\right) dt \\ &\le C_0 r^{-k-1} + \frac{2}{re'}\int_0^{\infty}\left(\frac{2(u+c_{\mathfrak{H}})}{re'}\right)^k G(u)\,du = C_1 r^{-k-1} \end{aligned}$$

($c_{\mathfrak{H}}$ is defined by (3.5)). By (3.1) the function $re(t)+h(t)$, $t\in[0,\delta]$ is monotone for large r, hence the change of variables $u = re(t)+h(t)$ in J_r^* is well defined and

$$t = \frac{u-h(t)}{r(e'+\beta(t))}, \qquad du = re'\nu(t)\,dt,$$

where

$$\nu(t) = 1 + (e')^{-1}\left(t\beta'(t) + \beta(t) + \frac{h'(t)}{r}\right).$$

Since $\nu(t) = 1 + o(1)$ and $t = (re')^{-1}(u - h^*(0)) + o(r^{-1})$, we see that

$$J_r^* = (e'r)^{-k-1}\left(\int_{h^*(0)}^{\infty}(u-h^*(0))^k g(0,u)du + o(1)\right).$$

Now come back to the proof of (3.6), for $t_1 > t_0$ and $e' > 0$ it has the form

$$I_0^+ = (e'r)^{-k-1}q^{(k)}\int_0^{\infty} u^k g(0, u + h^*(0))\,du + o(r^{-k-1}).$$

First of all truncate terms of order $o(r^{-k-1})$ from the integral I_0^+:

$$\left|\int_{\delta}^{t_1} q(t)g(t, re(t)+h(t))\,dt\right| \le C_2 G(C_3 r\delta - C_2) = o(r^{-k-1}),$$

$$\left|\int_0^{\delta}\alpha(t)t^k g(t, re(t)+h(t))\,dt\right| \le \max_{t\in(0,\delta)}|\alpha(t)|J_r = o(r^{-k-1}),$$

and

$$\left|\int_0^{\delta} t^k(g(0, re(t)+h(t)) - g(t, re(t)+h(t)))\,dt\right| \le \delta J_r = o(r^{-k-1}).$$

Therefore, $I_0^+ = q^{(k)}J_r^* + o(r^{-k-1})$. □

3.5. Projections of the homogeneous components of order 0. Now under the assumptions of Subsection 3.2 consider the integrals

$$I_1^{\pm} = I_1^{\pm}(r; q, e, h) = \pm\int_{t_0}^{t_1} q(t)(\operatorname{sign}(re(t)+h(t)) - \operatorname{sign}(e(t)))\,dt,$$

(again, "+" corresponds to $t_0 < t_1$, "−" corresponds to $t_0 > t_1$) and

$$I_1 = I_1(r; q, e, h) = \int_{t_1}^{t_2} q(t)(\operatorname{sign}(re(t) + h(t)) - \operatorname{sign}(e(t)))\, dt, \quad t_0 \in (t_1, t_2).$$

LEMMA 3.2. *The relations*

$$\lim_{r\to\infty} \sup_{h\in\mathfrak{H}} |r^{k+1} I_1^{\pm} - W^{\pm}| = 0, \tag{3.7}$$

are valid, where

$$W^{\pm} = \begin{cases} 0 & \text{if } \pm h^*(t_0)e' \geq 0, \\ \dfrac{2q^{(k)}\operatorname{sign}(-e')}{k+1}\left(\dfrac{-h^*(t_0)}{re'}\right)^{k+1} & \text{if } \mp h^*(t_0)e' < 0. \end{cases}$$

If r is large enough and $h^*(t_0)e' > 0$, then not only $W^+ = 0$, moreover, $\operatorname{sign}(re(t) + h(t)) = \operatorname{sign}(e(t))$ and $I_1^+ = 0$. By Lemma 3.2,

$$I_1 = \frac{2q^{(k)}\operatorname{sign}(-e')}{k+1}\left(\frac{-h^*(t_0)}{re'}\right)^{k+1} + o(r^{-k-1}).$$

3.6. Proof of Lemma 3.2. We prove (3.7) for the symbol "+" (i.e. for I_1^+) for the case $e' > 0$ and $h^*(t_0) < 0$, other cases are similar or even simpler; obviously,

$$W^+ = -2q^k |h^*(t_0)/e'|^{k+1}/(k+1).$$

Since the function $\operatorname{sign}(re(t) + h(t)) - \operatorname{sign}(e(t))$ is non-zero if and only if

$$t \in \Gamma(r) = \{e(t)h(t) < 0\} \cap \{|e(t)| < |h(t)|r^{-1}\},$$

we have that

$$\operatorname{sign}(re(t) + h(t)) - \operatorname{sign}(e(t)) \equiv -2\operatorname{sign} e(t) = 2\operatorname{sign} h(t)$$

for $t \in \Gamma(r)$. From (3.5) it follows that $\Gamma(r) \subset (0, c_{\mathfrak{H}} r^{-1})$, from $e' > 0$ it follows that $\Gamma(r) \subset \{t : e(t) > 0\}$ for large r, therefore,

$$\Gamma(r) = \{t : e(t) > 0,\ h(t) < 0,\ r|e(t)| < |h(t)|\}.$$

Now we have

$$\begin{aligned} I_1^+ &= -2\int_{\Gamma(r)} q(t)\, dt = -2q^{(k)} \int_{\Gamma(r)} \tau^k d\tau + o(r^{-k-1}) \\ &= -2q^{(k)} \int_0^{|h^*(t_0)/re'|} \tau^k\, d\tau + o(r^{-k-1}) \\ &= \frac{-2q^{(k)}}{k+1}\left(\frac{|h^*(t_0)|}{re'}\right)^{k+1} + o(r^{-k-1}) = W^+ r^{k+1} + o(r^{-k-1}) \end{aligned}$$

and the lemma is proved. □

3.7. Corollaries and examples. Lemmas 3.1 and 3.2 are used in application throughout the paper for $k = 1$.

COROLLARY 3.7. *Let $k = 1$. If $h^*(t_0) = 0$, then $I_1^{\pm} = o(r^{-2})$ and*

$$\begin{aligned} I_0^{\pm}(r; e, e, h) &= \frac{1}{r^2|e'|} \int_{\mathbb{R}^\sigma} u\, g(t_0, u)\, du + o(r^{-2}), \\ I_0^{\pm}(r; e, -e, h) &= -\frac{1}{r^2|e'|} \int_{\mathbb{R}^{-\sigma}} u\, g(t_0, u)\, du + o(r^{-2}) \end{aligned}$$

The relations

$$\begin{aligned} I_1(r; e, \pm e, h) &= \mp r^{-2}|e'|^{-1}[h^*(t_0)]^2 + o(r^{-2}), \\ I_0(r; e, \pm e, h) &= \pm r^{-2}|e'|^{-1} \int_{-\infty}^{\infty} u\, g(t_0, u + h^*(t_0))\, du + o(r^{-2}) \end{aligned}$$

are valid for any $h^*(t_0)$.

In the following examples $[0, T] = [0, \pi]$. Examples 3.8, 3.9 are used for the Dirichlet BVP, Example 3.10 is used for the Neumann BVP.

EXAMPLE 3.8. Let $e(t) = \sin t$, $h^*(0) = h^*(\pi) = 0$. Then ([8])

$$\langle \sin t, g(t, \pm r \sin t + h(t)) \rangle_\pi = \pm \frac{1}{r^2} \int_{\mathbb{R}^\pm} u(g(0, u) + g(\pi, u))\, du + o(r^{-2}). \tag{3.8}$$

In particular, if $g(t, u) \equiv g(u)$, then

$$\langle \sin t, g(\pm r \sin t + h(t)) \rangle_\pi = \pm \frac{2}{r^2} \int_{\mathbb{R}^\pm} u\, g(u)\, du + o(r^{-2}). \tag{3.9}$$

EXAMPLE 3.9. Let $e(t) = \sin nt$, $h^*(0) = h^*(\pi) = 0$, where $n \geq 2$ is an integer. The formulas are different for even and odd values of n (the sign σ of the function e in the vicinity of the point π takes value "+" for odd n and the value "−" for the even ones), we formulate them for $n = 2, 3$ only, for simplicity we restrict ourself with the case $g(t, x) = g(x)$. For $n = 2$

$$\langle \sin 2t, g(t, \pm r \sin 2t + h(t)) \rangle_\pi = \frac{\pm 1}{2r^2} \int_{-\infty}^{\infty} u(g(u) + g(u + h^*(\pi/2)))\, du + o(r^{-2}), \tag{3.10}$$

$$\langle \sin 2t, \operatorname{sign}(r \sin 2t \pm h(t)) - \operatorname{sign}(\sin 2t) \rangle_\pi = -\frac{1}{2r^2}(h^*(\pi/2))^2 + o(r^{-2}); \tag{3.11}$$

for $n = 3$

$$\begin{aligned} \langle \sin 3t, g(t, \pm r \sin 3t + h(t)) \rangle_\pi &= \pm \frac{2}{3r^2} \int_{\mathbb{R}^\pm} u\, g(u)\, du + o(r^{-2}) \\ &\pm \frac{1}{3r^2} \int_{-\infty}^{\infty} u(g(u + h^*(\pi/3)) + g(u + h^*(2\pi/3)))\, du, \end{aligned} \tag{3.12}$$

([8]) Here and everywhere below $\langle \cdot, \cdot \rangle_\pi$ is a usual scalar product in $L^2(0, \pi)$.

$$(3.13)\quad \langle \sin 3t, \operatorname{sign}(r\sin 3t \pm h(t)) - \operatorname{sign}(\sin 3t)\rangle_\pi = -\frac{1}{3r^2}((h^*(\pi/3))^2 + h^*(2\pi/3))^2) + o(r^{-2}).$$

Example 3.10. Let $e(t) = \cos t$. Then

$$(3.14)\quad \langle \cos t, g(t, \pm r\cos t + h(t))\rangle_\pi = \pm\frac{1}{r^2}\int_{-\infty}^{\infty} u\, g(\pi/2, u + h^*(\pi/2))\, du + o(r^{-2}),$$

$$(3.15)\quad \langle \cos t, \operatorname{sign}(r\cos t \pm h(t)) - \operatorname{sign}(\cos t)\rangle_\pi = -\frac{1}{r^2}(h^*(\pi/2))^2 + o(r^{-2}).$$

4. Scalar Dirichlet BVP

4.1. The simplest case. Consider in this section the scalar Dirichlet BVP

$$(4.1)\qquad x'' + x + f(t,x) = 0, \quad x(0) = x(\pi) = 0.$$

The linear operator $A = -(d^2/dt^2)^{-1}$ with the boundary conditions $x(0) = x(\pi) = 0$ maps L^2 continuously to C^1, it is completely continuous in L^2. Solutions of (4.1) coincide with solutions of the operator equation $x = A(x + \mathfrak{f}x)$. The number 1 is an eigenvalue of A: $A(\sin t) = \sin t$. Let f satisfy (1.2), put $\overline{f}^{\pm} = \langle \sin t, f^{\pm}(t)\rangle_\pi$. If $\overline{f}^{\pm} \neq 0$ and the Landesman-Lazer condition $\overline{f}^{-}\overline{f}^{+} < 0$ is valid, then the index at infinity of the field $\Phi x = x - A(x + \mathfrak{f}x)$ in L^2 is well defined and non-zero, therefore, in particular, there exists at least one solution of (4.1). In this Section we formulate some statements on the solvability of (4.1) for the case $\overline{f}^{+}\overline{f}^{-} = 0$. Put

$$\mathcal{J}_1^{\pm} = \int_{\mathbb{R}^{\pm}} u(g(0,u) + g(\pi,u))\, du.$$

In Theorems 4.1 and 4.3 g satisfies (3.3) and G satisfies (3.4) for $k = 1$.

Theorem 4.1. *Let $\overline{f}^{-} = \overline{f}^{+} = 0$. If $\mathcal{J}_1^{+}\mathcal{J}_1^{-} > 0$, then* (4.1) *has at least one solution.*

Under the assumptions of Theorem 4.1 the index of infinity of the field Φ is non-zero by Theorem 2.1, in asymptotic formula (2.5) both numbers $k_\pm$ equal 2. From formula (3.8) it follows that $\alpha^{\pm} = \pm\mathcal{J}_1^{\pm}/\sqrt{\pi}$.

The condition $\overline{f}^{-} = \overline{f}^{+} = 0$ is equivalent to $\overline{b} \stackrel{\text{def}}{=} \langle \sin t, b(t)\rangle_\pi = 0 = \langle \sin t, a(t)\rangle_\pi$. The simplest example is $a(t) \equiv 0$, $\overline{b} = 0$. If $f(t,x) = b(t) + g(x)$, then the assumption $\mathcal{J}_1^{+}\mathcal{J}_1^{-} > 0$ has especially simple form. The next statement follows from (3.9).

COROLLARY 4.2. *Problem* (4.1) *has at least one solution if* $\overline{b} = 0$ *and*

$$\int_0^\infty u\,g(u)\,du \int_{-\infty}^0 u\,g(u)\,du > 0, \tag{4.2}$$

If g is even, then (4.2) never holds. For odd g it is valid if

$$\int_0^\infty u\,g(u)\,du \neq 0.$$

THEOREM 4.3. *If* $\overline{f}^- \neq 0$, $\overline{f}^+ = 0$ *and* $\mathcal{J}_1^+ \overline{f}^- < 0$, *then* (4.1) *has at least one solution.*

Under the assumptions of this theorem the values $k_\pm$ in (2.5) are different: $k_+ = 2$, $k_- = 0$; $\alpha^- = \overline{f}^-/\sqrt{\pi}$, $\alpha^+ = \mathcal{J}_1^+/\sqrt{\pi}$. The value α^+ is computed by (3.8).

4.2. The Dirichlet BVP, $n > 1$**.** Now let us consider the problem

$$x'' + n^2 x + f(t,x) = 0, \quad x(0) = x(\pi) = 0, \quad n > 1. \tag{4.3}$$

If $n = 1$, then $e(t) = \sin t$ and $e(t_0) = 0$ if and only if either $t_0 = 0$ or $t_0 = \pi$. And by the boundary conditions, $h(t_0) = h^\pm(t_0) = 0$.

For $n > 1$ this is not the case: $e(t) = \sin nt$ and $e(t_0) = 0$ not only for $t_0 = 0, \pi$, but also for the points $k\pi/n$, $k = 1, \dots, (n-1)$. Generically $h^\pm(k\pi/n) \neq 0$, these values take part in the answers, all the formulations become more cumbersome.

We present examples for $n = 2$ and $n = 3$ (an example for even n and an example for odd n) only. Let P_n be the orthogonal projector in L^2 onto the strict line containing the function $\sin nt$, let $Q_n x = x - P_n x$.

Suppose f has the form $f(t,x) = b(t) + g(x)$ and $g(x) \to 0$ as $x \to \pm\infty$. Let $P_n b = 0$. The Fredholm Alternative Lemma implies that the linear BVP

$$h'' + n^2 h = b(t), \quad h(0) = h(\pi) = 0$$

has a unique solution $h^* \in Q_n L^2$ ($F^+ = F^- = b(t)$, therefore, $h^+ = h^- = h^*$). Put

$$\mathcal{J}_2 = \int_{-\infty}^\infty u(g(u) + g(u + h^*(\pi/2)))\,du;$$

$$\mathcal{J}_3^\pm = 2\int_{\mathbb{R}^\pm} u\,g(u)\,du + \int_{-\infty}^\infty u(g(u + h^*(\pi/3)) + g(u + h^*(2\pi/3)))\,du.$$

THEOREM 4.4. *Let* $f(t,x) = b(t) + g(x)$, *let either* $n = 2$ *and* $\mathcal{J}_2 \neq 0$ *or* $n = 3$ *and* $\mathcal{J}_3^+ \mathcal{J}_3^- > 0$. *Then problem* (4.3) *has at least one solution.*

To prove this theorem one can use (3.10) and (3.12). If the function f satisfies (1.1), then in the formulas for $\mathcal{J}^\pm$ it is necessary to use additionally (3.11) and (3.13).

Now let $n = 2$, $a(t) \equiv a \neq 0$, $\langle b(t), \sin 2t\rangle_\pi + 2a = 0$. From the Fredholm Alternative Lemma and from the equality $P_n(b(t) + a\,\mathrm{sign}(\sin 2t)) = 0$ it follows that there exists $h = h^+ \in Q_n L^2$ such that $h'' + n^2 h = b(t) + a\,\mathrm{sign}(\sin 2t)$, $h(0) = h(\pi) = 0$.

Theorem 4.5. *Let $n = 2$, $f(t,x) = b(t) + a\,\mathrm{sign}(x) + g(x)$, and $a^{-1}\mathcal{J}_2 > (h^+(\pi/2))^2$. Then* (4.3) *has at least one solution.*

5. Vector Dirichlet BVP

5.1. Preliminary. Here we discuss the Dirichlet BVP in $\mathbb{R}^N$, $N > 1$:

$$\boldsymbol{x}'' + \boldsymbol{\mathcal{B}}\boldsymbol{x} + \boldsymbol{f}(t,\boldsymbol{x}) = \boldsymbol{0}, \qquad \boldsymbol{x}(0) = \boldsymbol{x}(\pi) = \boldsymbol{0}. \tag{5.1}$$

Throughout this section elements of $\mathbb{R}^N$ and functions to this space are boldfaced: $\boldsymbol{0} \in \mathbb{R}^N$, $\boldsymbol{x}: [0,\pi] \to \mathbb{R}^N$, $\boldsymbol{f}: [0,\pi] \times \mathbb{R}^N \to \mathbb{R}^N$, $\boldsymbol{\mathcal{B}}$ is an $N \times N$ matrix. Denote by $\langle\,\cdot\,,\,\cdot\,\rangle_N$ and $|\cdot|_N$ a product and the corresponding norm in $\mathbb{R}^N$; the formula

$$\langle \boldsymbol{x}, \boldsymbol{y}\rangle_{L^2} = \int_0^\pi \langle \boldsymbol{x}(t), \boldsymbol{y}(t)\rangle_N \, dt$$

defines the scalar product in the space $L^2 = L^2([0,\pi]; \mathbb{R}^N)$.

If $\boldsymbol{\mathcal{B}}$ has no eigenvalues of the type k^2 for integer $k > 0$, then the differential operator $\boldsymbol{x}'' + \boldsymbol{\mathcal{B}}\boldsymbol{x}$ with the Dirichlet boundary conditions $\boldsymbol{x}(0) = \boldsymbol{x}(\pi) = \boldsymbol{0}$ is continuously invertible in L^2 and (if $\boldsymbol{f}$ is bounded or sublinear) the index at infinity of the vector field $\boldsymbol{x} - (d^2/dt^2 + \boldsymbol{\mathcal{B}})^{-1}\boldsymbol{f}(t,\boldsymbol{x})$ is equal to either $+1$ or -1.

Let 1 be a simple eigenvalue of the matrix $\boldsymbol{\mathcal{B}}$, $\boldsymbol{\mathcal{B}}\boldsymbol{e} = \boldsymbol{e}$, $|\boldsymbol{e}|_N = 1$, let $\boldsymbol{\mathcal{B}}$ have not other eigenvalues of the form k^2 for integer k.

Suppose the nonlinearity $\boldsymbol{f}$ has the form

$$\boldsymbol{f}(t,\boldsymbol{x}) = \boldsymbol{G}(t,\boldsymbol{x}) + \boldsymbol{g}(t,\boldsymbol{x}),$$

where the function $\boldsymbol{G}$ is positively homogeneous of order 0:

$$\boldsymbol{G}(t, r\boldsymbol{x}) \equiv \boldsymbol{G}(t,\boldsymbol{x}), \quad r > 0;$$

and $\boldsymbol{g} \to \boldsymbol{0}$ at infinity. The function $\boldsymbol{G}$ has discontinuity at the origin (if it is nonconstant in $\boldsymbol{x}$), the function $\boldsymbol{g}$ must compensate this discontinuity. The simplest case is $\boldsymbol{G} \equiv \boldsymbol{b}(t)$, here $\boldsymbol{g}$ may be continuous.

The operator $\boldsymbol{A} = (-d^2/dt^2)^{-1}: L^2 \to L^2$ with the boundary conditions $\boldsymbol{x}(0) = \boldsymbol{x}(\pi) = \boldsymbol{0}$ is well defined. Its spectrum $\sigma(\boldsymbol{A})$ consists from zero and from the eigenvalues k^{-2}, each has the multiplicity N, adjoined vectors do not exist. The eigenfunction $\boldsymbol{e}\sin t$ corresponds to the eigenvalue 1 of the operator $\boldsymbol{A}\boldsymbol{\mathcal{B}}$, the operators $\boldsymbol{A}$ and $\boldsymbol{\mathcal{B}}$ commute. Let $\boldsymbol{\mathcal{B}}^*\boldsymbol{e}^* = \boldsymbol{e}^*$, where $\boldsymbol{\mathcal{B}}^*$ is the transposed matrix and $\|e^*\|_N = 1$. The function $\boldsymbol{e}^*\sin t$ is an eigenfunction of the operator $\boldsymbol{A}\boldsymbol{\mathcal{B}}^*$: $\boldsymbol{A}\boldsymbol{\mathcal{B}}^*(\boldsymbol{e}^*\sin t) = \boldsymbol{e}^*\sin t$. Put $\boldsymbol{P}\boldsymbol{x} = \sqrt{2/\pi}\langle \boldsymbol{e}^*\sin t, \boldsymbol{x}\rangle_{L^2}$ and $\boldsymbol{Q}\boldsymbol{x} = \boldsymbol{x} - \boldsymbol{P}\boldsymbol{x}$.

Consider the vector field

$$\Phi \boldsymbol{x}(t) = \boldsymbol{x} - \boldsymbol{A}(\boldsymbol{\mathcal{B}}\boldsymbol{x}(t) + \boldsymbol{f}(t, \boldsymbol{x}(t))). \tag{5.2}$$

If

$$\mu^{\pm} \stackrel{\text{def}}{=} \int_0^{\pi} \sin t \langle \boldsymbol{e}^*, \boldsymbol{G}(t, \pm r\boldsymbol{e} \sin t)\rangle_N \, dt \neq 0,$$

then (1.5) holds, the field Φ is once degenerate.

Twice degenerate cases appear if $\mu^+\mu^- = 0$.

Theorem 2.1 is formulated for the fields Φ of the form (2.1), different from (5.2). To study fields (5.2) it is possible either to reformulate Theorem 2.1 or to assume that the matrix $\boldsymbol{\mathcal{B}}$ is invertible. In the last case field (5.2) has the form (2.1): $\Phi\boldsymbol{x}(t) = \boldsymbol{x} - \boldsymbol{A}\boldsymbol{\mathcal{B}}(\boldsymbol{x}(t) + \boldsymbol{F}(\boldsymbol{x}))$, where $\boldsymbol{F}(\boldsymbol{x}) = \boldsymbol{\mathcal{B}}^{-1}\boldsymbol{f}(t, \boldsymbol{x}(t))$.

To study the field Φ using Theorem 2.1, it is necessary to find leading terms of the expressions

$$J^{\pm} = \int_0^{\pi} \sin t \langle \boldsymbol{e}^*, \boldsymbol{g}(t, \pm r\boldsymbol{e} \sin t + \boldsymbol{h}(t))\rangle_N \, dt$$

and

$$S^{\pm} = \int_0^{\pi} \sin t \langle \boldsymbol{e}^*, \boldsymbol{G}(t, \pm r\boldsymbol{e} \sin t + \boldsymbol{h}(t)) - \boldsymbol{G}(t, \pm \boldsymbol{e})\rangle_N \, dt.$$

Here $\boldsymbol{h}(t) = \boldsymbol{h}^{\pm}(t) + \widetilde{\boldsymbol{h}}(t)$, where $\boldsymbol{h}^{\pm}(t) = (\boldsymbol{I} - \boldsymbol{A}\boldsymbol{\mathcal{B}}\boldsymbol{Q})^{-1}\boldsymbol{A}\boldsymbol{Q}\boldsymbol{G}(t, \pm\boldsymbol{e})$, and $\|\widetilde{\boldsymbol{h}}\|_{C^1} \leq \varepsilon(r)$, where $\varepsilon(r) \to 0$ as $r \to \infty$. The values J^+ and S^+ must be analyzed if $\mu^+ = 0$, the values J^- and S^- must be analyzed if $\mu^- = 0$.

For scalar BVPs expansions for similar to $J^{\pm}$ and $S^{\pm}$ expressions are presented in Section 3. For the vector equations there are two very different cases. The simplest one is $\boldsymbol{G} \equiv \boldsymbol{b}(t)$, it is considered in the next subsection. In this case $S^{\pm} = 0$ and $J^{\pm}$ have leading terms of order r^{-2}.

For generic $\boldsymbol{G}$ the expressions $S^{\pm}$ contain leading terms of order r^{-1}, this case is a subject of another paper. We would only like to emphasize the formula

$$S^{\pm} = \frac{1}{r} \int_0^{\pi} \left\langle \boldsymbol{e}^*, \frac{\partial \boldsymbol{G}(t, \boldsymbol{x})}{\partial \boldsymbol{x}}\bigg|_{\boldsymbol{x}=\pm\boldsymbol{e}} \boldsymbol{h}^{\pm}(t) \right\rangle_N dt + o(r^{-1}),$$

it holds for continuous at $\boldsymbol{x} \neq \boldsymbol{0}$ and differentiable at $\boldsymbol{x} = \pm\boldsymbol{e}$ functions $\boldsymbol{G}(t, \boldsymbol{x})$.

Generic homogeneous functions $\boldsymbol{G}$ have discontinuities not only at the origin. For example, let $n = 2$, $\boldsymbol{x} = (x_1, x_2)$, and $\boldsymbol{f}(t, \boldsymbol{x}) = (f_1(t, x_1, x_2), f_2(t, x_1, x_2))$. If $f_1(t, x_1, x_2) = 2\pi^{-1} \arctan(x_1)$, then $f_1 = \operatorname{sign}(x_1) + g(x_1)$; the function $\operatorname{sign}(x_1)$ is discontinuous along the strict line $x_1 = 0$ on the plane (x_1, x_2).

5.2. The case $\boldsymbol{G} \equiv \boldsymbol{b}(t)$. Suppose

$$|\boldsymbol{g}(t, \boldsymbol{x}_1) - \boldsymbol{g}(t, \boldsymbol{x}_2)|_N \leq |\boldsymbol{x}_1 - \boldsymbol{x}_2|_N \, G(\min\{|\boldsymbol{x}_1|_N, |\boldsymbol{x}_2|_N\}), \quad \boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{R}^N.$$

We do not use similar assumptions to study the scalar case. Such difference between scalar and vector cases follows from the multiplace character of the functions $g_j(t, x_1, \dots, x_N)$: it is impossible to use the change of variables $re(t) + h(t) = u$ as this is done in the proof of Lemma 3.1 in Subsection 3.4.

Let

$$|\boldsymbol{g}(t,\boldsymbol{x})|_N \le G(|\boldsymbol{x}|_N), \qquad |\boldsymbol{g}(t_1,\boldsymbol{x}) - \boldsymbol{g}(t_2,\boldsymbol{x})|_N \le |t_1 - t_2|\, G(|\boldsymbol{x}|_N),$$

let $G\colon \mathbb{R}^+ \to \mathbb{R}^+$ decrease and satisfy (3.4) for $k = 1$.

In the above assumptions

$$J^{\pm} = \pm r^{-2}\mathfrak{J}^{\pm} + o(r^{-2}), \qquad \mathfrak{J}^{\pm} = \int_{\mathbb{R}^{\pm}} u\langle \boldsymbol{e}^*, (\boldsymbol{g}(0,u\boldsymbol{e}) + \boldsymbol{g}(\pi,u\boldsymbol{e}))\rangle_N \, du.$$

Let us formulate a simple result on the solvability of problem (5.1).

Theorem 5.1. *Let* $\boldsymbol{G} \equiv \boldsymbol{b}(t)$ *and* $\langle \sin t, \langle \boldsymbol{e}^*, \boldsymbol{b}(t)\rangle_N\rangle_\pi = 0$. *If* $\mathfrak{J}^+\mathfrak{J}^- > 0$, *then problem* (5.1) *has at least one solution.*

The assumption $\mathfrak{J}^+\mathfrak{J}^- > 0$ generalizes the assumption $\mathfrak{J}_1^+\mathfrak{J}_1^- > 0$ of Theorem 4.1.

6. Applications to bifurcations

6.1. General discussion. The possibility to compute the index at infinity gives a standard set of results on solvability, non-uniqueness, bifurcations etc. For example, if the index is non-zero, then the corresponding operator equation or BVP has at least one solution, some examples are presented above in Sections 4 and 5. If a singular point x_0 of a vector field is known, its index is well defined, and differs from the index at infinity, then there exists at least one singular point $x_1 \ne x_0$.

Suppose a linear operator A satisfies all the assumptions of Subsection 2.1 and consider the vector field

$$V_\lambda x = x - \lambda A x - AF(x;\lambda);$$

the parameter $\lambda \in \mathbb{R}$ is defined in a vicinity of the point $\lambda_0 = 1$. There exists an unbounded continuous branch of singular points (generically there are 2 such branches) of the field V_λ. The existence follows from the simplicity of the eigenvalue 1 of the operator A according to the Changing Index Principle [11] developed by Mark Krasnosel'skiĭ in the early 50's of the previous century. To prove the existence of unbounded branches, it is not necessary to compute the index at infinity of the vector field V_{λ_0}; the indices of the field V_λ for $\lambda < \lambda_0$ and for $\lambda > \lambda_0$ are known and different, therefore such branches exist. However, the information about the value of $\operatorname{ind}_\infty V_{\lambda_0}$ contains an essential additional information about bifurcation diagrams, i.e. about the possible number of the branches and about their geometry. The value $\operatorname{ind}_\infty V_{\lambda_0}$ may be computed with the use of presented above constructions.

If $\mathrm{ind}_\infty V_\lambda = 1$ for $\lambda \geq \lambda_0$ and $\mathrm{ind}_\infty V_\lambda = -1$ for $\lambda < \lambda_0$, then generically there exist 2 unbounded branches of singular points from the left of λ_0, each of them has the index -1, and a bounded branch with the index 1 for $|\lambda - \lambda_0| \leq \varepsilon$. Therefore there exist at least 3 singular points for $\lambda < \lambda_0$ close enough to λ_0. The simplest bifurcation diagram is presented at the left-hand part of Figure 1. If $\mathrm{ind}_\infty V_\lambda = 1$ for $\lambda > \lambda_0$ and $\mathrm{ind}_\infty V_\lambda = -1$ for $\lambda < \lambda_0$, but $\mathrm{ind}_\infty V_{\lambda_0} = 0$, then generically there exists an unbounded branch of the index -1 for $\lambda < \lambda_0$, there exists an unbounded branch of the index $+1$ for $\lambda > \lambda_0$, no singular points for $\lambda = \lambda_0$. The simplest bifurcation diagram for this case is presented at the right-hand part of Figure 1.

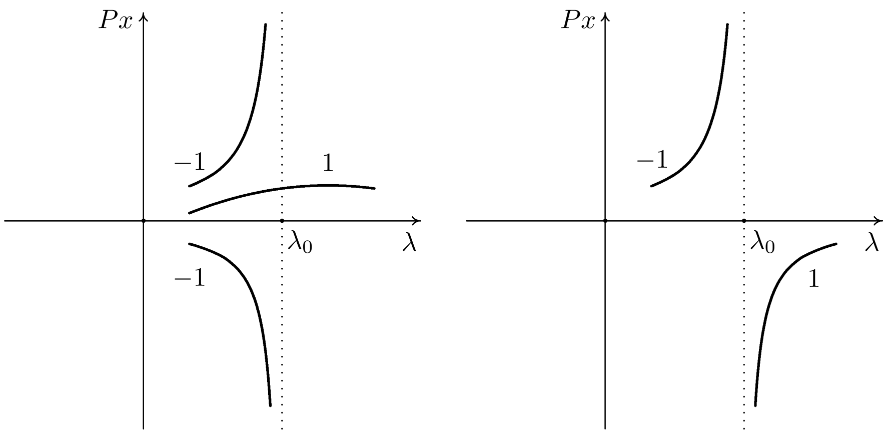

FIGURE 1. The simplest bifurcation diagrams for $x = \lambda Ax + AF(x;\lambda)$

More complex bifurcation diagrams are possible as well for these combinations of indices; additional branches of singular points may exist; generically, the total index of these branches equals 0 for any λ.

In the next subsection we present (without complete proofs) some illustrative results on bifurcations at infinity for Dirichlet BVPs. To prove them one can replace BVPs by equivalent operator equations of the type $x = Ax + AF(x;\lambda)$, $A = (-d^2/dt^2)^{-1}$, $F(x;\lambda) = f(t, x(t);\lambda)$. The linear part $x - Ax$ is degenerate and independent of the parameter, it is possible to apply Theorems 2.1 and 2.4 to study the vector fields $U_\lambda x = x - Ax - AF(x;\lambda)$.

6.2. Examples.

EXAMPLE 6.1. In this example $|\mathrm{ind}_\infty U_{\lambda_0}| = 1$, $\mathrm{ind}_\infty U_\lambda = 0$ for $\lambda \neq \lambda_0$. Such situation appears if the leading homogeneous nonlinearities of order 0 define the number $\mathrm{ind}_\infty U_\lambda = 0$ for $\lambda \neq \lambda_0$, for $\lambda = \lambda_0$ the leading terms are degenerate, and smaller terms define $\mathrm{ind}_\infty U_{\lambda_0} \neq 0$.

THEOREM 6.2. *Let λ_0 be an isolated zero of the function*

$$\overline{b}(\lambda) \stackrel{\text{def}}{=} \langle \sin t, b(t;\lambda)\rangle_\pi.$$

Suppose $|g(u;\lambda)| \le G(|u|)$, the function G satisfies (3.4) *for $k = 1$, and*

$$K \stackrel{\text{def}}{=} \int_0^\infty ug(u;\lambda_0)\,du \int_{-\infty}^0 ug(u;\lambda_0)\,du > 0. \tag{6.1}$$

Then λ_0 is an asymptotic bifurcation point for the problem

$$x'' + x + b(t;\lambda) + g(x;\lambda) = 0, \quad x(0) = x(\pi) = 0: \tag{6.2}$$

for any $\varepsilon > 0$ the set of all solutions x for all values of λ, $|\lambda - \lambda_0| < \varepsilon$ is unbounded in L^2. There exists an $\varepsilon > 0$ such that problem (6.2) *has at least two solutions for $\lambda \in \{|\lambda - \lambda_0| < \varepsilon,\ \lambda \ne \lambda_0\}$.*

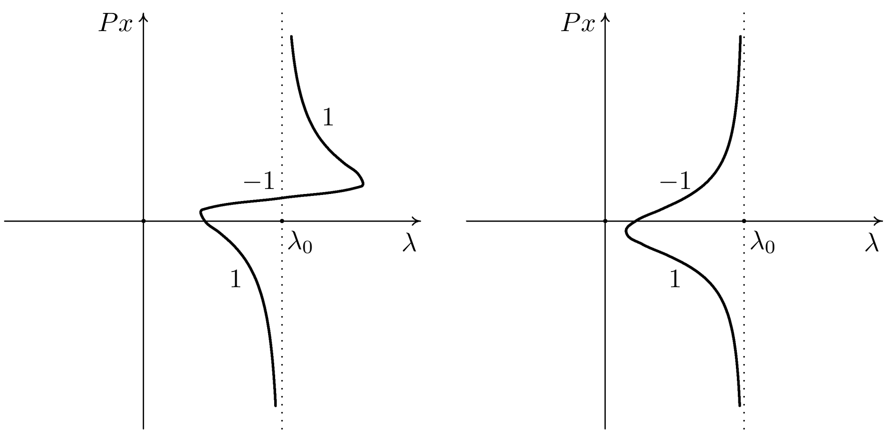

FIGURE 2. Bifurcation diagrams for equation (6.2)

The simplest bifurcation diagram that is possible under the assumptions of Theorem 6.2 if $\overline{b}(\lambda_0 + 0)\overline{b}(\lambda_0 - 0) < 0$ is drawn schematically at the left-hand part of Figure 2. If $\overline{b}(\lambda_0 + 0)\overline{b}(\lambda_0 - 0) > 0$, then the bifurcation diagram has another form.

At the right-hand part of Figure 2 there is a bifurcation diagram for (6.2) if instead of (6.1) the opposite inequality $K < 0$ holds. The Changing Index Principle is inapplicable: for all values of λ the index at infinity is well defined and is equal to 0.

From Theorem 2.4 (its assumptions are valid for the both choices of the sign) it follows that the robust zero λ_0 of the function $\overline{b}$ is an asymptotic bifurcation point for problem (6.2); for λ sufficiently close to λ_0 (either only for $\lambda > \lambda_0$ or only for $\lambda < \lambda_0$) there exist al least two solutions of (6.2).

Example 6.3. Bifurcations of the BVP

$$x'' + x + b(t) + g(x;\lambda) = 0, \quad x(0) = x(\pi) = 0, \quad \langle \sin t, b(t)\rangle_\pi = 0. \tag{6.3}$$

are defined by the functions

$$\gamma^\pm(\lambda) = \int_{\mathbb{R}^\pm} ug(u;\lambda)\,du. \tag{6.4}$$

Let $|g(u;\lambda)| \le G(|u|)$ and let G satisfy condition (3.4) for $k = 1$.

Theorem 6.4. *If either λ_0 is a robust zero of the function γ^+ and $\gamma^-(\lambda_0) \neq 0$ or λ_0 is a robust zero of the function γ^- and $\gamma^+(\lambda_0) \neq 0$, then λ_0 is an asymptotic bifurcation point for problem* (6.3).

Theorem 6.4 also follows from the Changing Index Principle. For example, let $(\lambda - \lambda_0)\gamma^+(\lambda) > 0$ for $\lambda \neq \lambda_0$ and $\gamma^-(\lambda_0) < 0$. For this case $\mathrm{ind}_\infty U_\lambda = 1$ for $\lambda > \lambda_0$ and $\mathrm{ind}_\infty U_\lambda = 0$ for $\lambda < \lambda_0$.

If λ_0 is a robust zero of both functions (6.4) simultaneously, then it is also an asymptotic bifurcation point for problem (6.3) (the index is undefined in the point λ_0).

If $\gamma^+(\lambda)\gamma^-(\lambda) > 0$, $\lambda \neq \lambda_0$, then the Changing Index Principle is applicable, $\mathrm{ind}_\infty U_\lambda = \mathrm{sign}(\gamma^+(\lambda)) \neq 0$ and $\gamma^+(\lambda + 0) = -\gamma^+(\lambda - 0)$.

If $\gamma^+(\lambda)\gamma^-(\lambda) < 0$, $\lambda \neq \lambda_0$, then $\mathrm{ind}_\infty U_\lambda = 0$ for $\lambda \neq \lambda_0$, the Changing Index Principle is inapplicable, but the value λ_0 is an asymptotic bifurcation point (6.3), this follows from Theorem 2.4.

Under the assumptions of Theorem 6.4 functions (6.4) may change their signs even if the function $g(x;\lambda)$ is independent of λ for $|x|$ large enough. This means that solutions of (6.3) appear from infinity when we change the values of the nonlinearity in a bounded w.r.t. x domain.

Example 6.5. Consider the problem

$$x'' + 4x + 3\lambda \sin t + g(x) = 0, \qquad x(0) = x(\pi) = 0. \tag{6.5}$$

Now the critical eigenfunction is $\sin 2t$, the linearized problem $x''+4x+3\lambda\sin t = 0$ has a unique solution $h^*(t) = -\lambda \sin t \perp \sin 2t$; obviously, $h^*(\frac{\pi}{2}) = -\lambda$. Suppose that $|g(x)| \le G(|x|)$ and (3.4) is valid for $k = 1$. The integrals

$$i_1 = \int_{-\infty}^{\infty} u\, g(u)\,du, \qquad i_2 = \int_{-\infty}^{\infty} g(u)\,du$$

converge absolutely; let $i_2 \neq 0$. The value $\lambda_0 = -2i_1/i_2$ of the parameter is a robust zero of the function

$$\delta(\lambda) = \int_{-\infty}^{\infty} u(g(u) + g(u-\lambda))\,du = 2i_1 + \lambda i_2,$$

by expansions (3.10) the sign of the function δ defines the sign of the index ± 1 of the vector field $x - A(4x + 3\lambda \sin t + g(x))$ at infinity. The value λ_0 is an asymptotic bifurcation point of this field and of problem (6.5).

The last example is very particular, it is interesting because only the amplitude of the non-resonant term $\sin t$ depends on the parameter λ.

7. Other boundary problems

7.1. The Dirichlet BVP on the domain with a smooth boundary. Theorems 2.1 and 2.4 are applicable to some classic BVPs for PDEs. Let Ω be a planar bounded domain, let its boundary $\partial\Omega$ be a smooth curve of the length ℓ. Consider the BVP

$$\Delta u + \lambda_0 u + b(x,y) + g(x,y,u) = 0, \quad u(x,y) = 0|_{(x,y)\in\partial\Omega},$$

where λ_0 is the leading eigenvalue of the Laplace operator $-\Delta$ with the Dirichlet boundary conditions. To compute the index at infinity of the field $u - (-\Delta)^{-1}(\lambda_0 u + b + g)$ it is convenient to analyze the asymptotics of the integrals

$$J^{\pm} = \int_\Omega e(x,y) g(x,y,\pm r e(x,y) + h(x,y))\, dx\, dy,$$
$$\Delta e + \lambda_0 e = 0, \quad e(x,y) = 0|_{(x,y)\in\partial\Omega}.$$

Let us choose a point $(x_0,y_0) \in \partial\Omega$ and let us parameterize the curve $\partial\Omega$ by the points $\varphi \in [0,\ell]$ preserving the length: $\Phi\colon [0,\ell] \to \partial\Omega$; $\Phi(0) = \Phi(\ell) = (x_0,y_0)$. Let $g(\Phi(\varphi),u)$ be the values of the function $g(x,y,u)$ for $(x,y) \in \partial\Omega$, let $\nu(\varphi)$ be the derivative of the function e in the point $\Phi(\varphi)$ toward the inner normal to $\partial\Omega$. In these notations

$$J^{\pm} = \pm r^{-2} \int_0^\ell (\nu(\varphi))^{-1}\, d\varphi \int_{\mathbb{R}^\pm} u\, g(\Phi(\varphi),u)\, du + o(r^{-2}).$$

If λ_0 is some other eigenvalue (not the leading one), then the formulas have much more cumbersome form (as well as for ODEs), the corresponding eigenfunction takes zero values not only on the curve $\partial\Omega$, but also in other points.

7.2. Non-power leading terms, the Dirichlet BVP in domains with nonsmooth boundaries. Asymptotic expansions may have close to (2.6) form

$$\langle e^*, F(\pm re + h)\rangle = \alpha^{\pm}\zeta^{\pm}(r) + o(\zeta^{\pm}), \tag{7.1}$$

where ζ^+, ζ^- are decreasing at $+\infty$ positive functions, $\alpha^{\pm}$ equals $+1$ or -1. Similar to Theorem 2.1 results make it possible to compute the index at infinity of fields satisfying (7.1) instead of (2.5).

Expansions (7.1) appear, for example, for the Dirichlet BVP on the square $\Omega = \{x, y \in [0, \pi]\}$. Denote by Au the inverse operator to the linear differential operator $-\Delta$ with the boundary conditions $u(x, y) = 0|_{(x,y)\in\partial\Omega}$. Singular points of the field $\Phi u = u - A(2u + g(x, y, u))$ coincide with the solutions of the BVP

$$u_{xx} + u_{yy} + 2u + f(x, y, u) = 0, \quad u(x, y) = 0|_{(x,y)\in\partial\Omega}.$$

The number 1 is the simple eigenvalue of $2Au$, it corresponds to the eigenfunction $e(x, y) = \sin x \sin y$. If $f(x, y, u) = b(x, y)+g(u)$, $|g(x)| \le G(|x|)$, and the integral

$$\int^{+\infty} u\, G(u) \,\ln u\, du$$

converges absolutely, then

$$\int_0^\pi \int_0^\pi e(x, y)g(\pm re(x, y) + h(x, y))\, dx\, dy = \pm\frac{4 \ln r}{r^2} \int_{\mathbb{R}^\pm} u\, g(u)\, du + o(r^{-2} \ln r).$$

Such expansions make it possible to compute the index at infinity of the field Φ and to prove various statements on the Dirichlet BVP on the square.

7.3. The Neumann problem. Consider the BVP

$$x'' + x + b(t) + g(t, x) = 0, \qquad x'(0) = x'(\pi) = 0. \tag{7.2}$$

The spectrum of the linear operator $-x''$ with these boundary conditions consists from the numbers $0, 1, 4, \dots,$ the eigenfunctions are $1, \cos t, \cos 2t, \dots.$ Let g satisfy (3.3) and let (3.4) hold for $k = 1$.

If $\langle \cos t, b(t)\rangle_\pi \ne 0$, then the index at infinity of the corresponding vector field

$$\Phi x = x - \left(-\frac{d^2}{dt^2} + 1\right)^{-1} (2x + b(t) + g(t, x))$$

is equal to 0. If $\langle \cos t, b(t)\rangle_\pi = 0$, then the linear problem $x'' + x + b(t) = 0$, $x'(0) = x'(\pi) = 0$, has a unique solution h^* satisfying $\langle \cos t, h^*(t)\rangle_\pi = 0$.

THEOREM 7.1. *Let* $\langle \cos t, b(t)\rangle_\pi = 0$ *and*

$$\int_{-\infty}^{\infty} u\, g(\pi/2, u + h^*(\pi/2))\, du \ne 0.$$

Then problem (7.2) *has at least one solution.*

Expansions (3.14) are useful to prove the last theorem. For the Neumann BVPs with nonlinearities of general type (1.1) necessary assumptions are more cumbersome; they include the right-hand side of (3.15).

7.4. The case $e' = 0$. Eigenfunctions of some BVP vanish together with their derivatives. The typical example is the BVP

$$x^{(\mathrm{IV})} - \lambda x = f(t,x), \quad x(0) = x'(0) = x(1) = x'(1) = 0.$$

If λ_0 is an eigenvalue of the differential operator $x^{(\mathrm{IV})}$ with these boundary condition, then various degenerate cases appear. Presented in this paper approach can be continued to study these cases and to compute the index at infinity.

To take into account weak nonlinearities it is necessary to analyse the asymptotics of the integrals $I_j^{\pm}$ and I_j, where $e(t) = e^{(s)}(t-t_0)^s + o((t-t_0)^s)$. The case $s > 1$ does not contain any additional difficulties, we would like to mention the expansion

$$\int_{t_0-\delta}^{t_0+\delta} e(t)\, g(t, re(t) + h(t))\, dt$$
$$= \frac{\theta r^{-1-\theta}}{|e^{(s)}|^{\theta}} \int_{-\infty}^{\infty} \operatorname{sign}(u)|u|^{\theta}\, g(t_0, u + h^*(t_0))\, du + o(r^{-1-\theta}),$$

$\theta = s^{-1}$, $h = h^* + \widetilde{h}$, $\widetilde{h} = o(1)$; the restriction on G has the opposite to (1.7) form

$$\int^{\infty} u^{\theta}\, G(u)\, du < \infty.$$

7.5. Nonlinearities with derivatives. Consider the Dirichlet BVP

$$y'' + y + \psi(t, y, y') = 0, \qquad y(0) = y(\pi) = 0 \tag{7.3}$$

and the vector field $\Psi x = x - Ax - \psi(t, Ax, (Ax)')$, where $A = -(d^2/dt^2)^{-1}$ with the Dirichlet boundary conditions. Any zero x of the field Ψ generates the solution $y = Ax$ of (7.3).

(1) Let $\psi(t,x,x') = f(t,x) + \chi(t,x,x')$. If $|\chi(t,x,x')| = o((|x| + |x'|)^{-2})$, then

$$\langle \sin t, \chi(t, r\sin t + h(t), r\cos t + h'(t))\rangle_{\pi} = o(r^{-2}),$$

the leading term of order r^{-2} in the expansions of projections is independent of χ, and all the results on solvability and bifurcations remain.

(2) Let $\psi(t,x,x') = f(t,x) + \chi(t,x')$, let $\chi(t,x') \to 0$ as $x' \to \infty$. Then it is possible to apply Lemma 3.1 with $e(t) = \cos t$ and $q(t) = \sin t$ to estimate the projection $d_r = \langle \sin t, \chi(t, r\cos t + h'(t))\rangle_{\pi}$. The function $\cos t$ equals zero on $[0,\pi]$ in the point $\pi/2$ only, therefore the number k from (3.2) is equal to 0, and $q^{(0)} = 1$. Therefore d_r has order r^{-1}:

$$d_r = \mu^* r^{-1} + o(r^{-1}), \qquad \mu^* = \int_{-\infty}^{\infty} \chi(\pi/2, u)\, du \neq 0,$$

and defines the leading terms. Let $f(t,x) = b(t) + a(t)\mathrm{sign}x + g(t,x)$, $\mu^+ = 0$, $\mu^- \neq 0$ ($\mu^\pm$ are defined by (1.4)), and

$$|\chi(t,y)| \le G(|y|), \qquad \int^\infty G(u)\,du < \infty, \qquad \lim_{|x|\to\infty} \sup_{t\in[0,\pi]} |x\, g(t,x)| \to 0. \tag{7.4}$$

Theorem 7.2. *If $\mu^*\mu^- < 0$, then* (7.3) *has at least one solution.*

If $\mu^*\mu^- > 0$, then $\mathrm{ind}_\infty \Psi = 0$ in L^2. From the last condition in (7.4) it follows that $\langle \sin t, g(t, r\sin t + h)\rangle_\pi = o(r^{-1})$.

Theorem 7.2 does not follow directly from Theorem 2.1: the vector field Ψ has the form different from $x - Ax - AF(x)$; some additional constructions are necessary.

(3) Let $\psi(t,x,x') = b(t) + g(x,x')$ and

$$\lim_{|x|+|x'|\to 0} |g(x,x')| = 0.$$

If $\overline{b} = \langle \sin t, b(t)\rangle_\pi \neq 0$, then $\mathrm{ind}_\infty \Psi = 0$. If $\overline{b} = 0$, then the index is defined by the leading terms of the expressions

$$G_r^\pm = \langle \sin t, g(\pm r\sin t + h(t), \pm r\cos t + h'(t))\rangle_\pi.$$

If at infinity $g(x,y) = R^{-\alpha} g_0(\varphi) + o(R^{-\alpha})$, $x + yi = Re^{\varphi i}$, then the asymptotics of $G_r^\pm$ are defined by the value of α; the behavior of $G_r^\pm$ is very close to the behavior of G_r for the case $e(t) \ge \varepsilon_0 > 0$ (see Section 1).

References

[1] J. M. Alonso and R. O. Ortega, *Unbounded solutions of semilinear equations at resonance*, Nonlinearity **6** (1996), 1099–1111.

[2] S. Fučik, *Solvability of Nonlinear Equations and Boundary Value Problems*, D. Reidel, Dordrecht, 1980.

[3] S. Fučik and A. Kufner, *Nonlinear Differential Equations*, Elsevier, NY, 1980.

[4] G. H. Hardy, J. E. Littlewood and G. Polia, *Inequalities*, Cambridge Univ. Press, 1988.

[5] A. M. Krasnosel'skiĭ, *A priori estimates of the norms of solutions of scalar inequalities*, Soviet Math. Dokl. **19** (1978), 1481–1485.

[6] ______, *Asymptotics of Nonlinearities and Operator Equations*, Birkhäuser, Basel, 1995.

[7] ______, *Asymptotic homogeneity of hysteresis operators*, Z. Angew. Math. Mech. **76** (1996), 313–316.

[8] A. M. Krasnosel'skiĭ and M. A. Krasnosel'skiĭ, *Vector fields in a product of spaces and applications to differential equations*, Differential Equations **33** (1997), 59–66.

[9] A. M. Krasnosel'skiĭ and J. Mawhin, *The index at infinity of some twice degenerate compact vector fields*, Discrete Contin. Dynam. Systems **1** (1995), 207–216.

[10] ______, *The index at infinity for some vector fields with oscillating nonlinearities*, Discrete Contin. Dynam. Systems **6** (2000), 165–174.

[11] M. A. Krasnosel'skiĭ and P. P. Zabreĭko, *Geometrical methods of nonlinear analysis*, A Series of Comprehensive Studies in Mathematics, vol. 263, Springer, 1984.

[12] A. C. Lazer and D. E. Leach, *Bounded perturbations of forced harmonic oscillators at resonance*, Ann. Mat. Pura Appl. **82** (1969), 49–68.

[13] J. Mawhin, *Topological Degree Methods in Nonlinear Boundary Value Problems*, AMS, 1971.

Manuscript received January 15, 2009

Alexander Krasnosel'skiĭ
Institute for Information Transmission Problems
19 B. Karetny
127994, GSP-4 Moscow, RUSSIA

E-mail address: sashaamk@iitp.ru