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INCOMPRESSIBILITY AND GLOBAL INVERSION

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Abstract. Given a local diffeomorphism $f\colon \mathbb{R}^n \to \mathbb{R}^n$, we consider certain incompressibility conditions on the parallelepiped $Df(x)([0,1]^n)$ which imply that the pre-image of an affine subspace is non-empty and has trivial homotopy groups. These conditions are then used to establish criteria for f to be globally invertible, generalizing in all dimensions the previous results of M. Sabatini.

1. Introduction

The question of deciding whether a locally invertible map admits a global inverse is one of obvious importance in mathematics. Hadamard's observation in [8] that a local homeomorphism $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is bijective if and only if it is *proper* (i.e. the pre-image of a compact set is compact) was very influential and related ideas eventually made their way into Riemannian Geometry and Nonlinear Analysis (see [2] and [5], for instance). Meanwhile, questions pertaining to the general area of Global Inversion arose in Algebra and Algebraic Geometry (the Jacobian Conjecture, see [7] and [9]), as well as Differential Equations and Dynamical Systems (the Markus–Yamabe Conjecture, see [3], [4], [6] and [20]). Global invertibility is also an important topic in applied disciplines such as Network Theory, Economics, and Numerical Analysis ([15] and [17]).

Our purpose in this paper is to use geometric methods in order to obtain new analytic conditions under which one can detect global injectivity and global

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invertibility. In particular, we extend to $\mathbb{R}^n$ the results of M. Sabatini [21] on global invertibility of *planar* local diffeomorphism.

2. Invertibility in all dimensions

We begin by recalling the results of M. Sabatini:

Theorem 2.1 ([21, Corollary 2.1]). *Let $f = (f_1, f_2) \in \mathrm{C}^1(\mathbb{R}^2, \mathbb{R}^2)$ be a local diffeomorphism. If*

$$\int_0^\infty \inf_{\|x\|=r} \|\nabla f_1(x)\|\, dr = \infty,$$

then f_1 assumes every real value and f is injective.

Theorem 2.2 ([21, Corollary 2.3]). *Let $f = (f_1, f_2) \in \mathrm{C}^1(\mathbb{R}^2, \mathbb{R}^2)$ be a local diffeomorphism. If*

$$\int_0^\infty \inf_{\|x\|=r} \frac{|\det Df(x)|}{\|\nabla f_2(x)\|}\, dr = \int_0^\infty \inf_{\|x\|=r} \frac{|\det Df(x)|}{\|\nabla f_1(x)\|}\, dr = \infty, \tag{2.1}$$

then f is a diffeomorphism of $\mathbb{R}^2$ onto $\mathbb{R}^2$.

The results above invite comparison with the classical Hadamard–Plastock theorem [18]. The later states that a Banach space local diffeomorphism $f\colon X \to X$ is bijective if

$$\sup_{x\in X} \|Df(x)^{-1}\| < \infty. \tag{2.2}$$

For instance, (2.1) implies the invertibility of the simple planar map $(x, y) \mapsto (x + y^3, y)$, a fact which is not covered by (2.2). In the finite-dimensional case the Hadamard–Plastock theorem has been substantially improved by S. Nollet and F. Xavier in [12] using degree theory. A special case of the main result in [12] states that a local diffeomorphism $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is bijective if, for every unit vector $v \in \mathbb{R}^n$,

$$\inf_{x\in\mathbb{R}^n} \|Df(x)^t v\| > 0. \tag{2.3}$$

The same simple map $(x, y) \mapsto (x + y^3, y)$ can be shown to be invertible via condition (2.3). For more on global invertibility see, for instance, [1], [6], [11]–[14], [21], [22], [24]–[26]. We highlight the approach in [1] where it is shown that the above mentioned result of [12] is a manifestation of a topological phenomenon. Namely, using intersection theory, it is shown in [1] that a local diffeomorphism $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is bijective if and only if the pre-image of every affine hyperplane is non-empty and *acyclic* (i.e. has the homology of a point). Hence there is a clear conceptual link between invertibility and the topology of the pre-images of affine subspaces. We also refer the reader to [23] for a recent survey of geometric and topological aspects of global invertibility.

An initial examination of [21] would seem to suggest that Sabatini's results reflect a purely two-dimensional phenomenon. For instance, this is the case with the Gutierrez global injectivity theorem [6], which does not extend to higher dimensions [22, Theorem 4]. Contrary to this expectation, we show in this paper that the results in [21] have a natural extension in all dimensions.

Letting $\wedge$ denote the wedge product of vectors in $\mathbb{R}^n$, the determinant of the Jacobian matrix can be expressed as

$$|\det Df(x)| = \Big\| \bigwedge_{1\le j\le n} \nabla f_j(x) \Big\| = |\det(\langle \nabla f_j(x), \nabla f_k(x)\rangle)_{1\le j,k\le n}|^{1/2}.$$

We will also consider the quantity

$$\Big\| \bigwedge_{\substack{1\le j\le n\\ j\ne i}} \nabla f_j(x) \Big\| = \Big| \det(\langle \nabla f_j(x), \nabla f_k(x)\rangle)_{\substack{1\le j,k\le n\\ j,k\ne i}} \Big|^{1/2}.$$

Hence, in the expression above, we take the wedge product of the rows of the Jacobian matrix with the i-th row removed. Our main result is the following.

THEOREM 2.3. *Let $n \ge 2$, $f = (f_1, \dots, f_n)\colon \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 local diffeomorphism, k an integer with $1 \le k \le n$, and H an affine subspace of codimension k. Assume that*

$$\int_0^\infty \inf_{\|x\|=r} \frac{\| \bigwedge_{1\le j\le n} \nabla f_j(x)\|}{\Big\| \bigwedge_{\substack{1\le j\le n\\ j\ne i}} \nabla f_j(x)\Big\|}\, dr = \infty, \tag{2.4}$$

for each $i = 1, \dots, k$. Then $f^{-1}(H)$ is non-empty and $\pi_j(f^{-1}(H)) = 0$ for $j = 0, \dots, n-k$. In particular, $f^{-1}(H)$ is non-empty and connected.

As with [21] and many works in this type of problem, our proof is based on the use of certain flows associated to the local diffeomorphism. However, our geometric idea to use the pullback of the standard hyperplane foliation and projecting the flows on the leaves is new.

As we consider hyperplanes of higher codimension, we have the following results establishing injectivity and invertibility which generalize Theorems 2.1 and 2.2, respectively.

COROLLARY 2.4. *Let $n \ge 2$ and $f = (f_1, \dots, f_n)\colon \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 local diffeomorphism such that, for each $i = 1, \dots, n-1$, we have*

$$\int_0^\infty \inf_{\|x\|=r} \frac{\| \bigwedge_{1\le j\le n} \nabla f_j(x)\|}{\Big\| \bigwedge_{\substack{1\le j\le n\\ j\ne i}} \nabla f_j(x)\Big\|}\, dr = \infty,$$

then f is injective.

Corollary 2.5. *Let $n \geq 2$ and $f = (f_1, \dots, f_n)\colon \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 local diffeomorphism such that, for each $i = 1, \dots, n$, we have*

$$\int_0^\infty \inf_{\|x\|=r} \frac{\| \bigwedge_{1\leq j\leq n} \nabla f_j(x)\|}{\left\| \bigwedge_{\substack{1\leq j\leq n \\ j\neq i}} \nabla f_j(x)\right\|}\, dr = \infty,$$

then f is bijective.

Using the estimate $\|v_1 \wedge \dots \wedge v_n\| \leq \|v_1\| \dots \|v_n\|$, we have the following consequence of Corollary 2.5.

Corollary 2.6. *A C^1 local diffeomorphism $f = (f_1, \dots, f_n)\colon \mathbb{R}^n \to \mathbb{R}^n$ is bijective provided that, for all $i = 1, \dots, n$,*

$$\inf_{x\in\mathbb{R}^n} \frac{|\det Df(x)|}{\prod_{1\leq j\leq n, j\neq i} \|\nabla f_j(x)\|} > 0. \tag{2.5}$$

We observe that Corollary 2.6 may be interpreted geometrically. Given an invertible linear transformation $A\colon \mathbb{R}^n \to \mathbb{R}^n$, the unit n-cube $[0,1]^n \subseteq \mathbb{R}^n$ is mapped by A into the n-parallelepiped $A\left([0,1]^n\right)$. The distance ℓ_i between the i-th pair of opposite $(n-1)$-faces of $A([0,1]^n)$ is given by:

$$\ell_i = \frac{\| \bigwedge_{1\leq j\leq n} Ae_j\|}{\left\| \bigwedge_{\substack{1\leq j\leq n \\ j\neq i}} Ae_j\right\|},$$

where $\{e_1, \dots, e_n\}$ is the canonical basis of $\mathbb{R}^n$. Taking $A = Df(x)$ we see that an uniform bound on the distance between opposite faces of $Df(x)([0,1]^n)$, i.e. a volume incompressibility condition, is equivalent to (2.5).

3. Geometric arguments

Recall the following estimate on the norm of two simple vectors ξ, $\eta \in \Lambda_*\mathbb{R}^n$,

$$\|\xi \wedge \eta\| \leq \|\xi\|\|\eta\|, \tag{3.1}$$

with equality if and only if $*\xi \wedge \eta = 0$. Here $*$ denotes the Hodge operator. Then (3.1) implies that

$$\|\nabla f_i(x)\| \geq \frac{\| \bigwedge_{1\leq j\leq n} \nabla f_j(x)\|}{\left\| \bigwedge_{\substack{1\leq j\leq n \\ j\neq i}} \nabla f_j(x)\right\|}.$$

Our approach will make use of the flows associated to the vector fields ∇f_i, for each i. The issue of completeness of vector fields appears naturally in the context of global invertibility (see [12], [18], [19] and [21]). The two lemmas below are certainly well known. The proof of Lemma 3.1 may be found, for instance, in [12, Lemma 2.2], while Lemma 3.2 deals with the simplest situation that arises in Morse Theory and the argument is the same one as given in [16, p. 113] (see also [10] and [20] for these results in a broader setting).

LEMMA 3.1. *Let $Z:\mathbb{R}^n \to \mathbb{R}^n$ be a non-vanishing smooth vector field which satisfies*

$$\int_0^\infty \min_{\|x\|=r} \|Z(x)\|\, dr = \infty.$$

Then the vector field $Z(x)/\|Z(x)\|^2$ is complete.

LEMMA 3.2. *Let M be a connected complete Riemannian manifold with metric g. If $h: M \to \mathbb{R}$ is smooth, $\nabla h(p)$ in non-vanishing for all $p \in M$ and $\nabla h/\|\nabla h\|^2$ is a complete vector field, then M is diffeomorphic to $\mathbb{R} \times h^{-1}(c)$. In particular, $h^{-1}(c)$ is non-empty and connected for every $c \in \mathbb{R}$*

We now provide the proof of our main result.

PROOF OF THEOREM 2.3. We start out by observing that f may be assumed to be smooth (see [22, p. 441]). Without lost of generality, we may also assume that the subspace H is parallel to the last $n-k$ axis. Indeed, consider a rotation $A \in SO(n)$ so that $A(H)$ is parallel to the last $n-k$ axis. We then establish the result for the local diffeomorphism $A \circ f: \mathbb{R}^n \to \mathbb{R}^n$. In view of the above geometric interpretation of (2.4) in terms of parallelepipeds, the quantity given by (2.4) will remain unchanged. For simplicity, let $H = \operatorname{span}\{e_i \mid i = n-k+1, \dots, n\}$. Consider $C = (c_1, \dots, c_n) \in Im(f)$, by an iteration process of restricting f onto pre-images of higher codimensional subspaces of $\mathbb{R}^n$, we will show that $C_H = (0, \dots, 0, c_{n-k+1}, \dots, c_n) \in H \cap Im(f)$ and in the process we will establish the desired topological properties of $f^{-1}(H)$.

Let $f_1: \mathbb{R}^n \to \mathbb{R}$ be the first component map of f and consider $f_1^{-1}(t)$. Note that $f_1^{-1}(t)$ is the pre-image of the affine hyperplanes with unit normal e_1 at distance t from the origin. For notational purposes, let $\mathcal{H}_1(t) := \{x \in \mathbb{R}^n \mid \langle x, e_1\rangle = t\}$ be the leaves of a codimension one foliation $\mathcal{F}_1$ of $\mathbb{R}^n$, note that $f_1^{-1}(t) = f^{-1}(\mathcal{H}_1(t))$. The integral in (2.4) ensures that,

$$\int_0^\infty \inf_{\|x\|=r} \|\nabla f_1(x)\|\, dr = \infty,$$

since

$$\|\nabla f_1(x)\| \geq \frac{\|\bigwedge_{1\leq j\leq n} \nabla f_j(x)\|}{\left\|\bigwedge_{\substack{1\leq j\leq n\\ j\neq 1}} \nabla f_j(x)\right\|}.$$

Therefore, by Lemma 3.1, $\nabla f_1/\|\nabla f_1\|^2$ is complete. Lemma 3.2 now implies that $f_1^{-1}(t) \times \mathbb{R}$ is diffeomorphic to $\mathbb{R}^n$. Since $\pi_j(A \times B) = \pi_j(A) \times \pi_j(B)$, we establish that $\pi_j(f_1^{-1}(t))$ is trivial for $j = 0, \dots, n-1$ and, in particular, $f_1^{-1}(t)$ is non-empty and connected for all $t \in \mathbb{R}$.

Next, consider the leaf of $\mathcal{F}_1$ containing C denoted by $H_1 = \mathcal{H}_1(c_1)$ and let $N_1 = f^{-1}(H_1)$ which is non-empty and connected.

Once this first step is accomplished, we consider an iteration of the argument above by restricting the map f to higher codimensional subspaces as follows.

Beginning with $m = 2$ until $m = k$, assume that we have done the process above $m-1$ times and obtained from $H_{m-1} = \mathcal{H}_1(c_1) \cap \ldots \cap \mathcal{H}_{m-1}(c_{m-1})$ the non-empty and connected set $N_{m-1} = f^{-1}(H_{m-1})$. Now take a codimension one foliation $\mathcal{F}_m$ of H_{m-1} where the leaves are the (intersection of) hyperplanes $\mathcal{H}_m(t)$. Note that $\mathcal{F}_m$ induces a foliation on $\mathbb{R}^n$. Let the component map f_m of f restricted to N_{m-1} be given by $g_m \colon N_{m-1} \to \mathbb{R}$. The gradient of g_m is

$$\nabla g_m(x) = \mathrm{Proj}_{T_x N_{m-1}}(\nabla f_m(x)).$$

Because $\nabla f_i(x) \perp T_x N_{m-1}$ for $i = 1, \ldots, m-1$, the decomposition $\nabla f_m(x) = \nabla g_m(x) + w_m(x)$, is such that $w_m(x) \in \mathrm{span}\{\nabla f_1(x), \ldots, \nabla f_{m-1}(x)\}$. Then,

$$\begin{aligned} &\left\| \bigwedge_{1\le j\le m-1} \nabla f_j(x) \wedge \nabla f_m(x) \bigwedge_{m+1\le j\le n} \nabla f_j(x) \right\| \\ &\quad = \left\| \bigwedge_{1\le j\le m-1} \nabla f_j(x) \wedge (\nabla g_m(x) + w_m(x)) \bigwedge_{m+1\le j\le n} \nabla f_j(x) \right\| \\ &\quad = \left\| \bigwedge_{1\le j\le m-1} \nabla f_j(x) \wedge \nabla g_m(x) \bigwedge_{m+1\le j\le n} \nabla f_j(x) \right\|, \end{aligned} \tag{3.2}$$

where we use the distributive properties of wedge product and the last term vanishes because it contains the wedge product of vectors in the same subspace. Inequality (3.1) implies that

$$\begin{aligned} &\left\| \bigwedge_{1\le j\le m-1} \nabla f_j(x) \wedge \nabla g_m(x) \wedge \bigwedge_{m+1\le j\le n} \nabla f_j(x) \right\| \\ &\qquad \le \|\nabla g_m(x)\| \left\| \bigwedge_{\substack{1\le j\le n\\ j\ne m}} \nabla f_j(x) \right\|. \end{aligned} \tag{3.3}$$

Combining (3.3) with (3.2), we obtain

$$\|\nabla g_m(x)\| \ge \frac{\left\| \bigwedge_{1\le j\le n} \nabla f_j(x) \right\|}{\left\| \bigwedge_{\substack{1\le j\le n\\ j\ne m}} \nabla f_j(x) \right\|}. \tag{3.4}$$

Considering the vector field $Y_m = \nabla g_m / \|\nabla g_m\|^2$, we note that Y_m may be globally defined by projecting the vector field $\nabla f_m / \|\nabla f_m\|^2$ onto the sets $f^{-1}(\mathcal{H}_{m-1}(t))$. Hence (3.4) can then be considered as a global condition and integrating both sides of (3.4) we obtain,

$$\int_0^\infty \inf_{\|x\|=r} \|\nabla g_m(x)\|\, dr \ge \int_0^\infty \inf_{\|x\|=r} \frac{\left\| \bigwedge_{1\le j\le n} \nabla f_j(x) \right\|}{\left\| \bigwedge_{\substack{1\le j\le n\\ j\ne m}} \nabla f_j(x) \right\|}\, dr = \infty.$$

Lemma 3.1 establishes that Y_m is complete and notice that the integral curves of Y_m will remain on the same level set of f_m. Next, by Lemma 3.2 we have that $g_m^{-1}(t) \times \mathbb{R}$ is diffeomorphic to N_{m-1} and thus $\pi_j(g_m^{-1}(t))$ is trivial for $j = 0, \dots, n-m$ and, in particular $g_m^{-1}(t)$ is non-empty and connected for all $t \in \mathbb{R}$.

Next, consider the particular leaf of $\mathcal{F}_m$ containing C, denoted by $H_m = H_{m-1} \cap \mathcal{H}_m(c_m)$. Now let

$$N_m = g_m^{-1}(c_m) = (f_1^{-1}(c_1) \cap \ldots \cap f_{m-1}^{-1}(c_{m-1})) \cap f_m^{-1}(c_m).$$

We repeat the above argument until $m = k$, thus showing that

$$N_k = f_1^{-1}(c_1) \cap \ldots \cap f_{n-1}^{-1}(c_k) = f^{-1}(H)$$

is non-empty and has the property that $\pi_j(f^{-1}(H))$ is trivial for $j = 0, \dots, n-k$. In particular, $f^{-1}(H)$ is non-empty and connected. □

Corollary 2.4 follows by observing that when $k = n - 1$, we obtain as a conclusion in Theorem 2.3 that the pre-image of a line is connected. Then a simple argument in [12, p. 24] shows that C is assumed exactly once, thus ensuring that f is injective. Corollary 2.5 follows from Corollary 2.4 and the observation that a codimension n subspace reduces to a point. Indeed, since the pre-image of a point is non-empty by Theorem 2.3, f is both injective and surjective.

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