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MULTIPLE SOLUTIONS FOR THE MEAN CURVATURE EQUATION

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Abstract. We perturb the mean curvature operator and find multiple critical points of functionals that are not even. As a consequence we find infinitely many solutions for a quasilinear elliptic equation. The generality of our results are also reflected in the relaxed hypotheses related to the behavior of the functions around zero and at infinity.

1. Introduction

In this paper we show that the number of solutions of the mean curvature equation

$$\begin{cases} -\mathrm{div}\left(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega. \end{cases} \tag{1.1}$$

increases as the parameter $\lambda > 0$ increases. We assume that $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous and there is a constant

(1.2) $c > 0$ and $g \in L^\infty(\Omega)$, $g > c$ such that

$$\lim_{u\to 0} \frac{f(x,u)}{|u|^{p-1}u} = g(x) \text{ uniformly in } x,\ 1 < p < \frac{N+2}{N-2}.$$

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Our main result reads as follows.

Theorem 1.1. *For every* $m \in \mathbb{N}$ *there is a* λ_0 *such that problem* (1.1) *has at least* m *solutions if* $\lambda > \lambda_0$. *Moreover, the solutions tend to* 0 *in* $W^{1,\infty}(\Omega)$ *as* $\lambda \to \infty$.

In [6] the authors proved the existence of only one positive solution of mountain pass type for λ large under the assumption that $f(s)/s$ is increasing in s. There is no need to assume such an assumption here and our f may depend on x. An interesting fact is that we use the technic that resemblances the perturbation from symmetry of [8] and [12] to prove the existence of multiple solutions for large λ, but f do not need to be odd, we just need f to be asymptotically odd. And to fall into an appropriate functional setting, we perturb the mean curvature operator.

There are many surfaces of constant mean curvature, which are unbounded. The surfaces need not to be C^2 up to the boundary, see [9] for a study on convex domains. A classical assumption to yielding to $C^{2,\alpha}(\overline{\Omega})$ solutions is $(N/(N-1))|H| \leq K$, where K is the mean curvature of the boundary $\partial\Omega$ and $f \equiv H$, see [10]. Non-convex domains are treated in [13].

Our result should be compared with those in BV functional setting, where critical points of the energy functional are found, but they need not to correspond to weak solutions of the mean curvature equation, see [5], [7].

Here we adopt a truncation of the mean curvature operator, like in [6]. We also use some ideas from [2], that allows us to find sequences of critical values of even functionals bounding the energy functional corresponding to (1.1). These critical values lead to weak solutions of (1.1).

The proof of our main result is splitted in a series of lemmas in the next section.

2. Proof of Theorem 1.1

It will be convenient for our purposes to define $\alpha(t) = 1/(\sqrt{1+t})$ and the truncation

$$\phi(t) = \begin{cases} \alpha(t) & \text{for } t \leq M, \\ \alpha(M) & \text{for } t \geq M. \end{cases}$$

for some constant $M > 0$. We will study the truncated problem

$$\begin{cases} -\operatorname{div}(\phi(|\nabla u|^2)\nabla u) = \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega. \end{cases} \tag{2.1}$$

Let $h(x,u) = f(x,u) - g(x)|u|^{p-1}u$. By condition (1.2) we have

$$\lim_{u \to 0} h(x,u)/|u|^p = 0 \quad \text{uniformly in } x.$$

We now start by constructing the perturbative scheme to treat problem (2.1). If $a > 0$ is sufficiently small then there is a constant $C_a > 0$ such that $|h(x,u)| \le C_a|u|^p$ for $|u| \le 2a$ and $C_a \to 0$ as $a \to 0$.

Let β_a be a C^∞ function such that $\beta_a(u) = 1$ for $|u| \le a$, $\beta_a(u) = 0$ for $|u| \ge 2a$, and $0 \le \beta_a \le 1$. Define $h_a(x,u) = \beta_a(u)h(x,u)$ for every $(x,u) \in \Omega \times \mathbb{R}$ and consider the problem

$$\tag{2.2} \begin{cases} -\operatorname{div}(\phi(\lambda^{2/(p-1)}|\nabla u|^2)\nabla u) & \\ \qquad = g(x)|u|^{p-1}u + \lambda^{p/(p-1)}h_a(x,\lambda^{-1/(p-1)}u) & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega. \end{cases}$$

LEMMA 2.1. *If u is a solution of* (2.2) *and* $\|\nabla u\|_{L^\infty} \le a\lambda^{1/(p-1)}$, *then* $v(x) = \lambda^{-1/(p-1)}u(x)$ *is a solution of* (2.1) *and then of* (1.1).

We define now our functional framework. Let

$$h_{a,\lambda}(x,u) = \lambda^{p/(p-1)}h_a(x,\lambda^{-1/(p-1)}u)$$

and

$$H_{a,\lambda}(u) = \int_0^u h_{a,\lambda}(x,s)ds.$$

Observe that

$$|h_{a,\lambda}(x,u)| \le C_a|u|^p, \qquad |H_{a,\lambda}(x,u)| \le \frac{C_a}{p+1}|u|^{p+1}$$

and

$$|H_{a,\lambda}(x,u)| \le \frac{2^{p+1}}{p+1}a^{p+1}C_a\lambda^{(p+1)/(p-1)}.$$

Define

$$\tag{2.3} c(a,\lambda) := \frac{C_a}{p+1}2^{p+1}a^{p+1}\lambda^{(p+1)/(p-1)}|\Omega|.$$

Let $\Phi(s) = \int_0^s \phi(t)\,dt$. The following expressions define functionals over $H_0^1(\Omega)$:

$$\begin{aligned} I_1(u) &= \frac{\alpha(M)}{2}\int_\Omega \Phi(|\nabla u|^2) - \frac{g(x)}{p+1}|u|^{p+1}, \\ I_2(u) &= \frac{1}{2}\int_\Omega \Phi(|\nabla u|^2) - \frac{g(x)}{p+1}|u|^{p+1} \end{aligned}$$

and

$$J_\lambda(u) = \frac{1}{2\lambda^{2/p-1}}\int_\Omega \Phi(\lambda^{2/p-1}|\nabla u|^2) - \int_\Omega \frac{g(x)}{p+1}|u|^{p+1} - \int_\Omega H_{a,\lambda}(x,u).$$

Then

$$I_1(u) - c(a,\lambda) \le J_\lambda(u) \le I_2(u) + c(a,\lambda).$$

Notice that if $v = \alpha(M)^{1/p-1}u$, then $I_1(v) = \alpha(M)^{(p+1)/(p-1)}I_2(u)$.

Notice that I_1 and I_2 are even functionals. We use the minimax variational methods to find multiple critical points of J_λ.

We define the sets

$$V_k = \mathrm{span}\{\varphi_1, \dots, \varphi_k\} \quad \text{and} \quad Z_k = \mathrm{span}\{\varphi_k, \varphi_{k+1}, \dots\}$$

by splitting $X := H_0^1(\Omega)$ into $(\varphi_k)_{k=1,2,\dots}$, which are the eigenfunctions of the Laplacian with $\|u\|_{L^2} = 1$. Throughout the paper, $\|\cdot\|$ represent the H_0^1 norm.

Lemma 2.2. *There are sequences $r_k > 0$ and $\rho_k > 0$ satisfying $\rho_k > r_k$, $\rho_{k+1} > \rho_k$, $r_{k+1} > r_k$ and $r_k \to \infty$ as $k \to \infty$,*

$$\max_{u\in V_k, \|u\|\geq \rho_k} I_2(u) < 0 \quad \textit{and} \quad \inf_{u\in Z_k, \|u\|=r_k} I_2(u) \to \infty.$$

Proof. Let $\beta_k = \sup_{u\in Z_k, \|u\|=1} (|\nabla u|^{p+1})^{1/(p+1)}$ then

$$\bigg(\int_\Omega |u|^{p+1}\bigg)^{1/(p+1)} \leq \beta_k \bigg(\int_\Omega |\nabla u|^2\bigg)^{1/2}.$$

Choose $r_k = (c\beta_k^{p+1})^{-1/(p-1)}$ if $u \in Z_k$ and $\|u\| = r_k$, where c was defined in (2.3), then

$$I_2(u) \geq \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{c}{p+1}\beta_k^{p+1}\bigg(\int_\Omega |\nabla u|^2\bigg)^{(p+1)/2}$$

and

$$I_2(u) \geq \bigg(\frac{1}{2} - \frac{1}{p+1}\bigg)(c\beta_k^{p+1})^{-2/(p-1)}.$$

By Lemma 3.8 of [14], one has $\beta_k \to 0$. Moreover, since V_k is finite dimensional

$$\bigg(\int_\Omega |u|^{p+1}\bigg)^{1/p+1} \geq c_k \bigg(\int_\Omega |\nabla u|^2\bigg)^{1/2}$$

for all $u \in V_k$ with $c_k \to 0$, we get

$$I_2(u) \leq \frac{1}{2}\int_\Omega |\nabla u|^2 - cc_k^{p+1}\bigg(\int_\Omega |\nabla u|^2\bigg)^{(p+1)/2} dx$$

then we have $I_2(u) < 0$ if

$$\bigg(\int_\Omega |\nabla u|^2\bigg)^{1/2} > \bigg(\frac{1}{2cc_k^{p+1}}\bigg)^{1/p-1}.$$

Without loss of generality we choose ρ_k such that $\rho_k > r_k$ and $\rho_{k+1} > \rho_k$. $\square$

Observe that

$$\max_{u\in V_k, \|u\|\geq \rho_k(\alpha(M)^{1/p-1})} I_1(u) < 0, \quad \text{and} \quad \inf_{u\in Z_k, \|u\|=r_k(\alpha(M)^{1/p-1})} I_1(u) \to \infty$$

Define

$$\begin{aligned} B_k^1 &= \{u \in V_k : \|u\| \le \rho_k \alpha(M)^{1/p-1}\}, & N_k^1 &= \{u \in Z_k : \|u\| = r_k \alpha(M)^{1/p-1}\}, \\ B_k^2 &= \{u \in V_k : \|u\| \le \rho_k\}, & N_k^2 &= \{u \in Z_k : \|u\| = r_k\}, \end{aligned}$$

and

$$\begin{aligned} \Lambda_1^i &= \{\psi \in C(B_1^i, X) : \psi \text{ is odd and } \psi|_{\partial B_1^i} = \mathrm{id}\}, \\ \Lambda_k^i &= \{\psi \in C(B_k^i, X) : \psi \text{ is odd, } \psi|_{\partial B_{k-1}^i} \in \Lambda_{k-1}^i \text{and } \psi|_{\partial B_k^i} = \mathrm{id}\}, \quad i = 1, 2. \end{aligned}$$

The following lemma appears in [12] and [14].

Lemma 2.3. *If $\psi \in C(B_k^i, X)$, ψ is odd and $\psi|_{\partial B_k^i} = \mathrm{id}$, then $\psi(B_k^i) \cap N_k^i \neq \emptyset$ for $i = 1, 2$.*

Define $c_k^i = \inf_{\psi \in \Lambda_k^i} \max_{u \in B_k^i} I_i(u)$. Take $\psi \in \Lambda_k^i$, by above lemma there exists $u^i \in B_k^i$ such that $\psi(u^i) \in N_k^i$. Then, for every $\psi \in \Lambda_k^i$,

$$c_k^i \ge \max_{u \in B_k^i} I_i(\psi(u)) \ge I_i(\psi(u)) \ge \inf_{v \in N_k^i} I_i(v) \to \infty.$$

implying

$$c_k^i \ge \inf_{v \in N_k^i} I_i(v) \to \infty.$$

For $\psi \in \Lambda_k^2$, define $\varphi(u) = \alpha(M)^{1/p-1} \psi(\alpha(M)^{-1/p-1} u)$ belongs to Λ_k^1 and then

$$I_1(\widetilde{\varphi}(u)) = \alpha(M)^{(p+1)/(p-1)} I_2(\psi(v))$$

where $v = \alpha(M)^{-1/p-1} u$. A calculation shows $c_k^1 = \alpha(M)^{(p+1)/(p-1)} c_k^2$.

Define

$$B_{k+1}^+ = \{u = v + t\varphi_{k+1} : v \in V_k,\ t \ge 0,\ \|u\| \le \rho_{k+1}\},$$

$$\begin{aligned} \Pi_k = \Big\{\phi \in C(B_{k+1}^+, H_0^1) : \phi|_{B_{k+1}^+ \cap V_k} &\text{ is odd, } \phi|_{B_k^+} \in \Lambda_k^2, \\ &\phi|_{\partial B_{k+1}^+ - B_k^2} = \mathrm{id},\ \max_{u \in \partial B_{k+1}^+} I_2(\phi(u)) < c_k^2 + 1/2\Big\}. \end{aligned}$$

Lemma 2.4. $\Pi_k \neq \emptyset$.

Proof. Let $\psi \in \Lambda_k^2$ such that $\max_{u \in B_k^2} I_2(\psi(u)) < c_k^2 + 1/2$. Extend ψ to a function $\widetilde{\psi}$ such that $\widetilde{\psi}|_{B_{k+1}^2 \cap V_k}$ is odd. This is possible since $\phi|_{\partial B_{k+1}^+ - B_k^2} = \mathrm{id}$. Therefore $\max_{u \in \partial B_{k+1}^+} I_2(\phi(u)) < c_k^2 + 1/2$. Thus, $\widetilde{\psi} \in \Pi_k$. $\square$

Lemma 2.5. *For a given* $K > 0$ *and* $N \in \mathbb{N}$ *there is* $m \geq N$ *such that* $c_{m+1} > c_m + K$.

Proof. By contradiction, if there is $N \in \mathbb{N}$ such that $c_{m+1} \leq c_m + K$ for every $m \geq N$. This implies $c_{N+q} \leq c_N + qK$ for every $q \geq 1$. This is a contradiction to the fact that $c_{N+q} \geq c(N+q)^\gamma$ for some $\gamma > 1$ and every sufficiently large q, see [1] or [12, p. 124]. □

Lemma 2.6. *There is a sequence of indexes* $k_1 < k_2 \dots$ *such that* $c^1_{k_1+1} - 1 > 1$, $c^2_{k_n+1} > c^2_{k_n} + 3$ *for* $n \geq 1$ *and* $c^1_{k_n+1} - 1 > d_{k_{n-1}} + 1$ *for* $n \geq 2$ *where*

$$d_{k_n} = \inf_{\phi \in \Pi_{k_n}} \max_{u \in B^+_{k_n+1}} I_2(\phi(u)).$$

Proof. The proof follows by induction and applying the previous lemma. □

Define $\widetilde{c}_{k_n} = \inf_{\phi \in \Pi_{k_n}} \max_{u \in B^+_{k_n+1}} J_\lambda(\phi(u))$. By our construction, $\widetilde{c}_{k_n} < d_{k_n} + 1$ and $\widetilde{c}_{k_n} > c^1_{k_n+1} - 1 > d_{k_{n-1}} + 1 > \widetilde{c}_{k_{n-1}}$ we get $\widetilde{c}_{k_n} > \widetilde{c}_{k_{n-1}}$. We remark that $\widetilde{c}_{k_i}$, depend on λ and a, but $c^i_{k_n}$ and d_{k_n} do not.

Lemma 2.7. *If* a *is small enough and* λ *is such that* $c(a,\lambda) = 1$, *then* $\widetilde{c}_{k_i}(\lambda)$ *is a critical value.*

Proof. Suppose on the contrary, then by the Deformation Lemma for every sufficiently small $\varepsilon > 0$ there exists $\eta := \eta_\varepsilon$ such that $\eta(J_\lambda^{\widetilde{c}_{k_i}+\varepsilon}) \subset J_\lambda^{\widetilde{c}_{k_i}-\varepsilon}$, we have used the notation $J^d_\lambda = \{u : J_\lambda(u) \leq d\}$ and $\eta = \mathrm{id}$ in $H^1_0 - \{\widetilde{c}_{k_i} - 2\varepsilon \leq J_\lambda(u) \leq \widetilde{c}_{k_i} + 2\varepsilon\}$. For $\phi \in \Pi_{k_i}$

$$\max_{u \in B^+_{k_i+1}} J_\lambda(\phi(u)) \leq \widetilde{c}_{k_i} + \varepsilon.$$

Take $u \in \partial B^+_{k_i+1}$, then $I_2(\phi(u)) < c_{k_i} + 1/2$ and

$$\begin{aligned} J_\lambda(\phi(u)) &\leq I_2(\phi(u)) + 1 \leq c^2_{k_i} + 3/2 \\ &\leq c^2_{k_i+1} + 3/2 - 3 \leq \alpha^{-(p+1)/(p-1)} c^1_{k_i+1} - 3/2 \\ &\leq \alpha^{-(p+1)/(p-1)} (\widetilde{c}_{k_i+1} - 1) - 3/2 \leq \widetilde{c}_{k_i} - 2\varepsilon \end{aligned}$$

if

$$\widetilde{c}_{k_i}\left(\frac{1}{\alpha^{(p+1)/(p-1)}} - 1\right) < \frac{3}{2} - \frac{1}{\alpha^{(p+1)/(p-1)}} - 2\varepsilon.$$

But $\widetilde{c}_{k_i} < d_{k_m} + 1$, then the right hand side of the expression above converges to zero as $M \to 0$, then there is $M_0 > 0$ such that

$$J_\lambda(\phi(u)) < \widetilde{c}_{k_i} - 2\varepsilon$$

for $M \leq M_0$ and every $i = 1, \dots, m$. Thus, $\eta \circ \phi(u) = \phi(u)$ and $\eta \circ \phi \in \Pi_{k_i+1}$, a contradiction. □

The following boundedness is standard, see e.g. [12].

LEMMA 2.8. *Assume $N \geq 3$. Let $q \in C(\mathbb{R})$ and $|q(u)| \leq k|u|^{p-1}$ for every $u \in \mathbb{R}$ and some constant $k > 0$. If $u \in H_0^1(\Omega)$ is a solution of the equation $-\operatorname{div}(\Phi(|\nabla u|^2)\nabla u) = q(u)$ in Ω, then there are constants $0 < M < M_0$ and $C = C(p, k, N, M_0) > 0$ such that*

$$\|u\|_{L^\infty(\Omega)} \leq C\|u\|_{L^p(\Omega)}^{2(p+1)/((N-2)(1-p)+4)}.$$

If $N = 2$, there are constants $0 < M < M_0$ and $C = C(\Omega, p, M_0) > 0$ and $\alpha = \alpha(p) > 0$ such that $\|u\|_{L^\infty(\Omega)} \leq C\|u\|_{L^p(\Omega)}^{\alpha}$.

PROOF OF THEOREM 1.1. Let $u := u_{k_i}$ a solution in level $\widetilde{c}_{k_i}$, then

$$\int_\Omega \Phi(|\nabla u|^2\lambda^{2/p-1})|\nabla u|^2 = \int_\Omega |u|^{p+1}g(x) + \int_\Omega h_{a,\lambda}(x,u)u,$$

$$\int_\Omega |\nabla u|^2 \geq \int_\Omega |u|^{p+1}g(x) - C_a\int_\Omega |u|^{p+1},$$

$$\begin{aligned}\widetilde{c}_{k_i} &= \frac{1}{2}\int_\Omega \Phi(|\nabla u|^2\lambda^{2/p-1}) - \frac{1}{p+1}\int_\Omega |u|^{p+1}g(x) - \int_\Omega H_{a,\lambda}(x,u)\\ &\geq \frac{\alpha(M)}{2}\int_\Omega |\nabla u|^2 - \frac{1}{p+1}\int_\Omega |u|^{p+1}g(x) - \frac{C_a}{p+1}\int_\Omega |u|^{p+1}.\end{aligned}$$

Then

$$\begin{aligned}\frac{\alpha(M)}{2}\int_\Omega |u|^{p+1}g(x) - C_a\alpha(M)\int_\Omega |u|^{p+1}&\\ - \frac{1}{p+1}\int_\Omega |u|^{p+1}g(x) - \frac{C_a}{p+1}\int_\Omega |u|^{p+1} &\leq \widetilde{c}_{k_i},\end{aligned}$$

$$\left(\frac{\alpha(M)}{2} - \frac{1}{p+1}\right)\int_\Omega |u|^{p+1}g(x) - C_a\left(\alpha(M) + \frac{1}{p+1}\right)\int_\Omega |u|^{p+1} \leq \widetilde{c}_{k_i} \leq d_{k_m}.$$

For $M \leq M_0$, $\int_\Omega |u|^{p+1} \leq c(m, M_0, g)$.

By Lemma 2.8, $\|u\|_{L^\infty(\Omega)} \leq c_1(m, M_0)$, and then $\|\nabla u\|_{L^\infty(\Omega)} \leq c_2(m, M_0)$.

Since $c(a,\lambda) = 1$, then $\lambda a^{1/p-1} = (2^{p+1}/(p+1)C_a)^{-1/p+1}$. Choose a sufficiently small (and this implies that λ is large) such that $\lambda a^{1/p-1} \geq c_2(m, M_0)$.□

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