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TRAVELING FRONT SOLUTIONS IN NONLINEAR DIFFUSION DEGENERATE FISHER-KPP AND NAGUMO EQUATIONS VIA THE CONLEY INDEX

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Abstract. Existence of one dimensional traveling wave solutions $u(x,t) := \phi(x-ct)$ at the stationary equilibria, for the nonlinear degenerate reaction-diffusion equation $u_t = [K(u)u_x]_x + F(u)$ is studied, where K is the density coefficient and F is the reactive part. We use the Conley index theory to show that there is a traveling front solutions connecting the critical points of the reaction-diffusion equations. We consider the nonlinear degenerate generalized Fisher-KPP and Nagumo equations.

1. Introduction

In this paper, we discuss the problem of existence of traveling front for the following diffusion reaction scalar equation:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K(u)\frac{\partial u}{\partial x}\right) + F(u), \quad x \in \mathbb{R},\ t \geq 0, \tag{1.1}$$

with $u(x,0) = u_0(x)$, $0 \leq u_0(x) \leq 1$, for all $x \in \mathbb{R}$, in the following two situations:

Case 1. Degenerate generalized Fisher-KPP equation: the diffusion function K and the reaction function F are defined on the interval $[0,1]$ and satisfy the

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conditions:

$$\text{(I)}\qquad \begin{cases} (1) & F(0) = F(1) = 0,\ F > 0 \text{ in } (0,1), \\ (2) & K, F \in C^1[0,1],\ F'(0) > 0 \text{ and } F'(1) < 0, \\ (3) & K(x) = 0 \text{ for } x = 0, 1 \text{ and } K > 0 \text{ in } (0,1). \end{cases}$$

Case 2. Degenerate generalized Nagumo equation: for a given real number $\alpha \in (0,1)$, the diffusion function K and the reaction function F are defined on the interval $[0,1]$ and satisfy the conditions:

$$\text{(II)}\qquad \begin{cases} (1) & F(0) = F(\alpha) = F(1) = 0,\ F < 0 \text{ in } (0,\alpha),\ F > 0 \text{ in } (\alpha,1), \\ (2) & K, F \in C^1[0,1],\ F'(0) < 0,\ F'(\alpha) > 0 \text{ and } F'(1) < 0, \\ (3) & K(x) = 0 \text{ for } x = 0, \alpha, 1 \text{ and } K > 0 \text{ in } (0,\alpha) \cup (\alpha,1). \end{cases}$$

The degeneracy condition on the diffusion coefficient term K imply that equation (1.1) is of parabolic type for all nonzero u of F and degenerates into an ODE at the zeros of F. Degenerate diffusion appears in models for biological invasion to take into account population density pressure. Equation (1.1) can be seen as a generalization of those arising as models for different biological, physical or chemical systems (see for example: [4], [22], [28], [29], [25], [23] and the references cited therein).

Although very detailed analysis of *traveling wave solutions* (t.w.s.) to the constant diffusion equation with nonlinear kinetic term already exists in the literature (see [21], [12], [11], [16]), this is not the case for generalized degenerate nonlinear Fisher-KPP or Nagumo reaction-diffusion equations. Since the pioneering work of R. A. Fisher [8] and A. Kolmogorov et al. [16] on t.w.s. for a constant diffusion equation with quadratic-like kinetic part, much research has been developed to try to extend this analysis to more general reaction-diffusion equations. Particular cases of a density dependent diffusion coefficient vanishing at one point ($u = 0$), with reactive part having the quadratic properties listed above, have been studied (see for example D. G. Aronson (1980) [2], J. D. Muray (1989) [23], De Pablo et al. (1991) [7], P. Grindrod et al. (1987) [10], Y. Hosono (1985) [14]). F. Sanchez-Garduno and P. K. Maini (1995 and 1997) ([28], [29]) considered the same equation (1.1) with a degeneracy at zero: $K(0) = 0$. Their results focus essentially on the existence of a critical value $c^* > 0$ of c for which the above equation has a t.w.s. of sharp type.

In this paper we study the existence of t.w.s. for equation (1.1) under the set of conditions (I) and (II). Using the Conley index, we give in the first a complete description of the t.w.s. for equation (1.1), connecting the critical points, in the cases of Fisher-KPP and Nagumo equations without degeneracy. When the diffusion term degenerate, by using the technique of D. Terman [32], we obtain the existence of traveling front solutions of equation (1.1), with their corresponding speeds, connecting the critical points zeros of F, in Fisher-KPP and Nagumo situations. The techniques we use are based on careful choices of isolating neighbourhood and application of the Conley index. Other application of the Conley index to systems of reaction-diffusion equations can be found for example in ([6], [9], [19], [20]).

We should mention that the equation studied in this paper is more general than in ([28], [29]). For example, equation (1.1), with $K(u) = u^m(1-u)^n(u-\alpha)^{2s}$ for an arbitrary positive constants m and n and an integer s is not covered by equations in [28] and [29]. Our method is topological and can be exploited to consider systems in $\mathbb{R}^2$ (see for example Gardner [9], [19], [20]). The advantage of topological methods is that knowledge on the boundary of isolating neighbourhoods can be used to get information about the structure of the differential equation in the interior.

A traveling front connecting two critical points v_0 to v_1 of (1.1) is a solution of (1.1) of the form

$$u(x,t) := \phi(x - ct) \tag{1.2}$$

satisfying the following properties:

(i) $\phi \in C^2(\mathbb{R})$, $\lim_{\zeta\to-\infty}\phi(\zeta) = v_0$, $\lim_{\zeta\to\infty}\phi(\zeta) = v_1$,
(ii) $\phi' < 0$, $v_1 \le \phi(\zeta) \le v_0$, $\zeta \in \mathbb{R}$,

where $c \in \mathbb{R}$ is the speed or velocity of the front.

A traveling wave connecting two critical points v_0 to v_1 of (1.1) is a solution of (1.1) in the form (1.2) satisfying only (i). By postulating a solution $u(x,t) = \phi(x - ct)$ of equation (1.1), where $\phi\colon \mathbb{R} \to \mathbb{R}$, $\xi \to \phi(\xi)$ is some function, one obtains the following ordinary differential equation:

$$K(\phi)\phi'' + c\phi' + K'(\phi)(\phi')^2 + F(\phi) = 0 \tag{1.3}$$

By setting $v = \phi'$, equation (1.3) can be written as the singular ODE system

$$\begin{cases} \phi' = v, \\ K(\phi)v' = -cv - K'(\phi)v^2 - F(\phi) \end{cases} \tag{1.4}$$

The singularity can be removed by introducing the parameter $\tau = \tau(\xi)$ into (1.4) such that

$$\frac{d\tau}{d\xi} = \frac{1}{K(\phi(\tau(\xi)))}$$

(see: [1], [30])

If we define $\phi(\xi) = \varphi(\tau(\xi))$ and $v(\xi) = w(\tau(\xi))$ and denote by dot the derivative with respect to τ, system (1.4) can be re-written as the following non singular system

$$\begin{cases} \dot\varphi = K(\varphi)w, \\ \dot w = -cw - K'(\varphi)w^2 - F(\varphi), \end{cases}$$

or

$$X' = f(X) \tag{1.5}$$

with $X = (x, y)$ and $f(x, y) = (K(x)y, -cy - K'(x)y^2 - F(x))$.

The paper is organized as follows. At the first, we consider equation (1.1) in the cases of Fisher-KPP and Nagumo equations, without degeneracy. To apply the Conley index theory, we construct an isolating neighbourhood for an interval of c values. Using the summation property of Conley index and invariance of the triangle $\Omega = \{(x, y) \in \mathbb{R}^2,\ 0 < x < 1,\ mx < y < 0\}$, m is a chosen negative constant, we show that there exist a number $c_0 > 0$ (the minimal speed of the fronts) such that, for every $c > c_0$, we have one traveling front solution of Fisher-KPP equation connecting the critical points $u_1 = (1, 0)$ to $u_0 = (0, 0)$. In the case of Nagumo equation, by using a theorem of Conley and Salamon (Propositions 5.4 and 5.5 in the Appendix), we show that there is a single speed c_1 for which a front connecting the two critical points having same Conley index, exists and we give a complete description of the set of wave like bounded solutions. Section 3 deals with equation (1.1) in the cases of degenerate generalized Fisher-KPP and Nagumo equations. We consider the equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left((K(u) + \mu)\frac{\partial u}{\partial x}\right) + F(u), \quad x \in \mathbb{R},\ t \geq 0, \tag{1.1_μ}$$

If, for each $\mu > 0$, exist a traveling front solutions for $(1.1)_\mu$, connecting a critical points v_1 to v_0, corresponding to a specific value $c_1(\mu) > 0$ of the speed c, we must then prove that as $\mu \to 0$, some subsequence of the solutions converges to a traveling front solution for the degenerate equation. We exploit the above property to extend some results on t.w.s. from nondegeneracy to degeneracy cases. Section 4 is a Conclusion and Section 5 is an Appendix.

In the Appendix, a brief review of some of the relevant portions of the Conley index theory and gradient like-systems is provided.

2. Basic facts about the equations

Let's show that the Fisher-KPP and Nagumo equations, without degeneracy are gradient like-equations and they have an isolating neighbourhood. Consider system (1.5) under the set of conditions (1) and (2) in (I) or (II), but instead of (3), suppose that $K > 0$ in $[0, 1]$.

2.1. A Lyapunov function. A property is said to be satisfied fast of a given point u if it is satisfied out side a neighbourhood of u. Let q, M, L, r a positive constants such that:

(iii) $\sup_x |K'| \leq M$, $\inf_x K \geq q$, $1 + \sup_x (F/K)' \leq L$, $\inf_x |F/K| > r$, fast of the critical points of (1.5).

One has

Proposition 2.1. *If $c \neq 0$, then*

$$V(x,y) := L\int_0^x K(t)F(t)\,dt + c^2\int_0^x \frac{F(t)}{K(t)}\,dt + cyF(x) + \frac{L}{2}K^2(x)y^2$$

is a Lyapunov function for the system (1.5).

Proof. On solutions of (1.5) one has

$$\frac{d}{dt}V(x,y) = -cK^2(x)\left\{y^2\left(L - \left(\frac{F(x)}{K(x)}\right)'\right) + \left(\frac{F(x)}{K(x)}\right)^2\right\}.$$

So, in view of (iii), $dV(x,y)/dt < 0$ if $c > 0$ and $dV(x,y)/dt > 0$ if $c < 0$.

It follows that if $c \neq 0$, the only bounded solutions are critical points and orbits connecting the critical points. In contrast, when $c = 0$, we have a Hamiltonian system. □

2.2. An isolating neighbourhood. In order to apply the Conley index theory to our problem, we need a compact invariant set. We want the set of bounded solutions S_c to be compact. The following results guarantee this (see Proposition 5.2 in the Appendix).

Proposition 2.2. *For every $c_0 > 0$ and $\delta_0 > 0$ there exists $\varepsilon > 0$ such that,*

$$\frac{d}{dt}V(x,y) \leq -\varepsilon\|f(x,y)\|$$

for $c \geq c_0$ and $\|(x,y) - (a,b)\| \geq \delta_0$, with (a,b) is an arbitrary critical point of (1.5).

Proof. Denotes by $\|(x,y)\| =: \sup(|x|,|y|)$ and let $\delta_0 > 0$ and $(x,y) \in \mathbb{R}^2$ such that $\|(x,y) - (a,b)\| \geq \delta_0$, for all critical points (a,b) of (1.5). One has

$$\frac{d}{dt}V(x,y) = -cK^2(x)\left\{y^2\left(L - \left(\frac{F(x)}{K(x)}\right)'\right) + \left(\frac{F(x)}{K(x)}\right)^2\right\}.$$

So, in view of (iii),

$$\frac{d}{dt}V(x,y) + \varepsilon|yK| \leq -cK^2y^2 - cF^2 + \varepsilon|y|K := P_1(x,y)$$

and

$$\frac{d}{dt}V(x,y) + \varepsilon|cy + K'y^2 + F| \\ \leq y^2(-cK^2 + \varepsilon M) + \varepsilon c|y| - cF^2 + \varepsilon|F| := P_2(x,y).$$

We want ε such that $P_1(x,y) \leq 0$ and $P_2(x,y) \leq 0$.

Case 1. If $|y| > \delta_0$,

$$P_1(x,y) = -cK^2\left\{y^2 + \left(\frac{F}{K}\right)^2\right\} + \varepsilon|y|K \leq (-cqy^2 + \varepsilon|y|)K.$$

So, $P_1(x,y) < 0$ if $\varepsilon < c_0 q\delta_0$.

$$P_2(x,y) = y^2(-ck^2 + \varepsilon M) + \varepsilon c|y| - cF^2 + \varepsilon|F|.$$

The discriminant of the polynomial in $|F|$ is

$$\Delta = \varepsilon^2 + \{-4c^2y^2K^2 + \varepsilon(4Mcy^2 + 4c^2|y|)\}.$$

So $\Delta < 0$ for ε sufficiently small and it follows that $P_2(x,y) < 0$ for $c \geq c_0$ and $|y| \geq \delta_0$.

Case 2. If $|y| < \delta_0$, then $|x| > \delta_0$ and $|x-\alpha| > \delta_0$ and $|x-1| > \delta_0$.

The discriminant of the polynomial $P_1(x,y)$ in $|y|$ is

$$\Delta_1 = (\varepsilon^2 - 4(cF)^2)K^2 \leq (\varepsilon^2 - 4(crq)^2)K^2.$$

So, $\Delta_1 < 0$ for ε sufficiently small and it follows that $P_1(x,y) < 0$.

Now, as $F^2 - \varepsilon|F| \geq (rq)^2 - \varepsilon rq$ it follows that

$$P_2(x,y) \leq -cy^2K^2 - c(rq)^2 + \varepsilon(y^2M + c|y| + rq),$$

thus $P_2(x,y) < 0$ for ε sufficiently small. We conclude the desired result. □

Proposition 2.3. *There exist an isolating neighbourhood N for the flow generated by* (1.5) *without degeneracy, for every c in an arbitrary fixed compact* $[c_1, c_2]$.

Proof. If $0 < c_1 \leq c_2$, denote by $B_{i,\delta}$ the closed ball of center the critical point u_i and of radius δ and by $H = \bigcup_i B_{i,\delta}$ and define constants

$$\beta_0(c) = \inf_H V, \quad \beta_1(c) = \sup_H V \quad \text{and} \quad \alpha(c) = \frac{\beta_1(c) - \beta_0(c)}{\varepsilon}.$$

with ε is defined in Proposition 2.2. In Conley [5, p. 30], it is shown that for every fixed $c > 0$, all bounded solutions are contained in the closed $\alpha(c)$-neighbourhood of H. If $\alpha = \sup_{c\in[c_1,c_2]} \alpha(c)$ and N_1 is the closed α-neighbourhood of H, one easily checks that N_1 is an isolating neighbourhood of S_c for every $c \in [c_1, c_2]$.

Now, observe that if $c \in [-c_2, -c_1]$, with $0 < c_1 \leq c_2$, by the following variables change, $u(t) = x(-t)$, $v(t) = -y(-t)$, system (1.5) becomes,

$$\begin{cases} u' = K(u)v, \\ v' = -(-c)v - K'(u)v^2 - F(u), \end{cases}$$

with $(-c) \in [c_1, c_2]$. So, by arguing as above, we get an isolating neighbourhood N_2 for S_c, for every $c \in [-c_2, -c_1]$.

If $c_1 < 0 < c_2$, then for $c = 0$, $V_0(x,y) := L\{\int_0^x K(t)F(t)\,dt + \frac{1}{2}K^2(x)y^2\}$ is a first integral of (1.5) and by an appropriate variables change, the system is Hamiltonian. The set S_0 of bounded solutions consists of the three critical points and the teadrop filled with periodic solutions. So, every compact neighbourhood of S_0 is an isolating neighbourhood of S_0.

Now an isolating neighbourhood N_3 of S_0 for $c = 0$ is also an isolating neighbourhood of S_c for $|c|$ sufficiently small.

Let $0 < \eta \ll 1$ be sufficiently small such that S_c has an isolating neighbourhood N_3, for every $|c| \leq \eta$ and denote by N_1 (resp. N_2) an isolating neighbourhood of S_c for $c \in [c_1, -\eta]$ (resp. $c \in [\eta, c_2]$).

It is clear that the rectangle $N := [-\overline{r}, \overline{r}] \times [-\overline{R}, \overline{R}]$ becomes an isolating neighbourhood for S_c for every $c \in [c_1, c_2]$ by choosing the constants $\overline{r}$ and $\overline{R}$ sufficiently large to insure, $N_i \subset N$, for $i = 1, 2, 3$. □

2.3. Existence of traveling fronts without degeneracy.

2.3.1. Fisher-KPP case. Consider equation (1.1) under the set of conditions (1) and (2) in (I), but instead of (3), we suppose that $K > 0$ in $[0, 1]$ and the goal is to look for traveling fronts connecting the critical points $u_0 = (0, 0)$ and $u_1 = (1, 0)$. At the first, let's calculate the Conley index $h(u_0)$ and $h(u_1)$ of u_0 and u_1. (For the definition of h, see the Appendix).

Proposition 2.4.

$$h(u_1) = \Sigma^1 \quad \text{and} \quad h(u_0) = \begin{cases} \Sigma^0 & \text{for } c > 0, \\ \Sigma^2 & \text{for } c < 0. \end{cases}$$

Proof. In view of Proposition 5.1 in the Appendix, the result follows easily from the form of the characteristic polynomials of the linearized equation of (1.5) around u_0 and u_1, respectively. □

Theorem 2.5. *For every $c > 0$, there is a traveling wave from u_1 to u_0.*

Proof. If not, we must have $S_c = \{u_0, u_1\}$ and therefore it's Conley index, $h(S_c) = \Sigma^0 \vee \Sigma^1$. As $h(S_{-c})$is either $\Sigma^2 \vee \Sigma^1$ or $\overline{0}$ (see [17]), $h(S_c) \neq h(S_{-c})$ which leads a contradiction with the continuation property. □

In the following, let's show that from a specific value of the speed, the traveling wave of Theorem 2.1 is a front.

Theorem 2.6. *If $c_0 = 2\sqrt{L_1 L_2}$, with $L_1 = \sup_{x\in[0,1]} F(x)/x$ and $L_2 = \sup_{x\in[0,1]}((xK(x))')$, then for all $c \geq c_0$, the Fisher-KPP equation* (1.1)*, without degeneracy has a traveling front from u_1 to u_0.*

Proof. We will show that every t.w.s. corresponding to $c \geq c_0$ is in the domain

$$\Omega_m = \{(x, y) \in \mathbb{R}^2,\ 0 < x < 1 \text{ and } mx < y < 0\},$$

where m is an arbitrary scalar satisfying,

$$m < 0 \quad \text{and} \quad L_2 m^2 + cm + L_1 < 0. \tag{2.1}$$

Denote by $u(\xi) = (x(\xi), y(\xi))$ a t.w.s. connecting u_1 to u_0. The solution u cannot intersect the line $x = 1$, because if it does, there exists ξ_0 such that $u(\xi_0) = (1, y_0)$. Since the Lyapunov function V is decreasing on nonconstant solutions,

$$V(u(\xi_0)) < V(u_1).$$

But

$$V(u(\xi_0)) = V(u_1) + \frac{L}{2}(K(1)y_0)^2$$

and leads a contradiction. Thus in a neighbourhood of u_1, $u(\xi)$ is in Ω_m.

Now let's show that $u(\xi)$ can't leave Ω_m. If it does, let ξ_1 denote the first value of ξ such that $u(\xi) \in \partial\Omega_m$ and let $(x_1, y_1) = u(\xi_1)$.

If $0 < x_1 < 1$ and $y_1 = 0$, then for $\xi < \xi_1$,

$$y(\xi) < 0, \quad \text{and} \quad \frac{y(\xi) - y_1}{\xi - \xi_1} > 0,$$

so $y_1' \geq 0$ and leads a contradiction with the fact that $y_1' = -F(x_1) < 0$ because $y(\xi_1) = y_1 = 0$ and F is positive on $]0, 1[$.

If $0 < x_1 < 1$ and $y_1 = mx_1$, then for $\xi < \xi_1$,

$$y(\xi) > mx(\xi)$$

(because $(x(\xi), y(\xi)) \in \Omega_m$), and

$$\frac{y(\xi) - y_1}{\xi - \xi_1} < m\frac{x(\xi) - x_1}{\xi - \xi_1},$$

so $y_1' \leq mx_1'$. But

$$\begin{cases} x_1' = K(x_1)y_1 = mx_1K(x_1), \\ y_1' = -cmx_1 - K'(x_1)(mx_1)^2 - F(x_1), \end{cases}$$

therefore

$$\begin{aligned} mx_1' - y_1' &= m^2x_1K(x_1) + cmx_1 + K'(x_1)(mx_1)^2 + F(x_1) \\ &= x_1\bigg(m^2(x_1K'(x_1) + K(x_1)) + cm + \frac{F(x_1)}{x_1}\bigg) \geq 0. \end{aligned}$$

Thus $L_2m^2 + cm + L_1 \geq 0$ and leads a contradiction with the choice of m. □

Remark 2.7. (a) Condition (2.1) is equivalent to $m_1 < m < m_2$, with $m_1 = (-c - \sqrt{c^2 - 4L_1L_2})/(2L_2)$, $m_2 = (-c + \sqrt{c^2 - 4L_1L_2})/(2L_2)$, m_1 is decreasing with respect to L_2 and $\Omega_m \subset \Omega_{m_1}$.

(b) Let $c_* = 2\sqrt{F'(0)K(0)}$. If $0 < c < c_*$, the eigenvalues λ_1 and λ_2 of $f'(u_0)$ are

$$\lambda_1 = \frac{1}{2}\{-c + i\sqrt{4F'(0)K(0) - c^2}\} \quad \text{and} \quad \lambda_2 = \frac{1}{2}\{-c - i\sqrt{4F'(0)K(0) - c^2}\}.$$

Hence u_0 is a (stable) focus and there is no traveling front $u(\xi)$ going to u_0 as $\xi \to \infty$.

COROLLARY 2.8. *If the functions F and K satisfy the conditions*

$$F(x) \leq xF'(0) \quad \textit{and} \quad (xK(x))' \leq K(0),\ 0 \leq x \leq 1,$$

then the minimal speed c_{0*} is given by $c_{0*} = 2\sqrt{F'(0)K(0)}$.

PROOF. The hypotheses of Corollary 2.8 imply that $c_0 \leq c_{0*}$, Remark 2.7(b) and Theorem 2.6 imply $c_0 = c_{0*}$. □

2.3.2. Nagumo case. Consider equation (1.1) under the set of conditions (1) and (2) in (II), but instead of (3), we suppose that $K > 0$ in $[0, 1]$. The goal now is to look for traveling fronts connecting the critical points $u_0 = (0, 0)$, $u_\alpha = (\alpha, 0)$ and $u_1 = (1, 0)$.

LEMMA 2.9.

$$h(u_0) = h(u_1) = \Sigma^1 \quad \textit{and} \quad h(u_\alpha) = \begin{cases} \Sigma^0 & \textit{for } c > 0, \\ \Sigma^2 & \textit{for } c < 0. \end{cases}$$

PROOF. As in Proposition 2.4 the result follows easily from the form of the characteristic polynomials of the linearized equations of (1.5) around u_0, u_α and u_1, respectively. □

PROPOSITION 2.10. *There exists $c_{00} > 0$ (the minimal speed) such that for all $c \geq c_{00}$ there is a traveling front connecting u_1 to u_α.*

PROOF. For initial data in $[\alpha, 1]$ we have a same situation as for Fisher-KPP case without degeneracy, so the result follows from the above Subsection. □

The previous Propositions 2.10 and 5.5 in the Appendix are used to show

THEOREM 2.11. *Suppose $\int_0^1 F(x)\,dx > 0$. There exists a unique $c_1 \in]\overline{c_{00}}, c_{00}[$ for which a t.w.s. of Nagumo equation without degeneracy occurs from u_1 to u_0, where $\overline{c_{00}}$ is an arbitrary sufficiently small value of the speed c.*

PROOF. *Existence*: Let's show that the hypotheses of Proposition 5.5 hold. As u_α is an attractor, the set $N_0 := N \setminus B_\varepsilon(u_\alpha)$ is an isolating neighbourhood of Nagumo equation without degeneracy, for every $c \in [\overline{c_{00}}, c_{00}]$, where N is the isolating neighbourhood defined in Proposition 2.3 and $B_\varepsilon(u_\alpha)$ denotes the open ball with center u_α and radius a sufficiently small $\varepsilon > 0$. The reason for this choice is that we do not want to have the previous connecting orbits from u_1 or u_0 to u_α. As we have a gradient-like system, for every $c \in [\overline{c_{00}}, c_{00}]$, $(\{u_0\}, \{u_1\})$ forms an attractor-repeller for the maximal invariant set S_c of N_0 and

$$S_{\overline{c_{00}}} = S_{c_{00}} = \{u_0\} \cup \{u_1\}.$$

We conclude the proof of existence by the following, Lemma 2.12. *Suppose $\int_0^1 F(x)\,dx > 0$. The connected simple systems* (CSS) *of* (1.5) *corresponding to c_{00} and $\overline{c_{00}}$ are different.*

Uniqueness: Denote by γ_{c_1} the connecting orbit corresponding to the value of c_1 and let $c \neq c_1$. If $c < c_1$ (for example), the eigenvector $\begin{pmatrix} 1 \\ \lambda_c^+/K(1) \end{pmatrix}$ $\Big($resp. $\begin{pmatrix} 1 \\ \lambda_c^-/K(0) \end{pmatrix}\Big)$ corresponding to the positive eigenvalue λ_c^+ of $f'(1)$ (resp. negative eigenvalue λ_c^- of $f'(0)$) are such that $\lambda_c^+ > \lambda_{c_1}^+$ and $\lambda_c^- < \lambda_{c_1}^-$. So, locally, the unstable manifold of u_1 corresponding to the value of c lies below γ_{c_1} whereas the stable manifold of u_0 lies above it. If there exist another connecting orbits γ_c for the value c, let t_1 be the first time for which γ_c intersect γ_{c_1} at a point (x_1, y_1). One has,

$$\begin{cases} x_1' = K(x_1)y_1, \\ y_1' = -cy_1 - K'(x_1)y_1^2 - F(x_1). \end{cases}$$

So, for t sufficiently close to t_1 with $t < t_1$, $y_{c_1}(t) > y_c(t)$ and

$$\frac{y_c(t) - y_1}{t - t_1} > \frac{y_{c_1}(t) - y_1}{t - t_1}.$$

When $t \to t_1$, we obtain $\dot{y}_{c_1}(t_1) \leq \dot{y}_c(t_1)$. By substitution in (1.5), we found, $c_1 \leq c$ which leads a contradiction. □

Proof of Lemma 2.12. From Remark 5.6 in the Appendix, we only have to show that for $c = c_{00}$, the unstable manifold $W^u_{c_{00}}(u_1)$ of u_1 lies above the stable manifold $W^s_{c_{00}}(u_0)$ of u_0 and for $c = \overline{c_{00}}$, $W^u_{\overline{c_{00}}}(u_1)$ lies below $W^s_{\overline{c_{00}}}(u_0)$ (see the proof in [15, p. 952]).

In fact, from Proposition 2.10, $W^u_{c_{00}}(u_1)$ coincides with the connecting front $\gamma_{1\alpha}$ from u_1 to u_α and because of $\gamma_{1\alpha}$ and uniqueness of the solutions, $W^s_{c_{00}}(u_0)$ can't intersect the segment $(\alpha, 1]$. Also $W^s_{c_{00}}(u_0)$ can't intersect the segment $(0, \alpha]$. In fact, $W^s_{c_{00}}(u_0)$ can't start from the attractor u_α at $-\infty$ and if $W^s_{c_{00}}(u_0)$ intersect the segment $(0, \alpha)$ at a point $(\beta, 0)$, the Lyapunov function satisfy $V(0,0) < V(\beta, 0)$, a contradiction with the fact that $V(0,0) = 0$ and $V(\beta, 0) < 0$. Consequently, $W^s_{c_{00}}(u_0)$ intersect the axis $x = 1$ for a finite time t_1 at a point $(0, y_1)$, with $y_1 < 0$.

We conclude that $W^u_{c_{00}}(u_1)$ lies above $W^s_{c_{00}}(u_0)$. To show that $W^u_{\overline{c_{00}}}(u_1)$ lies below $W^s_{\overline{c_{00}}}(u_0)$, remarks at the first that for $c = 0$ and $\int_0^1 F(x)\,dx > 0$, we have the following properties:

(a) As we a Hamiltonian system, the unstable manifold $W^u_0(u_0)$ coincide with the stable one $W^s_0(u_0)$ to form a homoclinic orbit around u_α and based in u_0.

(b) The unstable manifold $W^u_0(u_1)$ can't intersect the segment $(\alpha, 1)$ (see the proof of Theorem 2.6) and, from uniqueness of the solution, can't intersect the previous homoclinic orbit.

Consequently, $W_0^u(u_1)$ intersects the axis $x = 0$ for finite time t_0 at a point $(0, y_0)$, with $y_0 < 0$.

By continuity with respect to c, we deduce that, for a sufficiently small value $\overline{c_{00}}$ of c, $W^u_{\overline{c_{00}}}(u_1)$ intersects the axis $x = 0$ for finite time $\overline{t_0}$ at a point $(0, \overline{y_0})$, with $\overline{y_0} < 0$.

With this, we deduce that $W^u_{\overline{c_{00}}}(u_1)$ lies below $W^s_{\overline{c_{00}}}(u_0)$. □

COROLLARY 2.13. *If* $\int_0^1 F(t)\,dt > 0$ *then:*

(a) *for* $0 \le c \le c_1$ *there is no connection between* u_1 *and* u_α*,*
(b) *for* $c > c_1$ *there exists a connecting orbit* $C_{1\alpha}$ *from* u_1 *to* u_α*, with* $C_{1\alpha}$ *is a front for* $c \ge c_{00}$ *and an oscillation for* $c_1 < c < \overline{c_*}$*, with* $\overline{c_*} = 2\sqrt{F'(\alpha)K(\alpha)}$*,*
(c) *for all* $c > 0$*, there exist a connecting orbit* $C_{0\alpha}$ *from* u_0 *to* u_α*.*

PROOF. (a) If there exists a connection γ between u_1 and u_α, by an argument as in proof of the uniqueness in the previous theorem, γ must intersect the connection γ_{c_1} between u_1 and u_0 and leads a contradiction.

(b) Suppose the unstable manifold of u_1, $W_c^u(u_1)$ corresponding to the value of c is bounded for $t \to \infty$. So its ω-limit set is a critical point.

In fact by a similar argument as in proof of the uniqueness of c_1, $W_c^u(u_1)$ can't intersect γ_{c_1}. Also, by an argument as in the proof of Theorem 2.6, it can't intersect the lines $x = 1$ and $x = 0$.

Now, if $W_c^u(u_1)$ is not bounded, there exist $(t_n)_n \subset \mathbb{R}^+$ going to ∞ such that $y(t_n) \to \infty$ and $V(x(t_n), y(t_n)) \to \infty$.

As $t_n \ge 0$, $V(x(t_n), y(t_n)) < V(u_1)$, a contradiction.

So the ω-limit set of $W_c^u(u_1)$ is u_α. The nature of $C_{1\alpha}$ for $c \ge c_{00}$ and for $c < \overline{c_*}$ derive from Proposition 2.10 and Remark 2.7(b), respectively.

(c) Suppose $W_c^u(u_0)$ is bounded for $t \to \infty$. So, it's ω-limit set is a critical point. As V is decreasing on orbits of equation (1.5) and $V(u_\alpha) < V(u_0) < V(u_1)$, the ω-limit set of $W_c^u(u_0)$ is u_α.

In fact for the boundedness of $W_c^u(u_0)$, an argument as in the proof of Theorem 2.6 shows that $W_c^u(u_0)$ can't intersect the lines $x = 1$ and $x = 0$. Now, if $W_c^u(u_0)$ is not bounded, there exist $(t_n)_n \subset \mathbb{R}^+$ going to ∞ such that $y(t_n) \to \pm\infty$ and $V(x(t_n), y(t_n)) \to \infty$. As $t_n \ge 0$, $V(x(t_n), y(t_n)) < V(x(0), y(0))$, a contradiction. □

Collecting the previous results we have

THEOREM 2.14. *Suppose* $\int_0^1 F(t)\,dt > 0$ *and let* S_c *denote the set of bounded solutions of equation* (1.5) *in Nagumo case without degeneracy. Then there exist* $0 < c_1 < c_{00}$ *such that:*

(a) *for* $c \ge c_{00}$*,* $S_c = \{\gamma_{0\alpha}, \gamma_{1\alpha}\}$*, with* γ_{ij} *is a front connecting* u_i *to* u_j*,*

(b) *for* $c_1 < c < c_{00}$, $S_c = \{C_{0\alpha}, C_{1\alpha}\}$, $C_{1\alpha}$ *is a connecting orbit from* u_1 *to* u_α *which is an oscillation for* $c < c_{\overline{*}}$ *and neither oscillation nor front for* $c \geq c_{\overline{*}}$,

(c) *for* $c = c_1$, $S_{c_1} = \{C_{0\alpha}, \gamma_{10}\}$, c_1 *is unique,*

(d) *for* $0 < c < c_1$, $S_c = \{C_{0\alpha}, u_1\}$,

(e) *for* $c = 0$, S_0 *consists of a variety of periodic orbits an orbit homoclinic to* u_0 *and* u_1.

Proof. The assertions (a)–(d) follows from Proposition 2.10, Theorem 2.11 and Corollary 2.13. For (e), at $c = 0$ we have a Hamiltonian system. For the set of bounded solutions, see for example [13, Chapter 14]. □

Remark 2.15. If $\int_0^1 F(t)\,dt < 0$, analogous reasoning lead to a similar results as in Theorem 2.14 for $c < 0$.

3. Existence of traveling fronts with degeneracy

3.1. Fisher-KPP case. Consider system (1.1) under the set of conditions (I). Theorem 2.6 now implies that, for any given $\mu \in (0,1]$ and for all $c > 2\sqrt{L_1 L_2}$, with $L_1 = \sup_{x\in[0,1]} F(x)/x$ and $L_2 = L_{2\mu} = \sup_{x\in[0,1]}(\mu + (xK(x))')$, the Fisher-KPP equation $(1.1)_\mu$:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\bigg((K(u)+\mu)\frac{\partial u}{\partial x}\bigg) + F(u), \quad x \in \mathbb{R},\ t \geq 0,$$

has a traveling front $\gamma_{10}(c,\mu)$ from u_1 to u_0. Furthermore, $\gamma_{10}(c,\mu)$ is in the domain Ω_m, where m is an arbitrary scalar satisfying (2.1). In view of Remark 2.7, Ω_m is bounded with respect to $\mu \in [0,1]$ and we can found a constant $m_1 < 0$, (for example the one given in Remark 2.7 corresponding to L_2 for $\mu = 0$) such that $\gamma_{10}(c,\mu) \subset [0,1] \times [m_1, 0]$, for any $\mu \in [0,1]$.

Let μ_n be a sequence of positive numbers tending to 0. For each n, we get a connecting orbit $\gamma_n = \gamma_n(c)$ from u_1 to u_0 in the flow generated by $(1)_{\mu_n}$. The closure $\mathrm{cl}(\gamma_n) = \{u_1\} \cup \gamma_n \cup \{u_0\}$ is a compact subset of $[0,1] \times [m_1, 0]$, so it has a convergent subsequence also labelled by $\mathrm{cl}(\gamma_n)$ on the Hausdorff metric on the compact subsets of $[0,1] \times [m_1, 0]$. The limit $\gamma_0^*(c)$ is nonempty, compact, connected and invariant for the flow generated by equation (2.1) (see Reineck [24, Lemma 3.8]). It is clear that the corresponding solution $\{u_n(t)\}$ converges uniformly with respect to t on any compact set of $\mathbb{R}$ to a functional $u^*(t)$ connecting u_1 to u_0 which is invariant in the $\mu = 0$ flow of equation (1.5). Then we deduce that $u^*(t)$ is a solution.

We conclude,

Theorem 3.1. *If* $c_0 = 2\sqrt{L_1 L_2}$, *with* $L_1 = \sup_{x\in[0,1]} F(x)/x$ *and* $L_2 = \sup_{x\in[0,1]}(x(1+K(x)))'$, *then for all* $c \geq c_0$, *the Fisher-KPP equation* (1.1), *with degeneracy has a traveling front from* u_1 *to* u_0.

3.2. Nagumo case. For Nagumo equation, analogous methodology as in Fisher-KPP case can be applied. Consider system (1.1) under the set of conditions (II). Theorem 2.11 now implies that for any given $\mu \in (0,1]$, there exist a positive constants $c_{00} = c_{00}(\mu)$ and $\overline{c_{00}} = \overline{c_{00}}(\mu)$ and a unique $c_1(\mu) \in]\overline{c_{00}}, c_{00}[$, uniformly bounded with respect to μ in $(0,1]$ for which a traveling wave from u_1 to u_0 occurs for the Nagumo equation $(1.1)_\mu$. Furthermore $c_1(\mu) > 0$ if and only if $\int_0^1 F(t)\,dt > 0$.

Let (μ_n) be a sequence of positive numbers tending to 0. For each n, we get a connecting orbit γ_n from u_1 to u_0 in the flow generated by $(1.1)_{\mu_n}$.

In view of Remark 2.7(a), the closure $\mathrm{cl}(\gamma_n) = \{u_1\} \cup \gamma_n \cup \{u_0\}$ is a compact subset of $[0,1] \times [m_1, -m_1]$ for a sufficiently negative constant m_1. So the sequence $\mathrm{cl}(\gamma_n)$ has a convergent subsequence also labelled by $\mathrm{cl}(\gamma_n)$ on the Hausdorff metric on compact subsets of $[0,1] \times [m_1, -m_1]$. The limit γ_0^* is nonempty, compact, connected and invariant for the flow generated by equation (1.5) (see Reineck [24, Lemma 3.8]). It is clear that the corresponding solution $\{u_n(t)\}$ converge uniformly on any compact set of $\mathbb{R}$ to a functional $u^*(t)$ connecting u_1 to u_0 which is invariant in the $\mu = 0$ flow. It follows that $u^*(t)$ is a solution of equation (1.5).

Uniqueness of the connection γ_0^* and the corresponding single value c_1^* of the speed c can be obtained using a similar arguments as in the proof of Theorem 2.5.

About the sign of c_1^* and the type of γ_0^*, following a similar methodology to that given in [30], we can precise the nature of γ_0^* and verify that $c_1^* \neq 0$. Arguing as in Corollary 2.13, and collecting the above results, we obtain

Theorem 3.2. *Suppose $\int_0^1 F(t)\,dt > 0$. Then there exist a unique value $c_1^* > 0$ of the speed c such that the degenerate generalized Nagumo equation has:*

(a) *for $c > c_1^*$, a t.w.s. $\gamma_{1\alpha}$ connecting u_1 to u_α,*
(b) *for $c = c_1^*$, a unique t.w.s. γ_{10}, connecting u_1 to u_0,*
(c) *for $0 < c < c_1^*$ no connection between u_1 and u_α.*

4. Conclusion

By Conley index theory, we give an alternative proof of a complete description of the set of bounded solutions for the nondegenerate Fisher-KPP and Nagumo equations. In the degenerate case, we showed that the technique of David Terman (see Reineck [24]) is applicable. We have shown the passage of existence of t.w.s. connecting the critical points of the system from the nondegenerate case to the cases of double degenerate Fisher-KPP equation and triple degenerate Nagumo equation. Here we generalize some results obtained by F. Sanchez-Garduno and P. K. Maini [28] in 1997.

Essentially our results for the degenerate equations are a perturbation of nondegenerate situation. Some natural extensions to our work are the analysis of existence of t.w.s. for a coupled nonlinear degenerate reaction-diffusion equation. We leave these for future consideration.

5. Appendix

In this section, a brief survey of some of the relevant portions of the Conley index theory and gradient like-system are provided. Basic references for this material are ([5], [3], [26], [27], [31]).

5.1. Conley index. Let $\varphi: X \times \mathbb{R} \to X$ be a flow on a locally compact topological space. A compact set $N \subset X$ is an isolating neighbourhood if its maximal invariant set is contained strictly in its interior, i.e.

$$\operatorname{Inv}(N, \varphi) := \{x \in N \mid \varphi(x, R) \subset N\} \subset \operatorname{Int}(N).$$

If $S = \operatorname{Inv}(N, \varphi)$ for some isolating neighbourhood N, then S is called an *isolated invariant set* (i.i.s.).

The Conley index studies i.i.s. S; the essential tool for this study being an *index pair* for S: a compact pair (N, L) satisfying the following axioms:

(a) $N \setminus L$is a neighbourhood of S and $\operatorname{cl}(N \setminus L)$ is an isolating neighbourhood for S.
(b) L is positively invariant in N: if $x \in L$, $\varphi(x, [0, t]) \subseteq N$, then $\varphi(x, t) \in L$.
(c) L is an exit set for N: if $x \in N$, $\varphi(x, (0, \infty)) \not\subseteq N$, then there exists a $t > 0$ such that $\varphi(x, [0, t]) \subseteq N$, $\varphi(x, t) \in L$.

Given S an i.i.s. with index pair (N, L), the Conley index of S denoted by $h(S)$ is the homotopy type of the pointed topological space $(N/L, L)$,

$$h(S) = [N/L, L].$$

A subset A of a compact invariant set S in X is called an attractor (relative to S) if there is a neighbourhood U of A in S such that $\omega(U) = A$.

The dual repeller R of A in S is defined by $R := \{x \in S \mid \omega(x) \cap A = \emptyset\}$. The couple (A, R) is called attractor-repeller pair.

In the case of isolated critical points, the following standard result will be used to determine the appropriate Conley index.

Proposition 5.1. *If x_0 is a hyperbolic critical point with unstable manifold $W^u(x_0)$ of dimension n, then $\{x_0\}$ is an isolated invariant set and*

$$h(x_0) = \Sigma^n, \quad \textit{the pointed } n\textit{-sphere.}$$

The following result gives an isolated neighbourhood:

Proposition 5.2 ([5]). *Suppose*

$$\frac{d}{dt}x = f(x)$$

a differential equation on $\mathbb{R}^n$ *and let* $V(x)$ *be a smooth function on* $\mathbb{R}^n$. *Suppose there is a compact set* $K \subset \mathbb{R}^n$ *and a constant* $\varepsilon > 0$ *such that, for* $x \in \mathbb{R}^n \setminus K$,

$$\frac{d}{dt}V(x(t)) \leq -\varepsilon \|f(x(t))\|.$$

Then the set of bounded solutions of the equation is compact (*in particular it is isolated, so has an index*).

We state the continuation theorem for the Conley index.

Let $\varphi^\mu : X \times \mathbb{R} \to X$, $\mu \in \Lambda$, be a continuously parameterized family of flows, where the parameter space Λ is a compact locally contractible, connected metric space. The parameterized flow corresponding to the family φ^μ is the continuous flow,

$$\Phi: X \times \mathbb{R} \times \Lambda \to X \times \Lambda, \quad (x, t, \mu) \mapsto \Phi(x, t, \mu) := (\varphi^\mu\ (x, t), \mu).$$

An i.i.s. S_μ for φ^μ for $\mu \in [\mu_1, \mu_2] \subset \Lambda$ are said to be related by continuation if we can find an isolated neighbourhood N in the product space $X \times \Lambda$ that is isolating for Φ, for each $\mu \in [\mu_1, \mu_2]$, and

$$\mathrm{Inv}(N^\mu, \varphi^\mu) = S_\mu, \quad \text{where } N^\mu := N \cap (X \times \{\mu\}).$$

Proposition 5.3 (Continuation property). *Let* S_{μ_0} *and* S_{μ_1} *be i.i.s. that are related by continuation. Then* $h(S_{\mu_0}) \approx h(S_{\mu_1})$.

The following statement is fundamental to many applications of the Conley theory.

Proposition 5.4 (Summation property). *Assume* $S = S_0 \cup S_1$ *is an isolated invariant set where* S_0 *and* S_1 *are disjoint invariant sets. Then*

$$h(S) = h(S_0) \vee h(S_1).$$

For certain applications, especially in global bifurcation problems, the Conley index is not enough. We need a more refined concept that will distinguish between i.i.s. that have the same Conley index. The required concept is that of a *connected simple system*.

5.2. Connected simple system. In what follows S denotes an i.i.s. and N an isolating neighbourhood. Let (N, L) be an index pair for S, then the homotopy type of the pointed space is usually referred to as the Conley index. There exists however, a finer version of the index which will be used in the proof. First a definition.

A *connected simple system* consist of a collection I_0 of pointed spaces along with a collection I_m of homotopy classes of maps between these such that:

(1) $\hom(X,X') = \{[f] \in [X,X']/[f] \in I_m\}$ is nonempty and consists of a single element for each ordered pair X, X' of spaces in I_0,

(2) if X, X', $X'' \in I_0, [f] \in \hom(X,X')$ and $[f'] \in \hom(X',X'')$, then $[f' \circ f] \in \hom(X,X'')$,

(c) $\hom(X,X) = \{[1_X]\}$ for all $X \in I_m$.

Recall ([5], [15] [20]) that the Conley index of S forms a connected simple system, where $I_0 = \{(N/L,[L])/(N,L)$ is an index pair for $S\}$ and I_m consists of maps defined by the flow between the elements of I_0. The connected simple system of the Conley index of S is denoted by $I(S)$.

Proposition 5.5 (Conley [5], Salamon [27]). *If (A_μ, A^*_μ) is an attractor-repeller pair for the i.i.s. S_μ, which continues for $\mu \in [\mu_1,\mu_2] \subset \mathbb{R}$ and*

$$S_{\mu_1} = A_{\mu_1} \cup A^*_{\mu_1}, \quad S_{\mu_2} = A_{\mu_2} \cup A^*_{\mu_2}$$

but $I(S_{\mu_1})$ is not the same as $I(S_{\mu_2})$, then for some $\mu \in \,]\mu_1,\mu_2[$, there exists a connecting orbit from A^*_μ to A_μ.

Remark 5.6. The standard example where the above proposition is applied is as follows (see [15, p. 951]). Suppose that one works in the plane $\mathbb{R}^2$ and the flow depends on a single parameter μ and the attractor-repeller pair consists of two saddles (of index 1). Suppose one can find values of the parameter μ such that for the first value, the unstable manifold of the one saddle lies above the stable manifold of the other, while for the second value it lies below it. The Proposition then gives the existence of a saddle connection for some intermediate value of μ.

5.3. Gradient-like system. Let $V\colon X \to \mathbb{R}$ be a continuous function on a locally compact Banach space. The flow φ on X is called *gradient-like with respect to* V if, for all $t > 0$, $\varphi(x,t) \neq x$ imply $V(\varphi(x,t)) < V(x)$ (or $V(\varphi(x,t)) > V(x)$).

Thus V is strictly decreasing (or increasing) on nonconstant solutions; in particular φ can't have any periodic or homoclinic orbits.

Proposition 5.7. *Let φ be gradient-like flow in $\mathbb{R}^n$ with respect to V, with isolated critical points. If $\varphi(x,\mathbb{R}^+)$ (resp. $\varphi(x,\mathbb{R}^-)$) is nonconstant and bounded positive (resp. negative) orbit, then the omega limit set $\omega(x)$ (resp. alpha limit set $\alpha(x)$) is a single critical point.*

Proof. We indicate the proof for positive orbits. For every $x \in \mathbb{R}^n$, the map $t \to V(\varphi(x,t))$ is decreasing and $\varphi(x,\mathbb{R}^+)$ is bounded. So, $\lim_{t\to\infty} V(\varphi(x,t)) = l$

exist and thus for every $y \in \omega(x)$, with $y = \lim_{n\to\infty} \varphi(x, t_n)$,

$$V(y) = V\Big(\lim_{n\to\infty} \varphi(x, t_n)\Big) = \lim_{n\to\infty} V(\varphi(x, t_n)) = l.$$

Also, for all $t > 0$,

$$V(\varphi(y, t)) = \lim_{n\to\infty} V(\varphi(x, t + t_n)) = l,$$

therefore $\varphi(y, t) = y$ and y is a critical point.

As $\omega(x)$ is connect and the critical points are isolated, $\omega(x)$ is a single point.□

In a gradient like system, the only bounded solutions are critical points and orbits connecting them.

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