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CONLEY INDEX AND HOMOLOGY INDEX BRAIDS IN SINGULAR PERTURBATION PROBLEMS WITHOUT UNIQUENESS OF SOLUTIONS

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Abstract. We define the concept of a Conley index and a homology index braid class for ordinary differential equations of the form

$$\dot{x} = F_1(x), \tag{E}$$

where $\mathcal{M}$ is a C^2-manifold and F_1 is the principal part of a *continuous vector field* on $\mathcal{M}$. This allows us to extend our previously obtained results from [5] on singularly perturbed systems of ordinary differential equations

$$\begin{aligned} \varepsilon\dot{y} &= f(y,x,\varepsilon), \\ \dot{x} &= h(y,x,\varepsilon) \end{aligned} \tag{E_ε}$$

on $Y \times \mathcal{M}$, where Y is a finite dimensional Banach space and $\mathcal{M}$ is a C^2-manifold, to the case where the vector field in (E_ε) is continuous, but not necessarily locally Lipschitzian.

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1. Introduction

Let $\mathcal{M}$ be a finite dimensional (boundaryless) second countable paracompact differentiable manifold of class C^2. Consider the ordinary differential equation

$$\dot{x} = F_1(x) \tag{1.1}$$

where F_1 is the principal part of a vector field $F\colon \mathcal{M} \to T(\mathcal{M})$, i.e. for $x \in \mathcal{M}$, $F(x) = (x, F_1(x))$ where $F_1(x) \in T_x(\mathcal{M})$. If F is a locally Lipschitzian vector field on $\mathcal{M}$ then (1.1) generates a local flow on $\mathcal{M}$ and the classical Conley index theory applies.

However, in some applications the right hand side of (1.1) is merely *continuous*. In such cases the Cauchy problem for equation (1.1) does not necessarily have unique solutions, so (1.1) does not generate a flow and the classical Conley index theory cannot be applied.

In this paper we present an extension of the Conley index theory to the case of ordinary differential equations of the type (1.1) with a merely *continuous* right hand side. For every isolating neighborhood N relative to F we define an index $h(f, N)$ and show that all properties of the classical Conley index theory hold in this more general setting. In addition, we show that the index depends only on the isolated invariant set in question and not on the choice of its isolating neighborhood. This generalizes some results from the paper [8] to the (technically more involved) manifold case.

In addition, we also provide an extension of the (co)homology index braid theory to this more general case.

As an application of this theory we show that all results of our previous paper [5] continue to hold under some more general assumptions on the nonlinearities involved.

2. Graded module braids

In this section we recall some basic notions from the theory of graded module braids. For more details, see [7].

Recall that a *strict partial order* on a set P is a relation $\prec \subset P \times P$ which is irreflexive and transitive. As usual, we write $x \prec y$ instead of $(x, y) \in \prec$. The symbol $<$ will be reserved for the less-than-relation on $\mathbb{R}$.

For the rest of this paper, unless specified otherwise, let P be a fixed finite set and $\prec$ be a fixed strict partial order on P.

A set $I \subset P$ is called *a $\prec$-interval* if whenever i, j, $k \in P$, i, $k \in I$ and $i \prec j \prec k$, then $j \in I$. By $\mathcal{I}(\prec)$ we denote the set of all $\prec$-intervals in P.

An *adjacent n-tuple of $\prec$-intervals* is a sequence $(I_j)_{j=1}^n$ of pairwise disjoint $\prec$-intervals whose union is a $\prec$-interval and such that, whenever $j < k$, $p \in I_j$ and $p' \in I_k$, then $p' \not\prec p$ (i.e. $p \prec p'$ or else p and p' are not related by $\prec$). By

$\mathcal{I}_n(\prec)$ we denote the set of all adjacent n-tuples of $\prec$-intervals. If I, $J \in \mathcal{I}(\prec)$ are such that (I, J), $(J, I) \in \mathcal{I}_2(\prec)$, we say that I and J are *noncomparable*.

For the rest of t his paper we fix a (commutative) ring Γ. We write IJ instead of $I \cup J$ and similarly for more than two intervals.

Definition 2.1. For each $J \in \mathcal{I}(\prec)$ and $q \in \mathbb{Z}$, let $G_q(J)$ be a Γ-module and for each $(I, J) \in \mathcal{I}_2(\prec)$ and $q \in \mathbb{Z}$ let

$$i_{I,J,q}\colon G_q(I) \to G_q(IJ), \quad p_{I,J,q}\colon G_q(IJ) \to G_q(J), \quad \partial_{I,J,q}\colon G_q(J) \to G_{q-1}(I)$$

be given maps. The family $\mathcal{G}(\prec)$ of all these modules $G_q(I)$ and all these maps $i_{I,J,q}$, $p_{I,J,q}$ and $\partial_{I,J,q}$ is called a *graded homology* Γ*-module braid over* $\prec$ if the following conditions are satisfied:

(a) the sequence

$$\longrightarrow G_q(I) \xrightarrow{i_{I,J,q}} G_q(IJ) \xrightarrow{p_{I,J,q}} G_q(J) \xrightarrow{\partial_{I,J,q}} G_{q-1}(I) \longrightarrow$$

is exact;

(b) whenever I, $J \in \mathcal{I}(\prec)$ are noncomparable, then $p_{J,I,q} \circ i_{I,J,q} = \mathrm{Id}|_{G_q(I)}$;

(c) whenever $(I, J, K) \in \mathcal{I}_3(\prec)$, the following diagram

$$\begin{tikzcd}
{} \arrow[d, bend right] & {} \arrow[dl] \arrow[dr] & {} \arrow[d, bend left] \\
G_q(I) \arrow[dr, "i_{I,J,q}"] \arrow[dd, bend right, "i_{I,JK,q}"'] & & G_{q+1}(K) \arrow[dl, "\partial_{IJ,K,q+1}"'] \arrow[dd, bend left, "\partial_{J,K,q+1}"] \\
& G_q(IJ) \arrow[dl, "i_{IJ,K,q}"'] \arrow[dr, "p_{I,J,q}"] & \\
G_q(IJK) \arrow[dr, "p_{I,JK,q}"] \arrow[dd, bend right, "p_{IJ,K,q}"'] & & G_q(J) \arrow[dl, "i_{J,K,q}"'] \arrow[dd, bend left, "\partial_{I,J,q}"] \\
& G_q(JK) \arrow[dl, "p_{J,K,q}"'] \arrow[dr, "\partial_{I,JK,q}"] & \\
G_q(K) \arrow[dr, "\partial_{IJ,K,q}"] \arrow[dd, bend right, "\partial_{J,K,q}"'] & & G_{q-1}(I) \arrow[dl, "i_{I,J,q-1}"'] \arrow[dd, bend left, "i_{I,JK,q-1}"] \\
& G_{q-1}(IJ) \arrow[dl, "p_{I,J,q-1}"'] \arrow[dr, "i_{IJ,K,q-1}"] & \\
G_{q-1}(J) \arrow[dr] \arrow[d, bend right] & & G_{q-1}(IJK) \arrow[dl] \arrow[d, bend left] \\
{} & {} & {}
\end{tikzcd}$$

commutes.

Let $\mathcal{G}(\prec)$ be a graded homology Γ-module braid over $\prec$ and $k \in \mathbb{N}_0$. The collection $\mathcal{G}_k(\prec)$ of the Γ-modules

$$G_{q-k}(J), \quad q \in \mathbb{Z},\ J \in \mathcal{I}(\prec),$$

and the maps $i_{I,J,q-k}$, $p_{I,J,q-k}$ and $\partial_{I,J,q-k}$, for $(I, J) \in \mathcal{I}_2(\prec)$ and $q \in \mathbb{Z}$, is a graded homology Γ-module braid over $\prec$ called *the shift to the left by* k *of* $\mathcal{G}(\prec)$.

Let $\mathcal{G} = \mathcal{G}(\prec)$ and $\widetilde{\mathcal{G}} = \widetilde{\mathcal{G}}(\prec)$ be graded homology Γ-module braids over $\prec$. Suppose $\theta := (\theta_q(J))_{q\in\mathbb{Z},J\in\mathcal{I}(\prec)}$ is a family $\theta_q(J)\colon G_q(J) \to \widetilde{G}_q(J)$ of Γ-module homomorphisms such that, for all $(I,J) \in \mathcal{I}_2(\prec)$, the diagram

$$\begin{tikzcd}
{} \arrow[r] & G_q(I) \arrow[r, "i_{I,J,q}"] \arrow[d, "\theta_q(I)"] & G_q(IJ) \arrow[r, "p_{I,J,q}"] \arrow[d, "\theta_q(IJ)"] & G_q(J) \arrow[r, "\partial_{I,J,q}"] \arrow[d, "\theta_q(J)"] & G_{q-1}(I) \arrow[r] \arrow[d, "\theta_{q-1}(I)"] & {} \\
{} \arrow[r] & \widetilde{G}_q(I) \arrow[r, "\widetilde{i}_{I,J,q}"'] & \widetilde{G}_q(IJ) \arrow[r, "\widetilde{p}_{I,J,q}"'] & \widetilde{G}_q(J) \arrow[r, "\widetilde{\partial}_{I,J,q}"'] & \widetilde{G}_{q-1}(I) \arrow[r] & {}
\end{tikzcd}$$

commutes. Then we say that θ is a *morphism from $\mathcal{G}$ to $\widetilde{\mathcal{G}}$* and we write $\theta\colon \mathcal{G} \to \widetilde{\mathcal{G}}$. If each $\theta_q(J)$ is an isomorphism, then we say that θ is an *isomorphism* and that $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ are *isomorphic* graded homology Γ-module braids over $\prec$.

Definition 2.2. For each $J \in \mathcal{I}(\prec)$ and $q \in \mathbb{Z}$, let $G^q(J)$ be a Γ-module and for each $(I,J) \in \mathcal{I}_2(\prec)$ and $q \in \mathbb{Z}$ let

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& G^q(IJ) \arrow[dl, "i_{IJ,K,q}"'] \arrow[dr, "p_{I,J,q}"] & \\
G^q(IJK) \arrow[dr, "p_{I,JK,q}"] \arrow[dd, bend right, "p_{IJ,K,q}"'] & & G^q(J) \arrow[dl, "i_{J,K,q}"'] \arrow[dd, bend left, "\partial_{I,J,q}"] \\
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G^q(K) \arrow[dr, "\partial_{IJ,K,q}"] \arrow[dd, bend right, "\partial_{J,K,q}"'] & & G^{q+1}(I) \arrow[dl, "i_{I,J,q+1}"'] \arrow[dd, bend left, "i_{I,JK,q+1}"] \\
& G^{q+1}(IJ) \arrow[dl, "p_{I,J,q+1}"'] \arrow[dr, "i_{IJ,K,q+1}"] & \\
G^{q+1}(J) \arrow[dr] \arrow[d, bend right] & & G^{q+1}(IJK) \arrow[dl] \arrow[d, bend left] \\
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Let $\mathcal{G}(\prec)$ be a graded cohomology Γ-module braid over $\prec$ and $k \in \mathbb{N}_0$. The collection $\mathcal{G}^k(\prec)$ of the Γ-modules

$$G^{q-k}(J), \quad q \in \mathbb{Z},\ J \in \mathcal{I}(\prec),$$

and the maps $i_{I,J,q-k}$, $p_{I,J,q-k}$ and $\partial_{I,J,q-k}$, for $(I,J) \in \mathcal{I}_2(\prec)$ and $q \in \mathbb{Z}$, is a graded cohomology Γ-module braid over $\prec$ called *the shift to the left by k of* $\mathcal{G}(\prec)$.

Let $\mathcal{G} = \mathcal{G}(\prec)$ and $\widetilde{\mathcal{G}} = \widetilde{\mathcal{G}}(\prec)$ be graded cohomology Γ-module braids over $\prec$. Suppose $\theta := (\theta_q(J))_{q\in\mathbb{Z}, J\in\mathcal{I}(\prec)}$ is a family $\theta_q(J)\colon G(J) \to \widetilde{G}(J)$ of Γ-module homomorphisms such that, for all $(I,J) \in \mathcal{I}_2(\prec)$, the diagram

$$\begin{array}{ccccccccccc}
\longrightarrow & G^q(I) & \xrightarrow{i_{I,J,q}} & G^q(IJ) & \xrightarrow{p_{I,J,q}} & G^q(J) & \xrightarrow{\partial_{I,J,q}} & G^{q+1}(I) & \longrightarrow \\
& \downarrow{\scriptstyle \theta_q(I)} & & \downarrow{\scriptstyle \theta_q(IJ)} & & \downarrow{\scriptstyle \theta_q(J)} & & \downarrow{\scriptstyle \theta_{q+1}(I)} & \\
\longrightarrow & \widetilde{G}^q(I) & \xrightarrow[\widetilde{i}_{I,J,q}]{} & \widetilde{G}^q(IJ) & \xrightarrow[\widetilde{p}_{I,J,q}]{} & \widetilde{G}^q(J) & \xrightarrow[\widetilde{\partial}_{I,J,q}]{} & \widetilde{G}^{q+1}(I) & \longrightarrow
\end{array}$$

commutes. Then we say that θ is a *morphism from $\mathcal{G}$ to $\widetilde{\mathcal{G}}$* and we write $\theta\colon \mathcal{G} \to \widetilde{\mathcal{G}}$. If each $\theta_q(J)$ is an isomorphism, then we say that θ is an *isomorphism* and that $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ are *isomorphic* graded cohomology Γ-module braids over $\prec$.

We define a category $\mathcal{B}$ whose objects are all the graded homology (resp. cohomology) Γ-modules braids over $\prec$. Given objects $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ in $\mathcal{B}$ let $\mathrm{Mor}_{\mathcal{B}}(\mathcal{G}, \widetilde{\mathcal{G}})$ be the set of all morphisms from $\mathcal{G}$ to $\widetilde{\mathcal{G}}$.

Given objects $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ in $\mathcal{B}$ we say $\mathcal{G}$ is related to $\widetilde{\mathcal{G}}$, and write $\mathcal{G} \sim \widetilde{\mathcal{G}}$, if and only if $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ are isomorphic graded (co)homology Γ-module braids over $\prec$. It is obvious that $\sim$ is a equivalence relation in $\mathcal{B}$. Given $\mathcal{G}$ in $\mathcal{B}$ let $[\mathcal{G}]$ denote the equivalence class of $\mathcal{G}$.

Note that if $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ are isomorphic graded homology (resp. cohomology) braids then so are $\mathcal{G}_k$ and $\widetilde{\mathcal{G}}_k$ (resp. $\mathcal{G}^k$ and $\widetilde{\mathcal{G}}^k$) for all $k \in \mathbb{N}_0$. Thus the shift operation

$$[\mathcal{G}]_k = [\mathcal{G}_k], \tag{2.1}$$

resp.

$$[\mathcal{G}]^k = [\mathcal{G}^k] \tag{2.2}$$

is well defined on isomorphism classes of graded homology (resp. cohomology) braids.

3. Approximation of continuous vector field on manifolds

Throughout this paper let $\mathcal{M}$ be a (boundaryless) second countable paracompact differentiable manifold of class C^2 modeled on some finite-dimensional

Banach space E. Let $T(\mathcal{M})$ denote the tangent bundle of $\mathcal{M}$. Whitney Imbedding Theorem implies that there is a finite dimensional normed space $\mathbf{E}$ and an imbedding $\mathbf{e}\colon \mathcal{M} \to \mathbf{E}$ of class C^2. We define the metric $d_{\mathcal{M}}$ on $\mathcal{M}$ such that $\mathbf{e}$ is an isometry.

Using the notation from [5] let $\Gamma = \Gamma^{\mathcal{M}}\colon T(\mathcal{M}) \to \mathbf{E}$ be the map given by $\Gamma(x, \underline{u}) = D^{\mathcal{M}}\mathbf{e}(x)(\underline{u})$, $(x, \underline{u}) \in T(\mathcal{M})$. [5, Subsections 3.1, 3.3 and Section 4] imply that Γ is continuous.

We now state the following basic approximation result.

Proposition 3.1. *Let $F\colon \mathcal{M} \to T(\mathcal{M})$ be a continuous vector field. Then for every $\epsilon \in]0, \infty[$ there is a C^1-vector field $G\colon \mathcal{M} \to T(\mathcal{M})$ such that*

$$\sup_{x\in\mathcal{M}} |\Gamma(G(x)) - \Gamma(F(x))|_{\mathbf{E}} < \epsilon.$$

Proof. For every chart $\alpha\colon U \to V \subset E$ of $\mathcal{M}$ the map

$$U \to \mathcal{L}(E, \mathbf{E}), \quad x \mapsto D(\mathbf{e} \circ \alpha^{-1})(\alpha(x))$$

is continuous, therefore locally bounded. It follows that there exists an atlas $(\alpha_i\colon U_i \to V_i)_{i\in I}$ of $\mathcal{M}$ such

$$C_i := \sup_{x\in U_i} \|D(\mathbf{e} \circ \alpha_i^{-1})(\alpha_i(x))\|_{\mathcal{L}(E,\mathbf{E})} < \infty, \quad i \in I. \tag{3.1}$$

Moreover, we may assume that the covering $(U_i)_{i\in I}$ is locally finite and that there is a C^2-partition of unity $(\phi_i\colon \mathcal{M} \to \mathbb{R})_{i\in I}$ subordinated to the covering $(U_i)_{i\in I}$.

Let $i \in I$ be arbitrary and set $F^i = F|_{U_i}$. The set $T(\mathcal{M})|_{U_i} = \bigcup_{x\in U_i}\{x\} \times T_x(\mathcal{M})$ is open in $T(\mathcal{M})$ and the map χ_{α_i} given by

$$\chi_{\alpha_i}\colon \bigcup_{x\in U_i} (\{x\} \times T_x(\mathcal{M})) \to \alpha_i(U_i) \times E, \quad (x, \underline{u}) \mapsto (\alpha_i(x), \underline{u}(\alpha_i))$$

is a homeomorphism from $T(\mathcal{M})|_{U_i}$ to $V_i \times E$. Actually χ_{α_i} is a C^1-diffeomorphism in the sense that χ_{α_i} is a C^1-map from $T(\mathcal{M})|_{U_i}$ to $V_i \times E$ and $\chi_{\alpha_i}^{-1}$ is a C^1-map from $V_i \times E$ to $T(\mathcal{M})$. Analogous remarks apply to $\alpha_i\colon U_i \to V_i$. In particular, the map $\widetilde{F}^i := \pi_2 \circ \chi_{\alpha_i} \circ F^i \circ \alpha_i^{-1}\colon V_i \to E$ is continuous, where $\pi_2\colon V_i \times E \to E$ is the projection on the second component. It follows that there is a C^1-map $\widetilde{G}^i\colon V_i \to E$ such that

$$\sup_{y\in V_i} |\widetilde{G}^i(y) - \widetilde{F}^i(y)|_E < \epsilon/(2C_i).$$

For every $x \in U_i$ let $G_1^i(x)$ be the uniquely defined element $\underline{u}$ of $T_x(\mathcal{M})$ such that $\underline{u}(\alpha_i) = \widetilde{G}^i(\alpha_i(x))$. This defines a map $G_1^i\colon U_i \to T(\mathcal{M})$ such that $G_1^i(x) \in T_x(\mathcal{M})$ for each $x \in U_i$.

For each $x \in \mathcal{M}$ define

$$G_1(x) = \sum_{i \in I} \phi_i(x) G_1^i(x).$$

This is actually a finite sum in $T_x(\mathcal{M})$ and so $G_1(x)$ is a well-defined element of $T_x(\mathcal{M})$. Now define the map $G\colon \mathcal{M} \to T(\mathcal{M})$ by $G(x) = (x, G_1(x))$, $x \in \mathcal{M}$. It follows that G is a vector field on $\mathcal{M}$. We will prove that G is of class C^1. Let $x_0 \in \mathcal{M}$ be arbitrary. Then there is an open neighbourhood W of x_0 in $\mathcal{M}$, a chart $\gamma\colon W \to \widetilde{W}$ and a finite subset J of I such that $\phi_i(x) = 0$ for all $i \in I \setminus J$ and all $x \in W$. It is enough to prove that $H_\gamma := \chi_\gamma \circ G|_W \circ \gamma^{-1}\colon \widetilde{W} \to \widetilde{W} \times E$ is of class C^1. However, for $y \in \widetilde{W}$,

$$H_\gamma(y) = \Big(y, \sum_{i \in J} \phi_i(\gamma^{-1}(y)) D(\gamma \circ \alpha_i^{-1})(\alpha_i(\gamma^{-1}(y))).\widetilde{G}^i(\alpha_i(\gamma^{-1}(y))) \Big)$$

and this expression clearly shows that H_γ is of class C^1.

Note that, for every $x \in \mathcal{M}$, $F(x) = (x, F_1(x))$ where $F_1(x) \in T_x(\mathcal{M})$. For every $i \in I$ the definition of $\widetilde{F}^i$ implies that, for every $x \in U_i$,

$$F_1(x)(\alpha_i) = \widetilde{F}^i(\alpha_i(x)).$$

It follows that

$$\begin{aligned}\Gamma(F(x)) &= D^{\mathcal{M}}\mathbf{e}(x).F_1(x) = D(\mathbf{e} \circ \alpha_i^{-1})(\alpha_i(x)).F_1(x)(\alpha_i)\\ &= D(\mathbf{e} \circ \alpha_i^{-1})(\alpha_i(x)).\widetilde{F}^i(\alpha_i(x))\end{aligned}$$

so

$$\Gamma(F(x)) = D(\mathbf{e} \circ \alpha_i^{-1})(\alpha_i(x)).\widetilde{F}^i(\alpha_i(x)).$$

Analogously,

$$\Gamma(G(x)) = D(\mathbf{e} \circ \alpha_i^{-1})(\alpha_i(x)).\widetilde{G}^i(\alpha_i(x)).$$

It follows that, for every $x \in \mathcal{M}$,

$$\begin{aligned}|\Gamma(G(x)) - \Gamma(F(x))|_{\mathbf{E}} &= \Big| \sum_{i \in I} \phi_i(x) D(\mathbf{e} \circ \alpha_i^{-1})(\alpha_i(x)).(\widetilde{G}^i(\alpha_i(x)) - \widetilde{F}^i(\alpha_i(x)))|_{\mathbf{E}}\\ &\le \sum_{i \in I} \phi_i(x) |D(\mathbf{e} \circ \alpha_i^{-1})(\alpha_i(x)).(\widetilde{G}^i(\alpha_i(x)) - \widetilde{F}^i(\alpha_i(x)))|_{\mathbf{E}}\\ &\le \sum_{i \in I} \phi_i(x) C_i \cdot \epsilon/(2C_i) = \epsilon/2.\end{aligned}$$

Thus

$$\sup_{x \in \mathcal{M}} |\Gamma(G(x)) - \Gamma(F(x))|_{\mathbf{E}} < \epsilon.$$

The proposition is proved. $\square$

Lemma 3.2. *Let N be a compact subset of $\mathcal{M}$ and F and F^κ, $\kappa \in \mathbb{N}$, be continuous vector fields on $\mathcal{M}$. Assume that* $\sup_{x\in N} |\Gamma(F^\kappa(x)) - \Gamma(F(x))|_{\mathbf{E}} \to 0$ *as $\kappa \to \infty$. Let $J \subset \mathbb{R}$ be an arbitrary interval. For every $\kappa \in \mathbb{N}$, let $x_\kappa\colon J \to N$ satisfy the equation*

$$\dot{x}_\kappa(t) = \dot{x}_\kappa^{\mathcal{M}}(t) = F_1^\kappa(x_\kappa(t)), \quad t \in J.$$

Then a subsequence of $(x_\kappa)_{\kappa\in\mathbb{N}}$ converges in $\mathcal{M}$, uniformly on compact subsets of J, to a function $x\colon J \to N$ satisfying the equation

$$\dot{x}(t) = F_1(x(t)), \quad t \in J.$$

Proof. For $\kappa \in \mathbb{N}$ let $y_\kappa = \mathbf{e} \circ x_\kappa$. It follows that

$$y_\kappa'(t) = \Gamma(F^\kappa(x_\kappa(t))), \quad t \in J.$$

An application of the Arzelà–Ascoli theorem shows that a subsequence of $(y_\kappa)_{\kappa\in\mathbb{N}}$, again denoted by $(y_\kappa)_{\kappa\in\mathbb{N}}$, converges in $\mathbf{E}$, uniformly on compact subsets of J, to a continuous function $y\colon J \to \mathbf{e}(N)$. Since $\mathbf{e}$ is a homeomorphism of $\mathcal{M}$ onto $\mathbf{e}(\mathcal{M})$ there is a unique map $x\colon J \to N$ with $y = \mathbf{e} \circ x$, x is continuous into $\mathcal{M}$ and $(x_\kappa)_{\kappa\in\mathbb{N}}$ converges to x in $\mathcal{M}$, uniformly on compact subsets of J.

For $\kappa \in \mathbb{N}$ and t, $t_0 \in J$ we have

$$y_\kappa(t) = y_\kappa(t_0) + \int_{t_0}^t \Gamma(F^\kappa(x_\kappa(s)))\,\mathrm{d}s.$$

Letting $\kappa \to \infty$ we conclude that

$$y(t) = y(t_0) + \int_{t_0}^t \Gamma(F(x(s)))\,\mathrm{d}s.$$

Proceeding as in the proof of [5, Proposition 4.6] we obtain that x is differentiable into $\mathcal{M}$ and

$$\dot{x}(t) = F_1(x(t)), \quad t \in J.$$

This completes the proof. □

An ordinary differential equation

$$\dot{x} = F_1(x)$$

generates a local (semi)flow on $\mathcal{M}$, provided the vector field $F\colon \mathcal{M} \to \mathcal{TM}$ is locally Lipschitzian. However, even merely continuous vector fields can still define local (semi)flows. This is e.g. the case for a continuous vector field obtained from an originally locally Lipschitzian vector field by a transformation via a C^1-diffeomorphism.

Therefore the following definition is natural:

Definition 3.3. Let F be a continuous vector field on $\mathcal{M}$ and π be a local semiflow on $\mathcal{M}$. We say that π *is generated by* F if for every intervall $J \subset \mathbb{R}$ and every function $x\colon J \to \mathcal{M}$, x is a solution of π if and only if x is differentiable on J and

$$\dot{x}(t) = F_1(x(t)), \quad t \in J.$$

Given $x \in \mathcal{M}$ and $\alpha \in]0,\infty[$ we denote by $B_\alpha(x)$ the set of all $y \in \mathcal{M}$ with $d_{\mathcal{M}}(y,x) \leq \alpha$. Since $(\mathcal{M}, d_{\mathcal{M}})$ is locally compact, $B_\alpha(x)$ is compact for α small enough (depending on x).

Lemma 3.4. *Let $\overline{a} \in \mathcal{M}$ be arbitrary and $\delta \in]0,\infty[$ be such that $N := B_{2\delta}(\overline{a})$ is compact. Let π be a local semiflow on $\mathcal{M}$ generated by the continuous vector field F on $\mathcal{M}$. Let $C \in]0,\infty[$ be arbitrary with $C \geq \sup_{x \in N} |\Gamma(F(x))|_{\mathbf{E}}$. Define $\tau = \delta/C$. Let $\overline{x} \in \mathcal{M}$ be arbitrary with $d_{\mathcal{M}}(\overline{x}, \overline{a}) \leq \delta$. Then $\overline{x}\pi\tau$ is defined and $\overline{x}\pi\,[0,\tau] \subset N$.*

Proof. Since N is compact, π does not explode in N. Thus if the assertion of the lemma does not hold, then there exists a smallest $r \in [0,\tau]$ such that $\overline{x}\pi r$ is defined and $d_{\mathcal{M}}(\overline{x}\pi r, \overline{a}) = 2\delta$. It follows that $\overline{x}\pi\,[0,r] \subset N$ and $0 < r < \tau$. Let $y(t) = \mathbf{e}(\overline{x}\pi t)$ for $t \in [0,r]$. It follows that

$$d_{\mathcal{M}}(\overline{x}\pi r, \overline{x}) = |y(r) - y(0)|_{\mathbf{E}} = \left| \int_0^r \Gamma(F(\overline{x}\pi s))\,\mathrm{d}s \right|_{\mathbf{E}} \leq Cr < C\tau = \delta$$

and so

$$2\delta = d_{\mathcal{M}}(\overline{x}\pi r, \overline{a}) \leq d_{\mathcal{M}}(\overline{x}\pi r, \overline{x}) + d_{\mathcal{M}}(\overline{x}, \overline{a}) < \delta + \delta = 2\delta.$$

This contradiction concludes the proof. $\square$

We now obtain the basic

Theorem 3.5. *Let F^κ, $\kappa \in \mathbb{N}_0$, be continuous vector fields on $\mathcal{M}$ and π_κ, $\kappa \in \mathbb{N}_0$, be local semiflows on $\mathcal{M}$. Suppose that π_κ is generated by F^κ for $\kappa \in \mathbb{N}_0$. In addition, assume that, for every compact subset N of $\mathcal{M}$,*

$$\sup_{x \in N} |\Gamma(F^\kappa(x)) - \Gamma(F^0(x))|_{\mathbf{E}} \to 0 \quad \text{as } \kappa \to \infty.$$

Under these hypotheses, $\pi_\kappa \to \pi_0$ as $\kappa \to \infty$.

We need the following lemmas:

Lemma 3.6. *Assume the hypotheses of Theorem 3.5. Let $\kappa_0 \in \mathbb{N}$ be arbitrary and $(\overline{a}_\kappa)_{\kappa \geq \kappa_0}$ be a sequence in $\mathcal{M}$ and $a_0 \in \mathcal{M}$ with $\overline{a}_\kappa \to \overline{a}_0$ in $\mathcal{M}$ as $\kappa \to \infty$. Let N be compact in $\mathcal{M}$ and $\tau \in]0,\infty[$ be such that $\overline{a}_\kappa\pi_\kappa\tau$ is defined and $\overline{a}_\kappa\pi_\kappa\,[0,\tau] \subset N$ for all $\kappa \geq \kappa_0$. Then $\overline{a}_0\pi_0\tau$ is defined and $\sup_{t \in [0,\tau]} d_{\mathcal{M}}(\overline{a}_\kappa\pi_\kappa t, \overline{a}_0\pi_0 t) \to 0$ as $\kappa \to \infty$.*

Proof. Define $x_\kappa(t) = \overline{a}_\kappa\pi_\kappa t$ for $\kappa \geq \kappa_0$ and $t \in [0,\tau]$. By Lemma 3.2 a subsequence of $(x_\kappa)_{\kappa\geq\kappa_0}$ converges in $\mathcal{M}$, uniformly on $[0,\tau]$, to a function $x\colon [0,\tau] \to N$ satisfying the equation

$$\dot{x}(t) = F_1^0(x(t)), \quad t \in J.$$

It follows from our assumption that x is a solution of π_0. Since $x(0) = \overline{a}_0$ we see that $\overline{a}_0\pi_0 t$ is defined and $\overline{a}_0\pi_0 t = x(t)$ for all $t \in [0,\tau]$.

This argument also proves that every subsequence of $(x_\kappa)_{\kappa\geq\kappa_0}$ converges to x in $\mathcal{M}$, uniformly on $[0,\tau]$. Therefore the full sequence $(x_\kappa)_{\kappa\geq\kappa_0}$ converges to x in $\mathcal{M}$, uniformly on $[0,\tau]$. This proves the lemma. □

Lemma 3.7. *Assume the hypotheses of Theorem* 3.5. *For every $\overline{a} \in \mathcal{M}$ there are δ, $\tau \in]0,\infty[$ such that for every $\overline{a}_0 \in \mathcal{M}$ with $d_{\mathcal{M}}(\overline{a}_0, \overline{a}) < \delta$ and every sequence $(\overline{a}_\kappa)_\kappa$ converging to $\overline{a}_0$ in $\mathcal{M}$ there is a $\kappa_0 \in \mathbb{N}$ such that both $\overline{a}_0\pi_0\tau$ and $\overline{a}_\kappa\pi_\kappa\tau$, $\kappa \geq \kappa_0$, are defined and* $\sup_{t\in[0,\tau]} d_{\mathcal{M}}(\overline{a}_\kappa\pi_\kappa t, \overline{a}_0\pi_0 t) \to 0$ *as $\kappa \to \infty$.*

Proof. Let $\overline{a} \in \mathcal{M}$ be arbitrary and $\delta \in]0,\infty[$ be such $N := B_{2\delta}(\overline{a})$ is compact. By our assumption there is a $C \in]0,\infty[$ such that

$$C \geq \sup_{\kappa\in\mathbb{N}_0} \sup_{x\in N} |\Gamma(F^\kappa(x))|_{\mathbf{E}}.$$

Let $\tau = \delta/C$. For every $\overline{a}_0 \in \mathcal{M}$ with $d_{\mathcal{M}}(\overline{a}_0, \overline{a}) < \delta$ and every sequence $(\overline{a}_\kappa)_\kappa$ converging to $\overline{a}_0$ in $\mathcal{M}$ there is a $\kappa_0 \in \mathbb{N}$ with $d_{\mathcal{M}}(\overline{a}_\kappa, \overline{a}) < \delta$ for $\kappa \geq \kappa_0$. Lemma 3.4 implies that both $\overline{a}_0\pi_0\tau$ and $\overline{a}_\kappa\pi_\kappa\tau$, $\kappa \geq \kappa_0$, are defined and $\overline{a}_0\pi_0\,[0,\tau] \subset N$ and $\overline{a}_\kappa\pi_\kappa\,[0,\tau] \subset N$ for $\kappa \geq \kappa_0$. Lemma 3.6 implies that $\sup_{t\in[0,\tau]} d_{\mathcal{M}}(\overline{a}_\kappa\pi_\kappa t, \overline{a}_0\pi_0 t) \to 0$ as $\kappa \to \infty$. □

We can now give a

Proof of Theorem 3.5. We must prove that whenever $\overline{x}_\kappa \to \overline{x}_0$ in $\mathcal{M}$, $t_\kappa \to t_0$ in $[0,\infty[$ as $\kappa \to \infty$ and $\overline{x}_0\pi_0 t_0$ is defined, then $\overline{x}_\kappa\pi_\kappa t_\kappa$ is defined for κ large enough and $\overline{x}_\kappa\pi_\kappa t_\kappa \to \overline{x}_0\pi_0 t_0$ in $\mathcal{M}$ as $\kappa \to \infty$.

Now, as $\overline{x}_0\pi_0 t_0$ is defined, there is a $b > t_0$, $b \in]0,\infty[$, such that $\overline{x}_0\pi_0 r$ is defined for all $r \in [0,b[$. Define

$$I := \{\, r \in [0,b[\mid \text{there exists an } \kappa_0 \in \mathbb{N} \text{ such that } \overline{x}_\kappa\pi_\kappa r \text{ is defined for } \kappa \geq \kappa_0 \\ \text{and } \sup_{s\in[0,r]} d_{\mathcal{M}}(\overline{x}_\kappa\pi_\kappa s, \overline{x}_0\pi_0 s) \to 0, \text{ as } \kappa \to \infty \,\}.$$

It is clear that $0 \in I$. Furthermore if $0 \leq r' < r$ and $r \in I$, then $r' \in I$. Let

$$\overline{r} := \sup I.$$

It follows that $\overline{r} \leq b$ and $[0,\overline{r}[\subset I$. An application of Lemma 3.7 with $\overline{a} := \overline{x}_0$ shows that $\overline{r} > 0$. We claim that $\overline{r} = b$. Suppose, on the contrary, that $\overline{r} < b$.

It follows that $\overline{x}_0\pi_0\overline{r}$ is defined. Let $\delta > 0$ and $\tau > 0$ be as in Lemma 3.7 with $\overline{a} := \overline{x}_0\pi_0\overline{r}$.

Choose $r \in \mathbb{R}$ with $0 < r < \overline{r} < r + \tau$ and $d_{\mathcal{M}}(\overline{x}_0\pi_0 r, \overline{x}_0\pi_0\overline{r}) < \delta$. We have that $r \in I$ so there exists an $\kappa_0 \in \mathbb{N}$ such that $\overline{x}_\kappa\pi_\kappa r$ is defined for all $\kappa \geq \kappa_0$ and

$$\sup_{s\in[0,r]} d_{\mathcal{M}}(\overline{x}_\kappa\pi_\kappa s, \overline{x}_0\pi_0 s) \to 0, \quad \text{as } \kappa \to \infty. \tag{3.2}$$

Set $\overline{a}_0 = \overline{x}_0\pi_0 r$, $\overline{a}_\kappa = \overline{a}_0$ for $\kappa < \kappa_0$ and $\overline{a}_\kappa := \overline{x}_\kappa\pi_\kappa r$ for $\kappa \geq \kappa_0$. Applying Lemma 3.7 and choosing κ_0 larger if necessary we see that both $\overline{a}_0\pi_0\tau$ and $\overline{a}_\kappa\pi_\kappa\tau$, $\kappa \geq \kappa_0$, are defined and

$$\sup_{t\in[0,\tau]} d_{\mathcal{M}}(\overline{a}_\kappa\pi_\kappa t, \overline{a}_0\pi_0 t) \to 0 \quad \text{as } \kappa \to \infty. \tag{3.3}$$

Formulas (3.2) and (3.3) imply that $\overline{x}_0\pi_0(r+\tau)$ and $\overline{x}_\kappa\pi_\kappa(r+\tau)$, $\kappa \geq \kappa_0$, are defined and

$$\sup_{s\in[0,r+\tau]} d_{\mathcal{M}}(\overline{x}_\kappa\pi_\kappa s, \overline{x}_0\pi_0 s) \to 0, \quad \text{as } \kappa \to \infty.$$

Thus $r + \tau \in I$, but $r + \tau > \overline{r}$, a contradiction, which proves that $\overline{r} = b$.

Since $t_0 \in [0, b[$, it follows that there is an $r \in [0, b[$ with $t_0 < r$ and $t_\kappa < r$ for all κ large enough. In particular $\overline{x}_0\pi_0 t_\kappa$ and $\overline{x}_\kappa\pi_\kappa t_\kappa$ are defined for κ large enough and

$$d_{\mathcal{M}}(\overline{x}_\kappa\pi_\kappa t_\kappa, \overline{x}_0\pi_0 t_\kappa) \to 0 \quad \text{as } \kappa \to \infty.$$

Since

$$d_{\mathcal{M}}(\overline{x}_0\pi_0 t_\kappa, \overline{x}_0\pi_0 t_0) \to 0 \quad \text{as } \kappa \to \infty,$$

we have that

$$d_{\mathcal{M}}(\overline{x}_\kappa\pi_\kappa t_\kappa, \overline{x}_0\pi_0 t_0) \to 0 \quad \text{as } \kappa \to \infty.$$

The proposition is proved. □

Given a continuous vector field F on $\mathcal{M}$ and $N \subset \mathcal{M}$ let $\mathrm{Sol}(F, N)$ be the set of all functions $x: \mathbb{R} \to N$ satisfying the equation

$$\dot{x}^{\mathcal{M}}(t) = F_1(x(t)), \quad t \in \mathbb{R}.$$

Lemma 3.2 immediately implies the following result.

PROPOSITION 3.8. *Let N be a compact subset of $\mathcal{M}$ and F and F^κ, $\kappa \in \mathbb{N}$, be continuous vector fields on $\mathcal{M}$ such that*

$$\sup_{x\in N} |\Gamma(F^\kappa(x)) - \Gamma(F(x))|_{\mathbf{E}} \to 0, \quad \text{as } \kappa \to \infty.$$

Set $\mathcal{T}_\kappa := \mathrm{Sol}(F^\kappa, N)$, $\kappa \in \mathbb{N}$, and $\mathcal{T} := \mathrm{Sol}(F, N)$. Then $\mathcal{T}_\kappa \to \mathcal{T}$ (in $C(\mathbb{R}, \mathcal{M})$) as $\kappa \to \infty$ (in the sense of [1]*).*

We conclude this section with the following result.

Corollary 3.9. *Let F be a continuous vector field on $\mathcal{M}$ and N be a compact subset of $\mathcal{M}$ such that* $\mathrm{Inv}_{\mathcal{T}}(N) \subset \mathrm{Int}_{\mathcal{M}}(N)$, *where* $\mathcal{T} = \mathrm{Sol}(F,N)$. *Then there is an* $\epsilon \in]0,\infty[$ *such that whenever* $\widetilde{F}$ *is a continuous vector field on* $\mathcal{M}$ *with* $\sup_{x\in N} |\Gamma(F(x)) - \Gamma(\widetilde{F}(x))|_{\mathbf{E}} < \epsilon$ *then* $\mathrm{Inv}_{\widetilde{\mathcal{T}}}(N) \subset \mathrm{Int}_{\mathcal{M}}(N)$, *where* $\widetilde{\mathcal{T}} = \mathrm{Sol}(\widetilde{F},N)$. *Let* $\epsilon(F,N)$ *be the supremum of all such numbers* ϵ.

Proof. This follows from Proposition 3.8 and [1, Proposition 2.14]. □

4. Conley index in the absence of uniqueness

We assume that the reader is familiar with the classical Conley index theory, as expounded in the monographs [6], [9] or [10].

In this section we give an extension of Conley index theory to the case of ordinary differential equations on $\mathcal{M}$ with a merely *continuous* right hand side. This extends some results from [8] to the manifold case.

Definition 4.1. Given a continuous vector field F on $\mathcal{M}$ and a compact subset N of $\mathcal{M}$ with $\mathrm{Inv}_{\mathcal{T}}(N) \subset \mathrm{Int}_{\mathcal{M}}(N)$, where $\mathcal{T} = \mathrm{Sol}(F,N)$, we define the *Conley index $h(F,N)$ of N relative to F* by

$$h(F,N) := h(\pi_G, \mathrm{Inv}_{\pi_G}(N))$$

where G is any C^1-vector field on $\mathcal{M}$ with $\sup_{x\in N} |\Gamma(G(x)) - \Gamma(F(x))|_{\mathbf{E}} < \epsilon(F,N)$ and π_G is the local (semi)flow on $\mathcal{M}$ generated by G.

A vector field G satisfying the above assumptions exists in view of Proposition 3.1. In view of Corollary 3.9, $\mathrm{Inv}_{\pi_G}(N) \subset \mathrm{Int}_{\mathcal{M}}(N)$, so $h(\pi_G, \mathrm{Inv}_{\pi_G}(N))$ is defined.

The following result shows that the above definition is independent of the choice of G:

Proposition 4.2. *Let N be a compact subset of $\mathcal{M}$. If G and G' are C^1-vector fields on $\mathcal{M}$ with*

$$\sup_{x\in N} |\Gamma(G(x)) - \Gamma(F(x))|_{\mathbf{E}} < \epsilon(F,N) \ \text{and} \ \sup_{x\in N} |\Gamma(G'(x)) - \Gamma(F(x))|_{\mathbf{E}} < \epsilon(F,N),$$

then

$$h(\pi_G, \mathrm{Inv}_{\pi_G}(N)) = h(\pi_{G'}, \mathrm{Inv}_{\pi_{G'}}(N)).$$

Proof. For $\theta \in [0,1]$ set $G_1^\theta := (1-\theta)G_1 + \theta G_1'$. It follows that $G^\theta \colon \mathcal{M} \to \mathcal{T}(\mathcal{M})$, $x \mapsto (x, G_1^\theta(x))$ is a C^1-vector field on $\mathcal{M}$ and $\sup_{x\in N} |\Gamma(G^\theta(x)) - \Gamma(F(x))|_{\mathbf{E}} < \epsilon(F,N)$ for all $\theta \in [0,1]$.

For $\theta \in [0,1]$ let π_θ be the local (semi)flow on $\mathcal{M}$ generated by G^θ and let $(\theta_n)_n$ be an arbitrary sequence in $[0,1]$ converging to some $\theta \in [0,1]$. We claim that $\pi_{\theta_n} \to \pi_\theta$ as $n \to \infty$. Since the map $\mathcal{M} \times \mathbb{R} \to T(\mathcal{M})$, $(x,\theta) \mapsto G^\theta(x)$ is continuous (even of class C^1), it follows that the map $\mathcal{M} \times \mathbb{R} \to \mathbf{E}$, $(x, \theta) \mapsto \Gamma(G^\theta(x))$ is continuous. In particular, for every compact subset M of $\mathcal{M}$

$$\sup_{x \in M} |\Gamma(G^{\theta_n}(x)) - \Gamma(G^\theta(x))|_{\mathbf{E}} \to 0 \quad \text{as } n \to \infty.$$

Hence Theorem 3.5 proves our claim. Now compactness of N and Conley index continuation principle, see, e.g. [10, Theorem I.12.2], imply that the Conley index $h(\pi_\theta, \mathrm{Inv}_{\pi_{G^\theta}}(N))$ is defined and independent of $\theta \in [0, 1]$. This proves the proposition. □

The number $\epsilon(F, N)$ also depends on the Banach space $\mathbf{E}$ and the embedding $\mathbf{e}: \mathcal{M} \to \mathbf{E}$ and we should write $\epsilon(F, N, \mathbf{E}, \mathbf{e})$ instead of $\epsilon(F, N)$ to stress this dependence. However, we claim that the Conley index $h(F, N)$ is independent of $\mathbf{E}$ or $\mathbf{e}$. This follows from the following

PROPOSITION 4.3. *Let* $\mathbf{E}$ *and* $\widetilde{\mathbf{E}}$ *be Banach spaces and* $\mathbf{e}: \mathcal{M} \to \mathbf{E}$ *and* $\widetilde{\mathbf{e}}: \mathcal{M} \to \widetilde{\mathbf{E}}$ *be* C^2*-embeddings. Let* N *be compact in* $\mathcal{M}$*. Then there is a* $C = C(N) \in]0, \infty[$ *such that for all* $x \in N$ *and all* $\underline{v} \in T_x(\mathcal{M})$

$$|D^{\mathcal{M}} \widetilde{\mathbf{e}}(x).\underline{v}|_{\widetilde{\mathbf{E}}} \le C |D^{\mathcal{M}} \mathbf{e}(x).\underline{v}|_{\mathbf{E}}.$$

Let F and N be as in Definition 4.1 and define

$$\widetilde{\epsilon} := \min\{\epsilon(F, N, \mathbf{E}, \mathbf{e}), C^{-1} \epsilon(F, N, \widetilde{\mathbf{E}}, \widetilde{\mathbf{e}})\}.$$

By Proposition 3.1 there is a C^1-vector field G on $\mathcal{M}$ such that

$$\sup_{x \in N} |D^{\mathcal{M}} \mathbf{e}(x).G_1(x) - D^{\mathcal{M}} \mathbf{e}(x).F_1(x)|_{\mathbf{E}} < \widetilde{\epsilon} \le \epsilon(F, N, \mathbf{E}, \mathbf{e}).$$

Proposition 4.3 implies that

$$\sup_{x \in N} |D^{\mathcal{M}} \widetilde{\mathbf{e}}(x).G_1(x) - D^{\mathcal{M}} \widetilde{\mathbf{e}}(x).F_1(x)|_{\widetilde{\mathbf{E}}} < \epsilon(F, N, \widetilde{\mathbf{E}}, \widetilde{\mathbf{e}})$$

so that, according to Definition 4.1, $h(F, N)$ is equal to $h(G, \mathrm{Inv}_{\pi_G}(N))$ both relative to the pair $(\mathbf{E}, \mathbf{e})$ and to the pair $(\widetilde{\mathbf{E}}, \widetilde{\mathbf{e}})$. This shows our claim.

PROOF OF PROPOSITION 4.3. Since $\mathbf{e}^{-1}$ is a C^2-map from the submanifold $\mathbf{e}(\mathcal{M})$ to $\mathcal{M}$ it follows that $h: \mathbf{e}(\mathcal{M}) \to \widetilde{\mathbf{E}}$, $h = \widetilde{\mathbf{e}} \circ \mathbf{e}^{-1}$ is of class C^2. Thus, by well known results there is an open set U in $\mathbf{E}$ containing $\mathbf{e}(\mathcal{M})$ and an extension of h to a C^2-map from U to $\widetilde{\mathbf{E}}$, denoted h again. Let

$$C = C(N) = \sup_{y \in \mathbf{e}(N)} |Dh(y)|_{\mathcal{L}(\mathbf{E}, \widetilde{\mathbf{E}})} + 1. \tag{4.1}$$

Since $\mathbf{e}(N)$ is compact in $\mathbf{E}$ it follows that $C \in]0, \infty[$. Now let $x \in N$ and $\underline{v} \in T_x(\mathcal{M})$ be arbitrary. Let $\alpha: V \to \widetilde{V} \subset E$ be an arbitrary chart at x. By definition of $D^{\mathcal{M}} \mathbf{e}$ we have

$$D^{\mathcal{M}} \mathbf{e}(x).\underline{v} = D(\mathbf{e} \circ \alpha^{-1})(y).\underline{v}(\alpha).$$

Here, $y = \alpha(x)$. In the same way

$$D^{\mathcal{M}}\widetilde{\mathbf{e}}(x).\underline{v} = D(\widetilde{\mathbf{e}} \circ \alpha^{-1})(y).\underline{v}(\alpha).$$

Since $\widetilde{\mathbf{e}} \circ \alpha^{-1} = h \circ (\mathbf{e} \circ \alpha^{-1})$ the chain rule shows that

$$D(\widetilde{\mathbf{e}} \circ \alpha^{-1})(y).\underline{v}(\alpha) = Dh(\mathbf{e}(x))(D(\mathbf{e} \circ \alpha^{-1})(y).\underline{v}(\alpha))$$

so

$$D^{\mathcal{M}}\widetilde{\mathbf{e}}(x).\underline{v} = Dh(\mathbf{e}(x))(D^{\mathcal{M}}\mathbf{e}(x).\underline{v}). \tag{4.2}$$

Now (4.1) and (4.2) imply the assertion of the proposition. □

The index $h(F, N)$ depends only on the isolated invariant set $\mathrm{Inv}_{\mathcal{T}}(N)$.

Proposition 4.4. *Let F be a continuous vector field on $\mathcal{M}$ and let N', N'' be a compact subsets of $\mathcal{M}$ such that $\mathrm{Inv}_{\mathcal{T}'}(N') \subset \mathrm{Int}_{\mathcal{M}}(N')$, $\mathrm{Inv}_{\mathcal{T}''}(N'') \subset \mathrm{Int}_{\mathcal{M}}(N'')$ and $\mathrm{Inv}_{\mathcal{T}'}(N') = \mathrm{Inv}_{\mathcal{T}''}(N'')$, where $\mathcal{T}' = \mathrm{Sol}(F, N')$ and $\mathcal{T}'' = \mathrm{Sol}(F, N'')$. Then*

$$h(F, N') = h(F, N'').$$

Proof. Suppose $h(F, N') \neq h(F, N'')$. Choose a sequence $(G^\kappa)_{\kappa\in\mathbb{N}}$ of C^1-vector fields on $\mathcal{M}$ such that $\sup_{x\in\mathcal{M}} |\Gamma(G^\kappa(x)) - \Gamma(F(x))|_{\mathbf{E}} \to 0$ as $\kappa \to \infty$. Let $\pi_\kappa := \pi_{G^\kappa}$. Definition 4.1 implies that

$$h(\pi_\kappa, \mathrm{Inv}_{\pi_\kappa}(N')) \neq h(\pi_\kappa, \mathrm{Inv}_{\pi_\kappa}(N'')) \quad \text{for all } \kappa \text{ large enough.}$$

Taking a subsequence and exchanging N' with N'', if necessary, we may thus assume that

$$\mathrm{Inv}_{\pi_\kappa}(N') \setminus \mathrm{Inv}_{\pi_\kappa}(N'') \neq \emptyset \quad \text{for all } \kappa \in \mathbb{N}.$$

Therefore for every $\kappa \in \mathbb{N}$ there is an $x_\kappa \in \mathrm{Sol}(G^\kappa, N')$ with $x_\kappa(0) \notin \mathrm{Int}_{\mathcal{M}}(N'')$. An application of Lemma 3.2 yields an $x \in \mathrm{Sol}(F, N')$ with $x(0) \notin \mathrm{Int}_{\mathcal{M}}(N'')$. Hence $\mathrm{Inv}_{\mathcal{T}}(N') \neq \mathrm{Inv}_{\mathcal{T}}(N'')$, a contradiction. □

Recall that $\overline{0}$ is the homotopy type of any pointed one-point set. The index just defined is nontrivial:

Proposition 4.5. *Let F be a continuous vector field on $\mathcal{M}$ and N be a compact subset of $\mathcal{M}$ such that $\mathrm{Inv}_{\mathcal{T}}(N) \subset \mathrm{Int}_{\mathcal{M}}(N)$, where $\mathcal{T} = \mathrm{Sol}(F, N)$. Suppose that $h(F, N) \neq \overline{0}$. Then $\mathrm{Inv}_{\mathcal{T}}(N) \neq \emptyset$.*

Proof. Choose a sequence $(G^\kappa)_{\kappa\in\mathbb{N}}$ of C^1-vector fields on $\mathcal{M}$ such that

$$\sup_{x\in N} |\Gamma(G^\kappa(x)) - \Gamma(F(x))|_{\mathbf{E}} \to 0 \quad \text{as } \kappa \to \infty.$$

By Definition 4.1, $h(\pi_{G^\kappa}, \mathrm{Inv}_{\pi_{G^\kappa}}(N)) \neq \overline{0}$ for all κ large enough, so by Conley index theory $\mathrm{Inv}_{\pi_{G^\kappa}}(N) \neq \emptyset$ for all such κ. An application of Lemma 3.2 now shows that $\mathrm{Inv}_{\mathcal{T}}(N) \neq \emptyset$. □

We also have the following property:

PROPOSITION 4.6. *Let F and F' be continuous vector fields on $\mathcal{M}$ and N be a compact subset of $\mathcal{M}$ such that* $\mathrm{Inv}_{\mathcal{T}}(N) \subset \mathrm{Int}_{\mathcal{M}}(N)$, *where* $\mathcal{T} = \mathrm{Sol}(F, N)$. *Assume that* $\sup_{x\in N} |\Gamma(F'(x)) - \Gamma(F(x))|_{\mathbf{E}} < \epsilon(F, N)$. *Then*

$$h(F, N) = h(F', N).$$

PROOF. By Corollary 3.9, $\mathrm{Inv}_{\mathcal{T}'}(N) \subset \mathrm{Int}_{\mathcal{M}}(N)$, where $\mathcal{T}' = \mathrm{Sol}(F', N)$. Thus $\epsilon(F', N)$ is defined (and positive). Choose a C^1-vector field G on $\mathcal{M}$ such that

$$\sup_{x\in N} |\Gamma(G(x)) - \Gamma(F'(x))|_{\mathbf{E}} < \min(\epsilon(F', N), \epsilon(F, N) - \widetilde{\epsilon}),$$

where $\widetilde{\epsilon} := \sup_{x\in N} |\Gamma(F(x)) - \Gamma(F'(x))|_{\mathbf{E}}$. Hence

$$\sup_{x\in N} |\Gamma(G(x)) - \Gamma(F'(x))|_{\mathbf{E}} < \epsilon(F', N) \text{ and } \sup_{x\in N} |\Gamma(G(x)) - \Gamma(F(x))|_{\mathbf{E}} < \epsilon(F, N),$$

so

$$h(F', N) = h(\pi_G, \mathrm{Inv}_{\pi_G}(N)) = h(F, N).$$

The proposition follows. □

As a corollary to Proposition 4.6 we obtain the following version of Conley index continuation property:

COROLLARY 4.7. *Let (Λ, d) be a metric space, N be a compact subset of $\mathcal{M}$ and $(F_\lambda)_{\lambda\in\Lambda}$ be a family of continuous vector fields on $\mathcal{M}$ such that the map*

$$\Lambda \times N \to \mathbf{E}, \quad (\lambda, x) \mapsto \Gamma(F_\lambda(x))$$

is continuous. For each $\lambda \in \Lambda$ assume that $\mathrm{Inv}_{\mathcal{T}_\lambda}(N) \subset \mathrm{Int}_{\mathcal{M}}(N)$, *where* $\mathcal{T}_\lambda = \mathrm{Sol}(F_\lambda, N)$. *Then the map $\lambda \mapsto h(F_\lambda, N)$ is locally constant. In particular, if Λ is connected then the Conley index $h(F_\lambda, N)$ is independent of $\lambda \in \Lambda$.*

PROOF. Let $C(N, \mathbf{E})$ be the space of all continuous functions from N to $\mathbf{E}$ endowed with the supremum norm. Since N is compact, our hypotheses imply that the map $\Phi\colon \Lambda \to C(N, \mathbf{E})$, $\lambda \mapsto (\Gamma \circ F_\lambda)|_N$, is continuous. Thus, for every $\lambda_0 \in \Lambda$ there exists an $\delta \in]0, \infty[$ such that $d(\lambda, \lambda_0) < \delta$ implies $\sup_{x\in N} |\Gamma(F_\lambda(x)) - \Gamma(F_{\lambda_0}(x))|_{\mathbf{E}} < \epsilon(F_{\lambda_0}, N)$, i.e., by Proposition 4.6, $h(F_\lambda, N) = h(F_{\lambda_0}, N)$. In other words, the map $\lambda \mapsto h(F_\lambda, N)$ is locally constant. The proof is complete. □

If a local semiflow π on $\mathcal{M}$ is generated by a continuous vector field F, then we have two definitions of Conley index: the classical definition and the one we have defined in this paper. We now show that these definitions coincide:

Proposition 4.8. *Let π be a local semiflow on $\mathcal{M}$ and F be a continuous vector field on $\mathcal{M}$. Suppose that π is generated by F. A compact set $N \subset \mathcal{M}$ is an isolating neighborhood relative to π if and only if* $\mathrm{Inv}_{\mathcal{T}}(N) \subset \mathrm{Int}_{\mathcal{M}}(N)$, *where* $\mathcal{T} = \mathrm{Sol}(F, N)$. *In this case,*

$$h(\pi, \mathrm{Inv}_\pi(N)) = h(F, N).$$

Proof. The first assertion of the proposition is clear. To prove the second assertion note that, by Proposition 3.1 there is a sequence $(F^\kappa)_\kappa$ of C^1-vector fields on $\mathcal{M}$ such that $\sup_{x\in\mathcal{M}} |\Gamma(F^\kappa(x) - \Gamma(F(x)))|_{\mathbf{E}} \to 0$ as $\kappa \to \infty$. Therefore, by Proposition 4.6, $h(F^\kappa, N) = h(F, N)$ for κ large enough. By Definition 4.1 and Proposition 4.2 we have that $h(F^\kappa, N) = h(\pi_{F^\kappa}, \mathrm{Inv}_{\pi_{F^\kappa}}(N))$ for all $\kappa \in \mathbb{N}$. Finally, by Theorem 3.5, compactness of N and Conley index continuation principle, we have that $h(\pi_{F^\kappa}, \mathrm{Inv}_{\pi_{F^\kappa}}(N)) = h(\pi, \mathrm{Inv}_\pi(N))$ for all κ large enough. All this implies the second assertion of the proposition. □

We now show that the Conley index just defined is invariant with respect to conjugation. More precisely, let $\widetilde{\mathcal{M}}$ be a finite dimensional (boundaryless) second countable paracompact differentiable manifold of class C^2 modeled on a Banach space $\widetilde{E}$ and $\Phi\colon \mathcal{M} \to \widetilde{\mathcal{M}}$ be a C^1-diffeomorphism with inverse $\Phi^{-1}\colon \widetilde{\mathcal{M}} \to \mathcal{M}$. Let F be a continuous vector field on $\mathcal{M}$. Whenever $x\colon I \subset \mathbb{R} \to \mathcal{M}$ is a solution of

$$\dot{x} = F_1(x),$$

then, by the chain rule, $\widetilde{x} = \Phi \circ x\colon I \to \widetilde{\mathcal{M}}$ is a solution of

$$\dot{\widetilde{x}} = \widetilde{F}_1(\widetilde{x}),$$

where $\widetilde{F}$ is the continuous vector field on $\widetilde{\mathcal{M}}$ given by

$$\widetilde{F} = T\Phi \circ F \circ \Phi^{-1}.$$

This implies that whenever N is a compact subset of $\mathcal{M}$ such that $\mathrm{Inv}_{\mathcal{T}}(N) \subset \mathrm{Int}_{\mathcal{M}}(N)$, where $\mathcal{T} = \mathrm{Sol}(F, N)$, then $\widetilde{N} := \Phi(N)$ is a compact subset of $\widetilde{\mathcal{M}}$ such that $\mathrm{Inv}_{\widetilde{\mathcal{T}}}(\widetilde{N}) \subset \mathrm{Int}_{\widetilde{\mathcal{M}}}(\widetilde{N})$, where $\widetilde{\mathcal{T}} = \mathrm{Sol}(\widetilde{F}, \widetilde{N})$.

Proposition 4.9. *Under the above assumptions on Φ, F and N,*

$$h(F, N) = h(\widetilde{F}, \widetilde{N}). \tag{4.3}$$

Proof. By [5, Subsection 3.1], whenever $\mathcal{N}$ is a C^1-manifold, Y is a Banach space and $f\colon \mathcal{N} \to Y$ is a C^1-map, then for all $x \in \mathcal{N}$ and all $\underline{u} \in T_x(\mathcal{N})$,

$$T_x f(\underline{u})(\beta) = D^{\mathcal{N}} f(x).\underline{u} \tag{4.4}$$

where $\beta = \mathrm{Id}_Y$ is the identity chart on Y.

Define $\widetilde{\epsilon} := \min(\epsilon(F, N, \mathbf{E}, \mathbf{e}), \epsilon(\widetilde{F}, \widetilde{N}, \mathbf{E}, \widetilde{\mathbf{e}}))$, where $\widetilde{\mathbf{e}}: \widetilde{\mathcal{M}} \to \mathbf{E}$ is given by $\widetilde{\mathbf{e}} := \mathbf{e} \circ \Phi^{-1}$. Let G be a C^1-vector field on $\mathcal{M}$ with

$$\sup_{x \in N} |\Gamma(G(x)) - \Gamma(F(x))|_{\mathbf{E}} < \widetilde{\epsilon}$$

and let π_G be the local (semi)flow on $\mathcal{M}$ generated by G. Since $\widetilde{\epsilon} \le \epsilon(F, N)$, it follows that

$$h(F, N) = h(\pi_G, \operatorname{Inv}_{\pi_G}(N)). \tag{4.5}$$

Define

$$\widetilde{x}\widetilde{\pi}t := \Phi((\Phi^{-1}(\widetilde{x}))\pi_G t),$$

where $\widetilde{x} \in \widetilde{\mathcal{M}}$ and $t \in [0, \infty[$ are such that $(\Phi^{-1}(\widetilde{x}))\pi_G t$ is defined. It follows that $\widetilde{\pi}$ is the local (semi)flow generated on $\widetilde{\mathcal{M}}$ by the equation

$$\dot{\widetilde{x}} = \widetilde{G}_1(\widetilde{x})$$

where $\widetilde{G}$ is the (in general, merely) continuous vector field on $\widetilde{\mathcal{M}}$ given by

$$\widetilde{G} = T\Phi \circ G \circ \Phi^{-1}.$$

Since Conley index is invariant under (semi)flow conjugation, we have

$$h(\pi_G, \operatorname{Inv}_{\pi_G}(N)) = h(\widetilde{\pi}, \operatorname{Inv}_{\widetilde{\pi}}(\widetilde{N})). \tag{4.6}$$

Proposition 4.8 implies that

$$h(\widetilde{\pi}, \operatorname{Inv}_{\widetilde{\pi}}(\widetilde{N})) = h(\widetilde{G}, \widetilde{N}). \tag{4.7}$$

Note that $T\widetilde{\mathbf{e}} \circ \widetilde{F} = T\mathbf{e} \circ T(\Phi^{-1}) \circ T\Phi \circ F \circ \Phi^{-1}$ so

$$T\widetilde{\mathbf{e}} \circ \widetilde{F} = T\mathbf{e} \circ F \circ \Phi^{-1}. \tag{4.8}$$

In the same way

$$T\widetilde{\mathbf{e}} \circ \widetilde{G} = T\mathbf{e} \circ G \circ \Phi^{-1}. \tag{4.9}$$

Defining $\widetilde{\Gamma}: T(\widetilde{\mathcal{M}}) \to \mathbf{E}$ by $\widetilde{\Gamma}(\widetilde{x}, \underline{\widetilde{u}}) = D^{\widetilde{\mathcal{M}}}\widetilde{\mathbf{e}}(\widetilde{x})(\underline{\widetilde{u}})$ for $(\widetilde{x}, \underline{\widetilde{u}}) \in T(\widetilde{\mathcal{M}})$ we thus obtain from (4.4), (4.8) and (4.9) that

$$\widetilde{\Gamma} \circ \widetilde{F} = \Gamma \circ F \circ \Phi^{-1} \tag{4.10}$$

and

$$\widetilde{\Gamma} \circ \widetilde{G} = \Gamma \circ G \circ \Phi^{-1}. \tag{4.11}$$

Thus

$$\sup_{\widetilde{x} \in \widetilde{N}} |\widetilde{\Gamma}(\widetilde{G}(\widetilde{x})) - \widetilde{\Gamma}(\widetilde{F}(\widetilde{x}))|_{\mathbf{E}} = \sup_{x \in N} |\Gamma(G(x)) - \Gamma(F(x))|_{\mathbf{E}} < \widetilde{\epsilon}$$

and so Proposition 4.6 implies

$$h(\widetilde{G}, \widetilde{N}) = h(\widetilde{F}, \widetilde{N}). \tag{4.12}$$

Now (4.5)–(4.7) and (4.12) imply (4.3). □

5. (Co)homology index braids in the absence of uniqueness

In this section will assume that the reader is familiar with the papers [2], [3], [4], [11].

We will now present an extension of the (co)homology index braid theory to the case considered in Section 4.

Proposition 5.1. *Let F be a continuous vector field on $\mathcal{M}$ and N be a compact subset of $\mathcal{M}$ such that $\mathrm{Inv}_{\mathcal{T}}(N) \subset \mathrm{Int}_{\mathcal{M}}(N)$, where $\mathcal{T} := \mathrm{Sol}(F, N)$. Let F^κ, $\kappa \in \mathbb{N}$, be continuous vector fields on $\mathcal{M}$ such that*

$$\sup_{x\in N} |\Gamma(F^\kappa(x)) - \Gamma(F(x))|_{\mathbf{E}} \to 0 \quad \text{as } \kappa \to \infty$$

and set $\mathcal{T}_\kappa := \mathrm{Sol}(F^\kappa, N)$, $\kappa \in \mathbb{N}$. Suppose $(M_p)_{p\in P}$ be a $\prec$-ordered $\mathcal{T}$-Morse decomposition. For each $p \in P$, let V_p be a closed subset of N such that $M_p = \mathrm{Inv}_{\mathcal{T}}(V_p) \subset \mathrm{Int}_{\mathcal{M}}(V_p)$. Moreover, for every $I \in \mathcal{I}(\prec)$, let V_I be a closed subset of N such that

$$M(I) = \bigcup_{(i,j)\in I\times I} \mathrm{CS}_{\mathcal{T}}(M_i, M_j) = \mathrm{Inv}_{\mathcal{T}}(V_I) \subset \mathrm{Int}_{\mathcal{M}}(V_I).$$

Then there exists a $\kappa_0 \in \mathbb{N}$ such that for all $\kappa \geq \kappa_0$,

$$M_{\kappa,p} := \mathrm{Inv}_{\mathcal{T}_\kappa}(V_p) \subset \mathrm{Int}_{\mathcal{M}}(V_p)$$

and $(M_{\kappa,p})_{p\in P}$ is a $\prec$-ordered $\mathcal{T}_\kappa$-Morse decomposition. Moreover, for every $I \in \mathcal{I}(\prec)$,

$$M_\kappa(I) := \bigcup_{(i,j)\in I\times I} \mathrm{CS}_{\mathcal{T}_\kappa}(M_{\kappa,i}, M_{\kappa,j}) = \mathrm{Inv}_{\mathcal{T}_\kappa}(V_I) \subset \mathrm{Int}_{\mathcal{M}}(V_I).$$

Proof. Since $\mathcal{T}$ and $\mathcal{T}_\kappa$, $\kappa \in \mathbb{N}$, are compact, translation and cut-and-glue invariant, an application of Proposition 3.8 and [2, Theorem 3.3] completes the proof. □

The last result clearly implies the following proposition.

Proposition 5.2. *Let F be a continuous vector field on $\mathcal{M}$ and N be a compact subset of $\mathcal{M}$ such that $\mathrm{Inv}_{\mathcal{T}}(N) \subset \mathrm{Int}_{\mathcal{M}}(N)$, where $\mathcal{T} := \mathrm{Sol}(F, N)$. Suppose $(M_p)_{p\in P}$ is a $\prec$-ordered $\mathcal{T}$-Morse decomposition. For each $p \in P$, let V_p be a closed subset of N such that $M_p = \mathrm{Inv}_{\mathcal{T}}(V_p) \subset \mathrm{Int}_{\mathcal{M}}(V_p)$. Moreover, for every $I \in \mathcal{I}(\prec)$, let V_I be a closed subset of N such that*

$$M(I) := \bigcup_{(i,j)\in I\times I} \mathrm{CS}_{\mathcal{T}}(M_i, M_j) = \mathrm{Inv}_{\mathcal{T}}(V_I) \subset \mathrm{Int}_{\mathcal{M}}(V_I).$$

Then there is an $\epsilon \in]0,\infty[$ *such that whenever* $\widetilde{F}$ *is a continuous vector field on* $\mathcal{M}$ *with* $\sup_{x\in N} |\Gamma(F(x)) - \Gamma(\widetilde{F}(x))|_{\mathbf{E}} < \epsilon$ *then* $\mathrm{Inv}_{\widetilde{\mathcal{T}}}(N) \subset \mathrm{Int}_{\mathcal{M}}(N)$*,*

$$\widetilde{M_p} := \mathrm{Inv}_{\widetilde{\mathcal{T}}}(V_p) \subset \mathrm{Int}_{\mathcal{M}}(V_p) \quad \textit{for every } p \in P$$

and $(\widetilde{M_p})_{p\in P}$ *is a* $\prec$*-ordered* $\widetilde{\mathcal{T}}$*-Morse decomposition. For every* $I \in \mathcal{I}(\prec)$*,*

$$\widetilde{M}(I) := \bigcup_{(i,j)\in I\times I} \mathrm{CS}_{\widetilde{\mathcal{T}}}(\widetilde{M_i}, \widetilde{M_j}) = \mathrm{Inv}_{\widetilde{\mathcal{T}}}(V_I) \subset \mathrm{Int}_{\mathcal{M}}(V_I),$$

where $\widetilde{\mathcal{T}} = \mathrm{Sol}(\widetilde{F}, N)$*. By* $\epsilon(F, N, (V_p)_{p\in P})$ *we denote the supremum of all such numbers* ϵ*.*

PROPOSITION 5.3. *Under the assumptions and notations of Proposition* 5.2 *let* G *and* G' *be* C^1*-vector fields on* $\mathcal{M}$ *with*

$$\sup_{x\in N} |\Gamma(G(x)) - \Gamma(F(x))|_{\mathbf{E}} < \epsilon(F, N, (V_p)_{p\in P}),$$
$$\sup_{x\in N} |\Gamma(G'(x)) - \Gamma(F(x))|_{\mathbf{E}} < \epsilon(F, N, (V_p)_{p\in P}).$$

Let π_G *(resp.* $\pi_{G'}$*) be the local (semi)flow on* $\mathcal{M}$ *generated by* G *(resp.* G'*) and* $K_G = \mathrm{Inv}_{\pi_G}(N)$ *(resp.* $K_{G'} = \mathrm{Inv}_{\pi_{G'}}(N)$*). For each* $p \in P$*, define* $M_{p,G} = \mathrm{Inv}_{\pi_G}(V_p)$ *(resp.* $M_{p,G'} = \mathrm{Inv}_{\pi_{G'}}(V_p)$*). Then the homology index braids* $\mathcal{H}(\pi_G, K_G, (M_{p,G})_{p\in P})$ *and* $\mathcal{H}(\pi_{G'}, K_{G'}, (M_{p,G'})_{p\in P})$ *are isomorphic and the cohomology index braids* $\mathcal{CH}(\pi_G, K_G, (M_{p,G})_{p\in P})$ *and* $\mathcal{CH}(\pi_{G'}, K_{G'}, (M_{p,G'})_{p\in P})$ *are isomorphic.*

PROOF. The proof is completely analogous to the proof of Proposition 4.2 except that, instead of Conley index continuation principle we use [3, Theorem 3.7]. □

We introduce the following definition.

DEFINITION 5.4. Given a continuous vector field F on $\mathcal{M}$, a compact subset N of $\mathcal{M}$ with $\mathrm{Inv}_{\mathcal{T}}(N) \subset \mathrm{Int}_{\mathcal{M}}(N)$, where $\mathcal{T} = \mathrm{Sol}(F, N)$, a $\prec$-ordered $\mathcal{T}$-Morse decomposition $(M_p)_{p\in P}$ and a family $(V_p)_{p\in P}$ of closed subsets of N such that $M_p = \mathrm{Inv}_{\mathcal{T}}(V_p) \subset \mathrm{Int}_{\mathcal{M}}(V_p)$, $p \in P$, we define *the homology index braid class of* $(F, N, (V_p)_{p\in P})$ by

$$\overline{\mathcal{H}}(F, N, (V_p)_{p\in P}) := [\mathcal{H}(\pi_G, K_G, (M_{p,G})_{p\in P})]$$

and *the cohomology index braid class of* $(F, N, (V_p)_{p\in P})$ by

$$\overline{\mathcal{CH}}(F, N, (V_p)_{p\in P}) := [\mathcal{CH}(\pi_G, K_G, (M_{p,G})_{p\in P})],$$

where G is any C^1-vector field on $\mathcal{M}$ with

$$\sup_{x\in N} |\Gamma(G(x)) - \Gamma(F(x))|_{\mathbf{E}} < \epsilon(F, N, (V_p)_{p\in P}),$$

π_G is the local (semi)flow on $\mathcal{M}$ generated by G, $K_G = \mathrm{Inv}_{\pi_G}(N)$ and $M_{p,G} = \mathrm{Inv}_{\pi_G}(V_p)$, $p \in P$. A vector field G satisfying the above assumptions exists in view of Proposition 3.1.

Let $k \in \mathbb{N}_0$. We define *the homology (resp. cohomology) index braid class of* $(F, N, (V_p)_{p\in P})$ *shifted to left by* k by

$$\begin{aligned} \overline{\mathcal{H}}_k(F, N, (V_p)_{p\in P}) &:= \overline{\mathcal{H}}(F, N, (V_p)_{p\in P})_k, \\ \text{resp. } \overline{\mathcal{CH}}_k(F, N, (V_p)_{p\in P}) &:= \overline{\mathcal{CH}}(F, N, (V_p)_{p\in P})_k, \end{aligned}$$

(cf. (2.1) and (2.2).)

Remark. Proposition 5.3 shows that the concepts defined in Definition 5.4 are independent of the choice of the vector field G. An argument analogous to that following the statement of Proposition 4.3 shows that $\overline{\mathcal{H}}(F, N, (V_p)_{p\in P})$ and $\overline{\mathcal{CH}}(F, N, (V_p)_{p\in P})$ are independent of $\mathbf{E}$ and $\mathbf{e}$.

Moreover, $\overline{\mathcal{H}}(F, N, (V_p)_{p\in P})$ and $\overline{\mathcal{CH}}(F, N, (V_p)_{p\in P})$ depend only on the $\prec$-ordered $\mathcal{T}$-Morse decomposition $(M_p)_{p\in P}$:

Proposition 5.5. *Let F be a continuous vector field on $\mathcal{M}$ and let N', N'' be compact subsets of $\mathcal{M}$ and such that $\mathrm{Inv}_{\mathcal{T}'}(N') \subset \mathrm{Int}_{\mathcal{M}}(N')$, $\mathrm{Inv}_{\mathcal{T}''}(N'') \subset \mathrm{Int}_{\mathcal{M}}(N'')$ and $\mathrm{Inv}_{\mathcal{T}'}(N') = \mathrm{Inv}_{\mathcal{T}''}(N'')$, where $\mathcal{T}' = \mathrm{Sol}(F, N')$ and $\mathcal{T}'' = \mathrm{Sol}(F, N'')$. Let $(M_p)_{p\in P}$ be a $\prec$-ordered $\mathcal{T}$-Morse decomposition and, for $p \in P$ let V'_p and V''_p be closed subsets of N' and N'' resp. such that $M_p = \mathrm{Inv}_{\mathcal{T}'}(V'_p) \subset \mathrm{Int}_{\mathcal{M}}(V'_p)$ and $M_p = \mathrm{Inv}_{\mathcal{T}''}(V''_p) \subset \mathrm{Int}_{\mathcal{M}}(V''_p)$. Then*

$$\begin{aligned} \overline{\mathcal{H}}(F, N', (V'_p)_{p\in P}) &= \overline{\mathcal{H}}(F, N'', (V''_p)_{p\in P}), \\ \overline{\mathcal{CH}}(F, N', (V'_p)_{p\in P}) &= \overline{\mathcal{CH}}(F, N'', (V''_p)_{p\in P}). \end{aligned}$$

Proof. Analogous to the proof of Proposition 4.4. □

We also have the following

Proposition 5.6. *Let F and F' be continuous vector fields on $\mathcal{M}$ and N be a compact subset of $\mathcal{M}$ such that $\mathrm{Inv}_{\mathcal{T}}(N) \subset \mathrm{Int}_{\mathcal{M}}(N)$, where $\mathcal{T} = \mathrm{Sol}(F, N)$ and let $(M_p)_{p\in P}$ be a $\prec$-ordered $\mathcal{T}$-Morse decomposition. For each $p \in P$ let V_p be a closed subset of N such that $M_p = \mathrm{Inv}_{\mathcal{T}}(V_p) \subset \mathrm{Int}_{\mathcal{M}}(V_p)$. Assume that $\sup_{x\in N} |\Gamma(F'(x)) - \Gamma(F(x))|_{\mathbf{E}} < \epsilon(F, N, (V_p)_{p\in P})$. Then*

$$\begin{aligned} \overline{\mathcal{H}}(F, N, (V_p)_{p\in P}) &= \overline{\mathcal{H}}(F', N, (V_p)_{p\in P}), \\ \overline{\mathcal{CH}}(F, N, (V_p)_{p\in P}) &= \overline{\mathcal{CH}}(F', N, (V_p)_{p\in P}). \end{aligned}$$

Proof. Analogous to the proof of Proposition 4.6. □

As a corollary we obtain the following version of the continuation property for (co)homology index braids:

COROLLARY 5.7. *Let (Λ, d) be a metric space, N be a compact subset of $\mathcal{M}$ and let $(F_\lambda)_{\lambda\in\Lambda}$ be a family of continuous vector fields on $\mathcal{M}$ such that the map*

$$\Lambda \times N \to \mathbf{E},\ (\lambda, x) \mapsto \Gamma(F_\lambda(x))$$

is continuous. For each $\lambda \in \Lambda$ assume that $\mathrm{Inv}_{\mathcal{T}_\lambda}(N) \subset \mathrm{Int}_{\mathcal{M}}(N)$ and the family $(M_{p,\lambda})_{p\in P}$ is a $\prec$-ordered $\mathcal{T}_\lambda$-Morse decomposition, where $\mathcal{T}_\lambda = \mathrm{Sol}(F_\lambda, N)$. For each $p \in P$ let V_p be a closed subset of N such that $M_{p,\lambda} = \mathrm{Inv}_{\mathcal{T}_\lambda}(V_p) \subset \mathrm{Int}_{\mathcal{M}}(V_p)$. Under these assumptions, the maps $\lambda \mapsto \overline{\mathcal{H}}(F_\lambda, N, (V_p)_{p\in P})$ and $\lambda \mapsto \overline{\mathcal{CH}}(F_\lambda, N, (V_p)_{p\in P})$ are locally constant. In particular, if Λ is connected then the homology index braid class $\overline{\mathcal{H}}(F_\lambda, N, (V_p)_{p\in P})$ and the cohomology index braid class $\overline{\mathcal{CH}}(F_\lambda, N, (V_p)_{p\in P})$ are independent of $\lambda \in \Lambda$.

PROOF. Analogous to the proof of Corollary 4.7, but using Proposition 5.6 instead of Proposition 4.6. □

PROPOSITION 5.8. *Let π be a local semiflow on $\mathcal{M}$ and F be a continuous vector field on $\mathcal{M}$. Suppose that π is generated by F. Let $N \subset \mathcal{M}$ be a compact set which is an isolating neighborhood relative to π and $K := \mathrm{Inv}_\pi(N)$. Given a $\prec$-ordered $\mathcal{T}$-Morse decomposition $(M_p)_{p\in P}$ of K relative to π and a family $(V_p)_{p\in P}$ of closed subsets of N such that $M_p = \mathrm{Inv}_\pi(V_p) \subset \mathrm{Int}_{\mathcal{M}}(V_p)$, $p \in P$, we have*

$$\begin{aligned}\overline{\mathcal{H}}(F, N, (V_p)_{p\in P}) &= [\mathcal{H}(\pi, K, (M_p)_{p\in P})],\\ \overline{\mathcal{CH}}(F, N, (V_p)_{p\in P}) &= [\mathcal{CH}(\pi, K, (M_p)_{p\in P})].\end{aligned}$$

PROOF. Analogous to the proof of Proposition 4.8. □

PROPOSITION 5.9. *Let $\widetilde{\mathcal{M}}$, Φ, F and N be as in Proposition 4.9. Let $(M_p)_{p\in P}$ be a $\prec$-ordered $\mathcal{T}$-Morse decomposition and for each $p \in P$ let V_p be a closed subset of N such that $M_p = \mathrm{Inv}_{\mathcal{T}}(V_p) \subset \mathrm{Int}_{\mathcal{M}}(V_p)$, $p \in P$. For each $p \in P$, define $\widetilde{V}_p := \Phi(V_p)$ and $\widetilde{M}_p := \mathrm{Inv}_{\widetilde{\mathcal{T}}}(\widetilde{V}_p)$. Then $\widetilde{M}_p = \mathrm{Inv}_{\widetilde{\mathcal{T}}}(\widetilde{V}_p) \subset \mathrm{Int}_{\mathcal{M}}(\widetilde{V}_p)$ for $p \in P$ and $(\widetilde{M}_p)_{p\in P}$ is a $\prec$-ordered $\widetilde{\mathcal{T}}$-Morse decomposition. Moreover,*

$$\begin{aligned}\overline{\mathcal{H}}(F, N, (V_p)_{p\in P}) &= \overline{\mathcal{H}}(\widetilde{F}, \widetilde{N}, (\widetilde{V}_p)_{p\in P}),\\ \overline{\mathcal{CH}}(F, N, (V_p)_{p\in P}) &= \overline{\mathcal{CH}}(\widetilde{F}, \widetilde{N}, (\widetilde{V}_p)_{p\in P}).\end{aligned}$$

PROOF. Analogous to the proof of Proposition 4.9. □

6. A singular perturbation result in the absence of uniqueness

In this section we will apply the index theories developed in the preceding sections to extend results from our previous paper [5].

Consider the following assumptions:

Hypothesis 6.1.

(a) *Y is a finite dimensional normed linear space, $\overline{\varepsilon} \in \,]0,\infty[$ is arbitrary, Z_0 is open in $Y \times \mathcal{M}$ and $W_0 := Z_0 \times [0,\overline{\varepsilon}]$.*
(b) *$f\colon W_0 \to Y$ and $h\colon W_0 \to T(\mathcal{M})$ are maps such that, for each $\varepsilon \in \,]0,\overline{\varepsilon}]$, $f(\,\cdot\,,\varepsilon)$ and $h(\,\cdot\,,\varepsilon)$ are continuous.*
(c) *For $((y,x),\varepsilon) \in W_0$, $h((y,x),\varepsilon) = (x, h_1((y,x),\varepsilon))$ with $h_1((y,x),\varepsilon) \in T_x(\mathcal{M})$.*
(d) *$\phi\colon \mathcal{M} \to Y$ is a C^1-map such that for all $x \in \mathcal{M}$, $(\phi(x),x) \in Z_0$ and $f((\phi(x),x),0) = 0$.*
(e) *The map $f(\,\cdot\,,0)$ is of class C^1 and the map $h(\,\cdot\,,0)$ is continuous.*
(f) *For every $(y,x) \in Z_0$ the map f is continuous at $((y,x),0)$ and for every $x \in \mathcal{M}$, the map h is continuous at $((\phi(x),x),0)$.*

Hypothesis 6.2. *a_0, $b_0 \in \mathbb{R}$ are such that $a_0 < 0 < 1 < b_0$ and $B\colon \mathcal{M} \times \,]a_0,b_0[\, \to \mathcal{L}(Y,Y)$ is a continuous map such that $B(x,\lambda)$ is hyperbolic for every $(x,\lambda) \in \mathcal{M} \times [0,1]$, $B(x,0) = Df((\phi(x),x),0)$ and $B(x,1) = \overline{B}$ for every $x \in \mathcal{M}$, where $\overline{B} \in \mathcal{L}(Y,Y)$ has Morse-index $k \in \mathbb{N}_0$.*

Here, for normed spaces Z_1 and Z_2, $\mathcal{L}(Z_1,Z_2)$ is the normed space of all bounded linear maps from Z_1 to Z_2.

Remark. Note that Hypothesis 6.1 relaxes [5, Hypothesis 4.1] and Hypothesis 6.2 relaxes [5, Hypothesis 4.2].

For every $\varepsilon \in \,]0,\overline{\varepsilon}]$, consider the ordinary differential equation

$$\varepsilon\dot{y} = f((y,x),\varepsilon), \qquad \dot{x} = h_1((y,x),\varepsilon). \tag{6.1}$$

As in [5], for each $\varepsilon \in \,]0,\overline{\varepsilon}]$ equation (6.1) is interpreted as the ordinary differential equation

$$\dot{w} = F_1^{\varepsilon}(w)$$

where F_ε is the unique vector field on the manifold Z_0 such that for every $w = (y,x) \in Z_0$ and every chart β of Z_0 at (y,x) of the form $\beta = \mathrm{Id}_U \times \alpha$, with U open in Y, $y \in U$ and $\alpha \in \mathrm{Chart}_x(\mathcal{M})$, the principal part $F_1^{\varepsilon}(y,x)$ of $F_\varepsilon(y,x)$ has the form

$$F_1^{\varepsilon}(y,x)(\beta) = ((1/\varepsilon)f(y,x,\varepsilon), h_1(y,x,\varepsilon)(\alpha)).$$

By our assumptions, F_ε is a continuous vector field on Z_0.

Consider the "limiting" ordinary differential equation

$$\dot{x} = h_1((\phi(x),x),0) \tag{6.2}$$

and let $F_0\colon \mathcal{M} \to T(\mathcal{M})$ be the unique vector field on the manifold $\mathcal{M}$ such that for every $x \in \mathcal{M}$, $F_0(x) = (x, h_1((\phi(x),x),0))$. Note that F_0 is a continuous vector field on $\mathcal{M}$.

Given $M \subset \mathcal{M}$ and $\eta \in \,]0, \infty[$ define

$$[M]_\eta^\phi := \{(y, x) \in Z_0 \mid x \in M \text{ and } |y - \phi(x)|_Y \le \eta\}.$$

We also define

$$\mathcal{T}(M) = \mathrm{Sol}(F_0, M) \quad \text{and} \quad \mathcal{T} = \mathcal{T}(\mathcal{M}) = \mathrm{Sol}(F_0, \mathcal{M}).$$

We can now state the main result of this section.

THEOREM 6.3. *Assume Hypotheses* 6.1 *and* 6.2. *Let N be a compact subset of $\mathcal{M}$ with $\mathrm{Inv}_{\mathcal{T}}(N) \subset \mathrm{Int}_{\mathcal{M}}(N)$. Then there is an $\eta_0 \in \,]0, \infty[$ such that for every $\eta \in \,]0, \eta_0]$, there exists an $\varepsilon_0 = \varepsilon_0(\eta) \in \,]0, \overline{\varepsilon}]$ such that for every $\varepsilon \in \,]0, \varepsilon_0]$, $\mathrm{Inv}_{\mathcal{T}_\varepsilon}([N]_\eta^\phi) \subset \mathrm{Int}_{Y \times \mathcal{M}}([N]_\eta^\phi)$ and*

$$h(F_\varepsilon, [N]_\eta^\phi) = \Sigma^k \wedge h(F_0, N),$$

where $\mathcal{T}_\varepsilon = \mathcal{T}_{\varepsilon,N,\eta} := \mathrm{Sol}(F_\varepsilon, [N]_\eta^\phi)$. In addition, let $(M_p)_{p \in P}$ be a $\prec$-ordered $\mathcal{T}(N)$-Morse decomposition. For each $p \in P$, let V_p be a closed subset of N such that $M_p = \mathrm{Inv}_{\mathcal{T}}(V_p) = \mathrm{Inv}_{\mathcal{T}(N)}(V_p) \subset \mathrm{Int}_{\mathcal{M}}(V_p)$. For every $\eta \in \,]0, \infty[$, every $\varepsilon \in \,]0, \overline{\varepsilon}]$ and every $p \in P$, define

$$M_{p,\varepsilon} = M_{p,\varepsilon,V_p,\eta} := \mathrm{Inv}_{\mathcal{T}_\varepsilon}([V_p]_\eta^\phi).$$

Then, for every $\eta \in \,]0, \eta_0]$, there is an $\overline{\varepsilon}_0 = \overline{\varepsilon}_0(\eta) \in \,]0, \overline{\varepsilon}]$ such that for every $\varepsilon \in \,]0, \overline{\varepsilon}_0]$, the family $(M_{p,\varepsilon})_{p \in P}$ is a $\prec$-ordered $\mathcal{T}_\varepsilon$-Morse decomposition,

$$\begin{aligned} \overline{\mathcal{H}}(F_\varepsilon, [N]_\eta^\phi, ([V_p]_\eta^\phi)_{p \in P}) &= \overline{\mathcal{H}}_k(F_0, N, (V_p)_{p \in P}), \\ \overline{\mathcal{CH}}(F_\varepsilon, [N]_\eta^\phi, ([V_p]_\eta^\phi)_{p \in P}) &= \overline{\mathcal{CH}}^k(F_0, N, (V_p)_{p \in P}). \end{aligned}$$

Theorem 6.3 extends [5, Theorem 4.3] to the case of continuous vector fields.

We prove Theorem 6.3 by modifying the corresponding arguments of [5, proof of Theorem 4.3] and using the index theory developed in the previous sections.

Actually, the arguments in [5] are somewhat flawed: they are only valid under more stringent assumptions, e.g. that the map $h\colon W_0 \to T(\mathcal{M})$, $((y, x), \varepsilon) \mapsto h((y, x), \varepsilon)$ is locally Lipschitzian (in all variables). For this reason, we employ here a homotopy different from the one used in [5, (4.8)], see equation (6.4) below.

First of all, define the map $\tau\colon \mathcal{M} \to \mathbb{R}$ by

$$\tau(x) = \sup\{\, \rho \in [0, \infty[\mid B_\rho(\phi(x)) \times \{x\} \subset Z_0 \,\}, \quad x \in \mathcal{M}.$$

Here $B_\rho(a)$ is the open ball in Y at $a \in Y$ with radius ρ. Since Z_0 is open in $Y \times \mathcal{M}$ and $(\phi(x), x) \in Z_0$ for every $x \in \mathcal{M}$, it follows that, for every $x \in \mathcal{M}$, $\tau(x) > 0$. The definition of τ also implies that

$$Z_1 := \bigcup_{x \in \mathcal{M}} B_{\tau(x)}(\phi(x)) \times \{x\} \subset Z_0.$$

Now the fact that Y is finite dimensional and so closed bounded subsets of Y are compact easily implies that the map τ is lower semicontinuous. As a consequence we obtain that Z_1 is open in $Y \times \mathcal{M}$.

Thus, replacing Z_0 by Z_1 if necessary, we may assume without loss of generality that

Hypothesis 6.4. *Whenever $(y,x) \in Z_0$ and $\mu \in [0,1]$, then*

$$(\phi(x) + \mu(y - \phi(x)), x) \in Z_0.$$

Proposition 6.5. *Let $\widetilde{Z}_0$ be the set of all $(u,x) \in Y \times \mathcal{M}$ such that $(u + \phi(x), x) \in Z_0$. Then $\widetilde{Z}_0$ is open in $Y \times \mathcal{M}$. The map $\Phi\colon Z_0 \to \widetilde{Z}_0$ defined by $\Phi(y,x) = (u,x) := (y - \phi(x), x)$ is a C^1-diffeomorphism with inverse $\Phi^{-1}\colon \widetilde{Z}_0 \to Z_0$ given by $\Phi^{-1}(u,x) = (y,x) := (u + \phi(x), x)$. For $\varepsilon \in \,]0,\overline{\varepsilon}]$, consider the differential equation:*

$$\varepsilon\dot{u} = \widetilde{f}((u,x),\varepsilon), \qquad \dot{x} = \widetilde{h}_1((u,x),\varepsilon), \tag{6.3}$$

where, for $((u,x),\varepsilon) \in \widetilde{W_0} := \widetilde{Z}_0 \times [0,\overline{\varepsilon}]$,

$$\begin{aligned} \widetilde{f}((u,x),\varepsilon) &= f((u+\phi(x),x),\varepsilon) - \varepsilon D^{\mathcal{M}}\phi(x).h((u+\phi(x),x),\varepsilon), \\ \widetilde{h}_1((u,x),\varepsilon) &= h_1((u+\phi(x),x),\varepsilon). \end{aligned}$$

Let $\widetilde{F}_\varepsilon$ denote the unique vector field on the manifold $\widetilde{Z}_0$ such that for every $(u,x) \in \widetilde{Z}_0$ and every chart β of $\widetilde{Z}_0$ at (u,x) of the form $\beta = \mathrm{Id}_U \times \alpha$, with U open in Y, $u \in U$ and $\alpha \in \mathrm{Chart}_x(\mathcal{M})$, the principal part $\widetilde{F}_1^\varepsilon(u,x)$ of $\widetilde{F}_\varepsilon(u,x)$ has the form

$$\widetilde{F}_1^\varepsilon(u,x)(\beta) = ((1/\varepsilon)\widetilde{f}(u,x,\varepsilon), \widetilde{h}_1(u,x,\varepsilon)(\alpha)).$$

Then $\widetilde{F} = T\Phi \circ F \circ \Phi^{-1}$.

Proof. This is a simple calculation using [5, Section 3]. $\square$

Remark 6.6. It follows from Propositions 4.9, 5.9 and 6.5 that we may and will assume without loss of generality that $\phi = 0$ in Hypothesis 6.1. We will also write $[M]_\eta$ for $[M]_\eta^\phi$, i.e.

$$[M]_\eta := \{(y,x) \in Z_0 \mid x \in M \text{ and } |y|_Y \le \eta\}.$$

For each $\varepsilon \in \,]0,\overline{\varepsilon}]$ and $\lambda \in [0,1]$, consider the differential equation:

$$\begin{aligned} \varepsilon\dot{y} &= (1-\lambda)(f((y,x),\varepsilon) - Df((0,x),0)y) + B(x,\lambda)y =: F(y,x,\varepsilon,\lambda), \\ \dot{x} &= h_1(((1-\lambda)y,x),\varepsilon). \end{aligned} \tag{6.4}$$

In view of Hypothesis 6.4, for each $\varepsilon \in \,]0,\overline{\varepsilon}]$ and $\lambda \in [0,1]$ the right-hand side of equation (6.4) is defined and there is a unique vector field $\widehat{F}_{\varepsilon,\lambda}$ on Z_0 such that for every $(y,x) \in Z_0$ and every chart β of Z_0 at (y,x) of the form $\beta = \mathrm{Id}_U \times \alpha$, with U open in Y, $y \in U$ and $\alpha \in \mathrm{Chart}_x(\mathcal{M})$, the principal part $\widehat{F}_1^{\varepsilon,\lambda}(y,x)$ of $\widehat{F}_{\varepsilon,\lambda}(y,x)$ has the form

$$\widehat{F}_1^{\varepsilon,\lambda}(y,x)(\beta) = ((1/\varepsilon)F(y,x,\varepsilon,\lambda), h_1(((1-\lambda)y,x),\varepsilon)(\alpha)).$$

By our assumptions, $\widehat{F}_{\varepsilon,\lambda}$ is a continuous vector field on Z_0.

Furthermore, for $\varepsilon \in [0,\overline{\varepsilon}]$ there is a unique vector field $\widehat{F}_\varepsilon$ on $\mathcal{M}$ such that for every $x \in \mathcal{M}$ the principal part $\widehat{F}_1^\varepsilon(x)$ of $\widehat{F}_\varepsilon(x)$ has the form

$$\widehat{F}_1^\varepsilon(x) = h_1((0,x),\varepsilon).$$

Again it follows that $\widehat{F}_\varepsilon$ is a continuous vector field on $\mathcal{M}$. Note that $\widehat{F}_0 = F_0$.

Let the normed space $\mathbf{E}$ and the imbedding $\mathbf{e}\colon \mathcal{M} \to \mathbf{E}$ be as in Section 3.

It follows that $\widehat{\mathbf{e}}\colon Z_0 \to Y \times \mathbf{E}$, $(y,x) \mapsto (y, \mathbf{e}(x))$ is a C^2-embedding. Let $\widehat{\Gamma} = \Gamma^{Z_0}\colon T(Z_0) \to Y \times \mathbf{E}$ be the map defined as in Section 3, but with respect to Z_0 and $\widehat{\mathbf{e}}$ rather than $\mathcal{M}$ and $\mathbf{e}$. In other words, $\widehat{\Gamma}$ is given by

$$\widehat{\Gamma}((y,x),\underline{w}) = D^{Z_0}(\widehat{\mathbf{e}})(y,x).\underline{w}, \quad (y,x) \in Z_0,\ \underline{w} \in T_{(y,x)}(Z_0).$$

Now the fact that the map $r\colon [0,1] \times Z_0 \to Z_0$, $(\lambda,(y,x)) \mapsto ((1-\lambda)y,x)$ is defined and continuous implies that, for each $\varepsilon \in]0,\overline{\varepsilon}]$, the map $[0,1] \times Z_0 \to T(Z_0)$, $(\lambda,(y,x)) \mapsto F_{\varepsilon,\lambda}(y,x)$ is continuous, so

(6.5) the map $[0,1] \times Z_0 \to Y \times \mathbf{E}$, $(\lambda,(y,x)) \mapsto \widehat{\Gamma}(F_{\varepsilon,\lambda}(y,x))$ is continuous.

We will need the following result proved in [5].

Proposition 6.7 ([5, Proposition 4.6]). *Let $g\colon W_0 \to T(\mathcal{M})$ be a map such that*

(a) *for each $\varepsilon \in]0,\overline{\varepsilon}]$, $g(\cdot,\varepsilon)$ is continuous,*
(b) *g is continuous at $((0,x),0)$ for every $x \in \mathcal{M}$,*
(c) *for each $((u,x),\varepsilon) \in W_0$,*

$$g((u,x),\varepsilon) = (x, g_1((u,x),\varepsilon)) \text{ with } g_1((u,x),\varepsilon) \in T_x(\mathcal{M}).$$

Let M be compact in $\mathcal{M}$. Then there is an $\eta_1' \in]0,\infty[$ and an $\varepsilon' \in]0,\overline{\varepsilon}]$ such that $[M]_{\eta_1'} \subset Z_0$ and

$$\sup\{|D^{\mathcal{M}}\mathbf{e}(x).g_1((u,x),\varepsilon))|_{\mathbf{E}} \mid |u|_Y \le \eta_1',\ x \in M,\ \varepsilon \in]0,\varepsilon']\} < \infty.$$

For each $n \in \mathbb{N}$, let $\varepsilon_n \in]0,\varepsilon']$, a_n, $b_n \in [0,1]$, $u_n\colon \mathbb{R} \to Y$ and $x_n\colon \mathbb{R} \to M$ be such that $\varepsilon_n \to 0$, $\sup_{n\in\mathbb{N}} \sup_{t\in\mathbb{R}} |u_n(t)|_Y \le \eta_1'$ and for every $n \in \mathbb{N}$, x_n is differentiable into $\mathcal{M}$ and $((u_n(t),x_n(t)),\varepsilon_n) \in W_0$. Moreover, assume that one of the following alternatives holds:

(i) *$\lim_{n\to\infty} u_n(t) = 0$ for all $t \in \mathbb{R}$ and $\dot{x}_n(t) = g_1((a_n u_n(t), x_n(t)), b_n\varepsilon_n)$ for all $n \in \mathbb{N}$ and $t \in \mathbb{R}$;*
(ii) *$\dot{x}_n(t) = \varepsilon_n g_1((a_n u_n(t), x_n(t)), b_n\varepsilon_n)$ for all $n \in \mathbb{N}$ and $t \in \mathbb{R}$.*

Then there is a subsequence of $(x_n)_n$ *which converges in* $(\mathcal{M}, d_{\mathcal{M}})$*, uniformly on compact subsets of* $\mathbb{R}$*, to a function* $x\colon \mathbb{R} \to M$ *which is differentiable into* $\mathcal{M}$ *and such that, in case* (i)*,*

$$\dot{x}(t) = g_1((0, x(t)), 0), \quad t \in \mathbb{R}$$

and, in case (ii)*,*

$$\dot{x}(t) = 0, \quad t \in \mathbb{R}.$$

Define the maps $T_1\colon W_0 \to Y$ and $T_2\colon Z_0 \to Y$ by

$$\begin{aligned} T_1((y,x),\varepsilon) &= f((y,x),\varepsilon) - f((y,x),0), && ((y,x),\varepsilon) \in W_0, \\ T_2(y,x) &= f((y,x),0) - f((0,x),0) - Df((0,x),0)(y), && (y,x) \in Z_0. \end{aligned}$$

Since $f((0,x),0) = 0$ for all $x \in \mathcal{M}$ it follows that

$$f((y,x),\varepsilon) = T_1((y,x),\varepsilon) + T_2(y,x) + Df((0,x),0)(y), \quad ((y,x),\varepsilon) \in W_0.$$

The following result is the analogue of [5, Lemma 4.8] with the same proof.

Lemma 6.8. *Let* M *be compact in* $\mathcal{M}$*. Then there is an* $\eta_2' \in \,]0,\infty[$ *such that* $[M]_{\eta_2'} \subset Z_0$ *and whenever* $x \in M$*,* $\lambda \in [0,1]$ *and* $y\colon \mathbb{R} \to Y$ *is a solution of the equation*

$$\dot{y} = (1-\lambda)T_2(y,x) + B(x,\lambda)y$$

lying in $[M]_{\eta_2'}$*, then* $y \equiv 0$.

Let $M \subset \mathcal{M}$ be compact and $\overline{\eta} = \overline{\eta}(M) \in \,]0,\infty[$ be such that $[M]_{\overline{\eta}} \subset Z_0$. For $\varepsilon \in [0,\overline{\varepsilon}]$ let $\widehat{\mathcal{T}}_\varepsilon(M)$ be the set of functions $\sigma\colon \mathbb{R} \to Y \times \mathcal{M}$ such that $\sigma(t) = (0, x(t))$, $t \in \mathbb{R}$ where $x \in \mathrm{Sol}(\widehat{F}_\varepsilon, M)$. Moreover, for $\eta \in \,]0,\overline{\eta}]$, $\varepsilon \in \,]0,\overline{\varepsilon}]$ and $\lambda \in [0,1]$, set

$$\widehat{\mathcal{T}}(M,\eta,\varepsilon,\lambda) = \mathrm{Sol}(\widehat{F}_{\varepsilon,\lambda}, [M]_\eta).$$

Lemma 6.9. *For* $\varepsilon \in [0,\overline{\varepsilon}]$ *the set* $\widehat{\mathcal{T}}_\varepsilon(M)$ *is compact in* $C(\mathbb{R} \to Y \times \mathcal{M})$ *and translation and cut-and-glue invariant. Moreover, for* $\eta \in \,]0,\overline{\eta}]$*,* $\varepsilon \in \,]0,\overline{\varepsilon}]$ *and* $\lambda \in [0,1]$*, the set* $\widehat{\mathcal{T}}(M,\eta,\varepsilon,\lambda)$ *is compact in* $C(\mathbb{R} \to Y \times \mathcal{M})$ *and translation and cut-and-glue invariant.*

Proof. Since $\mathrm{Sol}(\widehat{F}_\varepsilon, M)$, $\varepsilon \in [0,\overline{\varepsilon}]$, and $\mathrm{Sol}(\widehat{F}_{\varepsilon,\lambda}, [M]_\eta)$, $\eta \in \,]0,\overline{\eta}]$, $\varepsilon \in \,]0,\overline{\varepsilon}]$ and $\lambda \in [0,1]$, are translation and cut-and-glue invariant sets, the result follows. □

Proposition 6.10. *Let* M *be compact in* $\mathcal{M}$*. Then there is an* $\eta' = \eta'(M) \in \,]0,\overline{\eta}(M)]$ *such that whenever* $\eta \in \,]0,\eta']$*,* $(\varepsilon_\kappa)_\kappa$ *is a sequence in* $]0,\overline{\varepsilon}]$ *converging to* 0 *and* $(\lambda_\kappa)_\kappa$ *is an arbitrary sequence in* $[0,1]$ *then* $\mathcal{T}_\kappa \to \mathcal{T}_0 = \widehat{\mathcal{T}}_0(M)$*, where*

$$\mathcal{T}_\kappa = \widehat{\mathcal{T}}(M,\eta,\varepsilon_\kappa,\lambda_\kappa), \quad \kappa \in \mathbb{N}.$$

Proof. Let $\eta' = \min(\eta_1', \eta_2')$, where η_1' and ε' are as in Proposition 6.7 with $g = h$ and η_2' is as in Lemma 6.8. Let $\eta \in \,]0,\eta']$ be arbitrary. It is enough to prove that whenever $(\varepsilon_n)_n$ is a sequence in $]0, \varepsilon']$ converging to 0, $(\lambda_n)_n$ is a sequence in $[0, 1]$ converging to $\lambda \in [0, 1]$ and $(\sigma_n)_n$ is a sequence such that, for each $n \in \mathbb{N}$, $\sigma_n \in \mathrm{Sol}(\widehat{F}_{\varepsilon_n,\lambda_n}, [M]_\eta)$ and $\sigma_n(t) =: (y_n(t), x_n(t))$, $t \in \mathbb{R}$ then (i) $(y_n)_n$ converges to $y \equiv 0$ in Y, uniformly on $\mathbb{R}$ and (ii) $(x_n)_n$ has a subsequence converging in $(\mathcal{M}, d_{\mathcal{M}})$, uniformly on compact subsets of $\mathbb{R}$, to an $x \in \mathrm{Sol}(F_0, M)$.

Suppose (i) is not true. Then by Lemma 6.9 and passing to a subsequence if necessary, we may assume that there is a $\delta \in]0, \infty[$ such that $|y_n(0)|_Y \geq \delta$ for all $n \in \mathbb{N}$. Define functions $v_n: \mathbb{R} \to Y$ and $\xi_n: \mathbb{R} \to \mathcal{M}$, $n \in \mathbb{N}$, by

$$v_n(t) = y_n(\varepsilon_n t), \quad \xi_n(t) = x_n(\varepsilon_n t), \quad t \in \mathbb{R}.$$

It follows that

$$\dot{\xi}_n(t) = \varepsilon_n h_1(((1 - \lambda_n)v_n(t), \xi_n(t)), \varepsilon_n), \quad n \in \mathbb{N}, \ t \in \mathbb{R}.$$

An application of Proposition 6.7 (with $g = h$) shows that, by passing to subsequences if necessary, we may assume that $(\xi_n)_n$ converges in $(\mathcal{M}, d_{\mathcal{M}})$, uniformly on compact subsets of $\mathbb{R}$, to a constant $\overline{\xi} \in M$. We also have that

$$\begin{aligned} \dot{v}_n(t) = {} & (1 - \lambda_n)T_1((v_n(t), \xi_n(t)), \varepsilon_n) \\ & + (1 - \lambda_n)T_2(v_n(t), \xi_n(t)) + B(\xi_n(t), \lambda_n)v_n(t), \end{aligned} \tag{6.6}$$

for $t \in \mathbb{R}$. By our assumptions

$$\lim_{\varepsilon \to 0^+} \sup_{(y,x) \in [M]_\eta} |T_1((y, x), \varepsilon)|_Y = 0. \tag{6.7}$$

Since, for each $t \in \mathbb{R}$, $\{ v_n(t) \mid n \in \mathbb{N} \}$ lies in a compact subset of Y, it follows from (6.7), (6.6) and Arzelà–Ascoli Theorem, passing to subsequences if necessary, that $(v_n)_n$ converges in Y, uniformly on compact subsets of $\mathbb{R}$ to a function $v: \mathbb{R} \to Y$ which is is differentiable into Y and

$$\dot{v}(t) = (1 - \lambda)T_2(v(t), \overline{\xi}) + B(\overline{\xi}, \lambda)v(t), \quad t \in \mathbb{R}.$$

It follows from Lemma 6.8 that $v = 0$, a contradiction as $|v(0)|_Y \geq \delta$. This shows that (i) is satisfied.

Now (i) and an application of Proposition 6.7 with $g = h$ shows that there is a subsequence of $(x_n)_n$ which converges in $(\mathcal{M}, d_{\mathcal{M}})$, uniformly on compact subsets of $\mathbb{R}$, to a function $x: \mathbb{R} \to M$ which is differentiable into $\mathcal{M}$ and such that

$$\dot{x}(t) = h_1((0, x(t)), 0), \quad t \in \mathbb{R}.$$

Thus $x \in \mathrm{Sol}(F_0, M)$. This proves (ii). □

Corollary 6.11. *Let N be as in Theorem* 6.3. *Let $\eta' = \eta'(N)$ be as in Proposition* 6.10 *with $M = N$. Then for every $\eta \in \,]0,\eta']$ there is an $\varepsilon_1(\eta) \in \,]0,\overline{\varepsilon}]$ such that for every $\varepsilon \in \,]0,\varepsilon_1(\eta)]$ and for every $\lambda \in [0,1]$ the set $[N]_\eta$ is such that $\operatorname{Inv}_{\widehat{\mathcal{T}}(N,\eta,\varepsilon,\lambda)}([N]_\eta) \subset \operatorname{Int}_{Y\times\mathcal{M}}([N]_\eta)$.*

Proof. If the corollary is not true, then there is an $\eta \in \,]0,\eta']$ and sequences $(\varepsilon_\kappa)_\kappa$ and $(\lambda_\kappa)_\kappa$ in $]0,\overline{\varepsilon}]$ and $[0,1]$ respectively such that $(\varepsilon_\kappa)_\kappa$ converges to zero and $[N]_\eta$ is such that

$$\operatorname{Inv}_{\mathcal{T}_\kappa}([N]_\eta) \not\subset \operatorname{Int}_{Y\times\mathcal{M}}([N]_\eta), \quad \text{for all } \kappa \in \mathbb{N},$$

where $\mathcal{T}_\kappa = \widehat{\mathcal{T}}(N,\eta,\varepsilon_\kappa,\lambda_\kappa)$. Set $\mathcal{T}_0 = \widehat{\mathcal{T}}_0(N)$.

It follows from Proposition 6.10 that $\mathcal{T}_\kappa \to \mathcal{T}_0$. Since $\operatorname{Inv}_{\mathcal{T}_0}([N]_\eta) = \{0\} \times \operatorname{Inv}_{\mathcal{T}}(N) \subset \operatorname{Int}_{Y\times\mathcal{M}}([N]_\eta)$, it follows from [1, Proposition 2.4] that, for all $\kappa \in \mathbb{N}$ large enough, $\operatorname{Inv}_{\mathcal{T}_\kappa}([N]_\eta) \subset \operatorname{Int}_{Y\times\mathcal{M}}([N]_\eta)$, a contradiction which proves the corollary. $\square$

Corollary 6.12. *Let N and $(V_p)_{p\in P}$ be as in Theorem* 6.3. *Let $\eta' = \eta'(N)$ be as in Proposition* 6.10 *with $M = N$. For all $\eta \in \,]0,\infty[$, $\varepsilon \in \,]0,\overline{\varepsilon}]$, $\lambda \in [0,1]$ and every $p \in P$, define*

$$M_{p,\varepsilon,\lambda} := \operatorname{Inv}_{\widehat{\mathcal{T}}(N,\eta,\varepsilon,\lambda)}([V_p]_\eta).$$

Then for every $\eta \in \,]0,\eta']$ there is an $\varepsilon_2(\eta) \in \,]0,\overline{\varepsilon}]$ such that for all $\varepsilon \in \,]0,\varepsilon_2(\eta)]$ and $\lambda \in [0,1]$ the family $(M_{p,\varepsilon,\lambda})_{p\in P}$ is a $\prec$-ordered $\widehat{\mathcal{T}}(N,\eta,\varepsilon,\lambda)$-Morse decomposition and for every $p \in P$, $M_{p,\varepsilon,\lambda} \subset \operatorname{Int}_{Y\times\mathcal{M}}([V_p]_\eta)$.

Proof. If the corollary is not true, then there is an $\eta \in \,]0,\eta']$ and sequences $(\varepsilon_\kappa)_\kappa$ and $(\lambda_\kappa)_\kappa$ in $]0,\overline{\varepsilon}]$ and $[0,1]$ respectively such that $(\varepsilon_\kappa)_\kappa$ converges to zero and, for every $\kappa \in \mathbb{N}$, either the family $(M_{p,\varepsilon_\kappa,\lambda_\kappa})_{p\in P}$ is not a $\prec$-ordered $\widehat{\mathcal{T}}(N,\eta,\varepsilon_\kappa,\lambda_\kappa)$-Morse decomposition or else, for some $p \in P$, the set $[V_p]_\eta$ is such that $M_{p,\varepsilon_\kappa,\lambda_\kappa} \not\subset \operatorname{Int}_{Y\times\mathcal{M}}([V_p]_\eta)$.

For $\kappa \in \mathbb{N}$ set $\mathcal{T}_\kappa = \widehat{\mathcal{T}}(N,\eta,\varepsilon_\kappa,\lambda_\kappa)$. Moreover, set $\mathcal{T}_0 = \widehat{\mathcal{T}}_0(N)$.

Our hypotheses imply that $(\{0\} \times M_p)_{p\in P}$ is a $\prec$-ordered $\mathcal{T}_0$-Morse decomposition. Moreover, for every $p \in P$,

$$\operatorname{Inv}_{\mathcal{T}_0}([V_p]_\eta) = \{0\} \times M_p \subset \operatorname{Int}_{Y\times\mathcal{M}}([V_p]_\eta).$$

Now, by Proposition 6.10, $\mathcal{T}_\kappa \to \mathcal{T}_0$. Therefore, it follows from [2, Theorem 3.3] that, for all $\kappa \in \mathbb{N}$ large enough, the family $(M_{p,\varepsilon_\kappa,\lambda_\kappa})_{p\in P}$ is a $\prec$-ordered $\mathcal{T}_\kappa$-Morse decomposition and, for all $p \in P$, the set $[V_p]_\eta$ is such that $M_{p,\varepsilon_\kappa,\lambda_\kappa} \subset \operatorname{Int}_{Y\times\mathcal{M}}([V_p]_\eta)$, a contradiction which proves the corollary. $\square$

LEMMA 6.13. *Let N be as in Theorem 6.3. Let $\eta'(N)$ be as in Proposition 6.10 with $M = N$. For every $\eta \in]0, \eta'(N)]$, let $\varepsilon_1(\eta)$ be as in Corollary 6.11. Whenever $\eta \in]0, \eta'(N)]$ and $\varepsilon \in]0, \varepsilon_1(\eta)]$, then*

$$h(\widehat{F}_{\varepsilon,1}, [N]_\eta) = \Sigma^k \wedge h(\widehat{F}_\varepsilon, N). \tag{6.8}$$

PROOF. By our assumptions, the map $[0, \overline{\varepsilon}] \times \mathcal{M} \to T(\mathcal{M})$, $(\varepsilon, x) \mapsto \widehat{F}_\varepsilon(x)$, is continuous at $(0, x)$, for each $x \in \mathcal{M}$. Moreover, $\widehat{F}_0 = F_0$. It follows that there an $\widehat{\varepsilon}_1 \in]0, \overline{\varepsilon}]$ such that, for all $\varepsilon \in [0, \widehat{\varepsilon}_1]$, $\sup_{x\in N} |\Gamma^{\mathcal{M}}(\widehat{F}_\varepsilon(x)) - \Gamma^{\mathcal{M}}(F_0(x))|_{\mathbf{E}} < \epsilon(F_0, N)$. Let $\eta \in]0, \eta'(N)]$ and $\varepsilon \in]0, \min(\varepsilon_1(\eta), \widehat{\varepsilon}_1)]$ be arbitrary. Proposition 4.6 implies that $h(\widehat{F}_\varepsilon, N)$ is defined and

$$h(\widehat{F}_\varepsilon, N) = h(F_0, N). \tag{6.9}$$

Moreover, Corollary 6.11 implies that

$$\mathrm{Inv}_{\widehat{\mathcal{T}}(N,\eta,\varepsilon,1)}([N]_\eta) \subset \mathrm{Int}_{Y\times\mathcal{M}}([N]_\eta).$$

Hence $h(\widehat{F}_{\varepsilon,1}, [N]_\eta)$ is defined. Let G be a C^1-vector field on the manifold $\mathcal{M}$ such that

$$\sup_{x\in N} |D^{\mathcal{M}}\mathbf{e}(x).G_1(x) - D^{\mathcal{M}}\mathbf{e}(x).\widehat{F}_1^\varepsilon(x)|_{\mathbf{E}} < \min(\epsilon(\widehat{F}_\varepsilon, N), \epsilon(\widehat{F}_{\varepsilon,1}, [N]_\eta)),$$

where $\widehat{F}_1^\varepsilon$ is the principal part of $\widehat{F}_\varepsilon$. Thus

$$h(\widehat{F}_\varepsilon, N) = h(\pi_G, K_G), \tag{6.10}$$

where π_G is the local (semi)flow on $\mathcal{M}$ generated by G and $K_G = \mathrm{Inv}_{\pi_G}(N)$.

Let G' be the C^1-vector field on the manifold $Y \times \mathcal{M}$ given by $G'(u, x) = (\varepsilon^{-1}\overline{B}u, G(x))$, for $(u, x) \in Y \times \mathcal{M}$ and let $\pi_{G'}$ be the local (semi)flow on $Y \times \mathcal{M}$ generated by G'. It follows that

$$\sup_{(u,x)\in[N]_\eta} |\widehat{\Gamma}(G'(x)) - \widehat{\Gamma}(\widehat{F}_{\varepsilon,1}(x))|_{Y\times\mathbf{E}} < \epsilon(\widehat{F}_{\varepsilon,1}, [N]_\eta)$$

and so

$$h(\widehat{F}_{\varepsilon,1}, [N]_\eta) = h(\pi_{G'}, K_{G'}), \tag{6.11}$$

where $K_{G'} = \mathrm{Inv}_{\pi_{G'}}([N]_\eta)$. Notice that $\pi_{G'} = \pi_\varepsilon \times \pi_G$, where π_ε is the (semi)flow generated by the linear differential equation

$$\varepsilon\dot{y} = \overline{B}y.$$

Since $\overline{B}$ is hyperbolic with Morse-index k, it follows that $\{0\} = \mathrm{Inv}_{\pi_\varepsilon}(D_\eta)$, with $D_\eta = \{y \in Y \mid |y|_Y \le \eta\}$, and $h(\pi_\varepsilon, \{0\}) = \Sigma^k$. Thus

$$h(\pi_{G'}, K_{G'}) = \Sigma^k \wedge h(\pi_G, K_G). \tag{6.12}$$

Now, formulas (6.9), (6.10), (6.11) and (6.12) imply formula (6.8). □

We can now give a

Proof of Theorem 6.3. Let N be as in Theorem 6.3 and $\widehat{\varepsilon}_1$ be as in the proof of Lemma 6.13. Let $\eta'(N)$ and for every $\eta \in]0,\eta'(N)]$ let $\eta_1(\eta)$ be as in Corollary 6.11. Set $\eta_0 = \eta'(N)$ and $\varepsilon_0(\eta) = \min(\varepsilon_1(\eta), \widehat{\varepsilon}_1)$ for $\eta \in]0,\eta_0]$. Let $\eta \in]0,\eta_0]$ and $\varepsilon \in]0,\varepsilon_0(\eta)]$ be arbitrary.

By Corollary 6.11

$$\operatorname{Inv}_{\widehat{\mathcal{T}}(N,\eta,\varepsilon,\lambda)}([N]_\eta) \subset \operatorname{Int}_{Y\times\mathcal{M}}([N]_\eta), \quad \text{for every } \lambda \in [0,1]. \tag{6.13}$$

Now (6.13), (6.5) and Corollary 4.7 imply

$$h(F_\varepsilon, [N]_\eta) = h(\widehat{F}_{\varepsilon,0}, [N]_\eta) = h(\widehat{F}_{\varepsilon,1}, [N]_\eta).$$

Lemma 6.13 implies that $h(\widehat{F}_{\varepsilon,1}, [N]_\eta) = \Sigma^k \wedge h(F_0, N)$ so

$$h(F_\varepsilon, [N]_\eta) = \Sigma^k \wedge h(F_0, N).$$

This proves the first part of Theorem 6.3.

Now let M_p and V_p, $p \in P$ be as in Theorem 6.3. There is an $\widehat{\varepsilon}_2 \in]0,\overline{\varepsilon}]$ with $\widehat{\varepsilon}_2 \le \widehat{\varepsilon}_1$ such that, for $\varepsilon \in]0,\widehat{\varepsilon}_2]$, $\sup_{x\in N} |\Gamma^{\mathcal{M}}(\widehat{F}_\varepsilon(x)) - \Gamma^{\mathcal{M}}(F_0(x))|_{\mathbf{E}} < \epsilon(F_0, N, (V_p)_{p\in P})$. For $\eta \in]0,\eta_0]$ let $\varepsilon_2(\eta)$ be as in Corollary 6.12.

Set $\overline{\varepsilon}_0(\eta) = \min(\varepsilon_0(\eta), \varepsilon_2(\eta), \widehat{\varepsilon}_2)$, $\eta \in]0,\eta_0]$. Let $\eta \in]0,\eta_0]$ and $\varepsilon \in]0,\overline{\varepsilon}_0(\eta)]$ be arbitrary.

Proposition 5.6 implies that $\overline{\mathcal{H}}(\widehat{F}_\varepsilon, N, (V_p)_{p\in P})$ and $\overline{\mathcal{CH}}(\widehat{F}_\varepsilon, N, (V_p)_{p\in P})$ are defined and

$$\begin{aligned} \overline{\mathcal{H}}(\widehat{F}_\varepsilon, N, (V_p)_{p\in P}) &= \overline{\mathcal{H}}(F_0, N, (V_p)_{p\in P}) \\ \overline{\mathcal{CH}}(\widehat{F}_\varepsilon, N, (V_p)_{p\in P}) &= \overline{\mathcal{CH}}(F_0, N, (V_p)_{p\in P}). \end{aligned} \tag{6.14}$$

Using (6.5) together with Corollary 6.12 and Corollary 5.7 we see that

$$\overline{\mathcal{H}}(F_\varepsilon, [N]_\eta, ([V_p]_\eta)_{p\in P}), \overline{\mathcal{CH}}(F_\varepsilon, [N]_\eta, ([V_p]_\eta)_{p\in P}),$$

$$\overline{\mathcal{H}}(\widehat{F}_{\varepsilon,1}, [N]_\eta, ([V_p]_\eta)_{p\in P}) \text{ and } \overline{\mathcal{CH}}(\widehat{F}_{\varepsilon,1}, [N]_\eta, ([V_p]_\eta)_{p\in P})$$

are defined and

$$\begin{aligned} \overline{\mathcal{H}}(F_\varepsilon, [N]_\eta, ([V_p]_\eta)_{p\in P}) &= \overline{\mathcal{H}}(\widehat{F}_{\varepsilon,1}, [N]_\eta, ([V_p]_\eta)_{p\in P}), \\ \overline{\mathcal{CH}}(F_\varepsilon, [N]_\eta, ([V_p]_\eta)_{p\in P}) &= \overline{\mathcal{CH}}(\widehat{F}_{\varepsilon,1}, [N]_\eta, ([V_p]_\eta)_{p\in P}). \end{aligned} \tag{6.15}$$

Let $\widetilde{\epsilon} := \min(\epsilon(\widehat{F}_\varepsilon, N, (V_p)_{p\in P}), \epsilon(\widehat{F}_{\varepsilon,1}, [N]_\eta, ([V_p]_\eta)_{p\in P})$ and G be a C^1-vector field on the manifold $\mathcal{M}$ such that

$$\sup_{x\in N} |D^{\mathcal{M}}\mathbf{e}(x).G_1(x) - D^{\mathcal{M}}\mathbf{e}(x).\widehat{F}_1^\varepsilon(x)|_{\mathbf{E}} < \widetilde{\epsilon},$$

where $\widehat{F}_1^\varepsilon$ is the principal part of $\widehat{F}_\varepsilon$. Thus

$$\begin{aligned} \overline{\mathcal{H}}(F_0, N, (V_p)_{p\in P}) &= [\mathcal{H}(\pi_G, K_G, (M_{p,G})_{p\in P})], \\ \overline{\mathcal{CH}}(F_0, N, (V_p)_{p\in P}) &= [\mathcal{CH}(\pi_G, K_G, (M_{p,G})_{p\in P})], \end{aligned} \tag{6.16}$$

where π_G is the local (semi)flow on $\mathcal{M}$ generated by G and $K_G = \mathrm{Inv}_{\pi_G}(N)$ and $M_{p,G} = \mathrm{Inv}_{\pi_G}(V_p)$, $p \in P$.

Let G' be the C^1-vector field on the manifold $Y \times \mathcal{M}$ given by $G'(u,x) = (\varepsilon^{-1}\overline{B}u, G(x))$, for $(u,x) \in Y \times \mathcal{M}$ and let $\pi_{G'}$ be the local (semi)flow on $Y \times \mathcal{M}$ generated by G'. It follows that

$$\sup_{(u,x)\in[N]_\eta} |\widehat{\Gamma}(G'(x)) - \widehat{\Gamma}(\widehat{F}_{\varepsilon,1}(x))|_{Y\times\mathbf{E}} < \epsilon(\widehat{F}_{\varepsilon,1}, [N]_\eta\,, ([V_p]_\eta)_{p\in P})$$

and so

$$\begin{aligned} \overline{\mathcal{H}}(\widehat{F}_{\varepsilon,1}, [N]_\eta\,, ([V_p]_\eta)_{p\in P}) &= [\mathcal{H}(\pi_{G'}, K_{G'}, (M_{p,G'})_{p\in P})], \\ \overline{\mathcal{CH}}(\widehat{F}_{\varepsilon,1}, [N]_\eta\,, ([V_p]_\eta)_{p\in P}) &= [\mathcal{CH}(\pi_{G'}, K_{G'}, (M_{p,G'})_{p\in P})], \end{aligned} \tag{6.17}$$

where $K_{G'} = \mathrm{Inv}_{\pi_{G'}}([N]_\eta)$ and $M_{p,G'} = \mathrm{Inv}_{\pi_{G'}}(V_p)$, $p \in P$. Notice that $\pi_{G'} = \pi_\varepsilon \times \pi_G$, where π_ε is the (semi)flow generated by the linear differential equation

$$\varepsilon\dot{y} = \overline{B}y.$$

The (semi)flow π_ε is clearly conjugate to the product semiflow $\pi_\varepsilon^- \times \pi_\varepsilon^+$ where π_ε^- resp. π_ε^+ is the (semi)flow on a finite-dimensional normed space Y^- resp. Y^+ generated by the linear differential equation

$$\varepsilon\dot{y} = B^-y \quad \text{resp. } \varepsilon\dot{y} = B^+y$$

where $B^- \in \mathcal{L}(Y^-, Y^-)$ resp. $B^+ \in \mathcal{L}(Y^+, Y^+)$ is a linear operator with all eigenvalues having negative resp. positive real parts. Thus $\pi_{G'}$ is conjugate to the (semi)flow $(\pi_G \times \pi_\varepsilon^+) \times \pi_\varepsilon^-$. Now, [5, Theorem 2.2] implies that the (co)homology index braid of $(\pi_{G'}, K_{G'}, (M_{p,G'})_{p\in P})$ is isomorphic to the (co)homology index braid of $(\pi_G \times \pi_\varepsilon^+, K_G \times \{0_{Y^+}\}, (M_{p,G} \times \{0_{Y^+}\})_{p\in P})$.

Since $k = \dim Y^+$, an application of [4, Theorem 3.1] and [11, Theorem 4.1] implies

$$\begin{aligned} \mathcal{H}(\pi_{G'}, K_{G'}, (M_{p,G'})_{p\in P}) &= \mathcal{H}_k(\pi_G, K_G, (M_{p,G})_{p\in P}), \\ \mathcal{CH}(\pi_{G'}, K_{G'}, (M_{p,G'})_{p\in P}) &= \mathcal{CH}^k(\pi_G, K_G, (M_{p,G})_{p\in P}). \end{aligned} \tag{6.18}$$

Now, formulas (6.14)–(6.18) imply

$$\begin{aligned} \overline{\mathcal{H}}(F_\varepsilon, [N]_\eta\,, ([V_p]_\eta)_{p\in P}) &= [\mathcal{H}_k(\pi_G, K_G, (M_{p,G})_{p\in P})] = \overline{\mathcal{H}}_k(F_0, N, (V_p)_{p\in P}), \\ \overline{\mathcal{CH}}(F_\varepsilon, [N]_\eta\,, ([V_p]_\eta)_{p\in P}) &= [\mathcal{CH}^k(\pi_G, K_G, (M_{p,G})_{p\in P})] = \overline{\mathcal{CH}}^k(F_0, N, (V_p)_{p\in P}). \end{aligned}$$

The theorem is proved. □

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