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A DIRECT PROOF OF EXISTENCE OF WEAK SOLUTIONS TO ELLIPTIC PROBLEMS

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Abstract. We provide a direct proof of existence and uniqueness of weak solutions to a broad family of strongly nonlinear elliptic equations with lower-order terms. The leading part of the operator satisfies general growth conditions settling the problem in the framework of fully anisotropic and inhomogeneous Musielak–Orlicz spaces generated by an N-function $M\colon \Omega\times\mathbb{R}^d\to[0,\infty)$. Neither ∇_2 nor Δ_2 conditions are imposed on M. Our results cover among others problems with anisotropic polynomial, Orlicz, variable exponent, and double phase growth.

1. Introduction

In this paper we investigate the following strongly nonlinear problem

$$\begin{cases} -\operatorname{div}\big(\mathcal{A}(x,\nabla u)+\Phi(u)\big)+b(x,u)=\operatorname{div}F & \text{in } \Omega,\\ u(x)=0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded Lipschitz domain in $\mathbb{R}^d$, $d>1$. The growth and coercivity of the vector field $\mathcal{A}$ are assumed to be controlled by a generalized anisotropic N-function $M\colon\Omega\times\mathbb{R}^d\to[0,\infty)$, $\Phi\in L^\infty\big(\Omega,\mathbb{R}^d\big)$ is a continuous vector field, b is a function satisfying a sign condition without any extra growth conditions

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