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# ON A CLASS OF HAUSDORFF MEASURE OF CARTESIAN PRODUCT SETS IN METRIC SPACES

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Abstract. In this paper we study, in a separable metric space, a class of Hausdorff measures $\mathcal{H}_\mu^{q,\xi}$ defined using a measure $\mu$ and a premeasure $\xi$. We discuss a Hausdorff structure of product sets. Weighted Hausdorff measures $\mathcal{W}_\mu^{q,\xi}$ appear as an important tool when studying the product sets. When $\mu$ and $\xi$ satisfy the doubling condition, we prove that $\mathcal{H}_\mu^{q,\xi} = \mathcal{W}_\mu^{q,\xi}$. As an application, the case where $\xi$ is defined as the Hausdorff function is considered.

## 1. Introduction

Let $(\mathbb{X}, \rho)$ and $(\mathbb{X}', \rho')$ be two separable metric spaces and let $\mathcal{P}(\mathbb{X})$ denote the family of Borel probability measures on $\mathbb{X}$. For $\mu \in \mathcal{P}(\mathbb{X})$ and $a > 1$, we write

$$P_a(\mu) = \limsup_{r \searrow 0} \left( \sup_{x \in \operatorname{supp} \mu} \frac{\mu(B(x, ar))}{\mu(B(x, r))} \right).$$

Now, we say that the measure $\mu$ satisfies the doubling condition if there exists $a > 1$ such that $P_a(\mu) < \infty$. It is easily seen that the exact value of the parameter $a$ is unimportant since $P_a(\mu) < \infty$, for some $a > 1$ if and only if $P_a(\mu) < \infty$, for all $a > 1$. Also, we denote by $\mathcal{P}_D(\mathbb{X})$ the family of Borel probability measures on $\mathbb{X}$ which satisfy the doubling condition. As classical examples of measures

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