Topological **M**ethods in **N**onlinear **A**nalysis
Volume 61, No. 2, 2023, 591–610
DOI: 10.12775/TMNA.2021.058

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REALIZATION OF A GRAPH AS THE REEB GRAPH OF A HEIGHT FUNCTION ON AN EMBEDDED SURFACE

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Abstract. We show that for a given finite graph G without loop edges and isolated vertices, there exists an embedding of a closed orientable surface in $\mathbb{R}^3$ such that the Reeb graph of the associated height function has the structure of G. In particular, this gives a positive answer to the corresponding question posed by Masumoto and Saeki in 2011. We also give a criterion for a given surface to admit such a realization of a given graph, and study the problem in the class of Morse functions and in the class of round Morse–Bott functions. In the case of realization up to homeomorphism, the height function can be chosen Morse–Bott; we estimate from below the number of its critical circles and the number of its isolated critical points in terms of the graph structure.

1. Introduction and main results

A continuous function $f\colon X \to \mathbb{R}$ defines the *Reeb graph* R_f [16] as a topological space obtained by contracting the contours (connected components of the level sets) of f to points, endowed with the quotient topology. For a smooth function f with a finite number of critical values on a closed manifold M, its Reeb graph R_f has the structure of a finite graph G (considered as a 1-dimensional CW complex, i.e. allowing multiple edges and loop edges), with vertices corresponding

2020 *Mathematics Subject Classification.* Primary: 57R40, 53A05; Secondary: 58K05, 57R45.

Key words and phrases. Reeb graph; height function; Morse–Bott function; orientable surface; embedding.