EMBEDDABILITY OF JOINS AND PRODUCTS OF POLYHEDRA

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ABSTRACT. We present a short proof of S. Parsa’s theorem that there exists a compact \( n \)-polyhedron \( P \), \( n \geq 2 \), non-embeddable in \( \mathbb{R}^{2n} \), such that \( P \ast P \) embeds in \( \mathbb{R}^{4n+2} \). This proof can serve as a showcase for the use of geometric cohomology. We also show that a compact \( n \)-polyhedron \( X \) embeds in \( \mathbb{R}^m \), \( m \geq 3(n+1)/2 \), if either

- \( X \ast K \) embeds in \( \mathbb{R}^{m+2k} \), where \( K \) is the \((k-1)\)-skeleton of the \(2k\)-simplex; or
- \( X \ast L \) embeds in \( \mathbb{R}^{m+2k} \), where \( L \) is the join of \( k \) copies of the 3-point set; or
- \( X \) is acyclic and \( X \times (\text{triod})^k \) embeds in \( \mathbb{R}^{m+2k} \).

1. Introduction

It was shown by Flores, van Kampen and Grünbaum [9] that every \( n \)-dimensional join of \( k_i \)-skeleta of \((2k_i + 2)\)-simplexes does not embed into \( \mathbb{R}^{2n} \) (see also [11, Examples 3.3, 3.5], [12], [20]). Some other \( k_i \)-polyhedra with this property are constructed in [12].

As noted by S. Parsa [15], it is implicit in a paper by Bestvina, Kapovich and Kleiner [5] that if compact polyhedra \( P^m \) and \( Q^m \) both have non-zero mod 2 van Kampen obstruction, then \( P \ast Q \) does not embed in \( \mathbb{R}^{2(n+m+1)} \). An \( n \)-dimensional polyhedron, non-embeddable in \( \mathbb{R}^{2n} \) but with vanishing mod 2 van Kampen obstruction was constructed by the author for each \( n \geq 2 \) [11], settling

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